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State-Feedback Control Design for Polynomial Discrete-Time Systems obtained via Second-Order Runge-Kutta Discretization ⋆

Priscila F. S. Guedes * **Márcio J. Lacerda** ** **Erivelton Nepomuceno** ***

* *Control and Modelling Group (GCOM), Department of Electrical Engineering, Federal University of São João del-Rei - UFSJ, São João del-Rei, MG, Brazil.*
E-mail: pri12_guedes@hotmail.com
** *Centre for Communications Technology School of Computing and Digital Media London Metropolitan University, UK.*
E-mail: m.lacerda@londonmet.ac.uk
*** *Centre for Ocean Energy Research Department of Electronic Engineering Maynooth University, Ireland.*
E-mail: erivelton.nepomuceno@mu.ie

Abstract: This paper addresses the state-feedback control problem for the class of state-polynomial discrete-time systems. The continuous-time polynomial nonlinear model is discretized by the second-order Runge-Kutta method. The Lyapunov theory and the exponential stability were employed to derive the conditions. The sum of squares formulation was used to check the constraints. Two approaches are presented, the first makes use of the Lyapunov function to recover the gain matrices. While the second formulation allows the design of rational state feedback control gains. We evaluated the impact of the step size used in the discretization process in the results. Numerical experiments were used to illustrate the potential of the proposed technique.

Keywords: State-feedback control, State polynomial systems, Second-order Runge-Kutta method, Computer arithmetic.

1. INTRODUCTION

A great variety of dynamical systems can be represented by continuous-time nonlinear models, that is, they can be written by a set of differential equations. For a better understanding of their behaviour and computational analysis, it is necessary to discretize these continuous-time models, transforming them into discrete-time systems (Ardourel and Jebeile, 2017; Hammel et al., 1987; Sauer et al., 1997; Galias, 2013; Lozi, 2013; Zhuang et al., 2018). There are several discretization methods, with methods based on the expansion of the Taylor series being one of the widely used, such as the Runge-Kutta family methods (Butcher and Goodwin, 2008; Quarteroni et al., 2010).

Depending on the continuous-time model to be discretized, nonlinear systems can arise. Concerning nonlinear systems, there are several tools to provide stability certificates and control design conditions (Khalil, 2002; Vidyasagar, 1993; Takagi and Sugeno, 1985). When nonlinear systems that depend polynomially on the states are studied, techniques based on the sum of squares (SOS) method (Papachristodoulou and Prajna, 2002, 2005) can be employed. The sum of squares formulation has been extensively used to tackle different problems such as estimating the region of attraction of polynomial systems (Topcu et al., 2010; Valmorbida and Anderson, 2017), control design (Nasiri et al., 2018; Ferreira et al., 2020), filter design Lacerda et al. (2015), and robust stability analysis Prajna et al. (2005) for instance.

In Gui et al. (2019), a class of chaotic systems under a state-feedback controller is described as a polynomial model, and the optimal control problem for the class of chaotic systems is transformed into a state-dependent linear matrix inequality, a viable solution to this problem is the application the sum-of-squares programming method. SOS is also used in Ramos et al. (2018), wherein the chaotic Lorenz system with parametric uncertainties was used for the switched control design to choose a polynomial state feedback gain that minimizes the time derivative of a polynomial Lyapunov function.

In Ebenbauer and Allgöwer (2006), the use of semidefinite programming and the sum of squares decomposition is used to solve the obtained stability analysis and control design dissipation inequalities in a numerically reliable and efficient way. In Saat et al. (2012) a state feedback controller with an integrator was proposed to stabilize discrete-time polynomial systems with norm-bounded uncertainties. The state-feedback control problem for state-polynomial discrete-time linear parameter varying systems is addressed in Lacerda et al. (2022).

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This article proposes new conditions for the design of state-feedback controllers for state-polynomial discrete-time systems. A continuous-time model is considered and then discretized via the second-order Runge-Kutta method. The original model is converted into a discrete-time system that depends on the step size employed in the discretization process. The polynomial discrete-time system is used to design a state-feedback controller that will be able to stabilize the closed-loop system. The exponential stability is considered to derive the design conditions and to measure the impact of the discretization step size on the performance of the closed-loop system. Two formulations are introduced. The first formulation makes use of the Lyapunov matrix to recover the state-feedback gain generating a polynomial state-feedback control law. On the other hand, the second formulation proposed in this paper uses a slack variable to design the state-feedback controller that may be rational on the states. Numerical experiments are employed to illustrate the influence of the discretization step size in the exponential stability of the closed-loop system.

The paper is organized as follows. The preliminaries are presented in Section 2. The main results are developed in Section 3. Section 4 illustrates the effectiveness of the proposed approach through a numerical experiment, and Section 5 concludes the paper.

Notation: The set $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and the set $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ matrices with real entries. The operator $diag(A,B)$ indicates a block diagonal matrix composed of matrices A and B. $M > (<)\ 0$ indicates a positive (negative) definite matrix. The symbol $(^T)$ indicates transpose and $\star$ represents a block induced by symmetry in a symmetric matrix. $\Sigma[x]$ is the set of sum of squares polynomials in variable x.

2. PRELIMINARIES

2.1 Second-Order Runge-Kutta Method (RK2)

Runge-Kutta methods are a family of iterative methods used as approximations for solutions of ordinary differential equations (Butcher and Goodwin, 2008). It is particularly useful for problems where analytical solutions are challenging or impossible to obtain. Moreover, it allows for the step-by-step approximation of the solution, breaking down the continuous problem into discrete steps that can be solved by computers. Consider an initial value problem described as

$$\dot{x} = f(x), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, and $f(x) \in \mathbb{R}^n$ is the vector field. The RK2 method uses an iterative process to estimate the value of x at discrete time points based on the given initial condition. The step-size, denoted as h (a small positive value), determines the size of each time increment.

$$x_{k+1} = x_k + \frac{h}{2}\left(K_1 + K_2\right), \tag{2}$$

where

$$K_1 = f(x_k),$$
$$K_2 = f(x_k + hK_1).$$

To better illustrate the idea behind the RK2 method, a simple example is given.

Example 1. Consider the following continuous-time system:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x^2 + y. \end{cases} \Longrightarrow \begin{cases} \dot{x} = f(x,y) = y, \\ \dot{y} = g(x,y) = x^2 + y. \end{cases} \tag{3}$$

For K_1 we have:

$$K_{1_x} = f(x_k, y_k) = y_k,$$
$$K_{1_y} = g(x_k, y_k) = x_k^2 + y_k.$$

Likewise, for K_2 we have:

$$\begin{aligned} K_{2_x} &= f(x_k + hk_{1_x}, y_k + hk_{1_y}) \\ &= f(x_k + hy_k, y_k + h(x_k^2 + y_k)) \\ &= y_k + h(x_k^2 + y_k), \\ K_{2_y} &= g(x_k + hk_{1_x}, y_k + hk_{1_y}) \\ &= g(x_k + hy_k, y_k + h(x_k^2 + y_k)) \\ &= (x_k + hy_k)^2 + y_k + h(x_k^2 + y_k) \\ &= x_k^2 + 2hx_ky_k + h^2y_k^2 + y_k + hx_k^2 + hy_k. \end{aligned}$$

Therefore, using equation (2), the discretization of the continuous time system using the RK2 method can be given by

$$\begin{cases} x_{k+1} = x_k + hy_k + \frac{1}{2}h^2(x_k^2 + y_k), \\ y_{k+1} = y_k + h(x_k^2 + y_k) + \frac{1}{2}h^2(x_k^2 + y_k) + h^2x_ky_k + \frac{1}{2}h^3y_k^2. \end{cases}$$

Remark 1. Note that the discretization process via RK2 of the continuous-time polynomial system given in (3), also results in a state polynomial system in discrete-time. Moreover, the discretization step-size h appears in the equations of the system.

Remark 2. It is important to emphasize that the equation presented in (2) is general to represent the continuous-time system and its respective discretization by the second-order Runge-Kutta method. Example 1 shows equation (2) for a specific case in which there was a need to adapt the general case to this example of two variables. For this example, there is the discretization for the variables x and y, with their respective Ks that depend on system (3) as a whole.

In addition to the second-order Runge-Kutta, there are other discretization methods, such as the third-order Runge-Kutta and the fourth-order Runge-Kutta method (Quarteroni and Saleri, 2006; Quarteroni et al., 2010; Hussain et al., 2016; Kennedy and Carpenter, 2019), which will be described next.

Considering the initial value problem (1), the third-order Runge-Kutta (RK3) that does not depend explicitly on time can be expressed by:

$$x_{k+1} = x_k + \frac{h}{6}\left(K_1 + 4K_2 + K_3\right), \tag{4}$$

where

$$K_1 = f(x_k),$$
$$K_2 = f(x_k + \frac{1}{2}hK_1),$$
$$K_3 = f(x_k + 2hK_2 - hK_1).$$

For the fourth-order Runge-Kutta (RK4), the discretized system reads

$$x_{k+1} = x_k + \frac{h}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right), \tag{5}$$

where

$$\begin{aligned} K_1 &= f(x_k), \\ K_2 &= f(x_k + \frac{h}{2}K_1), \\ K_3 &= f(x_k + \frac{h}{2}K_2), \\ K_4 &= f(x_k + hK_3). \end{aligned} \quad (6)$$

Remark 3. For the system presented in Example 1, the discretization process with RK3 and RK4 would result in a greater number of terms for the discrete-time system, when compared with RK2.

2.2 System Model

Consider the following state-polynomial system obtained from the discretization of the continuous-time model through the RK2

$$x_{k+1} = A(x_k)x_k + B(x_k)u_k, \quad (7)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^{n_u}$ is the control input and k is an integer indicating the time instant. The matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n_u}$ contain elements that depend polynomially on the state variables.

The main objective of this paper is to design a control law ensuring the exponential stability of the closed-loop system. To achieve this end, the following polynomial state-feedback control law is considered

$$u_k = K(x_k)x_k, \quad (8)$$

where $K(x_k) \in \mathbb{R}^{n_u\times n}$ is a polynomial matrix. Taking into account the state-feedback controller (8) in the system (7), the closed-loop system is

$$x_{k+1} = \widetilde{A}(x_k)x_k, \quad (9)$$

with $\widetilde{A}(x_k) = A(x_k) + B(x_k)K(x_k)$.

In this paper, the conditions will be derived from the Lyapunov theory and SOS certificates will be employed to guarantee the nonnegativeness of the polynomial constraints. A multivariable polynomial $F(x_1,x_2,\dots,x_n)$ of degree $2d$ is SOS, if it can be written according to

$$F(x_1,x_2,\dots,x_n) = \sum_{i=1}^{m} f_i^2(x_1,x_2,\dots,x_n), \quad (10)$$

where each polynomial $f_i(x_1,x_2,\dots,x_n)$ has degree lower or equal to d. Equation (10) is semi-positive definite and can be written as

$$F(x) = z^T Q z, \quad (11)$$

where z is a vector containing monomials of degree up to d of $(x_1,x_2,\dots,x_n)$.

3. MAIN RESULTS

A sufficient condition to design a state-feedback controller that depends polynomially on the states is presented in the next Theorem.

Theorem 1. If there exist matrices $P \in \mathbb{R}^{n\times n}$, polynomial matrices $Z(x_k) \in \mathbb{R}^{n_u\times n}$, and a positive scalar γ such that

$$M - \varepsilon I \in \Sigma[x],$$

holds with

$$M = \begin{bmatrix} \gamma^2 P & PA(x_k)^T + Z(x_k)^T B(x_k)^T \\ \star & P \end{bmatrix}, \quad (12)$$

then, the discrete-time state-polynomial closed-loop system (9) is exponentially stable and the polynomial state-feedback controller is given by

$$K(x_k) = Z(x_k)P^{-1}. \quad (13)$$

Proof. Consider the Lyapunov function $V(x_k) = x_k^T P^{-1} x_k$. To guarantee the exponential stability one must ensure that $\Delta V < (\gamma^2 - 1)V(x_k)$, or simply

$$V(x_{k+1}) - V(x_k) < (\gamma^2 - 1)V(x_k),$$

that reads

$$V(x_{k+1}) - \gamma^2 V(x_k) < 0,$$

or

$$x_{k+1}^T P^{-1} x_{k+1} - x_k^T \gamma^2 P^{-1} x_k < 0.$$

Considering the dynamics of the closed-loop system (9) one has,

$$x_k^T \widetilde{A}(x_k)^T P^{-1} \widetilde{A}(x_k) x_k - x_k^T \gamma^2 P^{-1} x_k < 0,$$

that is equivalent to

$$\gamma^2 P^{-1} - \widetilde{A}(x_k)^T P^{-1} \widetilde{A}(x_k) > 0.$$

By applying the Schur complement yields

$$\begin{bmatrix} \gamma^2 P^{-1} & \widetilde{A}(x_k)^T P^{-1} \\ \star & P^{-1} \end{bmatrix} > 0.$$

Pre- and post-multiplying the last condition by $diag(P,P)$ results in

$$\begin{bmatrix} \gamma^2 P & P\widetilde{A}(x_k)^T \\ \star & P \end{bmatrix} > 0.$$

Considering the change of variables $Z(x_k) = K(x_k)P$ one gets

$$\begin{bmatrix} \gamma^2 P & PA(x_k)^T + Z(x_k)^T B(x_k)^T \\ \star & P \end{bmatrix} > 0.$$

As the last constraint contains polynomial matrices in terms of the state variables, the SOS technique is employed to test the conditions. In this sense, the last condition is changed by the one presented in Theorem 1.

Remark 4. The main drawback with the conditions proposed in Theorem 1 is the fact that the state-feedback controller is recovered from the Lyapunov matrix that does not depend on the state vector x_k. However, the gain still is a polynomial function of the state vector x_k, once the matrix $Z(x_k)$ is also employed to recover it.

The next result presents a condition that does not employ the Lyapunov matrix to recover the controller and allows the design of controllers that depend rationally on x_k.

Theorem 2. If there exist matrices $P \in \mathbb{R}^{n\times n}$, polynomial matrices $Z(x_k) \in \mathbb{R}^{n_u\times n}$, $X(x_k) \in \mathbb{R}^{n\times n}$, and a positive scalar γ such that

$$\Psi - \varepsilon I \in \Sigma[x],$$

holds with

$$\Psi = \begin{bmatrix} X(x_k) + X(x_k)^T - P & X(x_k)^T A(x_k)^T + Z(x_k)^T B(x_k)^T \\ \star & \gamma^2 P \end{bmatrix}, \quad (14)$$

then, the discrete-time state-polynomial system (9) is exponentially stable, and the rational state-feedback controller is given by

$$K(x_k) = Z(x_k)X(x_k)^{-1}. \quad (15)$$

Proof. Consider the Lyapunov function $V(x_k) = x_k^T P^{-1} x_k$. To guarantee the exponential stability, one must ensure that $\Delta V <$

$(\gamma^2-1)V(x_k)$. Considering the dynamics of the closed-loop system (9) one has,

$$\gamma^2 P^{-1} - \widetilde{A}(x_k)^T P^{-1}\widetilde{A}(x_k) > 0.$$

Which can be rewritten as

$$P^{-1} - \widetilde{A}(x_k)^T P^{-1}\gamma^{-2}PP^{-1}\widetilde{A}(x_k) > 0.$$

Applying Schur complement yields

$$\begin{bmatrix} P^{-1} & \widetilde{A}(x_k)^T P^{-1} \\ \star & \gamma^2 P^{-1} \end{bmatrix} > 0,$$

Pre- and post-multiplying the last inequality by $diag(X(x_k)^T, P)$ and $diag(X(x_k), P)$ results in

$$\begin{bmatrix} X(x_k)^T P^{-1} X(x_k) & X(x_k)^T \widetilde{A}(x_k)^T \\ \star & \gamma^2 P \end{bmatrix} > 0,$$

By exploiting the inequality

$$X(x_k)^T P^{-1} X(x_k) \geq X(x_k) + X(x_k)^T - P,$$

and considering the change of variables $Z(x_k) = K(x_k)X(x_k)$ one gets

$$\begin{bmatrix} X(x_k) + X(x_k)^T - P & X(x_k)^T A(x_k)^T + B(x_k)^T Z(x_k)^T \\ \star & \gamma^2 P \end{bmatrix} > 0.$$

Then the proof is complete, and the controller gain can be recovered by (15). Following the same strategy employed in Theorem 1, the SOS formulation is used to provide a nonnegative certificate for the design condition.

Remark 5. Both Theorem 1 and Theorem 2 employ a constant matrix P, this is to guarantee that the Lyapunov function $V(x_k) = x_k^T P^{-1} x_k$ is radially unbounded. If a polynomial matrix $P(x_k)$ is employed, it restricts the conditions to a local context.

4. NUMERICAL EXPERIMENTS

To illustrate the potential of the proposed method some numerical experiments are considered. The routines were implemented in Matlab R2014a using the SOSTOOLS (Papachristodoulou et al., 2013) and the solver SeDuMi (Sturm, 1999).

Consider the Sprott F system (Sprott, 1994)

$$\begin{cases} \dot{x}_1 = x_2 + x_3, \\ \dot{x}_2 = -x_1 + \frac{1}{2}x_2, \\ \dot{x}_3 = -x_1^2 - x_3. \end{cases} \quad (16)$$

The discretization of the continuous-time system (16) using the second-order Runge-Kutta method gives

$$\begin{aligned} x_{1_{k+1}} &= x_{1_k} + hx_{2_k} + hx_{3_k} - \frac{1}{2}h^2(x_{1_k} + x_{3_k} - x_{1_k}^2) - \frac{1}{4}h^2 x_{2_k}, \\ x_{2_{k+1}} &= x_{2_k} - hx_{1_k} - \frac{1}{2}h^2(x_{2_k} + x_{3_k}) - \frac{3}{8}h^2 x_{2_k} + \frac{1}{4}h^2 x_{1_k}, \quad (17) \\ x_{3_{k+1}} &= x_{3_k} + hx_{1_k}^2 - hx_{3_k} + \frac{1}{2}h^3 x_{2_k}^2 + h^3 x_{2_k}x_{3_k} + \frac{1}{2}h^3 x_{3_k}^2 \\ &\quad - \frac{1}{2}h^2 x_{1_k}^2 + h^2 x_{1_k}x_{2_k} + h^2 x_{1_k}x_{3_k} + \frac{1}{2}h^2 x_{3_k}. \end{aligned}$$

A possible state-space representation for the discretized system (17) is

$$x_{k+1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} x_k \quad (18)$$

where the elements a_{ij} are given as follows

$$\begin{aligned} a_{11} &= 1 - \frac{1}{2}h^2 + \frac{1}{2}h^2 x_{1_k}, \\ a_{12} &= h - \frac{1}{4}h^2, \\ a_{13} &= h - \frac{1}{2}h^2, \\ a_{21} &= -h + \frac{1}{4}h^2, \\ a_{22} &= 1 - \frac{1}{2}h - \frac{3}{8}h^2, \\ a_{23} &= -\frac{1}{2}h^2, \\ a_{23} &= hx_{1_k} - \frac{1}{2}h^2 x_{1_k}, \\ a_{23} &= \frac{1}{2}h^3 x_{2_k} + h^2 x_{1_k}, \\ a_{23} &= 1 - h + h^3 x_{2_k} + \frac{1}{2}h^3 x_{3_k} + h^2 x_{1_k} + \frac{1}{2}h^2. \end{aligned} \quad (19)$$

Figure 1 shows the trajectory of x_{1_k} for the open-loop system, considering different step sizes with initial condition $x_0 = [0.1\ 0.1\ 0.1]^T$. It is possible to observe that after some time, the trajectories present different behavior, this divergence is caused by the accumulated error, because of the finite precision of the computer. The difference in colors in the x_1 curve highlights the difference in the trajectory with the change in the step size.

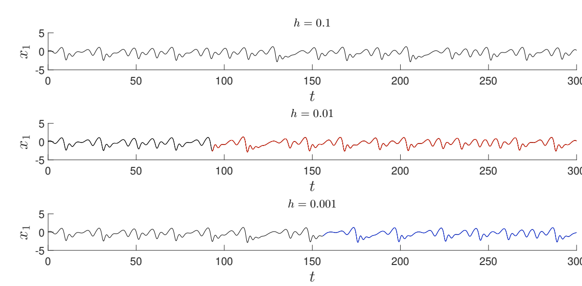

Fig. 1. Trajectory of x_{1_k} for the open-loop system, different step sizes h, and initial condition $x_0 = [0.1\ 0.1\ 0.1]^T$.

To test the conditions, the matrix $B = [0\ 1\ 0]^T$ is considered. By using Theorem 1 with $\varepsilon = 10^{-5}$ a polynomial state-feedback controller that stabilizes the system can be obtained. Table 1 shows the results obtained for γ^2, considering different degrees for the matrix $Z(x_k)$. It was possible to verify that as the degree of $Z(x_k)$ increases, we can find smaller values for γ^2 when considering $h = 0.1$. However, the degree of the polynomial matrix $Z(x_k)$ did not influence the results for $h = 0.01$ and $h = 0.001$.

Figure 2 depicts the trajectories for the second state of the closed-loop system when considering $\gamma^2 = 0.85$ and $h = 0.1$ for matrix $Z(x_k)$ up the degree four. In this case, 100 randomly generated initial conditions in the interval $[-15, 15]$ were considered. It can be seen that all the trajectories converge to the origin.

When considering an initial condition $[15\ -8\ 12]^T$, the trajectory of the third state of the closed-loop system is depicted in Figure 3, for $Z(x_k)$ up to degree four and different values of

Table 1. γ^2 variation for different degrees of the polynomial matrix $Z(x_k)$ and different step sizes h.

Step size (h)	Degree of polynomial matrix $Z(x_k)$		
	$[0:4]$	$[0:5]$	$[0:6]$
0.1	$0.85 \leq \gamma^2 \leq 1$	$0.84 \leq \gamma^2 \leq 1$	$0.83 \leq \gamma^2 \leq 1$
0.01	$0.99 \leq \gamma^2 \leq 1$	$0.99 \leq \gamma^2 \leq 1$	$0.99 \leq \gamma^2 \leq 1$
0.001	$\gamma^2 = 1$	$\gamma^2 = 1$	$\gamma^2 = 1$

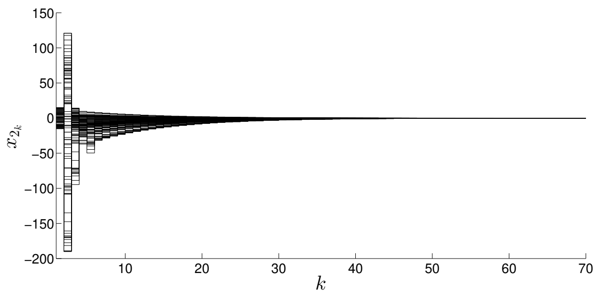

Fig. 2. State trajectories for 100 randomly generated initial conditions with the state-feedback control law given in (8), and $\gamma^2 = 0.85$ and $h = 0.1$ for matrix $Z(x_k)$ up the degree four.

γ^2. It can be observed that the smaller γ^2, the faster the system converges to the origin.

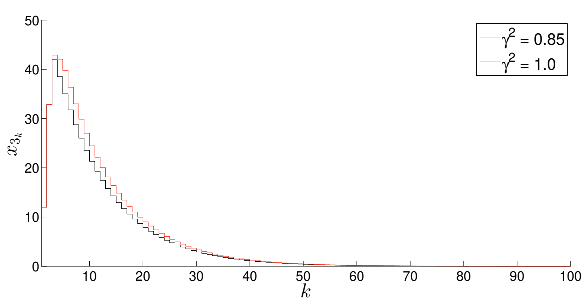

Fig. 3. State trajectories for $\gamma^2 = 0.85$ and $\gamma^2 = 1.0$ with the state-feedback control law given in (8) and initial condition $x_0 = [15 \ -8 \ 12]^T$.

Theorem 2 is also able to stabilize the system. Table 2 shows the results obtained for γ^2, considering different degrees for the matrices $Z(x_k)$ and $X(x_k)$. We did not report the step size $h = 0.001$ in this Table because the solutions were feasible only for $\gamma^2 = 1$ in all the scenarios. It is possible to observe that smaller degrees on the matrix $Z(x_k)$ in Theorem 2 can generate the same results of Theorem 1 by increasing the degree of the matrix $X(x_k)$. Although, for the matrix $Z(x_k)$ with polynomial degree up to 6, the increase in the degree of the matrix $X(x_k)$ made no difference in the values obtained for γ^2.

Figure 4 displays the trajectories of the second state for 100 randomly generated initial conditions in the interval $[-15, 15]$. The controller designed with Theorem 2 considering $\gamma^2 = 0.85$ and $h = 0.1$ for matrix $Z(x_k)$ up to degree four and matrix $X(x_k)$ up to degree two was used. As expected, all the trajectories always converge to the origin.

Considering an initial condition $x_0 = [9 \ 13 \ 11]^T$, the trajectory of the third state of the closed-loop system is shown in Figure 5,

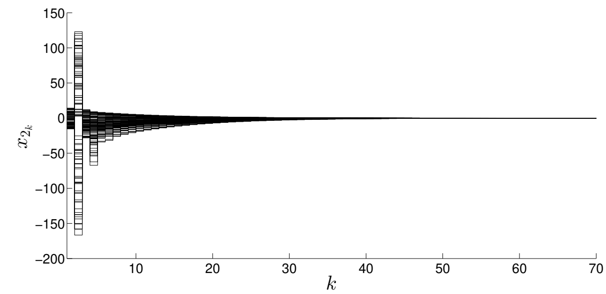

Fig. 4. State trajectories for 100 randomly generated initial conditions with the state-feedback control law given in (15), $\gamma^2 = 0.85$ and $h = 0.1$ for matrix $Z(x_k)$ up to degree four and matrix $X(x_k)$ up to degree two.

for $Z(x_k)$ up to degree five, $X(x_k)$ up to degree two and $h = 0.1$. As it can be seen, for $\gamma = 0.84$ the system response is faster than the two other cases considered.

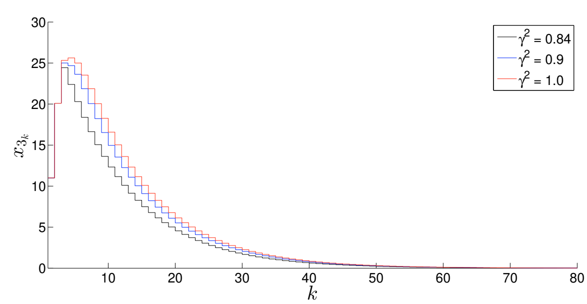

Fig. 5. State trajectories for the third-state considering $\gamma^2 = 0.84$, $\gamma^2 = 0.9$ and $\gamma^2 = 1.0$ with the state-feedback control law given in (15) and initial condition $x_0 = [9 \ 13 \ 11]^T$. Considering matrix $Z(x_k)$ up to degree five and matrix $X(x_k)$ up to degree two.

5. CONCLUSION

This work presented new conditions for designing state feedback controllers for discrete-time state-polynomial systems. The discrete-time system is obtained by discretizing the original continuous-time system using the second-order Runge-Kutta method. Two formulations are presented, the first based on the use of the Lyapunov matrix to recover the polynomial control gain, and the second making use of a slack variable to design the rational controllers. Exponential stability was used to obtain design conditions in both cases. Theorem 2 presented the best results due to the greater degree of complexity of the controller used when compared to Theorem 1. As future research, the authors are investigating the impact of the step size h in different performance metrics such as the ℓ_2 gain.

Table 2. γ^2 variation for different degrees of the polynomial matrices $X(x_k)$ and $Z(x_k)$, and different step sizes h.

$Z(x_k) = [0:4]$			
Step size (h)	Degree of polynomial matrix $X(x_k)$		
	$[0:2]$	$[0:3]$	$[0:4]$
0.1	$0.85 \leq \gamma^2 \leq 1$	$0.84 \leq \gamma^2 \leq 1$	$0.83 \leq \gamma^2 \leq 1$
0.01	$0.99 \leq \gamma^2 \leq 1$	$0.99 \leq \gamma^2 \leq 1$	$0.99 \leq \gamma^2 \leq 1$
$Z(x_k) = [0:5]$			
Step size (h)	Degree of polynomial matrix $X(x_k)$		
	$[0:2]$	$[0:3]$	$[0:4]$
0.1	$0.84 \leq \gamma^2 \leq 1$	$0.83 \leq \gamma^2 \leq 1$	$0.83 \leq \gamma^2 \leq 1$
0.01	$0.99 \leq \gamma^2 \leq 1$	$0.99 \leq \gamma^2 \leq 1$	$0.99 \leq \gamma^2 \leq 1$
$Z(x_k) = [0:6]$			
Step size (h)	Degree of polynomial matrix $X(x_k)$		
	$[0:2]$	$[0:3]$	$[0:4]$
0.1	$0.83 \leq \gamma^2 \leq 1$	$0.83 \leq \gamma^2 \leq 1$	$0.83 \leq \gamma^2 \leq 1$
0.01	$0.99 \leq \gamma^2 \leq 1$	$0.99 \leq \gamma^2 \leq 1$	$0.99 \leq \gamma^2 \leq 1$

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