Cohomology and extensions of representations of groups with normal Engel subgroups

Edward Kissin and Victor S. Shulman

June 28, 2023

Abstract

Let λ, U be representations of a group G with a normal subgroup N. The paper studies the first cohomology group $\mathcal{H}^1(G, \lambda, U)$ under various spectral type conditions imposed on the restrictions of λ, U to N. We assume often that N is an Engel group and examine various decompositions of the extension $\mathfrak{e}(\lambda, U, \xi)$ of λ by U associated with non-trivial (λ, U)-cocycles ξ.

Aiming at applications to double extensions and the theory of J-unitary group representations on indefinite metric spaces, we describe (λ, U)-cocycles when $G = D \ltimes N$ is the semidirect product, D is an Engel group and the restrictions of λ, U to N are $\chi\mathbf{1}$ for some character χ on N (such pairs of representations form in a sense a base class in the variety of all pairs of representations). Our description is complete if $G = \mathcal{D} \ltimes \mathbb{R}^n$, where $\mathcal{D}$ is the group of all diagonal matrices with positive entries, or $G = \mathcal{D} \ltimes T$ is the group of all upper triangular matrices $d = (d_{ij})$ with $d_{ii} > 0$ and T is its subgroup of all matrices with $d_{ii} = 1$, or $G = SO(2) \ltimes \mathbb{R}^2$.

Ethical Statement/Conflict of Interest statement: The manuscript is submitted only to this journal. It is original and does not contain any plagiarism. There is no conflict of interest.

Data availability statement: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Keywords: group representations, extension of representations, cohomology group related to representations, Engel groups, normal subgroups, semidirect product of groups.

1 Introduction and preliminaries

The paper is devoted to the study of spatial extensions of non-unitary representations of certain groups. A representation π of a group G on a Hilbert space X is a *spatial* extension of a representation λ of G by a representation U of G, if X contains a π-invariant subspace L such that the restriction $\pi|_L$ is equivalent to λ, while the representation that π induces on $X \ominus L$ is equivalent to U. A natural instrument for the study of spatial extensions is the technique of (λ, U)-cohomologies.

The case when U is unitary is especially important for the study of representations that are unitary with respect to an indefinite scalar product - the J-unitary representations in indefinite metric spaces (the Krein spaces). Assuming additionally that λ is finite-dimensional, one comes to representations on the Pontryagin Π_k-spaces. Seminal results in this area were obtained by M.A.Naimark and R.S.Ismagilov (see [23] and references therein). They outlined a wide range of further studies and raised many important questions. The subject was further developed by

Araki [1], Ismagilov [9, 10, 11], Kissin and Shulman [12], Naimark [21, 22], Sakai [25] and others (additional references and detailed information can be found in [15]). J-unitary representations were also considered in relation to the study of various problems in the quantum theory [3, 19, 27, 28] and the theory of derivations of C*-algebras implemented by symmetric unbounded operators [17].

This paper is a continuation of papers [13] and [16], where we studied extensions of representations of nilpotent groups, and found some triviality conditions for cohomologies of groups with normal Engel subgroups. Here we refine and extend many of these results and apply them to obtain various types of decomposition for extensions of representations. We also describe some important classes of cohomologies of semidirect product of groups: $G = D \ltimes N$, where D is an Engel group (this class of groups contains, in particular, many solvable groups). In turn, this allows us to analyse the structure of the extensions of the representations λ by U for these groups.

As in the case of nilpotent groups, where the results on (λ, U)-cocycles and extensions obtained in [13] were crucial in establishing and classifying in [14] various types of decomposition of their J-unitary representations on Π_k-spaces, the results of the present paper are essential for the study of J-unitary representations of semidirect products of groups on Π_k-spaces.

Throughout the paper G is a *connected locally compact group* with a *connected closed normal subgroup* N, and λ, U are *weakly continuous representations* of G on Hilbert spaces L and $\mathfrak{H}$. By $B(\mathfrak{H}, L)$ we denote the space of all bounded operators from $\mathfrak{H}$ to L. Let $K(G, N)$ be the minimal closed subgroup of G containing all commutators $ghg^{-1}h^{-1}$, $g \in G$, $h \in N$. Set $G^{[1]} = K(G, G)$, $G^{[2]} = K(G, G^{[1]}), ..., G^{[n]} = K(G, G^{[n-1]})$. A group G is *nilpotent* if $G^{[n]} = \{e\}$ for some n. Each $h \in G$ defines a map ad_h on G: $\mathrm{ad}_h(g) = ghg^{-1}h^{-1}$ for $g \in G$. An element h of G is *Engel* if, for each $g \in G$, there is $n_g \in \mathbb{N}$ such that $\mathrm{ad}_h^{n_g}(g) = e$. A group G is an *Engel group*, if it consists of Engel elements. Nilpotent groups are Engel groups, while solvable ones are not always Engel.

For any map μ of G, we denote by μ^N its restriction to N.

As we only study 1-cohomology, we write for brevity *cohomology*, *cocycles*, *coboundaries* instead of 1-*cohomology*, 1-*cocycles*, 1-*coboundaries*. A weakly continuous map ξ: $G \to B(\mathfrak{H}, L)$ is

$$\text{a } (\lambda, U)\text{-}\textit{cocycle} \text{ if } \xi(gh) = \lambda(g)\xi(h) + \xi(g)U(h) \text{ for } g, h \in G; \tag{1.1}$$

$$\text{a } (\lambda, U)\text{-}\textit{coboundary} \text{ if } \xi(g) = \lambda(g)T - TU(g) \text{ for } g \in G \text{ and some } T \in B(\mathfrak{H}, L). \tag{1.2}$$

Let $\mathcal{Z}(G, \lambda, U)$ be the set of all (λ, U)-cocycles and $\mathcal{B}(G, \lambda, U)$ the set of all (λ, U)-coboundaries. Then $\mathcal{H}(G, \lambda, U) = \mathcal{Z}(G, \lambda, U)/\mathcal{B}(G, \lambda, U)$ is the 1-*cohomology group of G related to λ, U*. If $\dim L = 1$ and $\iota = \mathbf{1}_L$ then $\mathcal{H}(G, \iota, U)$ is the 1-cohomology group of U in classical sense ([2],[4],[5],[6]).

In Section 2 we develop a spectral criterion of triviality of the group $\mathcal{H}(G, \lambda, U)$ for pairs (G, N) (for nilpotent G it was established in [13]). In Theorem 2.13 we show that if λ and U are *sectionally spectrally disjoint* (Definition 2.4) at some Engel elements of N then $\mathcal{H}(G, \lambda, U) = 0$.

The importance of the notion of the sectional spectral disjointness is also demonstrated in Theorem 3.9. It states that if a finite-dimensional subspace L is invariant for the restriction π^N of a representation π of G to an Engel subgroup N and the representation $\pi^N|_L$ is sectionally spectrally disjoint with the representation that π^N induces on $X \ominus L$, then L is also π-invariant.

The main motivation for the study of (λ, U)-cocycles of G is that every such cocycle ξ defines an *extension* $\mathfrak{e}(\lambda, U, \xi)$ of λ by U on $X = L \oplus \mathfrak{H}$ by the formula

$$\mathfrak{e}(g) = \mathfrak{e}(\lambda, U, \xi)(g) = \begin{pmatrix} \lambda(g) & \xi(g) \\ 0 & U(g) \end{pmatrix}, \; g \in G. \tag{1.3}$$

It is well known that L has an $\mathfrak{e}$-invariant complement H, if and only if ξ is a (λ, U)-coboundary. Moreover, two extensions are similar if and only if the corresponding cocycles are cohomological.

Section 4 deals with the structure of $\mathfrak{e} = \mathfrak{e}(\lambda, U, \xi)$ when N is an Engel group. In Theorem 5.2 we show that if $G = N$ is nilpotent, then L always approximately splits $\mathfrak{e}$, i.e., there are $\mathfrak{e}$-invariant subspaces $(Y_i, Z_i)_{i=1}^{\infty}$ in X such that $X = Y_i \dotplus Z_i$, $Y_{i+1} \subsetneqq Y_i$, $Z_i \subsetneqq Z_{i+1}$ for all i, and $L \subseteq M = \cap_i Y_i$ with $\dim(M/L) < \infty$.

A representation μ of N is *monothetic* if the spectrum of each operator $\mu(h)$, $h \in N$, is a singleton. In this case there is a character χ of N (a continuous multiplicative map from N to $\mathbb{C} \setminus \{0\}$) such that $\mathrm{Sp}(\mu(h)) = \chi(h)$ for all $h \in N$. If N is an Engel group, then (see [13, Theorem 3.17]) each finite-dimensional representation of N decomposes in the direct sum of monothetic ones. Moreover, in our setting, where N is a normal subgroup of G, each finite-dimensional representation λ of G decomposes in the direct sum of representations whose restrictions to N are monothetic [16] (we call such representations of G N-monothetic). So the following question arises naturally:

Under what conditions all extensions $\mathfrak{e}(\lambda, U, \xi)$ of a pair (λ, U) (with finite-dimensional λ) decompose in the direct sum of extensions $\mathfrak{e}(\lambda_i, U_i, \xi_i)$ with N-monothetic λ_i?

For commutative G, the positive answer to this question was given in [13]. However, the result does not extend to all nilpotent groups; the corresponding representation of the Heisenberg group was constructed in [13]. We show in Theorem 5.5 that the needed decomposition exists if $U(N)$ is contained in the center of $U(G)$

In Sections 6–8 we consider a comparatively simple (but, in some sense, the base) class of pairs of representations (λ, U) that are not spectrally disjoint on N: N-trivial representations, i.e.,

$$\lambda(h) = \mathbf{1}_L \text{ and } U(h) = \mathbf{1}_{\mathfrak{H}} \text{ for } h \in N. \tag{1.4}$$

Such a condition allows us to present a wide range of non-trivial (λ, U)-cocycles for the semi-direct products of groups $G = D \ltimes N$ (especially, when D is an Engel group). These cocycles are described by analytic formulas. This, in turn, permits us to describe in detail (see Section 6.4) the structure of the corresponding classes of extensions $\mathfrak{e}(\lambda, U, \xi)$.

This detailed information is important for the study of J-unitary representations of groups on Pontryagin spaces. To construct such a representation one needs to have a (λ, U)-cocycle that satisfies an additional condition of *neutrality*:

$$-\xi(h)\xi(g^{-1})^* = \lambda(h)\gamma(g) - \gamma(hg) + \gamma(h)\lambda(g^{-1})^*, \ g, h \in G,$$

for some map γ: $G \to B(L)$ [11],[12]. To choose from the space of all (λ, U)-cocycles the neutral ones, we need to have a maximally transparent description of this space.

It is shown in Theorem 6.4 that, for λ, U satisfying (1.4), each cocycle is the sum of two cocycles:

$$\xi = \xi_\eta + \xi_\beta, \text{ where } \xi_\eta(dh) = \eta(d), \ \xi_\beta(dh) = \lambda(dh)\beta(\omega_{_N}(h)) \text{ for } d \in D, \ h \in N.$$

Here $\omega_{_N}$: $N \to \mathbb{R}^l$ is the natural epimorphism (the number $l \in \mathbb{N}$ depends on the structure of N) and β: $\mathbb{R}^l \to B(\mathfrak{H}, L)$ is a linear map called a (λ, U)-*cocyclic map.* The map η is just a (λ^D, U^D)-cocycle on D and can be considered separately. In this paper we concentrate on the study of the (λ, U)-cocyclic maps β. An explicit formula for the (λ, U)-cocyclic maps β is obtained in Section 6.3 for the case when D is Engel. In Section 6.4 we study the structure of the extensions $\mathfrak{e}(\lambda, U, \xi)$ for N-trivial representations λ, U. We obtain their full description for nilpotent D.

In Section 7 we study (λ, U)-cocyclic maps and cocycles on groups $G = D \ltimes \mathbb{R}^n$, where D are Engel subgroups of $GL(n, \mathbb{R})$. We give their partial description when D is the nilpotent group T_n of all upper triangular matrices with 1 on the diagonal. The picture is complete when D is the group

$\mathcal{D}$ of all diagonal matrices with positive entries, and when $G = SO(2) \ltimes \mathbb{R}^2$. We also consider the case when D is the solvable (but not Engel) group S_n of all upper triangular matrices with positive diagonal entries.

Finally, in Section 8 we give a full description of (λ, U)-cocyclic maps and cocycles on the group S_n considered as the semidirect product of its subgroups $\mathcal{D}$ and T_n.

2 Spectrally disjoint representations and group cohomology

In this section we develop a spectral criterion of triviality of $\mathcal{H}(G, \lambda, U)$ for a group G with a normal subgroup N. In Theorem 2.13 we show that if λ and U are sectionally spectrally disjoint at some Engel elements of N then $\mathcal{H}(G, \lambda, U) = 0$. Set $B(\mathfrak{H}) = B(\mathfrak{H}, \mathfrak{H})$ and $B(L) = B(L, L)$.

2.1 Spectral disjointness of representations

We say that operators $A \in B(L)$ and $B \in B(\mathfrak{H})$ are *spectrally disjoint* (see [13]) if

$$\mathrm{Sp}(A) \cap \mathrm{Sp}(B) = \varnothing.$$

Lemma 2.1 [Rosenblum (see, for example, Corollary 0.13 [24])] *Let operators A and B be spectrally disjoint. If $AY = YB$ for $Y \in B(\mathfrak{H}, L)$, or $YA = BY$ for $Y \in B(L, \mathfrak{H})$ then $Y = 0$.*

We say that a map $\lambda\colon G \to B(L)$ and a representation $U\colon G \to B(\mathfrak{H})$ are *spectrally disjoint* at $h \in G$, if the operators $U(h)$ and $\lambda(h)$ are spectrally disjoint.

Let $h \in N$. As $\mathrm{Sp}(U(h_g)) = \mathrm{Sp}(U(h))$ for $g \in G$, Lemma 2.1 yields in this case that

$$\begin{aligned} &1) \quad \lambda(h)Y = YU(h_g) \text{ for } Y \in B(\mathfrak{H}, L), \text{ implies } Y = 0, \text{ where } h_g = ghg^{-1}; \\ &2) \quad Y\lambda(h) = U(h_g)Y \text{ for } Y \in B(L, \mathfrak{H}), \text{ implies } Y = 0. \end{aligned} \tag{2.1}$$

Unless otherwise specified, the maps λ and U will be *representations* of G on L and $\mathfrak{H}$.

We often need to consider the case that U (or λ) has the upper triangular form. For example,

$$\mathfrak{H} = \oplus_{i=1}^{k} \mathfrak{H}_i, \; k \leq \infty, \text{ and } U = \begin{pmatrix} U_1 & U_{12} & U_{13} & \cdots \\ 0 & U_2 & U_{23} & \cdots \\ 0 & 0 & U_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \; U_{ij} = 0 \text{ for } i > j,$$

where U_i are representations of G on $\mathfrak{H}_i$. In this case we write $U = \backslash U_i]_{i=1}^{k}$.

The result below was obtained in [16]. For convenience of the reader we give its proof here.

Proposition 2.2 *Let $U = \backslash U_i]_{i=1}^{k}$, $k \leq \infty$, let λ be spectrally disjoint with each U_i at some $h^i \in N$ and ξ be a (λ, U)-cocycle. Let ξ^N be the restrictions of ξ to the normal subgroup N.*

If ξ^N is a (λ^N, U^N)-coboundary ($\xi(h) = \lambda(h)T - TU(h)$ for all $h \in N$ and some $T \in B(\mathfrak{H}, L)$), then ξ is a (λ, U)-coboundary with the same unique representing operator T:

$$\xi(g) = \lambda(g)T - TU(g) \text{ for } g \in G.$$

Proof. Set $\eta(g) = \xi(g) - (\lambda(g)T - TU(g))$ for $g \in G$. Then η is also a (λ, U)-cocycle of G and $\eta(h) = 0$ for $h \in N$. Hence, as $h_{g^{-1}} \in N$ for all $h \in N$, $g \in G$, we have from (1.1)

$$\begin{aligned} \eta(gh_{g^{-1}}) &= \lambda(g)\eta(h_{g^{-1}}) + \eta(g)U(h_{g^{-1}}) = \eta(g)U(h_{g^{-1}}) \text{ and} \\ \eta(gh_{g^{-1}}) &= \eta(gg^{-1}hg) = \eta(hg) = \lambda(h)\eta(g) + \eta(h)U(g) = \lambda(h)\eta(g), \text{ so that} \\ \eta(g)U(h_{g^{-1}}) &= \lambda(h)\eta(g) \text{ for } g \in G \text{ and } h \in N. \end{aligned} \tag{2.2}$$

We have that $\eta = \{\eta_i\}_{i=1}^k$, where $\eta_i = \eta|_{\mathfrak{H}_i}\colon G \to B(\mathfrak{H}_i, L)$. So, by (2.2), $\eta_1(g)U_1(h^1_{g^{-1}}) = \lambda(h^1)\eta_1(g)$ for $g \in G$. Since λ and U_1 are spectrally disjoint at $h^1 \in N$, we get from (2.1) that $\eta_1 = 0$.

Hence we have from (2.2) that $\eta_2(g)U_2(h_{g^{-1}}) = \lambda(h)\eta_2(g)$ for all $g \in G$ and $h \in N$. Since λ and U_2 are spectrally disjoint at $h^2 \in N$, it follows as above that η_2. Continuing this process, we get $\eta = 0$. So $\xi(g) = \lambda(g)T - TU(g)$ for all $g \in G$.

Let also $\xi(g) = \lambda(g)S - SU(g)$ for all $g \in G$ and some $S \in B(\mathfrak{H}, L)$. Set $R = T - S$. Then

$$\lambda(g)R = RU(g) \text{ for } g \in G. \tag{2.3}$$

Let us show that $R = 0$. As $\mathfrak{H} = \oplus_{i=1}^k \mathfrak{H}_i$, we have $R = \{R_i\}_{i=1}^k$ where $R_i = R|_{\mathfrak{H}_i} \in B(\mathfrak{H}_i, L)$. By (2.3), $\lambda(h^1)R_1 = R_1U_1(h^1)$. As $\lambda(h^1)$ is spectrally disjoint with $U_1(h^1)$, $R_1 = 0$ by Lemma 2.1. So $\lambda(h^2)R_2 = R_2U_2(h^2)$. By Lemma 2.1, $R_2 = 0$. Continuing this process, we get $R = 0$. So $T = S$. ■

Using Proposition 2.2, we obtain the following results.

Theorem 2.3 *Let $U = \backslash U_i]_{i=1}^k$, $k \le \infty$, and λ be spectrally disjoint with each U_i at some $h^i \in N$.*

(i) *If $\mathcal{H}(N, \lambda^N, U^N) = 0$ then $\mathcal{H}(G, \lambda, U) = 0$.*
(ii) *If $\mathcal{H}(N, \lambda^N, U_i^N) = 0$ for some i, then $\mathcal{H}(G, \lambda, U_i) = 0$.*
(iii) *If $k < \infty$ and $\mathcal{H}(G, \lambda, U_i) = 0$ for all $i \le k$, then $\mathcal{H}(G, \lambda, U) = 0$.*
(iv) *If $k < \infty$ and $\mathcal{H}(N, \lambda^N, U_i^N) = 0$ for all $i \le k$, then $\mathcal{H}(G, \lambda, U) = 0$.*

Proof. Part (i) follows from Proposition 2.2. Part (ii) follows from (i) for $k = 1$, and part (iii) was proved in Lemma 2.3 [13].

(iv) By (ii), $\mathcal{H}(G, \lambda, U_i) = 0$ for all $i \le k$. Hence $\mathcal{H}(G, \lambda, U) = 0$ by (iii). ■

A continuous map $\chi\colon G \to \mathbb{C}$ is *a character* if $\chi(gh) = \chi(g)\chi(h)$ for $g, h \in G$. We denote by G^* the group (under pointwise multiplication) of all characters on G. A character χ is *unitary*, if

$$|\chi(g)| = 1 \text{ for } g \in G, \text{ i.e., } \chi(g^{-1}) = \overline{\chi(g)}. \tag{2.4}$$

Following [13], we call the spectral condition in Theorem 2.3 by *sectional spectral disjointness*.

Definition 2.4 *We say that a representation λ of G (respectively, a character χ) is* ***sectionally*** *spectrally disjoint with a representation U of G at some $\{g^i\}_{i=1}^k$ in G, if $U = \backslash U_i]_{i=1}^k$, $k < \infty$, and λ (respectively, χ) is spectrally disjoint with each diagonal U_i at g^i:*

$$\mathrm{Sp}(\lambda(g^i)) \cap \mathrm{Sp}(U_i(g^i)) = \varnothing \quad (\textit{respectively, } \chi(g^i) \notin \mathrm{Sp}(U_i(g^i))).$$

Let $\mathfrak{H} = \oplus_{i=1}^\infty \mathfrak{H}_i$, all $\mathfrak{H}_i$ be U-invariant and $P_{\mathfrak{H}_i}$ be the projections on $\mathfrak{H}_i$, so that

$$U = \oplus_{i=1}^\infty U_i, \text{ where } U_i = U|_{\mathfrak{H}_i}. \tag{2.5}$$

Lemma 2.5 (i) *A map* $\xi\colon G \to B(\mathfrak{H}, L)$ *is a* (λ, U)*-cocycle if and only if all maps* $\xi_i = \xi|_{\mathfrak{H}_i}\colon G \to B(\mathfrak{H}_i, L)$ *are* (λ, U_i)*-cocycles.*

(ii) *Let* $\mathcal{H}(G, \lambda, U_i) = 0$ *for all* $i \in \mathbb{N}$. *Then a map* $\xi\colon G \to B(\mathfrak{H}, L)$ *is*

1) *a* (λ, U)*-cocycle if and only if all* ξ_i *are* (λ, U_i)*-coboundaries, i.e.,*

$$\xi_i = \lambda T_i - T_i U_i \text{ for some } T_i \in B(\mathfrak{H}_i, L); \tag{2.6}$$

2) *a coboundary if and only if there are* $\{T_i\}_{i=1}^{\infty}$ *satisfying* (2.6) *and* $\oplus_{i=1}^{\infty} T_i P_{\mathfrak{H}_i} \in B(\mathfrak{H}, L)$;

3) *if* λ *is sectionally spectrally disjoint with all* U_i, *then operators* $\{T_i\}_{i=1}^{\infty}$ *are unique.*

Proof. (i) As all $P_{\mathfrak{H}_i}$ commute with U, it follows from (1.1) that all ξ_i are (λ, U_i)-cocycles.

Conversely, let all ξ_i be (λ, U_i)-cocycles. Set $P_m = \oplus_{i=1}^{m} P_{\mathfrak{H}_i}$, $m < \infty$. Then $P_m U = U P_m$ and $\|x - P_m x\| \to 0$ for $x \in \mathfrak{H}$, as $m \to \infty$. We have that $\xi(g)P_m = (\xi_1(g), ..., \xi_m(g), 0, ...)$ are (λ, U)-cocycles. Hence ξ is a (λ, U)-cocycle, since for $x \in \mathfrak{H}$,

$$\begin{aligned} \xi(g_1 g_2)x &= \lim_{m\to\infty} \xi(g_1 g_2) P_m x = \lim_{m\to\infty} (\lambda(g_1)\xi(g_2)P_m x + \xi(g_1)U(g_2)P_m x) \\ &= \lambda(g_1)\xi(g_2)x + \xi(g_1)U(g_2)x \text{ for } g_1, g_2 \in G. \end{aligned}$$

(ii) 1) follows from (1.2) and (i).

2) If ξ is a (λ, U)-coboundary, $\xi = \lambda T - TU$ for some $T \in B(\mathfrak{H}, L)$. So $T_i = T P_{\mathfrak{H}_i}$ satisfy (2.6) and $T = \oplus_{i=1}^{\infty} T_i P_{\mathfrak{H}_i}$.

Conversely, let $T = \oplus_{i=1}^{\infty} T_i P_{\mathfrak{H}_i} \in B(\mathfrak{H}, L)$ and all T_i satisfy (2.6). Then for $g \in G$,

$$\xi(g)P_m = (\xi_1(g), ..., \xi_m(g), 0, ...) = \lambda(g)TP_m - TP_m U(g) = (\lambda(g)T - TU(g))P_m.$$

Hence ξ is a (λ, U)-coboundary, since for $x \in \mathfrak{H}$,

$$\xi(g)x = \lim_{m\to\infty} \xi(g)P_m x = \lim_{m\to\infty} (\lambda(g)T - TU(g))P_m x = \lambda(g)Tx - TU(g)x.$$

3) Let, for some i, there are $S_i, T_i \in B(\mathfrak{H}_i, L)$ such that

$$\xi_i(g) = \lambda(g)S_i - S_i U_i(g) = \lambda(g)T_i - T_i U_i(g) \text{ for } g \in G.$$

Set $R_i = T_i - S_i$. Then $\lambda(g)R_i = R_i U_i(g)$. As λ is sectionally spectrally disjoint with each U_i, repeating the argument in the proof of Proposition 2.2 that follows (2.3), we get $R_i = 0$. So $T_i = S_i$. ■

Corollary 2.6 *Let* $U = \oplus_{i=1}^{\infty} U_i$, *let* λ *be sectionally spectrally disjoint with each* U_i *at some elements in* N *and let all* $\mathcal{H}(N, \lambda^N, U_i^N) = 0$. *Then*

(i) $\mathcal{H}(G, \lambda, U_i) = 0$ *for all* $i \in \mathbb{N}$.

(ii) *A map* $\xi\colon G \to B(\mathfrak{H}, L)$ *is a* (λ, U)*-cocycle if and only if there are operators* $T_i \in B(\mathfrak{H}_i, L)$, $i \in \mathbb{N}$, *such that* $\xi|_{\mathfrak{H}_i} = (\lambda T_i - T_i U_i)|_{\mathfrak{H}_i}$. *The family of operators* $\{T_i\}$ *is defined uniquely.*

The map ξ *is a* (λ, U)*-coboundary if and only if, in addition, the operator* $\oplus_{i=1}^{\infty} T_i P_{\mathfrak{H}_i}$ *is bounded.*

Proof. (i) follows from Theorem 2.3(ii), and (ii) follows from (i) and Lemma 2.5(ii). ■

It follows from Corollary 2.6 that Theorem 2.3(iii) does not extend to the case $k = \infty$.

Let $\dim L = 1$, i.e., $L = \mathbb{C}e$. Each operator $T \in B(\mathfrak{H}, L)$ has the form $T = y \otimes e$:

$$(y \otimes e)x = (x, y)_{\mathfrak{H}} e \text{ for all } x \in \mathfrak{H} \text{ and some } y \in \mathfrak{H}. \tag{2.7}$$

If $T \in B(\mathfrak{H})$ and $\mu \in \mathbb{C}$ then, for $x, y \in \mathfrak{H}$,

$$\begin{aligned} &\mu(x \otimes e) = x \otimes \mu e = \overline{\mu} x \otimes e, \ (x \otimes e)T = (T^* x) \otimes e, \\ &(x \otimes e)^* = e \otimes x \in B(L, \mathfrak{H}), \ (x \otimes e)(e \otimes y) = (y, x)_{\mathfrak{H}} (e \otimes e) \in B(L). \end{aligned} \tag{2.8}$$

Each map ξ: $G \to B(\mathfrak{H}, L)$ has the form $\xi(g) = r(g) \otimes e$ for some map r: $G \to \mathfrak{H}$.

Denote by ι the trivial representation: $\iota(g) = \mathbf{1}_L$ for $g \in G$. Then Corollary 2.6 yields

Corollary 2.7 *Let $U = \oplus_{i=1}^{\infty} U_i$ and $\chi \in G^*$. Suppose that, for each $i \in \mathbb{N}$,*

$$\mathcal{H}(N, \chi^N \iota^N, U_i^N) = 0 \text{ and there is } h^i \in N \text{ such that } \chi(h^i) \notin \mathrm{Sp}(U_i(h^i)).$$

Then

(i) $\mathcal{H}(G, \chi\iota, U_i) = 0$ *for all* $i \in \mathbb{N}$.

(ii) *A map* $\xi(g) = r(g) \otimes e$ *is a* $(\chi\iota, U)$*-cocycle if and only if there are* $y_i \in \mathfrak{H}_i$ *such that*

$$r(g) = \oplus_{i=1}^{\infty} (\overline{\chi(g)} y_i - U_i^*(g) y_i) \in \mathfrak{H}, \ \text{i.e.,} \ \sum_{i=1}^{\infty} \left\| \overline{\chi(g)} y_i - U_i^*(g) y_i \right\|^2 < \infty \text{ for } g \in G. \tag{2.9}$$

(iii) ξ *is a* $(\chi\iota, U)$*-coboundary if and only if* $\sum_{i=1}^{\infty} y_i = y \in \mathfrak{H}$, *so that* $r(g) = \overline{\chi(g)} y - U^*(g) y$.

If N has the property (T) (ι^N is isolated in the space $\widehat{N}$ of all equivalence classes of irreducible unitary representations of N), then $\mathcal{H}(N, \iota^N, V) = 0$ for any unitary representation V of N (Proposition III.2.9 [6]). So Corollary 2.7 yields

Corollary 2.8 *Let $U = \oplus_{i=1}^{\infty} U_i$ be* ***unitary*** *and N have property (T). If $1 \notin \mathrm{Sp}(U_i(h^i))$ for all i and some $h^i \in N$, then $\mathcal{H}(G, \iota, U_i) = 0$ for $i \in \mathbb{N}$, and (2.9) with $\chi(g) \equiv 1$ describes (ι, U)-cocycles.*

2.2 Engel elements in groups and group cohomology

Recall that $h \in G$ is an Engel element if, for each $g \in G$, there is $n_g \in \mathbb{N}$ such that $\mathrm{ad}_h^{n_g}(g) = e$, where $\mathrm{ad}_h(g) = hgh^{-1}g^{-1}$. A group G is Engel if it consists of Engel elements. In particular, all nilpotent groups are Engel.

We will show that the sectional spectral disjointedness of λ and U at Engel elements of N is sufficient for the triviality of $\mathcal{H}(G, \lambda, U)$.

Lemma 2.9 *Each Engel element of a normal subgroup N is also an Engel element of G.*

Proof. Let h be an Engel element of N. For $g \in G$, $\mathrm{ad}_h(g) = ghg^{-1}h^{-1} = h_g h^{-1} \in N$. Hence $\mathrm{ad}_h^{n+1}(g) = \mathrm{ad}_h^n(\mathrm{ad}_h(g)) = \mathrm{ad}_h^n(h_g h^{-1})$ for all n. As h is an Engel element of N, $\mathrm{ad}_h^n(h_g h^{-1}) = e$ for some n. Hence $\mathrm{ad}_h^{n+1}(g) = e$. Thus h is an Engel element of G. ■

The following result proved in Corollary 2.9 [13] plays an important role below.

Theorem 2.10 *If λ and U are spectrally disjoint at an Engel element of G then $\mathcal{H}(G, \lambda, U) = 0$.*

For convenience we include here a shorter and more transparent proof than the original one.

Lemma 2.11 *Suppose that representations λ and U are spectrally disjoint at $h \in G$, and let η be a (λ, U)-cocycle with $\eta(h) = 0$. If $\eta(hgh^{-1}g^{-1}) = 0$ for some $g \in G$, then $\eta(g) = 0$.*

Proof. As $0 = \eta(e) = \eta(gg^{-1}) = \lambda(g)\eta(g^{-1}) + \eta(g)U(g^{-1})$, we have

$$\eta(g^{-1}) = -\lambda(g)^{-1}\eta(g)U(g)^{-1} \text{ for all } g \in G. \tag{2.10}$$

So $\eta(h^{-1}) = 0$.

By our assumption,

$$\begin{aligned}
0 = \eta(hgh^{-1}g^{-1}) &\overset{(1.1)}{=} \lambda(hgh^{-1})\eta(g^{-1}) + \eta(hgh^{-1})U(g^{-1}) \\
&\overset{(2.10)}{=} -\lambda(hgh^{-1})\lambda(g^{-1})\eta(g)U(g^{-1}) + (\lambda(h)\eta(gh^{-1}) + \eta(h)U(gh^{-1}))U(g^{-1}) \\
&= -\lambda(h)\lambda(gh^{-1}g^{-1})\eta(g)U(g^{-1}) + \lambda(h)(\lambda(g)\eta(h^{-1}) + \eta(g)U(h^{-1}))U(g^{-1}) \\
&= -\lambda(h)\lambda(gh^{-1}g^{-1})\eta(g)U(g^{-1}) + \lambda(h)\eta(g)U(h^{-1}))U(g^{-1}) \\
&= \lambda(h)(\eta(g)U(h^{-1}) - \lambda(gh^{-1}g^{-1})\eta(g))U(g^{-1}).
\end{aligned}$$

Thus

$$\eta(g)U(h^{-1}) = \lambda(gh^{-1}g^{-1})\eta(g) = \lambda((h^{-1})_g)\eta(g). \tag{2.11}$$

As $\mathrm{Sp}(U(h)) \cap \mathrm{Sp}(\lambda(h)) = \varnothing$ and

$$\mathrm{Sp}(U(h^{-1})) = \{z^{-1}\colon z \in \mathrm{Sp}(U(h))\} \text{ and } \mathrm{Sp}(\lambda(h^{-1})) = \{z^{-1}\colon z \in \mathrm{Sp}(\lambda(h))\},$$

we have $\mathrm{Sp}(U(h^{-1})) \cap \mathrm{Sp}(\lambda(h^{-1})) = \varnothing$. Hence $\eta(g) = 0$ by (2.11) and (2.1). ■

Corollary 2.12 *Let λ and U be spectrally disjoint at an Engel element $h \in G$. If a (λ, U)-cocycle η satisfies the condition $\eta(h) = 0$ then $\eta = 0$.*

Proof. Since h is an Engel element then, for each $g \in G$, there is $n = n_g$ such that in the chain $g_0 = g$, $g_k = \mathrm{ad}_h(g_{k-1})$ the element $g_n = e$. Thus $\eta(g_n) = 0$. On the other hand, if $\eta(g_k) = 0$ then $\eta(g_{k-1}) = 0$ by Lemma 2.11. Thus $\eta(g) = 0$. So $\eta = 0$. ■

The end of the proof of Theorem 2.10. Let λ and U be spectrally disjoint at an Engel element $h \in G$, and let ξ be a (λ, U)-cocycle. Consider the operator $R(X) = \lambda(h)X - XU(h)$ for $X \in B(\mathfrak{H}, L)$. By Rosenblum's Theorem (Theorem 0.12 [24]),

$$\mathrm{Sp}(R) = \{\alpha - \beta\colon \alpha \in \mathrm{Sp}(\lambda(h)),\ \beta \in \mathrm{Sp}(U(h))\}.$$

As λ and U are spectrally disjoint at h, $\mathrm{Sp}(\lambda(h)) \cap \mathrm{Sp}(U(h)) = \varnothing$. Thus $0 \notin \mathrm{Sp}(R)$, so that R is invertible. Hence there is T such that $\lambda(h)T - TU(h) = \xi(h)$.

Set $\eta(g) = \xi(g) - \lambda(g)T + TU(g)$. It is a (λ, U)-cocycle and $\eta(h) = 0$. Hence $\eta = 0$ by Corollary 2.12. Thus $\xi(g) = \lambda(g)T - TU(g)$. So $\mathcal{H}(G, \lambda, U) = 0$. The proof of Theorem 2.10 is complete. ■

We have the following refinement of Theorem 2.3.

Theorem 2.13 *Let λ be sectionally spectrally disjoint with U at Engel elements $\{h^i\}_{i=1}^{k}$ of N. Then $\mathcal{H}(G,\lambda,U)=0$.*

Proof. By the assumption, $U = \backslash U_i]_{i=1}^{k}$, $k<\infty$, and λ is spectrally disjoint with each diagonal U_i at h^i. By Theorem 2.10, $\mathcal{H}(N,\lambda^N,U_i^N)=0$ for all i. It suffices now to apply Theorem 2.3(iv). ■

Making use of Theorem 2.13, we simplify conditions in Corollaries 2.6 and 2.7.

Corollary 2.14 *Let $U=\oplus_{i=1}^{\infty}U_i$ and let χ be a character on G.*

(i) *If a representation λ is sectionally spectrally disjoint with each U_i at some Engel elements of N, then the conclusions of Corollary* 2.6 *hold.*

(ii) *If $\chi(h^i)\notin \mathrm{Sp}(U_i(h^i))$ for all $i\in\mathbb{N}$ and some Engel elements $h^i\in N$, then the conclusions of Corollary* 2.7 *hold.*

In (2.5) all subspaces $\mathfrak{H}_i$ are U-invariant. We now show that sometimes it suffices to assume that they are U^N-invariant.

Proposition 2.15 *Let $\mathfrak{H}=\oplus_{i=1}^{\infty}\mathfrak{H}_i$ and all $\mathfrak{H}_i$ be U^N-invariant, i.e., $U^N=\oplus_{i=1}^{\infty}U_i^N$. Suppose that $U^N|_{\mathfrak{H}_i}=\omega_i\mathbf{1}_{\mathfrak{H}_i}$, $i\in\mathbb{N}$, for some distinct characters ω_i on N. Then*

(i) *All $\mathfrak{H}_i$ are U-invariant, so that $U=\oplus_{i=1}^{\infty}U_i$.*

(ii) *If $\omega_i(h^i)\notin\mathrm{Sp}(\lambda(h^i))$ for each $i\in\mathbb{N}$ and some Engel elements $h^i\in N$, then $\mathcal{H}(G,\lambda,U_i)=0$ for all $i\in\mathbb{N}$. In other words, the results of Corollary* 2.6(ii) *hold.*

Proof. (i) Let $U=(U_{ij})$ be the block-matrix form with respect to the decomposition $\mathfrak{H}=\oplus_{i=1}^{\infty}\mathfrak{H}_i$. As $gh=h_gg$ for $h\in N$, $g\in G$, we have $U(g)U(h)=U(h_g)U(g)$. As U^N is block-diagonal,

$$U_{ij}(g)U_{jj}(h)=U_{ii}(h_g)U_{ij}(g) \text{ for } i\neq j \text{ and all } h\in N,\ g\in G. \tag{2.12}$$

For $g\in G$ and a character ω on N, set $\omega^g(h)=\omega(h_g)$ for $h\in N$. Then $C_\omega=\{g\in G\colon \omega^g=\omega\}$ is a group. If C_ω contains an open subset of G, it contains a neighbourhood of e in G. As G is connected, $G=C_\omega$. Thus either $C_\omega=G$, or C_ω contains no open subsets of G.

If $i\neq j$ in (2.12) then

$$U_{ij}(g)(\omega_i^g(h)-\omega_j(h))=0 \text{ for } g\in G \text{ and } h\in N. \tag{2.13}$$

If $C_{\omega_i}=G$ then $\omega_i^g=\omega_i$ for all $g\in G$. As $\omega_i\neq\omega_j$, it follows from (2.13) that $U_{ij}=0$.

Suppose that C_{ω_i} contains no open subsets of G. Let $U_{ij}(r)\neq 0$ for some $r\in G$. Then $\omega_i^r=\omega_j$ by (2.13). If also $U_{ij}(s)\neq 0$ for some $s\neq r$, then $\omega_i^s=\omega_j$. Hence $\omega_i^r=\omega_i^s$. Thus $\omega_i^{r^{-1}s}=\omega_i$, so that $r^{-1}s\in C_{\omega_i}$. So $s\in rC_{\omega_i}$. Hence $U_{ij}(g)=0$ if $g\notin rC_{\omega_i}$. As rC_{ω_i} contains no open subsets of G and U_{ij} is weakly continuous on G, we have $U_{ij}=0$. Thus all $\mathfrak{H}_i$ are U-invariant.

(ii) As $\mathrm{Sp}(\lambda(h^i)\cap\mathrm{Sp}(U_i(h^i))=\mathrm{Sp}(\lambda(h^i)\cap\{\omega_i(h^i)\}=\varnothing$, λ is spectrally disjoint with each U_i, $i\in\mathbb{N}$, at an Engel element $h^i\in N$. So $\mathcal{H}(G,\lambda,U_i)=0$ for all $i\in\mathbb{N}$, by Theorem 2.10. So the results stated in Corollary 2.6(ii) hold. ■

3 Invariant subspaces and elementary representations

Definition 3.1 *A map* μ: $N \to B(L)$ *is spectrally continuous at* $h \in N$ *if, for each neighbourhood* V *of* $\mathrm{Sp}(\mu(h))$ *in* $\mathbb{C}$, *there is a neighbourhood* W_V *of* h *in* N *with* $\mathrm{Sp}(\mu(h')) \subseteq V$ *for* $h' \in W_V$.

In other words, μ is spectrally continuous if the multivalued map $h \mapsto \mathrm{Sp}(\mu(h))$ is upper semicontinuous on N [18]. If μ is norm-continuous, it is spectral continuous ([7], p 53).

Let π be a representation of G on a Hilbert space X and let π^N have an invariant subspace L:

$$X = L \oplus \mathfrak{H},\ \pi = \begin{pmatrix} \lambda & \xi \\ \rho & U \end{pmatrix} \text{ and } \pi^N = \begin{pmatrix} \lambda^N & \xi^N \\ 0 & U^N \end{pmatrix}, \tag{3.1}$$

where λ^N and U^N are representations of N, and $\mathfrak{H} = L^\perp$. The following sufficient conditions for L to be also π-invariant were obtained in [16].

Theorem 3.2 ([16]) *Let in* (3.1) $U^N = \backslash U_i^N]_{i=1}^k$, $k < \infty$, *let* λ^N *be spectrally disjoint with each* U_i^N *at some* $h^i \in N$, *and let* λ^N *be spectrally continuous at all* h^i. *Then* $\rho = 0$, *so that* L *is* π*-invariant.*

Corollary 3.3 *Let conditions in Theorem* 3.2 *hold, so that* $\rho = 0$ *in* (3.1).

(i) *If* ξ^N *is a* (λ^N, U^N)*-coboundary then* ξ *is a* (λ, U)*-coboundary.*

(ii) *If* N *is an Engel group then* ξ *is a* (λ, U)*-coboundary.*

Proof. (i) As $\rho = 0$, ξ is a (λ, U)-cocycle. If ξ^N is a (λ^N, U^N)-coboundary then, as λ^N is sectionally spectrally disjoint with U^N, it follows from Proposition 2.2 that ξ is a (λ, U)-coboundary.

(ii) If N is an Engel group then ξ is a (λ, U)-coboundary by Theorem 2.13. ■

Suppose that λ^N in (3.1) has an upper triangular form with respect to a decomposition

$$L = \oplus_{i=1}^k L_i,\ k < \infty,\ \text{i.e., } \lambda^N = \backslash \lambda_i^N]_{i=1}^k. \tag{3.2}$$

Let all λ_i^N be pairwise spectrally disjoint and let they be sectionally spectrally disjoint with U^N.

Corollary 3.4 (i) *If all* λ_i^N *are spectrally continuous at each* $h \in N$, *then all subspaces* $\oplus_{i=1}^j L_i$, $j \le k$, *are* π*-invariant, so that*

$$\pi = \begin{pmatrix} \lambda & \xi \\ 0 & U \end{pmatrix},\ \text{and } \lambda = \pi|_L = \backslash \lambda_i]_{i=1}^k \text{ with respect to decomposition (3.2).} \tag{3.3}$$

If all L_i *in* (3.2) *are* λ^N*-invariant then all* L_i *are also* π*-invariant*:

$$\lambda = \oplus_{i=1}^k \lambda_i. \tag{3.4}$$

(ii) *Let* N *be an Engel group. Then there exist* π*-invariant subspaces* $\{L_i'\}_{i=1}^k$ *and* H *and an equivalent scalar product on* X *such that*

$$L = \oplus_{i=1}^k L_i',\ X = L \oplus H \text{ and } \pi = \oplus_{i=1}^k \pi_i \oplus \pi|_H,\ \text{where } \pi_i = \pi|_{L_i'}. \tag{3.5}$$

If all L_i *in* (3.2) *are* λ^N*-invariant then they are also* π*-invariant and* $L_i' = L_i$.

Proof. (i) Set $X_1 = L_2 \oplus ... \oplus L_k \oplus \mathfrak{H}$. With respect to the decomposition $X = L_1 \oplus X_1$,

$$\pi = \begin{pmatrix} \lambda_1 & \xi_1 \\ \rho_1 & \pi_1 \end{pmatrix}, \ \pi^N = \begin{pmatrix} \lambda_1^N & \xi_1^N \\ 0 & \pi_1^N \end{pmatrix}, \text{ where } \pi_1^N = \begin{pmatrix} \lambda_2^N & * & * & * \\ 0 & \ddots & * & * \\ 0 & 0 & \lambda_k^N & * \\ 0 & 0 & 0 & U^N \end{pmatrix}.$$

As λ_1^N is spectrally disjoint with all λ_i^N, $2 \le i \le k$, and is sectionally spectrally disjoint with U^N, it is sectionally spectrally disjoint with π_1^N. As λ_1^N is spectrally continuous on N, we get $\rho_1 = 0$ by Theorem 3.2, so that L_1 is π-invariant.

Consider now the representation π_1 on X_1. Repeating the above process, we get that L_2 is invariant for π_1. So $L_1 \oplus L_2$ is π-invariant. Continuing this process, we conclude the proof of (3.3).

If some L_i is λ^N-invariant, take it instead of L_1. As above, it is π-invariant. This proves (3.4).

(ii) Let N be Engel. As each λ_i^N is sectionally spectrally disjoint with U^N and with every λ_j^N, $j \neq i$, we have $\mathcal{H}(G, \lambda_i, \lambda_j) = \mathcal{H}(G, \lambda_i, U) = 0$ by Theorem 2.13. It was proved in Proposition 2.5 and Remark 2.6 [13] that in this case there are subspaces $\{L_i'\}_{i=1}^k$ and H satisfying (3.5). ■

Recall that N^* is the group of all characters on N. The notions below were introduced in [13].

Definition 3.5 *A representation μ of a group N on L is called*

(i) *a χ-**representation** for some $\chi \in N^*$, if $\mathrm{Sp}(\mu(h)) = \{\chi(h)\}$ for all $h \in N$.*

(ii) ***elementary** if, for some $k < \infty$ and some equivalent scalar product,*

$$L = \oplus_{i=1}^k L_{\chi_i} \text{ and } \mu = \backslash\mu_{\chi_i}]_{i=1}^k \text{ with respect to this decomposition,} \tag{3.6}$$

where each μ_{χ_i} is a χ_i-representation of N on L_{χ_i} for some $\chi_i \in N^*$ (χ_i may repeat). Set

$$\mathrm{sign}(\mu) = \cup_{i=1}^k \chi_i, \text{ i.e., } \mathrm{sign}(\mu) \text{ consists of all characters from } \{\chi_i\}_{i=1}^k \text{ taken once.}$$

The set $\mathrm{sign}(\mu)$ does not depend on the choice of triangularization of L in (3.6) [13], and each elementary representation μ of a group N is spectrally continuous at all $h \in N$ [16].

The structure of elementary representations of connected Engel groups is simpler.

Proposition 3.6 [13, Corollary 2.18] *Let μ be a representation of a connected Engel group on L.*

(i) *If μ is elementary then there is a scalar product on L such that*

$$L = \oplus_{\chi \in \mathrm{sign}(\mu)} L_\chi, \text{ where each } L_\chi \text{ is } \mu\text{-invariant, so that } \mu = \oplus_{\chi \in \mathrm{sign}(\mu)} \mu_\chi \tag{3.7}$$

and $\mu_\chi = \mu|_{L_\chi}$ is a χ-representation of the group.

(ii) *If $\dim L < \infty$ then μ is elementary, so that (3.7) holds.*

For non-Engel groups, decomposition (3.7) does not always exist (see Example 3.10).

Remark 3.7 *A χ-representation μ of N is sectionally spectrally disjoint with U^N if and only if χ is sectionally spectrally disjoint with U^N. If $\chi \neq \omega \in N^*$ then ω is spectrally disjoint with μ.*

Corollary 3.4 and Remark 3.7 yield

Corollary 3.8 *Let the representation* $\lambda^N = \pi^N|_L$ *in* (3.1) *be elementary:*

$$L = \oplus_{i=1}^k L_i,\ k < \infty,\ \text{and } \lambda^N = \backslash \lambda_i^N]_{i=1}^k, \tag{3.8}$$

where λ_i^N *are* χ_i*-representations of* N *on* L_i, $\chi_i \in N^*$. *Let* $\{\chi_i\}_{i=1}^k$ *be distinct and sectionally spectrally disjoint with* U^N. *Then*

(i) L *is* π*-invariant and* $\lambda := \pi|_L = \backslash \lambda_i]_{i=1}^k$ *with respect to decomposition* (3.8).

(ii) *If all* L_i *are* λ^N*-invariant* ($\lambda^N = \oplus_{i=1}^k \lambda_i^N$), *they are also* λ*-invariant:* $\lambda = \oplus_{i=1}^k \lambda_i$.

If N is a connected Engel group, Corollary 3.8 admits the following refinement.

Theorem 3.9 *Let* N *be an Engel group.*

(i) *Let* $\lambda^N = \pi^N|_L$ *in* (3.1) *be elementary, so that, by Proposition* 3.6,

$$L = \oplus_{\chi \in \text{sign}(\lambda^N)} L_\chi,\ \text{all } L_\chi \text{ are } \lambda^N\text{-invariant and } \lambda^N = \oplus_{\chi \in \text{sign}(\lambda^N)} \lambda_\chi^N$$

in some scalar product on L, *where* $\lambda_\chi^N = \lambda^N|_{L_\chi}$ *are* χ*-representations of* N.

If all $\chi \in \text{sign}(\lambda^N)$ *are sectionally spectrally disjoint with* U^N, *then all* L_χ *are* π*-invariant and there exist a* π*-invariant subspace* H *and an equivalent scalar product on* X *such that*

$$X = L \oplus H \text{ and } \pi = \left(\oplus_{\chi \in \text{sign}(\lambda^N)} \lambda_\chi\right) \oplus \pi|_H,\ \text{where } \lambda_\chi = \pi|_{L_\chi}.$$

(ii) *If* $\dim L < \infty$ *then* $\lambda^N = \pi^N|_L$ *in* (3.1) *is elementary by Proposition* 3.6. *So* (i) *holds, if all* $\chi \in \text{sign}(\lambda^N)$ *are sectionally spectrally disjoint with* U^N.

For each character χ_i in Corollary 3.8, $\chi_i(h) = \text{Sp}(\lambda_i(h))$ for $h \in N$. As λ_i is a representation,

$$\chi_i(ghg^{-1}) = \text{Sp}(\lambda_i(ghg^{-1})) = \text{Sp}(\lambda_i(g)\lambda_i(h)\lambda_i^{-1}(g)) = \text{Sp}(\lambda_i(h)) = \chi_i(h) \text{ for } g \in G. \tag{3.9}$$

We illustrate Theorem 3.9 with the following example. Consider the connected groups

$$G = \left\{ g(a,b,x) = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} : a,b \in \mathbb{R}_+,\ x \in \mathbb{R} \right\} \text{ and } N = \{g(a,a,x) \in G\}. \tag{3.10}$$

Then G is a solvable and N is a normal commutative subgroup of G (N is Engel). For $\chi \in G^*$,

$$\chi(g) = \chi_{z,u}(g(a,b,x)) = a^z b^u \text{ and } \chi_{z,u}^N(g(a,a,x)) = a^{z+u} \text{ for some } z,u \in \mathbb{C}.$$

Let λ be a representation of G on X, $\dim X = n$. By the Lie theorem, $X = \oplus_{i=1}^n X_i$ and $\lambda = \backslash \lambda_i]_{i=1}^n$, where $\dim X_i = 1$ and $\lambda_i = \chi_i \mathbf{1}_{X_i}$. Even if all $\{\chi_i\}_{i=1}^n$ are different, λ does not necessarily decompose in the sum of χ_i-representations, as G is not an Engel group.

Set $s_i = z_i + u_i$. Then $\{\delta_i = \chi_i^N\}_{i=1}^n$, are characters on N: $\delta_i(g(a,a,x)) = \chi_i^N(g(a,a,x)) = a^{s_i}$ and $\text{sign}(\lambda^N)$ consists of all distinct characters from the set $\{\delta_i\}_{i=1}^n$. By Theorem 3.9,

$$X = \sum_{\delta \in \text{sign}(\lambda^N)} X_\delta,\ \text{each } X_\delta \text{ is } \lambda\text{-invariant and } \lambda^N|_{X_\delta} \text{ is a } \delta\text{-representation of } N. \tag{3.11}$$

Example 3.10 (i) Let $X = \mathbb{C}^2$. The representation $\lambda(g) = g$ of G on X is elementary with characters $\chi_{1,0}$, $\chi_{0,1}$. However, X does not decompose in the sum of λ-invariant subspaces, since $\mathrm{sign}(\lambda^N) = \{\delta\}$, where $\delta = \chi_1^N = \chi_2^N$. Thus $X = X_\delta$ in (3.11).

(ii) The representation λ: $g(a,b,x) \to \begin{pmatrix} a^2 & a^2 - b^3 \\ 0 & b^3 \end{pmatrix}$ on $\mathbb{C}^2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ is elementary with $\chi_{2,0}$, $\chi_{0,3}$. So $\mathrm{sign}(\lambda^N) = \{\delta_1, \delta_2\}$, where $\delta_i = \chi_i^N$, and

$$\mathbb{C}^2 = X_{\delta_1} \dotplus X_{\delta_2}, \text{ where } X_{\delta_1} = \mathbb{C}e_1 \text{ and } X_{\delta_2} = \mathbb{C}(-e_1 \oplus e_2)$$

are λ^N-invariant subspaces: $\lambda^N|_{X_{\delta_i}}$, $i = 1, 2$, are δ_i-representations, since

$$\lambda^N(g)e_1 = a^2 e_1 = \delta_1(g)e_1 \text{ and } \lambda^N(g)(-e_1 \oplus e_2) = a^3(-e_1 \oplus e_2) = \delta_2(g)(-e_1 \oplus e_2),$$

for $g = g(a,a,x) \in N$. It is easy to see that $X_{\delta_1}, X_{\delta_2}$ are also λ-invariant.

4 Cocycles on Engel and nilpotent groups

Let N be a connected locally compact group. The subgroup $N^{[1]}$ is normal and $\widehat{N} = N/N^{[1]}$ is connected, locally compact and commutative. Then ([20, Theorem 26]) $\widehat{N}$ is isomorphic to the direct product of a connected compact commutative subgroup K and $\mathbb{R}^l$ for some $l := l_{_N} \in \mathbb{N}$:

$$\widehat{N} \approx \mathbb{R}^l \times K \text{ and } (x,k)(y,k_1) = (x+y, kk_1) \text{ for } x, y \in \mathbb{R}^l,\ k, k_1 \in K. \tag{4.1}$$

We identify $\widehat{N}$ with $\mathbb{R}^l \times K$. Let p: $N \to \widehat{N}$ be the standard epimorphism and r: $(x,k) \mapsto x$ be the natural epimorphism of $\widehat{N}$ onto $\mathbb{R}^l$. Set

$$\omega_{_N} = r \circ p, \text{ so that } \omega_{_N}(h) = r(p(h)) = r(\widehat{h}) \text{ for } h \in N, \widehat{h} \in \widehat{N}. \tag{4.2}$$

Then $\omega_{_N}$: $N \to \mathbb{R}^l$ is a continuous epimorphism:

$$\omega_{_N}(hh_1) = r(\widehat{h}\,\widehat{h_1}) = r(\widehat{h}) + r(\widehat{h_1}) = \omega_{_N}(h) + \omega_{_N}(h_1). \tag{4.3}$$

Proposition 4.1 (i) *Let* $\chi, \theta \in N^*$ *and* λ *be a* χ*-representation of* N *on* L, $\dim L < \infty$. *Set*

$$\ker \xi = \cap_{h \in N} \ker \xi(h) \textit{ for a } (\lambda, \theta\mathbf{1}_{\mathfrak{H}})\textit{-cocycle } \xi. \tag{4.4}$$

1) *If* $\theta = \chi$ *then* $\dim(\mathfrak{H} \ominus \ker \xi) \le l \dim L$.

2) *If* $\theta \ne \chi$ *and* N *is Engel, then* $\dim(\mathfrak{H} \ominus \ker \xi) < \dim L$.

(ii) *A map* $0 \ne \xi$: $N \to B(\mathfrak{H}, L)$ *is a* $(\chi\mathbf{1}_L, \chi\mathbf{1}_{\mathfrak{H}})$*-cocycle if and only if there is a linear map* β: $\mathbb{R}^l \to B(\mathfrak{H}, L)$ *such that* $\xi(h) = \chi(h)\beta(\omega_{_N}(h))$ *for* $h \in N$.

Moreover, $l \ne 0$, $\dim_{\mathbb{R}}(\xi(N)) \le l$ *and* ξ *is not a coboundary.*

Proof. (i) Part 1) was proved in Proposition 3.10 [13].

(i) 2). By Remark 3.7, ϕ is spectrally disjoint with λ. As N is Engel, ξ is a $(\lambda, \phi\mathbf{1}_{\mathfrak{H}})$-coboundary by Theorem 2.10: $\xi = \lambda T - \phi T$, for some $T \in B(\mathfrak{H}, L)$. Let $\{e_i\}_{i=1}^n$ be a basis in L. Then $Tx = \sum_{i=1}^n (x, y_i)_{\mathfrak{H}} e_i$ for all $x \in \mathfrak{H}$ and some $y_i \in \mathfrak{H}$. Hence $\dim(\mathfrak{H} \ominus \ker T) \le n$. As $\ker(\xi(h)) = \ker(\lambda(h) - \phi(h)\mathbf{1}_L)T \supseteq \ker T$ for $h \in N$, we have $\ker \xi \supseteq \ker T$. So $\dim(\mathfrak{H} \ominus \ker \xi) \le \dim(\mathfrak{H} \ominus \ker T) \le n$.

(ii) Let ξ be a $(\chi\mathbf{1}_L, \chi\mathbf{1}_{\mathfrak{H}})$-cocycle. Clearly, ξ is a coboundary if and only if $\xi = 0$.

Let $\xi \neq 0$. Set $\alpha(h) = \xi(h)/\chi(h)$ for $h \in N$. Then $\alpha \neq 0$. As $\lambda = \chi\mathbf{1}_L$, $U = \chi\mathbf{1}_{\mathfrak{H}}$,

$$\alpha(th) \overset{(1.1)}{=} \chi(th)^{-1}(\lambda(t)\xi(h) + \xi(t)U(h)) = \alpha(h) + \alpha(t) \text{ for } t, h \in N. \tag{4.5}$$

Let $\omega_{_N}$ be the map defined in (4.2). As $B(\mathfrak{H}, L)$ is isomorphic to the Hilbert space $\mathfrak{H} \otimes L$, it follows from (4.5) and Corollary 2.2 [13] that there is a linear map β: $\mathbb{R}^l \to B(\mathfrak{H}, L)$ such that

$$\alpha(h) = \beta(\omega_{_N}(h)) \text{ for } h \in N,\ l \neq 0 \text{ and } \dim_{\mathbb{R}}(\alpha(N)) \leq l. \tag{4.6}$$

The converse statement follows from (4.3). ■

Let λ, U be representations of an Engel group N on L, $\mathfrak{H}$. From now on we assume that $\dim L < \infty$ and U is unitary. By Proposition 3.6 and (3.7), $\lambda = \oplus_{\chi\in\mathrm{sign}(\lambda)}\lambda_\chi$ is elementary. Set

$$\begin{gathered}\mathfrak{H}_\theta := \{x \in \mathfrak{H}\colon U(h)x = \theta(h)x \text{ for all } h \in N\} \text{ for } \theta \in N^*,\\ \Theta = \{\theta \in \ \mathrm{sign}(\lambda)\colon \mathfrak{H}_\theta \neq \{0\}\},\ \mathfrak{H}_\Theta = \oplus_{\theta\in\Theta}\mathfrak{H}_\theta \text{ and } \mathfrak{H}_\aleph = \mathfrak{H} \ominus \mathfrak{H}_\Theta.\end{gathered} \tag{4.7}$$

Then $\mathfrak{H} = \mathfrak{H}_\Theta \oplus \mathfrak{H}_\aleph$, all $\mathfrak{H}_\theta$ are U-invariant, and $U|_{\mathfrak{H}_\theta} = \theta\mathbf{1}_{\mathfrak{H}_\theta}$.

Let ξ be a (λ, U)-cocycle. With respect to decompositions (3.7) of L and (4.7) of $\mathfrak{H}$,

$$\xi = (\xi|_{\mathfrak{H}_\Theta}, \xi|_{\mathfrak{H}_\aleph}), \text{ where } \xi|_{\mathfrak{H}_\Theta} = (\xi^{\chi,\theta})_{\chi\in\ \mathrm{sign}(\lambda),\ \theta\in\Theta}, \tag{4.8}$$

$\xi^{\chi,\theta}$ are $(\lambda_\chi, \theta\mathbf{1}_{\mathfrak{H}_\theta})$-cocycles, $\xi|_{\mathfrak{H}_\aleph}$ is a $(\lambda, U|_{\mathfrak{H}_\aleph})$-cocycle. We say that λ and U are *eigen-disjoint* if

$$\Theta = \varnothing, \text{ i.e., } \mathfrak{H}_\Theta = \{0\}. \tag{4.9}$$

Theorem 2.13 and Proposition 4.1 yield

Corollary 4.2 (i) *If $\chi \in \mathrm{sign}(\lambda)$ and $\chi \neq \theta \in \Theta$, then $\xi^{\chi,\theta}$ is a $(\lambda_\chi, \theta\mathbf{1}_{\mathfrak{H}_\theta})$-coboundary.*

(ii) *If $\lambda_\chi = \chi\mathbf{1}_{L_\chi}$ and $\chi \in \Theta$, there is a linear map β_χ: $\mathbb{R}^l \to B(\mathfrak{H}_\chi, L_\chi)$ such that*

$$\xi^{\chi,\chi}(h) = \chi(h)\beta_\chi(\omega_{_N}(h)) \text{ for } h \in N.$$

(iii) *λ and $U|_{\mathfrak{H}_\aleph}$ are eigen-disjoint.*

(iv) *If all $\chi \in \mathrm{sign}(\lambda)$ are sectionally spectrally disjoint with $U|_{\mathfrak{H}_\aleph}$, then $\xi|_{\mathfrak{H}_\aleph}$ is a $(\lambda, U|_{\mathfrak{H}_\aleph})$-coboundary.*

The structure of $(\lambda_\chi, \chi\mathbf{1}_{\mathfrak{H}_\chi})$-cocycles $\xi^{\chi,\chi}$ on N when $\lambda_\chi \neq \chi\mathbf{1}_{L_\chi}$ can be complicated.

If N is nilpotent, the $(\lambda, U|_{\mathfrak{H}_\aleph})$-cocycle $\xi|_{\mathfrak{H}_\aleph}$ can be described in a general setting.

Theorem 4.3 *Let N be a nilpotent group and ξ be a (λ, U)-cocycle. There is a decomposition $\mathfrak{H}_\aleph = \oplus_{i=1}^n\mathfrak{H}_i$, $n \leq \infty$, such that all subspaces $\mathfrak{H}_i$ are U-invariant and all $(\lambda, U|_{\mathfrak{H}_i})$-cocycles $\xi|_{\mathfrak{H}_i}$: $N \to B(\mathfrak{H}_i, L)$ are coboundaries. If $n < \infty$, $\xi|_{\mathfrak{H}_\aleph}$ is a coboundary.*

Proof. By Corollary 4.2, λ and $U|_{\mathfrak{H}_\aleph}$ are eigen-disjoint. It follows from Proposition 3.3 [13] that $\mathfrak{H}_\aleph = \oplus_{i=1}^n\mathfrak{H}_i$, $n \leq \infty$, where all $\mathfrak{H}_i$ are U-invariant and each $\chi \in \mathrm{sign}(\lambda)$ is spectrally disjoint with all $U|_{\mathfrak{H}_i}$. As N is an Engel group, all $\mathcal{H}(N, \lambda_\chi, U|_{\mathfrak{H}_i}) = 0$ by Theorem 2.13. Thus each $(\lambda_\chi, U|_{\mathfrak{H}_i})$-cocycle is a coboundary. So all $(\lambda, U|_{\mathfrak{H}_i})$-cocycles are coboundaries: All $\xi|_{\mathfrak{H}_i}$ are coboundaries.

If $n < \infty$ then $\xi|_{\mathfrak{H}_\aleph}$ is a $(\lambda, U|_{\mathfrak{H}_\aleph})$-coboundary, as all $\xi|_{\mathfrak{H}_i}$ are coboundaries. ■

For $\lambda = \chi\mathbf{1}_L$ and an irreducible U, Theorem 4.3 was obtained in [5] (see p. 254 part (c)).

5 Extensions of representations and their decompositions.

Let π be a representation of G on X and L be a π-invariant subspace. We say that L *splits* π, if it has a π-invariant complement H, i.e.,

$$X = L \dotplus H \text{ and } \pi(g)H = H \text{ for all } g \in G.$$

We say that L *approximately splits* π, if there are π-invariant subspaces $(Y_i, Z_i)_{i=1}^{\infty}$ in X such that

$$X = Y_i \dotplus Z_i,\ Y_{i+1} \subsetneqq Y_i,\ Z_i \subsetneqq Z_{i+1} \text{ for all } i, \text{ and } L \subseteq M = \cap_i Y_i \text{ with } \dim(M/L) < \infty. \tag{5.1}$$

Let λ, U be representations of G on L and $\mathfrak{H}$. For a (λ, U)-cocycle ξ on G, the representation

$$\mathfrak{e}(g) = \mathfrak{e}(\lambda, U, \xi)(g) = \begin{pmatrix} \lambda(g) & \xi(g) \\ 0 & U(g) \end{pmatrix} \text{ of } G \text{ on } X = L \oplus \mathfrak{H} \tag{5.2}$$

is called *the extension of* λ *by* U *performed by* ξ. We will assume that U is *unitary* and $\dim L < \infty$.

It is well known that L splits $\mathfrak{e}$ if and only if ξ is a (λ, U)-coboundary: $\xi = \lambda T - TU$ for some $T \in B(\mathfrak{H}, L)$, so that $H = \{x - Tx\colon x \in \mathfrak{H}\}$.

Let in (5.2) $\mathfrak{H} = \oplus_{i=0}^{\infty} \mathfrak{H}_i$ and all $\mathfrak{H}_i$ be U-invariant. So $U = \oplus_{i=0}^{\infty} U_i$, where $U_i = U|_{\mathfrak{H}_i}$. Then

$$\xi = \{\xi_i\}_{i=0}^{\infty}, \text{ where } \xi_i = \xi|_{\mathfrak{H}_i}\colon G \to B(\mathfrak{H}_i, L) \text{ are } (\lambda, U_i)\text{-cocycles.}$$

Lemma 5.1 *If* $\dim \mathfrak{H}_0 < \infty$ *and* $\{\xi_i\}_{i=1}^{\infty}$ *are* (λ, U_i)*-coboundaries, then* L *approximately splits* $\mathfrak{e}$.

Proof. As $\xi_i = \lambda T_i - T_i U_i$, $T_i \in B(\mathfrak{H}_i, L)$, $1 \le i$, all $\widehat{\mathfrak{H}_i} = \{-T_i x + x\colon x \in \mathfrak{H}_i\}$ are $\mathfrak{e}$-invariant:

$$\mathfrak{e}(g)(-T_i x \oplus x) = (-\lambda(g) T_i x + \xi_i(g) x) \oplus U_i(g) x = -T_i U_i(g) x \oplus U_i(g) x \in \widehat{\mathfrak{H}_i}.$$

Thus $Y_i = L \oplus \mathfrak{H}_0 \oplus \left(\oplus_{k=i+1}^{\infty} \mathfrak{H}_i\right)$ and $Z_i = \sum_{k=1}^{i} \widehat{\mathfrak{H}_k}$ are $\mathfrak{e}$-invariant subspaces and (5.1) holds for $M = L \oplus \mathfrak{H}_0$. So L approximately splits $\mathfrak{e}$. ■

Theorem 5.2 *Let* $G = N$ *be nilpotent, let* λ, U *be representations of* N *on* $L, \mathfrak{H}$ *and* $\dim L < \infty$. *For any* (λ, U)*-cocycle* ξ, L *approximately splits* $\mathfrak{e} = \mathfrak{e}(\lambda, U, \xi)$.

Proof. We have from (3.7) that $\lambda = \oplus_{\chi \in \mathrm{sign}(\lambda)} \lambda_\chi$ is elementary. By (4.7) and (4.8), there is $\Theta \subseteq \mathrm{sign}(\lambda)$ such that $\mathfrak{H}_\Theta = \oplus_{\theta \in \Theta} \mathfrak{H}_\theta$, where $\mathfrak{H}_\theta$ are U-invariant subspaces,

$$\mathfrak{H} = \mathfrak{H}_\Theta \oplus \mathfrak{H}_\aleph,\ \xi = (\xi|_{\mathfrak{H}_\Theta}, \xi|_{\mathfrak{H}_\aleph}) \text{ and } \xi|_{\mathfrak{H}_\Theta} = (\xi^{\chi,\theta})_{\chi \in \mathrm{\ sign}(\lambda), \theta \in \Theta},$$

$\xi^{\chi,\theta}\colon \mathfrak{H}_\theta \to L_\chi$ are $(\lambda_\chi, \theta \mathbf{1}_{\mathfrak{H}_\theta})$-cocycles and $\xi|_{\mathfrak{H}_\aleph}$ is a $(\lambda, U|_{\mathfrak{H}_\aleph})$-cocycle. For $\theta \in \Theta$, set (cf. (4.4))

$$\ker \xi|_{\mathfrak{H}_\theta} = \cap_{\chi \in \mathrm{sign}(\lambda)} \ker \xi^{\chi,\theta} \subseteq \mathfrak{H}_\theta. \tag{5.3}$$

Set also $\mathfrak{H}_1 = \oplus_{\theta \in \Theta} \ker \xi|_{\mathfrak{H}_\theta}$ and $\mathfrak{H}_0 = \mathfrak{H}_\Theta \ominus \mathfrak{H}_1$. Then $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1 \oplus \mathfrak{H}_\aleph$ and

$$\mathfrak{H}_0 = \mathfrak{H}_\Theta \ominus (\oplus_{\theta \in \Theta} \ker \xi|_{\mathfrak{H}_\theta}) = \oplus_{\theta \in \Theta} (\mathfrak{H}_\theta \ominus \ker \xi|_{\mathfrak{H}_\theta}). \tag{5.4}$$

By Proposition 4.1, $\dim(\mathfrak{H}_\theta \ominus \ker \xi^{\chi,\theta}) < \infty$. Hence

$$\dim(\mathfrak{H}_\theta \ominus \ker \xi|_{\mathfrak{H}_\theta}) \overset{(5.3)}{=} \dim(\mathfrak{H}_\theta \ominus (\cap_{\chi \in \mathrm{sign}(\lambda)} \ker \xi^{\chi,\theta})) < \infty. \tag{5.5}$$

As $\operatorname{sign}(\lambda)$ is a finite set, it follows from (5.4), (5.5) that $\dim \mathfrak{H}_0 < \infty$.

By (4.7), λ and $U|_{\mathfrak{H}_0}$ are eigen-disjoint representations (see (4.9)). If N is nilpotent then, by Theorem 4.3, there is a decomposition $\mathfrak{H}_\aleph = \oplus_{i=2}^{\infty}\mathfrak{H}_i$, where all $\mathfrak{H}_i$ are U-invariant and $(\lambda, U|_{\mathfrak{H}_i})$-cocycles $\xi|_{\mathfrak{H}_i}\colon N \to B(\mathfrak{H}_i, L)$ are coboundaries. As $\xi|_{\mathfrak{H}_1} = 0$, all $\xi|_{\mathfrak{H}_i}$, $1 \le i < \infty$, are coboundaries. As $\dim \mathfrak{H}_0 < \infty$, it follows from Lemma 5.1 that L approximately splits $\mathfrak{e}$. ■

Let the normal subgroup N of G be an Engel group. By Proposition 3.6 and Theorem 3.9,

$$L = \oplus_{\chi\in\operatorname{sign}(\lambda^N)} L_\chi,\ \text{all } L_\chi \text{ are } \lambda\text{-invariant},\ \lambda = \oplus_{\chi\in\operatorname{sign}(\lambda^N)}\lambda_\chi \text{ with } \lambda_\chi = \lambda|_{L_\chi}, \tag{5.6}$$

and λ_χ^N are χ-representations of N on L_χ. For $F \subseteq \operatorname{sign}(\lambda^N)$, set

$$L_F = \oplus_{\chi\in F} L_\chi,\quad \lambda_F = \oplus_{\chi\in F}\lambda_\chi \text{ and } F^c = \operatorname{sign}(\lambda^N)\diagdown F.$$

Suppose that $\mathfrak{H} = \oplus_{j=1}^{m}\mathfrak{H}_j$, $m < \infty$, where all $\mathfrak{H}_j$ are U-invariant. Set $\Phi = \cup_{j=1}^{m} F_j$, where

$$F_j = \{\chi \in \operatorname{sign}(\lambda^N)\colon \chi \text{ is } \textbf{not} \text{ sectionally spectrally disjoint with } U^N|_{\mathfrak{H}_j}\}. \tag{5.7}$$

In [16] the following decomposition of the representation $\mathfrak{e}(\lambda, U, \xi)$ of G was obtained.

Theorem 5.3 *Let $F_j \cap F_k = \varnothing$ for $j \neq k$. Then there are operators $T_j \in B(\mathfrak{H}_j, L_{F_j^c})$ such that*

$$X = L_{\Phi^c} \dotplus X_1 \dotplus \dots \dotplus X_m \text{ is the direct sum of } \mathfrak{e}\text{-invariant subspaces}, \tag{5.8}$$

where $X_j = L_{F_j} \dotplus \widehat{\mathfrak{H}}_j$ and $\widehat{\mathfrak{H}}_j = \{-T_j x + x\colon x \in \mathfrak{H}_j\}$.

The representations $\mathfrak{e}|_{X_j}$ are similar to some extensions $\mathfrak{e}(\lambda_{F_j}, U|_{\mathfrak{H}_j}, \xi_j)$ of λ_{F_j} by $U|_{\mathfrak{H}_j}$.

Using Theorem 5.3, we will study the decomposition of $\mathfrak{e}(\lambda, U, \xi)$ in the sum of extensions with N-monothetic λ.

Definition 5.4 *We say that $\mathfrak{e}(\lambda, U, \xi)$ is N-**primary**, if λ^N is a χ-representation for some $\chi \in N^*$.*

If $\dim \mathfrak{H} < \infty$, extensions $\mathfrak{e}(\lambda, U, \xi)$ decompose in N-primary components by Proposition 3.6 and Theorem 3.9. In [13, Theorem 3.17] it was proved that if $G = N$ is commutative, the same is true when $\dim \mathfrak{H} = \infty$. Our next result is its generalization.

Recall that if M is a commutative group of unitary operators on $\mathfrak{H}$ then

$$\mathfrak{H} = \int_{M_u^*} \oplus\, \mathfrak{H}_\omega dP(\omega) \text{ and } T = \int_{M_u^*} \omega(T) dP(\omega) \text{ for } T \in M, \tag{5.9}$$

where M_u^* is the dual group of unitary characters on M and $P(\cdot)$ is a spectral measure on M_u^*.

Theorem 5.5 *Let λ and U be representations of G on L and $\mathfrak{H}$, let U be unitary, N be an Engel group and $\dim L < \infty$. If U maps N in the center of $U(G)$, then $\mathfrak{e} = \mathfrak{e}(\lambda, U, \xi)$ decomposes in the direct sum of N-primary components and a representation similar to unitary.*

Proof. Set $X = L \oplus \mathfrak{H}$. Let $\operatorname{sign}(\lambda^N) = \{\chi_i\}_{i=1}^n$. It follows from Lemma 2.12 [13] that, for each $i \in [1, n]$, there is $h_i \in N$ such that $\chi_i(h_i) \notin \{\chi_k(h_i)\colon\ k \in [1, n],\ k \neq i\}$. Let

$$\varepsilon_i = \frac{1}{2} \min\{|\chi_i(h_i) - \chi_k(h_i)|:\ k \in [1, n],\ k \neq i\} \text{ and } \varepsilon = \min\{\varepsilon_i\colon\ i \in [1, n]\}.$$

Then

$$|\chi_i(h_i) - \chi_k(h_i)| > \varepsilon \text{ for all } i \neq k \text{ in } [1, n]. \tag{5.10}$$

Let $M = U(N)$. Since M is commutative, (5.9) holds. For all $j \in [1, n]$, we set

$$\begin{aligned} \Omega_j = \{\omega \in M_u^*\colon\ &|\omega(U(h_i)) - \chi_i(h_i)| \geq \varepsilon \\ \text{for } 1 \leq i \leq j - 1, \text{ and } &|\omega(U(h_j)) - \chi_j(h_j)| < \varepsilon\}. \end{aligned} \tag{5.11}$$

Set also

$$\Omega_0 = \{\omega \in M_u^*\colon\ |\omega(U(h_i)) - \chi_i(h_i)| \geq \varepsilon \text{ for all } i \in [1, n]\}. \tag{5.12}$$

Then $\Omega_j \cap \Omega_k = \varnothing$ for all $j \neq k$, and $\cup_{j=0}^n \Omega_j = M_u^*$.

By (5.9), for all $j \in [0, n]$, the subspaces

$$\mathfrak{H}_j := \int_{\Omega_j} \oplus \mathfrak{H}_\omega dP(\omega) \text{ are } U\text{-invariant (some } \mathfrak{H}_j \text{ may be } \{0\}) \text{ and } \sum_{j=0}^n \oplus \mathfrak{H}_j = \mathfrak{H}.$$

Set $U_j = U|_{\mathfrak{H}_j}$. Let $\mathfrak{H}_j \neq 0$ and $1 \leq i < j \leq n$. As $\{h_k\}_{k=1}^n \subset N$, we have from (5.9) and (5.11)

$$\begin{aligned} \operatorname{Sp}(U_j(h_i)) &= \overline{\{\omega(U(h_i))\colon\ \omega \in \Omega_j\}} \subseteq \{z \in \mathbb{C}\colon\ |z - \chi_i(h_i)| > \varepsilon\} \\ \operatorname{Sp}(U_j(h_j)) &= \overline{\{\omega(U(h_j))\colon\ \omega \in \Omega_j\}} \subseteq \{z \in \mathbb{C}\colon\ |z - \chi_j(h_j)| < \varepsilon\}. \end{aligned}$$

So $\chi_i(h_i) \notin \operatorname{Sp}(U_j(h_i))$ for $i < j$. Thus χ_i is spectrally disjoint with U_j^N at h_i.

On the other hand, if $1 \leq j < i \leq n$ then, by (5.10), $|\chi_j(h_j) - \chi_i(h_j)| > \varepsilon$. So $\chi_i(h_j) \notin \operatorname{Sp}(U_j(h_j))$. Thus χ_i is spectrally disjoint with U_j^N at h_j for $j < i$.

Finally, for $i \in [1, n]$, we have from (5.9) and (5.12) that

$$\operatorname{Sp}(U_0(h_i)) = \overline{\{\omega(U(h_i))\colon\ \omega \in \Omega_0\}} \subseteq \{z \in \mathbb{C}\colon\ |z - \chi_i(h_i)| > \varepsilon\}.$$

So $\chi_i(h_i) \notin \operatorname{Sp}(U_0(h_i))$. Thus χ_i is spectrally disjoint with U_0^N at h_i. Hence each character $\chi_i \in \operatorname{sign}(\lambda^N)$ is spectrally disjoint with all representations U_j^N, $i \neq j \in [0, n]$.

For $j \in [0, n]$, consider the subsets F_j of $\operatorname{sign}(\lambda^N)$ defined in (5.7). Then $F_0 = \varnothing$ and, for each $j \in [1, n]$, either $F_j = \varnothing$, or $F_j = \{\chi_j\}$. Thus $F_j \cap F_k = \varnothing$ for all $j \neq k$ in $[0, n]$.

Set $\Phi^c = \operatorname{sign}(\lambda^N) \diagdown \cup_{j=0}^n F_j$. By Theorem 5.3, X is the direct sum of $\mathfrak{e}$-invariant subspaces:

$$X = \sum_{\chi \in \Phi^c} L_\chi \dotplus \sum_{j=0}^n X_j = \sum_{\chi \in \Phi^c} L_\chi \dotplus \sum_{j \in [0,n], F_j = \{\chi_j\}} X_j \dotplus \sum_{j \in [0,n], F_j = \varnothing} X_j, \tag{5.13}$$

where $X_j = L_{F_j} \dotplus \widehat{\mathfrak{H}_j}$ and $\widehat{\mathfrak{H}_j} = \{-T_j x + x\colon\ x \in \mathfrak{H}_j\}$ for some operators $T_j \in B(\mathfrak{H}_j, L_{F_j^c})$.

As it was mentioned after Definition 5.4, all representations $\mathfrak{e}|_{L_\chi}$, $\chi \in \Phi^c$, are N-primary.

If $F_j = \{\chi_j\}$ then $\mathfrak{e}|_{X_j} = \begin{pmatrix} \lambda_{\chi_j} & \eta_j \\ 0 & \widehat{U}_j \end{pmatrix}$, where the representation $\widehat{U}_j$ on $\widehat{\mathfrak{H}_j}$ is similar to U_j. So $\mathfrak{e}|_{X_j}$ is N-primary, since $\lambda_{\chi_j}^N$ is a χ_i-representation. If $F_j = \varnothing$ then $X_j = \widehat{\mathfrak{H}}_j$ and $\mathfrak{e}|_{X_j}$ is similar to U_j. Joining all such X_j, we complete the proof. ∎

Corollary 5.6 *Let* N *be an Engel group and let* $\dim L < \infty$. *Let* $\mathfrak{H} = \oplus_{i=1}^{\infty}\mathfrak{H}_i$ *and* $U^N = \oplus_{i=1}^{\infty}\omega_i \mathbf{1}_{\mathfrak{H}_i}$ *for distinct unitary* $\omega_i \in N^*$. *Then* $\mathfrak{e}(\lambda, U, \xi)$ *decomposes in the direct sum of* N*-primary components and a representation similar to unitary.*

Proof. By Proposition 2.15, all subspaces $\mathfrak{H}_i$ are U-invariant. So conditions (5.9) hold and the proof follows from Theorem 5.5. ■

Corollary 5.7 *Let* N *be commutative and compact and let* $\dim L < \infty$. *Any representation* $\mathfrak{e}(\lambda, U, \xi)$ *of* G *decomposes in the direct sum of* N*-primary components and a representation similar to unitary.*

Proof. As N is commutative and compact, all characters on N are unitary, so that $N^* = N_u^*$ is a discrete group of unitary characters. Since $\dim L < \infty$, λ^N is an elementary representation.

As N is compact, we can assume that U^N is a unitary representation of N. Hence $\mathfrak{H} = \oplus_{\omega_i \in \Omega}\mathfrak{H}_{\omega_i}$ and $U^N = \oplus_{\omega_i \in \Omega}\omega_i \mathbf{1}_{\mathfrak{H}_{\omega_i}}$ for some subset Ω of characters in N_u^*. Applying Corollary 5.6, we complete the proof. ■

6 Cocycles for (N, χ)-trivial representations of groups

Here we consider the simplest (and, in some sense, the base) class of (N, χ)-*trivial representations* that are not spectrally disjoint on N. Recall (see (1.4)) that λ, U are (N, χ)-trivial if

$$\lambda(h) = \chi(h)\mathbf{1}_L \text{ and } U(h) = \chi(h)\mathbf{1}_{\mathfrak{H}} \text{ for } h \in N \text{ and some } \chi \in N^*. \tag{6.1}$$

Characters $\chi \in N^*$ satisfying condition (6.1) are G-invariant:

$$\chi(h_g) = \chi(h) \text{ for } h \in N, g \in G, \text{ where } h_g = ghg^{-1} \in N, \text{ since} \tag{6.2}$$

$$\chi(h_g)\mathbf{1}_{\mathfrak{H}} = U(ghg^{-1}) = U(g)U(h)U(g)^{-1} \overset{(6.1)}{=} U(g)\chi(h)U(g)^{-1} = \chi(h)\mathbf{1}_{\mathfrak{H}}.$$

Condition (6.1) allows us to obtain a description of a wide range of non-trivial (λ, U)-cocycles for the semi-direct products of groups $G = D \ltimes N$ (especially, if D is Engel) without turning it into a trivial task. It also permits us to describe the structure of the corresponding extensions $\mathfrak{e}(\lambda, U, \xi)$.

Especially important for us are (N, χ_e)-trivial representations. We call them N-*trivial*:

$$\lambda(h) = \mathbf{1}_L \text{ and } U(h) = \mathbf{1}_{\mathfrak{H}} \text{ for } h \in N. \tag{6.3}$$

6.1 Inner representations of groups and (N, χ)-trivial representations

As N is normal in G, the subgroup $N^{[1]}$ is normal in G, the quotient groups $\widehat{G} := G/N^{[1]}$ and $\widehat{N} := N/N^{[1]}$ are connected locally compact and $\widehat{N}$ is a commutative normal subgroup of $\widehat{G}$. By Theorem 26 [20], $\widehat{N}$ is isomorphic to the direct product $\mathbb{R}^l \times K$ of a connected compact commutative subgroup K and $\mathbb{R}^l$ for some $l := l_N \in \mathbb{N}$.

Let p: $G \to \widehat{G}$ be the standard epimorphism. Identify $\widehat{N}$ with $\mathbb{R}^l \times K$ and with the commutative normal subgroup $p(N)$ of $\widehat{G}$. As in (4.2), let r: $(x, k) \mapsto x$ be the epimorphism of $\widehat{N}$ onto $\mathbb{R}^l$.

The subgroup K is normal in $\widehat{G}$. Indeed, let $k \in K$. As K is compact, $k^{n_i} \to a \in K$ for some powers $n_i \to \infty$ when $i \to \infty$. For $b \in \widehat{G}$, $bkb^{-1} \in \widehat{N}$ and $(bkb^{-1})^{n_i} = bk^{n_i}b^{-1} \to bab^{-1} \in \widehat{N}$. Hence

$r((bkb^{-1})^{n_i}) \to r(bab^{-1}) \in \mathbb{R}^l$, as $n_i \to \infty$. As $r((bkb^{-1})^{n_i}) \overset{(4.1)}{=} n_i r(bkb^{-1})$, we have $r(bkb^{-1}) = 0$. So $bkb^{-1} \in K$ and K is a normal subgroup of $\widehat{G}$.

Set $\widehat{g} = p(g)$ for $g \in G$. As in (4.2), let $\omega_{_N}\colon N \to \mathbb{R}^l$ be the continuous epimorphism defined by

$$\omega_{_N} = r \circ p|_N, \text{ so that } \omega_{_N}(h) = r(p(h)) = r(\widehat{h}) \text{ for } h \in N,\ \widehat{h} \in \widehat{N}.$$

For each $g \in G$, define the map τ_g on $\mathbb{R}^l = \omega_{_N}(N)$ by

$$\tau_g \omega_{_N}(h) = \omega_{_N}(h_g) = \omega_{_N}(ghg^{-1}) = r(\widehat{g}\widehat{h}\widehat{g}^{-1}) \text{ for } h \in N. \tag{6.4}$$

It is well-defined. Indeed, if $x = \omega_{_N}(h) = \omega_{_N}(f) \in \mathbb{R}^l$ for $h, f \in N$, then $r(\widehat{h}) \overset{(4.2)}{=} r(\widehat{f})$. So $\widehat{f} = \widehat{h}k$ for some $k \in K$, as $K = \ker r$. As $\widehat{g}k\widehat{g}^{-1} \in K$,

$$\tau_g x = \tau_g \omega_{_N}(f) \overset{(6.4)}{=} r(\widehat{g}\ \widehat{f}\ \widehat{g}^{-1}) = r(\widehat{g}\ \widehat{h}\ \widehat{g}^{-1}\widehat{g}k\widehat{g}^{-1}) = r(\widehat{g}\ \widehat{h}\ \widehat{g}^{-1}) \overset{(6.4)}{=} \tau_g \omega_{_N}(h).$$

Proposition 6.1 *The map $g \to \tau_g$ is a continuous representation of G on $\mathbb{R}^l$ and $N \subseteq \ker \tau$.*

Proof. As ω_N is an epimorphism, we have that $N \subseteq \ker \tau$, since for each $a \in N$,

$$\tau_a \omega_{_N}(h) = \omega_{_N}(aha^{-1}) \overset{(4.3)}{=} \omega_{_N}(a) + \omega_{_N}(h) - \omega_{_N}(a) = \omega_{_N}(h), \text{ i.e., } \tau_a = \mathbf{1}_{\mathbb{R}^l}.$$

Fix $h \in N$. The map $g \mapsto h_g = ghg^{-1}$ from G to N is continuous. Hence the map $g \mapsto \tau_g \omega_{_N}(h) = \omega_{_N}(ghg^{-1})$ from G to $\mathbb{R}^l$ is continuous.

Fix $g \in G$. The map $\delta_g\colon\ h \to ghg^{-1}$ on N is a continuous automorphism. So $\tau_g \omega_{_N}(h) = \omega_{_N}(\delta_g(h))$ for $h \in N$. Let V be an open set in $\mathbb{R}^l$. Then $\omega_{_N}^{-1}(V)$ is open in N, so that $\delta_g^{-1}(\omega_{_N}^{-1}(V))$ is open in N. As $\omega_{_N}$ is a quotient map, the set $W := \omega_{_N}(\delta_g^{-1}(\omega_{_N}^{-1}(V)))$ is open in $\mathbb{R}^l$ since the quotient maps are open. Hence τ_g is a continuous map, as

$$\tau_g(W) = \tau_g(\omega_{_N}(\delta_g^{-1}(\omega_{_N}^{-1}(V)))) = \omega_{_N}(\delta_g(\delta_g^{-1}(\omega_{_N}^{-1}(V))) = V.$$

For all $h, f \in N$, we have

$$\tau_g(\omega_{_N}(h) + \omega_{_N}(f)) = \omega_{_N}(ghfg^{-1}) = \omega_{_N}(ghg^{-1}) + \omega_{_N}(gfg^{-1}) = \tau_g(\omega_{_N}(h)) + \tau_g(\omega_{_N}(f)).$$

Thus τ_g is additive. So it is a $\mathbb{Q}$-linear map. As τ_g is continuous, it is a linear map on $\mathbb{R}^l$.

For $g, f \in G$,

$$\tau_g(\tau_f \omega_{_N}(h))) = \tau_g(\omega_{_N}(h_f)) = \omega_{_N}((h_f)_g) = \omega_{_N}(gfhf^{-1}g^{-1}) = \tau_{gf}\omega(h).$$

Thus τ is a continuous representation of G on $\mathbb{R}^l$. ■

In the usual way one defines the complexification $\widetilde{\tau}_g$ of the representation τ_g, $g \in G$:

$$\widetilde{\tau}_g z = \widetilde{\tau}_g(x + iy) = \tau_g x + i\tau_g y \text{ for } z = x + iy \in \mathbb{C}^l,\ x, y \in \mathbb{R}^l. \tag{6.5}$$

The representation $\widetilde{\tau}$ of G on $\mathbb{C}^l$ (the *inner* representation) plays a crucial part in what follows.

For a linear map F: $\mathbb{R}^l \to B(\mathfrak{H}, L)$, define its complexification $\widetilde{F}$: $\mathbb{C}^l \to B(\mathfrak{H}, L)$ in the usual way

$$\widetilde{F}(z) = \widetilde{F}(x + iy) = F(x) + iF(y) \text{ for } z = x + iy \in \mathbb{C}^l,\ x, y \in \mathbb{R}^l.$$

For representations λ, U of G on L and $\mathfrak{H}$, we define the representation $\kappa_{_{\lambda,U}}$ of G on $B(\mathfrak{H}, L)$ by

$$\kappa_{_{\lambda,U}}(g)T = \lambda(g)TU(g)^{-1} \text{ for } g \in G,\ T \in B(\mathfrak{H}, L).$$

In the study of (λ,U)-cocycles an important role is played by operators intertwining $\widetilde{\tau}$ with $\kappa_{_{\lambda,U}}$.

Definition 6.2 (i) *A linear map* $F\colon \mathbb{R}^l \to B(\mathfrak{H}, L)$ *is called* (λ, U)-***cocyclic*** *if its complexification* $\widetilde{F}$ *satisfies*

$$\lambda(g)\widetilde{F}(z) = \widetilde{F}(\widetilde{\tau}_g z)U(g) \text{ for } g \in G,\ z \in \mathbb{C}^l;\ \text{or, equivalently, } \kappa_{\lambda,U}(g)\widetilde{F} = \widetilde{F}\widetilde{\tau}_g. \tag{6.6}$$

(ii) *We say that a map* $\varphi\colon N \to B(\mathfrak{H}, L)$ *is* (λ, U)*-covariant if*

$$\varphi(h_g) = \kappa_{\lambda,U}(g)\varphi(h) = \lambda(g)\varphi(h)U(g)^{-1} \text{ for } h \in N, g \in G,\ \text{where } h_g = ghg^{-1}. \tag{6.7}$$

Proposition 6.3 *Let* λ, U *be* (N, χ)*-trivial representations of* G *and* $l = l_N$.

(i) *A map* $0 \neq \sigma\colon N \to B(\mathfrak{H}, L)$ *is a* (λ, U)*-covariant* $(\chi\mathbf{1}_L, \chi\mathbf{1}_{\mathfrak{H}})$*-cocycle if and only if there is a* (λ, U)*-cocyclic map* $\beta\colon \mathbb{R}^l \to B(\mathfrak{H}, L)$ *such that*

$$\sigma(h) = \chi(h)\beta(\omega_N(h)),\ h \in N,\ \text{where } \omega_N\colon N \to \mathbb{R}^l \text{ is defined in (4.2)}. \tag{6.8}$$

(ii) *If* ξ *is a* (λ, U)*-cocycle then* ξ^N *is a* (λ, U)*-covariant* $(\chi\mathbf{1}_L, \chi\mathbf{1}_{\mathfrak{H}})$*-cocycle on* N.

Proof. (i) If σ is a $(\chi\mathbf{1}_L, \chi\mathbf{1}_{\mathfrak{H}})$-cocycle, we have from Proposition 4.1 that there is a linear map $\beta\colon \mathbb{R}^l \to B(\mathfrak{H}, L)$ such that $\sigma(h) = \chi(h)\beta(\omega_N(h))$ for $h \in N$. If σ is (λ, U)-covariant,

$$\sigma(h_g) \overset{(6.7)}{=} \lambda(g)\sigma(h)U(g)^{-1} = \lambda(g)\chi(h)\beta(\omega_N(h))U(g)^{-1} \text{ and}$$
$$\sigma(h_g) = \chi(h_g)\beta(\omega_N(h_g)) \overset{(6.2)(6.4)}{=} \chi(h)\beta(\tau_g\omega_N(h)) \text{ for } g \in G.$$

Thus $\beta(\tau_g x) = \lambda(g)\beta(x)U(g)^{-1}$, $x \in \mathbb{R}^l$. Then $\widetilde{\beta}$ satisfies (6.6). So β is a (λ, U)-cocyclic map on N.

Conversely, if β is a (λ, U)-cocyclic map in (6.8), σ is a $(\chi\mathbf{1}_L, \chi\mathbf{1}_{\mathfrak{H}})$-cocycle by Proposition 4.1(ii). As $\widetilde{\beta}$ satisfies (6.6), we get, as above, that σ satisfies (6.7). So σ is (λ, U)-covariant.

(ii) As ξ is a (λ, U)-cocycle and λ, U are (N, χ)-trivial, ξ^N is a $(\chi\mathbf{1}_L, \chi\mathbf{1}_{\mathfrak{H}})$-cocycle. Then

$$\xi(gh) \overset{(1.1)}{=} \lambda(g)\xi(h) + \xi(g)U(h) \overset{(6.1)}{=} \lambda(g)\xi(h) + \xi(g)\chi(h),\ \text{and}$$
$$\xi(gh) \overset{(6.2)}{=} \xi(h_g g) \overset{(1.1)}{=} \lambda(h_g)\xi(g) + \xi(h_g)U(g) \overset{(6.1)(6.2)}{=} \chi(h)\xi(g) + \xi(h_g)U(g)$$

for $h \in N, g \in G$. Hence $\xi(h_g) = \lambda(g)\xi(h)U(g)^{-1}$, so that ξ^N is (λ, U)-covariant. ■

6.2 (λ, U)-cocycles on groups $G = D \ltimes N$

From now on we assume that $G = D \ltimes N$ is the semidirect product of connected subgroups D and N and that $l = l_N \neq 0$. So N is normal and each $g \in G$ can be uniquely written as $g = dh$ for some $d \in D, h \in N$. For $f = ct$, $c \in D, t \in N$,

$$gf = dhf = dfh_{f^{-1}} = dcth_{f^{-1}},\ \text{where } h_{f^{-1}} = f^{-1}hf. \tag{6.9}$$

Let λ, U be (N, χ)-trivial representations and $\xi\colon G \to B(\mathfrak{H}, L)$ be a (λ, U)-cocycle. Define the map $r\colon G \to N$ by $r(g) = r(dh) = h$ for $g \in G$, and set

$$\lambda'(g) = \lambda(g)/\chi(r(g)),\ U'(g) = U(g)/\chi(r(g)) \text{ and } \xi'(g) = \xi(g)/\chi(r(g)).$$

By (6.2) and (6.9),

$$r(gf) = r(f)f^{-1}r(g)f \text{ and } \chi(r(gf)) = \chi(r(f))\chi(f^{-1}r(g)f) = \chi(r(g))\chi(r(f)).$$

Hence λ' and U' are representations of G, ξ' is a (λ', U')-cocycle and, by (6.1),

$$\lambda'(g) = \lambda'(dh) = \lambda'(d) = \lambda(d),\ U'(g) = U'(dh) = U'(d) = U(d) \text{ for } d \in D.$$

So λ', U' are N-trivial representations. Clearly, they are representations of the quotient group $G/N \approx D$ and there is a one-to-one correspondence between all (N, χ)-trivial representations and N-trivial representations. There is also a one-to-one correspondence between the respective cocycles. So from now on we assume that λ and U are N-trivial (see (6.3)):

$$\lambda(dh) = \lambda(d) \text{ and } U(dh) = U(d), \text{ so that } \xi(dh) = \lambda(d)\xi(h) + \xi(d) \text{ for } d \in D, h \in N. \tag{6.10}$$

By (6.6), a linear map β: $\mathbb{R}^l \to B(\mathfrak{H}, L)$ is (λ, U)-cocyclic if its complexification $\widetilde{\beta}$: $\mathbb{C}^l \to B(\mathfrak{H}, L)$ satisfies

$$\lambda(d)\widetilde{\beta}(z) = \widetilde{\beta}(\widetilde{\tau}_d z)U(d) \text{ for } d \in D \text{ and } z \in \mathbb{C}^l. \tag{6.11}$$

Let ξ: $G \to B(\mathfrak{H}, L)$ be a weakly continuous map and ξ^D, ξ^N be its restrictions to D and N.

Theorem 6.4 *Consider the maps* $\eta = \xi^D$ *and* $\xi_\eta(dh) = \eta(d) = \xi(d)$. *The following are equivalent.*

(i) *The map* ξ: $G \to B(\mathfrak{H}, L)$ *is a* (λ, U)*-cocycle*;

(ii) η *is a* (λ^D, U^D)*-cocycle on* D, ξ^N *is a* (λ, U)*-covariant* $(\mathbf{1}_L, \mathbf{1}_{\mathfrak{H}})$*-cocycle on* N (*so that* $\xi^N(h) = \xi(h) = \beta(\omega_N(h))$ *for a* (λ, U)*-cocyclic map* β: $\mathbb{R}^l \to B(\mathfrak{H}, L)$) *and*

$$\xi = \xi_\beta + \xi_\eta,\ \textit{where } \xi_\beta(dh) = \lambda(d)\beta(\omega_N(h)) \textit{ for } d \in D, h \in N. \tag{6.12}$$

(iii) *The maps* ξ_β, ξ_η: $G \to B(\mathfrak{H}, L)$ *are* (λ, U)*-cocycles and* $\xi = \xi_\beta + \xi_\eta$.

Moreover, $\xi_\beta \neq 0$ *is not a* (λ, U)*-coboundary.*

Proof. (i) ⇒ (ii). Let ξ be a (λ, U)-cocycle. By Proposition 6.3, $\eta = \xi^D$ is a (λ^D, U^D)-cocycle and ξ^N is a (λ, U)-covariant $(\mathbf{1}_L, \mathbf{1}_{\mathfrak{H}})$-cocycle. By (6.10), $\xi = \xi_\beta + \xi_\eta$ and ξ_β is given in (6.12).

(ii) ⇒ (iii). As $\eta = \xi^D$ is weakly continuous on D, ξ_η is weakly continuous on G. Thus ξ_η is a (λ, U)-cocycle since, for $g = dh$, $f = ct$,

$$\xi_\eta(gf) \overset{(6.9)}{=} \eta(dc) \overset{(1.1)}{=} \lambda(d)\eta(c) + \eta(d)U(c) \overset{(6.10)}{=} \lambda(g)\xi_\eta(f) + \xi_\eta(g)U(f).$$

Let us show that ξ_β is a (λ, U)-cocycle. Since β, ω_N are continuous, ξ^N is continuous on N. As λ^D is weakly continuous on D, ξ_β is weakly continuous on G (see (6.12)). We also have

$$\begin{aligned} \xi_\beta(gf) &\overset{(6.9)}{=} \xi_\beta(dcth_{f^{-1}}) \overset{(6.12)}{=} \lambda(dc)\beta(\omega_N(th_{f^{-1}})) \overset{(6.2)}{=} \lambda(d)\lambda(c)(\beta(\omega_N(t)) + \beta(\omega_N(h_{f^{-1}}))) \\ &\overset{(6.10),(6.12)}{=} \lambda(g)\xi_\beta(f) + \lambda(g)\lambda(c)\beta(\omega_N(h_{f^{-1}})). \end{aligned} \tag{6.13}$$

Since $ch_{f^{-1}}c^{-1} = cf^{-1}hfc^{-1} = (ctc^{-1})^{-1}hctc^{-1}$,

$$\omega_N(ch_{f^{-1}}c^{-1}) \overset{(4.3)}{=} -\omega_N(ctc^{-1}) + \omega_N(h) + \omega_N(ctc^{-1}) = \omega_N(h). \tag{6.14}$$

As $U(f) = U(ct) = U(c)$ and β is a (λ, U)-cocyclic map, we have from (6.2) that

$$\lambda(c)\beta(\omega_N(h_{f^{-1}})) \overset{(6.11)}{=} \beta(\widetilde{\tau}_c\omega_N(h_{f^{-1}}))U(c) \overset{(6.4)}{=} \beta(\omega_N(ch_{f^{-1}}c^{-1}))U(f)$$
$$\overset{(6.14)}{=} \beta(\omega_N(h))U(f).$$

Substituting this in (6.13) and using (6.12), we get that ξ_β is a (λ, U)-cocycle.

(iii) $\Rightarrow$ (i) is evident.

Finally, if ξ_β is a (λ, U)-coboundary then, for some $T \in B(\mathfrak{H}, L)$, $\xi_\beta(h) \overset{(1.2)}{=} \lambda(h)T - TU(h) = 0$, as $\lambda(h) = \mathbf{1}_L$ and $U(h) = \mathbf{1}_{\mathfrak{H}}$ by (6.3). So $\xi_\beta = 0$ by (6.12). ■

Proposition 6.3 and Theorem 6.4 yield

Corollary 6.5 (i) *Each (λ, U)-covariant $(\mathbf{1}_L, \mathbf{1}_{\mathfrak{H}})$-cocycle σ uniquely extends to a (λ, U)-cocycle ξ vanishing on D: $\xi(dh) = \lambda(d)\sigma(h)$ for $d \in D, h \in N$.*

So there is a linear bijection Φ between the space of all (λ, U)-cocyclic maps β and the space $Z_N(G, \lambda, U)$ of all (λ, U)-cocycles vanishing on D: $\Phi(\beta)(dh) = \lambda(d)\beta(\omega_N(h))$.

(ii) *There is a linear bijection Ψ between the space of all (λ^D, U^D)-cocycles η on D and the space $Z_D(G, \lambda, U)$ of all (λ, U)-cocycles vanishing on N: $\Psi(\eta)(dh) = \eta(d)$.*

The following result presents a convenient decomposition of the cohomology group $\mathcal{H}(G, \lambda, U)$ in the direct sum of two subspaces. For the representations we deal with it can be considered as a concretization of the Lyndon-Serr-Hochschild cohomology sequence ([6, Section I.8], [8]).

Corollary 6.6 (i) *The set of all (λ, U)-cocycles $Z(G, \lambda, U) = Z_D(G, \lambda, U) \dotplus Z_N(G, \lambda, U)$.*

(ii) *$\mathcal{H}(G, \lambda, U) = E \dotplus F$, where the subspace E is naturally isomorphic to $\mathcal{H}(D, \lambda^D, U^D)$, while F is naturally isomorphic to the space M of all (λ, U)-cocyclic maps on N.*

Proof. (i) Clearly, $Z_D(G, \lambda, U) \cap Z_N(G, \lambda, U) = \{0\}$. The rest follows from Theorem 6.4.

(ii) If ξ is a (λ, U)-boundary, $\xi = \xi_\beta + \xi_\eta$ by Theorem 6.4, where ξ_β and ξ_η are boundaries. So, by Theorem 6.4, $\xi_\beta = 0$ and ξ is a (λ^D, U^D)-boundary. Thus $B(G, \lambda, U) = B(D, \lambda^D, U^D)$ and

$$\mathcal{H}(G, \lambda, U) = Z(G, \lambda, U)/B(G, \lambda, U) \sim Z(D, \lambda^D, U^D)/B(D, \lambda^D, U^D) \dotplus Z_N(G, \lambda, U)$$
$$\sim \mathcal{H}(D, \lambda^D, U^D) \dotplus M$$

which completes the proof. ■

In the next section we describe (λ, U)-cocyclic maps when D is Engel. The (λ^D, U^D)-cocycles η can be studied separately, since they only involve the group D. For a nilpotent D, they are described in Theorem 4.3. If $L = \mathbb{C}e$ and $\lambda^D = \iota$, their description is a classical problem (see [6]). In particular, for compact D, all (ι, U^D)-cocycles are coboundaries (Corollary III.2.1 [6]).

6.3 (λ, U)-cocyclic maps when D is an Engel group.

From now on λ, U are N-trivial and D is an *Engel group.* We also assume that U^D is unitary and $\dim L < \infty$. So, by (6.10), Proposition 3.6 and Corollary 2.18 [13],

$$\begin{aligned} &\lambda(dh) = \lambda(d),\ U(dh) = U(d) \text{ and } U(d^{-1}) = U(d)^*, && (6.15)\\ &L = \oplus_{\delta\in \mathrm{sign}(\lambda^D)} L_\delta \text{ and } \lambda^D = \oplus_{\delta\in \mathrm{sign}(\lambda^D)}\lambda_\delta, \text{ where} &&\\ &L_\delta = \{x \in L\colon \textstyle\prod_{i=1}^{n_\delta}(\lambda(d_i) - \delta(d_i)\mathbf{1}_L)x = 0 \text{ for all } \{d_i\}_{i=1}^{n_\delta} \text{ in } D\} && (6.16)\end{aligned}$$

are λ-invariant subspaces, $\lambda_\delta = \lambda^D|_{L_\delta}$ are δ-representations of D and $n_\delta = \dim L_\delta$.

By R_δ we denote the projection on L_δ, and by P_ϕ, $\phi \in D^*$, the projection on

$$\mathfrak{H}_\phi = \{x \in \mathfrak{H}\colon U(d)x = \phi(d)x,\ d \in D\}, \text{ so that } P_\phi U(d) = U(d)P_\phi = \phi(d)P_\phi. \quad (6.17)$$

Lemma 6.7 *If a linear manifold M is dense in $\mathfrak{H}_\phi^\perp$, then the linear manifold*

$$M' = \mathrm{span}\,\{U(d)x - \phi(d)x\colon x \in M,\ d \in D\}$$

is dense in $\mathfrak{H}_\phi^\perp$.

Proof. Let $\overline{M'} \neq \mathfrak{H}_\phi^\perp$. As $\mathfrak{H}_\phi^\perp$ is U-invariant, there is $0 \neq y \in \mathfrak{H}_\phi^\perp$ such that $0 = (U(d)x-\phi(d)x, y)$ for all $d \in D$, $x \in M$. Then $x\perp(U(d^{-1})y - \overline{\phi(d)}y)$. So $U(d^{-1})y - \overline{\phi(d)}y \in \mathfrak{H}_\phi$, as M is dense in $\mathfrak{H}_\phi^\perp$.

On the other hand, as $\mathfrak{H}_\phi^\perp$ is U-invariant, $U(d^{-1})y-\overline{\phi(d)}y \in \mathfrak{H}_\phi^\perp$. So $U(d^{-1})y = \overline{\phi(d)}y$ for $d \in D$. As ϕ is unitary, $U(d)y = \overline{\phi(d^{-1})}y = \phi(d)y$. Thus $y \in \mathfrak{H}_\phi$, a contradiction. So M' is dense in $\mathfrak{H}_\phi^\perp$. ∎

Let $\beta\colon \mathbb{R}^l \to B(\mathfrak{H}, L)$ be a (λ, U)-cocyclic map. Then $\beta = (\beta_\delta)_{\delta\in\mathrm{sign}(\lambda^D)}$, where

$$\beta_\delta = R_\delta\beta \text{ are } (\lambda_\delta, U)\text{-cocyclic maps and } \lambda_\delta(d)\widetilde{\beta}_\delta(u) \overset{(6.11)}{=} \widetilde{\beta}_\delta(\widetilde{\tau}_d u)U(d),\ u \in \mathbb{C}^l. \quad (6.18)$$

By Proposition 6.1, $N \subseteq \ker\tau$. So we write $\widetilde{\tau}$ instead of $\widetilde{\tau}^D$. As D is Engel, it follows from [13, Corollary 2.18] that there is a finite set $\mathrm{sign}(\widetilde{\tau})$ of characters on D such that

$$\begin{aligned} &\mathbb{C}^l = \sum_{\nu\in\mathrm{sign}(\widetilde{\tau})} Z_\nu,\ \text{each } Z_\nu \text{ is } \widetilde{\tau}\text{-invariant, } \widetilde{\tau}|_{Z_\nu} \text{ is a } \nu\text{-representations, where}&&\\ &Z_\nu = \left\{u \in \mathbb{C}^l\colon \textstyle\prod_{i=1}^{n_\nu}(\widetilde{\tau}_{d_i} - \nu(d_i)\mathbf{1}_{\mathbb{C}^l})u = 0 \text{ for all sets } \{d_i\}_{i=1}^{n_\nu} \text{ in } D\right\} && (6.19)\end{aligned}$$

and $n_\nu \leq \dim Z_\nu$. Denote by Q_ν the projection on the subspace Z_ν along $\sum_{\nu'\in\mathrm{sign}(\widetilde{\tau}),\nu'\neq\nu} Z_{\nu'}$.

For $\delta \in \mathrm{sign}(\lambda^D)$, set

$$\Omega_\delta = \{\nu \in \ \mathrm{sign}(\widetilde{\tau})\colon \mathfrak{H}_{\delta/\nu} \neq \{0\}\} \text{ and } L_\delta^1 = \{x \in L\colon \lambda(d)x = \delta(d)x,\ d \in D\}. \quad (6.20)$$

Theorem 6.8 *A linear map $\beta = (\beta_\delta)_{\delta\in\mathrm{sign}(\lambda^D)}\colon \mathbb{R}^l \to B(\mathfrak{H}, L)$ is (λ, U)-cocyclic if and only if*

$$\widetilde{\beta}_\delta(u) = \sum\nolimits_{\nu\in\Omega_\delta} b_\nu^\delta(Q_\nu u)P_{\delta/\nu} \text{ for } u \in \mathbb{C}^l, \quad (6.21)$$

where b_ν^δ, $\nu \in \Omega_\delta$, can be any linear map from Z_ν to $B(\mathfrak{H}_{\delta/\nu}, L_\delta)$ satisfying

$$\lambda_\delta(d)b_\nu^\delta(z) = \delta(d)b_\nu^\delta(\widetilde{\tau}_d z)/\nu(d) \text{ for } z \in Z_\nu \text{ and } d \in D. \quad (6.22)$$

If $\widetilde{\tau}|_{Z_\nu} = \nu\mathbf{1}_{Z_\nu}$ then b_ν^δ in (6.21) can be any linear map from Z_ν to $B(\mathfrak{H}_{\delta/\nu}, L_\delta^1)$.

Proof. Let β be (λ, U)-cocyclic. Fix $\delta \in \mathrm{sign}(\lambda^D)$. For $\nu \in \mathrm{sign}(\widetilde{\tau})$, set $\phi = \delta/\nu$. We will write $\widetilde{\beta}, \lambda, b_\nu, n$ instead of $\widetilde{\beta}_\delta, \lambda_\delta, b_\nu^\delta, n_\delta$.

Let us prove that $\widetilde{\beta}(z)|_{\mathfrak{H}_\phi^\perp} = 0$, $z \in Z_\nu$. Set $F_\nu(d) = \widetilde{\tau}_d - \nu(d)\mathbf{1}_{\mathbb{C}^l}$, $d \in D$, and consider the spaces

$$Z_\nu^0 = \{0\} \text{ and } Z_\nu^k = \{u \in \mathbb{C}^l\colon\ F_\nu(d)u \in Z_\nu^{k-1} \text{ for } d \in D\},\ k = 1, ..., n_\nu. \tag{6.23}$$

By (6.19), $Z_\nu^0 \subsetneqq Z_\nu^1 \subsetneqq ... \subsetneqq Z_\nu^{n_\nu} = Z_\nu$ and $F_\nu(d)Z_\nu^k \subseteq Z_\nu^{k-1}$ for $d \in D$. All Z_ν^k are $\widetilde{\tau}$-invariant since, for $x \in Z_\nu^k$, $\widetilde{\tau}_d x = F_\nu(d)x + \nu(d)x \in Z_\nu^k$. As $\widetilde{\beta}$ is linear, we have for $u \in \mathbb{C}^l$,

$$\widetilde{\beta}(F_\nu(d)u)U(d) = \widetilde{\beta}(\widetilde{\tau}_d u - \nu(d)u)U(d) \overset{(6.18)}{=} \lambda(d)\widetilde{\beta}(u) - \nu(d)\widetilde{\beta}(u)U(d). \tag{6.24}$$

If $z \in Z_\nu^1$ then $F_\nu(d)z = 0$, $d \in D$, by (6.23). So $\widetilde{\beta}(F_\nu(d)z) = 0$. Thus, by (6.24),

$$\lambda(d)\widetilde{\beta}(z) = \nu(d)\widetilde{\beta}(z)U(d), \text{ so that } (\lambda(d) - \delta(d)\mathbf{1}_L)\widetilde{\beta}(z) = \nu(d)\widetilde{\beta}(z)(U(d) - \phi(d)\mathbf{1}_{\mathfrak{H}})$$

for $d \in D$. Hence

$$0 \overset{(6.16)}{=} \left(\prod\nolimits_{i=1}^{n}(\lambda(d_i) - \delta(d_i)\mathbf{1}_L)\right)\widetilde{\beta}(z) = \widetilde{\beta}(z)\left(\prod\nolimits_{i=1}^{n}\nu(d_i)(U(d_i) - \phi(d_i)\mathbf{1}_{\mathfrak{H}})\right)$$

for each set $\{d_i\}_{i=1}^n$ in D. Hence $\widetilde{\beta}(z)\left(\prod\nolimits_{i=1}^{n}(U(d_i) - \phi(d_i)\mathbf{1}_{\mathfrak{H}})\right) = 0$. As

$$\mathrm{span}\left\{\left(\prod\nolimits_{i=1}^{n}(U(d_i) - \phi(d_i)\mathbf{1}_{\mathfrak{H}})\right)x\colon\ x \in \mathfrak{H}_\phi^\perp,\ \{d_i\}_{i=1}^n \text{ in } D\right\} \text{ is dense in } \mathfrak{H}_\phi^\perp$$

by Lemma 6.7, $\widetilde{\beta}(z)|_{\mathfrak{H}_\phi^\perp} = 0$ for all $z \in Z_\nu^1$.

Let $z \in Z_\nu^2$. Then $F_\nu(d)z \in Z_\nu^1$ for $d \in D$ by (6.23). So $\widetilde{\beta}(F_\nu(d)z)|_{\mathfrak{H}_\phi^\perp} = 0$ by the above argument. Hence, by (6.24), $\lambda(d)\widetilde{\beta}(z) = \nu(d)\widetilde{\beta}(z)U(d)$ for $d \in D$. Repeating the above proof, we get $\widetilde{\beta}(z)|_{\mathfrak{H}_\phi^\perp} = 0$ for all $z \in Z_\nu^2$. Continuing this process, we get that $\widetilde{\beta}(z)|_{\mathfrak{H}_\phi^\perp} = 0$ for each $k \geq 1$ and all $z \in Z_\nu^k$. As $Z_\nu = Z_\nu^{n_\nu}$ for $n_\nu \leq \dim Z_\nu$, we have that $\widetilde{\beta}(z)|_{\mathfrak{H}_\phi^\perp} = 0$ for $z \in Z_\nu$. Thus

$$\widetilde{\beta}(z) = \widetilde{\beta}(z)P_\phi \text{ for } z \in Z_\nu, \text{ so that } \widetilde{\beta}(Q_\nu u) = \widetilde{\beta}(Q_\nu u)P_\phi \text{ for } u \in \mathbb{C}^l. \tag{6.25}$$

If $\nu \notin \Omega_\delta$ then $\mathfrak{H}_\phi = \{0\}$, so that

$$\widetilde{\beta}(z) \overset{(6.25)}{=} \widetilde{\beta}(z)P_\phi = 0 \text{ for } z \in Z_\nu. \tag{6.26}$$

Let $\nu \in \Omega_\delta$. Then $P_\phi \neq 0$. As $\widetilde{\tau}_d z \in Z_\nu$ for $z \in Z_\nu$ and $d \in D$,

$$\lambda(d)\widetilde{\beta}(z)P_\phi \overset{(6.18)}{=} \widetilde{\beta}(\widetilde{\tau}_d z)U(d)P_\phi \overset{(6.17)}{=} \phi(d)\widetilde{\beta}(\widetilde{\tau}_d z)P_\phi. \tag{6.27}$$

For $\nu \in \Omega_\delta$, define the linear map b_ν from Z_ν into $B(\mathfrak{H}_{\delta/\nu}, L)$ by

$$b_\nu(z) = \widetilde{\beta}(z)|_{\mathfrak{H}_\phi} \text{ for } z \in Z_\nu, \text{ so that } \widetilde{\beta}(z) \overset{(6.25)}{=} \widetilde{\beta}(z)P_\phi = b_\nu(z)P_\phi. \tag{6.28}$$

Then (6.22) holds, since for $z \in Z_\nu$ and $d \in D$,

$$\lambda(d)b_\nu(z) = \lambda(d)\widetilde{\beta}(z)|_{\mathfrak{H}_\phi} \overset{(6.27)}{=} \phi(d)\widetilde{\beta}(\widetilde{\tau}_d z)|_{\mathfrak{H}_\phi} = \delta(d)b_\nu(\widetilde{\tau}_d z)/\nu(d).$$

By (6.19), $u = \sum_{\nu\in\mathrm{sign}(\widetilde{\tau})} Q_\nu u$ for $u \in \mathbb{C}^l$. Hence (6.21) holds, since

$$\widetilde{\beta}(u) = \sum\nolimits_{\nu\in\mathrm{sign}(\widetilde{\tau})}\widetilde{\beta}(Q_\nu u) \overset{(6.26)}{=} \sum\nolimits_{\nu\in\Omega_\delta}\widetilde{\beta}(Q_\nu u) \overset{(6.28)}{=} \sum\nolimits_{\nu\in\Omega_\delta} b_\nu(Q_\nu u)P_{\delta/\nu}.$$

Conversely, let linear maps $\{b_\nu\colon Z_\nu \to B(\mathfrak{H}_{\delta/\nu}, L)\}_{\nu\in\Omega_\delta}$ satisfy (6.22) and $\widetilde{\beta}$ be given in (6.21). Then

$$\lambda(d)\widetilde{\beta}(u) \overset{(6.21)}{=} \sum\nolimits_{\nu\in\Omega_\delta}\lambda(d)b_\nu(Q_\nu u)P_{\delta/\nu} \overset{(6.22)}{=} \sum\nolimits_{\nu\in\Omega_\delta}\frac{\delta(d)}{\nu(d)}b_\nu(\widetilde{\tau}_d Q_\nu u)P_{\delta/\nu}. \tag{6.29}$$

As each Z_ν is $\widetilde{\tau}$-invariant, $\widetilde{\tau}_d Q_\nu u = Q_\nu \widetilde{\tau}_d u$. By (6.17), $\frac{\delta(d)}{\nu(d)}P_{\delta/\nu} = P_{\delta/\nu}U(d)$. Hence

$$\lambda(d)\widetilde{\beta}(u) \overset{(6.29)}{=} \sum\nolimits_{\nu\in\Omega_\delta} b_\nu(Q_\nu\widetilde{\tau}_d u)P_{\delta/\nu}U(d) \overset{(6.21)}{=} \widetilde{\beta}(\widetilde{\tau}_d u)U(d)$$

for $u \in \mathbb{C}^l$ and $d \in D$. Thus (6.18) holds. Since it holds for all δ, β is (λ, U)-cocyclic.

Let $\widetilde{\tau}|_{Z_\nu} = \nu\mathbf{1}_{Z_\nu}$ for some $\nu \in \Omega_\delta$. Then (6.22) is equivalent to $\lambda_\delta(d)b_\nu^\delta(z) = \delta(d)b_\nu^\delta(z)$ for $z \in Z_\nu$, $d \in D$. So b_ν^δ can be any map from Z_ν to $B(\mathfrak{H}_{\delta/\nu}, L_\delta^1)$. ■

Let $\{e_n\}_{n=1}^l$ be a basis in $\mathbb{R}^l$. Suppose that, for $\delta_0 \in \mathrm{sign}(\lambda^D)$ and all $\nu \in \Omega_{\delta_0}$,

$$Z_\nu = \mathbb{C}f_\nu \text{ for } f_\nu \in \mathbb{C}^l, \text{ so that } Q_\nu e_n = a_{n\nu}f_\nu \text{ for } a_{n\nu} \in \mathbb{C}, \text{ and all } n = 1, ..., l. \tag{6.30}$$

If $b\colon Z_\nu \to B(\mathfrak{H}_{\delta/\nu}, L_\delta)$ is a linear map, $b(Q_\nu e_n) = a_{n\nu}b(f_\nu)$. So, for $u = (u_1, ..., u_l) \in \mathbb{C}^l$,

$$b(Q_\nu u) = \sum\nolimits_{n=1}^l u_n b(Q_\nu e_n) = \mu_\nu(u)b(f_\nu), \text{ where } \mu_\nu(u) = \sum\nolimits_{n=1}^l u_n a_{n\nu} \in \mathbb{C}. \tag{6.31}$$

Corollary 6.9 *If condition* (6.30) *hold, the linear map* $\beta_{\delta_0}\colon \mathbb{R}^l \to B(\mathfrak{H}, L_{\delta_0})$ *in* (6.21) *has form*

$$\widetilde{\beta}_{\delta_0}(u) = \sum\nolimits_{\nu\in\Omega_{\delta_0}}\mu_\nu(u)b_\nu(f_\nu)P_{\delta_0/\nu} \textit{ for } u \in \mathbb{C}^l,$$

where μ_ν *are defined in* (6.31) *and* $b_\nu(f_\nu)$ *can be any operator in* $B(\mathfrak{H}_{\delta_0/\nu}, L_{\delta_0}^1)$.

If $\lambda_{\delta_0} = \delta_0\mathbf{1}_L$ *then* $L_{\delta_0} = L_{\delta_0}^1$ *and each* $b_\nu(f_\nu)$, $\nu \in \Omega_{\delta_0}$, *can be any operator in* $B(\mathfrak{H}_{\delta/\nu}, L_{\delta_0})$.

6.4 (λ, U)-cocycles and extensions of N-trivial representations.

In this section we study the extensions $\mathfrak{e}(\lambda, U, \xi)$ when λ, U are N-trivial and D is Engel. Taking into account decompositions (6.16) and (6.19) of L and $\mathbb{C}^l$, set

$$\Phi = \cup_{\delta\in\mathrm{sign}(\lambda^D)}\Phi_\delta,\ \Phi_\delta = \{\phi = \frac{\delta}{\nu}\colon \nu \in \Omega_\delta\},\ \Theta = \{\delta \in \mathrm{sign}(\lambda^D)\colon \mathfrak{H}_\delta \neq \{0\}\},$$
$$\mathfrak{H}_\aleph = \mathfrak{H} \ominus \mathfrak{H}_{\Phi\cup\Theta}, \text{ where } \mathfrak{H}_{\Phi\cup\Theta} = \oplus_{\phi\in\Phi\cup\Theta}\mathfrak{H}_\phi, \text{ so that } \mathfrak{H} = \mathfrak{H}_{\Phi\cup\Theta} \oplus \mathfrak{H}_\aleph. \tag{6.32}$$

Then $\Phi_\delta = \{\phi = \frac{\delta}{\nu}\colon \nu \subseteq \mathrm{sign}(\widetilde{\tau}), \mathfrak{H}_\phi \neq \{0\}\}$ and $U|_{\mathfrak{H}_\phi} = \phi\mathbf{1}_{\mathfrak{H}_\phi}$.

Denote by R_δ, P_ϕ the projections on $L_\delta, \mathfrak{H}_\phi$. Let ξ be a (λ, U)-cocycle. By Theorem 6.4,

$$\xi = \xi_\beta + \xi_\eta, \text{ where } \xi_\beta(dh) = \lambda(d)\beta(\omega_N(h)),\ \xi_\eta(dh) = \eta(d) \text{ are } (\lambda, U)\text{-cocycles}. \tag{6.33}$$

Hence, with respect to decompositions (6.16) of L, (6.19) of $\mathbb{C}^l$ and (6.32) of $\mathfrak{H}$,

$$\xi = (\xi|_{\mathfrak{H}_{\Phi\cup\Theta}}, \xi|_{\mathfrak{H}_\aleph}), \text{ where } \xi|_{\mathfrak{H}_\aleph} \text{ is a } (\lambda, U|_{\mathfrak{H}_\aleph})\text{-cocycle},$$
$$\xi|_{\mathfrak{H}_{\Phi\cup\Theta}} = (\xi^{\delta,\phi})_{\delta\in\mathrm{sign}(\lambda^D),\phi\in\Phi\cup\Theta}, \text{ where } \xi^{\delta,\phi} \text{ are } (\lambda_\delta, \phi\mathbf{1}_{\mathfrak{H}_\phi})\text{-cocycles and } \lambda_\delta = \lambda|_{L_\delta},$$
$$\xi^{\delta,\phi}(dh) = \lambda_\delta(d)\beta^{\delta,\phi}(\omega_N(h)) + \eta^{\delta,\phi}(d), \text{ where } \beta^{\delta,\phi} = R_\delta\beta P_\phi \text{ and } \eta^{\delta,\phi} = R_\delta\eta P_\phi. \tag{6.34}$$

Corollary 6.10 (i) *If* $\phi \notin \Phi_\delta$ *then* $\beta^{\delta,\phi} = 0$. *If*

$$\phi \in \Phi_\delta \textit{ then } \beta^{\delta,\phi}(t) = b^\delta_\nu(Q_\nu t)P_\phi, \textit{ for } t \in \mathbb{R}^l, \textit{ where } \nu = \delta/\phi \textit{ and} \tag{6.35}$$

b^δ_ν: $Z_\nu \to B(\mathfrak{H}_{\delta/\nu}, L_\delta)$ *can be any linear map satisfying* (6.22).

If $\widetilde{\tau}|_{Z_\nu} = \nu \mathbf{1}_{Z_\nu}$ *then* b^δ_ν *can be any linear map from* Z_ν *to* $B(\mathfrak{H}_{\delta/\nu}, L^1_\delta)$.

(ii) *If* $\phi \neq \delta$ *then* $\eta^{\delta,\phi}$ *is a* $(\lambda_\delta, \phi\mathbf{1}_{\mathfrak{H}_\phi})$*-coboundary*:

$$\eta^{\delta,\phi}(d) = \lambda_\delta(d)T - \phi(d)T \textit{ for all } d \in D \textit{ and some } T \in B(\mathfrak{H}_\phi, L_\delta). \tag{6.36}$$

Let, as in (4.1), $D/D^{[1]} \approx \mathbb{R}^{l_D} \times K$. *If* $\phi = \delta$ *and* $\lambda_\delta = \delta\mathbf{1}_{L_\delta}$ *then, as in* (4.2), *there is a continuous epimorphism* $\omega_{_D}$: $D \to \mathbb{R}^{l_D}$ *and a linear map* β_δ: $\mathbb{R}^{l_D} \to B(\mathfrak{H}_\delta, L_\delta)$ *such that*

$$\eta^{\delta,\delta}(d) = \delta\,(d)\,\beta_\delta(\omega_{_D}(d)) \textit{ for } d \in D. \tag{6.37}$$

(iii) *The representations* λ^D, $U^D|_{\mathfrak{H}_\aleph}$ *are eigen-disjoint* (*see* (4.9)) *and* $\xi(dh)|_{\mathfrak{H}_\aleph} = \eta(d)|_{\mathfrak{H}_\aleph}$, *where* $\eta|_{\mathfrak{H}_\aleph}$ *is a* $(\lambda^D, U^D|_{\mathfrak{H}_\aleph})$*-cocycle.*

(iv) *Let* D *be nilpotent. Then* $\eta|_{\mathfrak{H}_\aleph}$ *is either a coboundary, or* $\mathfrak{H}_\aleph = \oplus_{i=1}^\infty \mathfrak{H}_i$, *where all subspaces* $\mathfrak{H}_i$ *are* U*-invariant and all maps* $\eta|_{\mathfrak{H}_i}$: $D \to B(\mathfrak{H}_i, L)$ *are* $(\lambda^D, U^D|_{\mathfrak{H}_i})$*-coboundaries.*

Proof. (i) follows from Theorem 6.8 since, for $t \in \mathbb{R}^l$,

$$\beta^{\delta,\phi}(t) = R_\delta\beta(t)P_\phi \overset{(6.21)}{=} \beta_\delta(t)P_\phi = \sum\nolimits_{\nu\in\Omega_\delta} b^\delta_\nu(Q_\nu t)P_{\delta/\nu}P_\phi \overset{(6.32)}{=} \sum\nolimits_{\theta\in\Phi_\delta} b^\delta_{\delta/\theta}(Q_\nu t)P_\theta P_\phi.$$

(ii) By (6.17), (6.32), $U(dh)|_{\mathfrak{H}_\phi} = \phi(d)\mathbf{1}_{\mathfrak{H}_\phi}$ for $\phi \in \Phi \cup \Theta$. So (ii) follows from Corollary 4.2.

(iii) Clearly, $\eta|_{\mathfrak{H}_\aleph}$ is a $(\lambda^D, U^D|_{\mathfrak{H}_\aleph})$-cocycle. By (4.9) and (6.32), λ^D and $U^D|_{\mathfrak{H}_\aleph}$ are eigen-disjoint. We have from (6.21) that $\beta|_{\mathfrak{H}_\aleph} = 0$. Hence, by (6.33), $\xi_\beta(dh)|_{\mathfrak{H}_\aleph} = 0$ and

$$\xi(dh)|_{\mathfrak{H}_\aleph} = \xi_\beta(dh)|_{\mathfrak{H}_\aleph} + \xi_\eta(dh)|_{\mathfrak{H}_\aleph} = \xi_\eta(dh)|_{\mathfrak{H}_\aleph} = \eta(d)|_{\mathfrak{H}_\aleph}.$$

(iv) Let N be nilpotent. As λ^D, $U^D|_{\mathfrak{H}_\aleph}$ are eigen-disjoint, Theorem 4.3 yields the proof. ■

If $\lambda_\delta \neq \delta\mathbf{1}_{L_\delta}$ in part (ii), the $(\lambda_\delta, \delta\mathbf{1}_{\mathfrak{H}_\delta})$-cocycle $\eta^{\delta,\delta}$ may have a complicated structure.

For $\delta \in \operatorname{sign}(\lambda^D)$ and $\phi \in \Phi \cup \Theta$, set $\ker\beta^{\delta,\phi} = \cap_{t\in\mathbb{R}^l} \ker\beta^{\delta,\phi}(t)$, $\ker\eta^{\delta,\phi} = \cap_{d\in D}\ker\eta^{\delta,\phi}(d)$,

$$\ker\xi^{\delta,\phi} = \ker\beta^{\delta,\phi} \cap \ker\eta^{\delta,\phi}, \;\; \ker\xi|_{\mathfrak{H}_\phi} = \cap_{\delta\in\operatorname{sign}(\lambda^D)}\ker\xi^{\delta,\phi}. \tag{6.38}$$

Theorem 6.11 *Let* λ, U *be* N*-trivial representations of* $G = D \ltimes N$ *and* ξ *be a* (λ, U)*-cocycle. Let* $X = L \oplus \mathfrak{H}$ *and* $\mathfrak{e} = \mathfrak{e}(\lambda, U, \xi) = \begin{pmatrix} \lambda & \xi \\ 0 & U \end{pmatrix}$. *Then*

(i) $\dim(\mathfrak{H}_{\Phi\cup\Theta} \ominus \mathfrak{H}_1) < \infty$, *where* $\mathfrak{H}_1 = \oplus_{\phi\in\Phi\cup\Theta}\ker\xi|_{\mathfrak{H}_\phi} = \ker\xi|_{\mathfrak{H}_{\Phi\cup\Theta}}$.

(ii) *If* D *is nilpotent then* L *approximately splits* $\mathfrak{e}$.

Proof. (i) Set $\mathfrak{H}_0 = \mathfrak{H}_{\Phi\cup\Theta} \ominus \mathfrak{H}_1$. Let us show that $\dim\mathfrak{H}_0 < \infty$. We have

$$\mathfrak{H}_0 = \mathfrak{H}_{\Phi\cup\Theta} \ominus \mathfrak{H}_1 \overset{(6.32)}{=} \oplus_{\phi\in\Phi\cup\Theta}(\mathfrak{H}_\phi \ominus \ker\xi|_{\mathfrak{H}_\phi}). \tag{6.39}$$

As D is Engel and $\eta^{\delta,\phi}$ is a $(\lambda_\delta, \phi\mathbf{1}_{\mathfrak{H}_\phi})$-cocycle on D, Proposition 4.1 implies $\dim(\mathfrak{H}_\phi \ominus \ker\eta^{\delta,\phi}) < \infty$.

Let $\phi \in \Phi_\delta$ and $\{e_j\}_{j=1}^{l}$ be a basis in $\mathbb{R}^l$. As $\beta^{\delta,\phi}$ is a linear map from $\mathbb{R}^l$ to $B(\mathfrak{H}_\phi, L_\delta)$,

$$\ker\beta^{\delta,\phi} = \cap_{t\in\mathbb{R}^l}\ker\beta^{\delta,\phi}(t) = \cap_{j=1}^{l}\ker\beta^{\delta,\phi}(e_j) \text{ and } \beta^{\delta,\phi}(e_j) \in B(\mathfrak{H}_\phi, L_\delta).$$

As $\dim L_\delta < \infty$, $\dim(\mathfrak{H}_\phi \ominus \ker\beta^{\delta,\phi}(e_j)) < \infty$ for each j. Therefore

$$\dim(\mathfrak{H}_\phi \ominus \ker\beta^{\delta,\phi}) = \dim(\mathfrak{H}_\phi \ominus \left(\cap_{j=1}^{l}\ker\beta^{\delta,\phi}(e_j)\right)) < \infty.$$

If $\phi \notin \Phi_\delta$ then, by (6.35), $\beta^{\delta,\phi} = 0$, so that $\ker\beta^{\delta,\phi} = \mathfrak{H}_\phi$ and $\dim(\mathfrak{H}_\phi \ominus \ker\beta^{\delta,\phi}) = 0$.

Let $\delta \in \operatorname{sign}(\lambda^D)$ and $\phi \in \Phi \cup \Theta$. By (6.38), $\ker\xi^{\delta,\phi} = \ker\beta^{\delta,\phi} \cap \ker\eta^{\delta,\phi}$. Hence, by the above, $\dim(\mathfrak{H}_\phi \ominus \ker\xi^{\delta,\phi}) < \infty$. Therefore, as $\operatorname{sign}(\lambda^D)$ is a finite set, for each $\phi \in \Phi \cup \Theta$,

$$\dim(\mathfrak{H}_\phi \ominus \ker\xi|_{\mathfrak{H}_\phi}) \overset{(6.38)}{=} \dim(\mathfrak{H}_\phi \ominus (\cap_{\delta\in\operatorname{sign}(\lambda^D)}\ker\xi^{\delta,\phi})) < \infty.$$

Hence, as $\Phi \cup \Theta$ is a finite set, it follows from (6.39) that $\dim\mathfrak{H}_0 < \infty$.

(ii) By (6.32), λ^D and $U^D|_{\mathfrak{H}_\aleph}$ are eigen-disjoint representations (see (4.9)). If D is nilpotent then, by Theorem 4.3, there is a decomposition $\mathfrak{H}_\aleph = \oplus_{i=2}^{\infty}\mathfrak{H}_i$, where all $\mathfrak{H}_i$ are U^D-invariant and $(\lambda^D, U^D|_{\mathfrak{H}_i})$-cocycles $\eta|_{\mathfrak{H}_i}$: $D \to B(\mathfrak{H}_i, L)$ are coboundaries. As $\xi(dh)|_{\mathfrak{H}_\aleph} = \eta(d)|_{\mathfrak{H}_\aleph}$, all $\xi|_{\mathfrak{H}_i}$, $2 \le i < \infty$, are coboundaries. We also have $\xi|_{\mathfrak{H}_1} = 0$, so that it is a coboundary. Then

$$X = L \oplus \mathfrak{H} \overset{(6.32)}{=} L \oplus \mathfrak{H}_{\Phi\cup\Theta} \oplus \mathfrak{H}_\aleph = L \oplus \mathfrak{H}_0 \oplus \mathfrak{H}_1 \oplus \mathfrak{H}_\aleph = L \oplus \mathfrak{H}_0 \oplus (\oplus_{i=1}^{\infty}\mathfrak{H}_i),$$

and, as $\dim\mathfrak{H}_0 < \infty$, we have from Lemma 5.1 that L approximately splits $\mathfrak{e}$. ■

Remark 6.12 *Note that*

$$X = L \oplus \mathfrak{H}_0 \oplus \mathfrak{H}_1 \oplus \mathfrak{H}_\aleph, \text{ where } \mathfrak{H}_0 = \oplus_{\phi\in\Phi\cup\Theta}(\mathfrak{H}_\phi \ominus \ker\xi|_{\mathfrak{H}_\phi}),\ \dim\mathfrak{H}_0 < \infty \text{ and } \xi|_{\mathfrak{H}_1} = 0.$$

The subspaces $\mathfrak{H}_1$ *and* $X \ominus \mathfrak{H}_1 = L \oplus \mathfrak{H}_0 \oplus \mathfrak{H}_\aleph$ *are* $\mathfrak{e}$*-invariant and* $\mathfrak{e}|_{\mathfrak{H}_1} = U|_{\mathfrak{H}_1}$. *So, without loss of generality, we may assume that all* $\ker\xi|_{\mathfrak{H}_\phi} = \{0\}$ *and* $\mathfrak{H}_\phi$, $\phi \in \Phi \cup \Theta$, *are finite-dimensional.*

6.5 (λ, U)-cocyclic maps and cocycles on groups $G = D \ltimes N$ when $\dim L = 1$.

We consider now the classical case: $\dim L = 1$, $L = \mathbb{C}e$, $\lambda = \rho\iota$ for $\rho \in G^*$, and

$$\operatorname{sign}(\lambda^D) = \{\delta_0 := \rho^D\},\ \lambda(dh) = \delta_0(d)\iota,\ U(dh) = U(d) \text{ and } U \text{ is unitary.} \tag{6.40}$$

Consider the conjugation antilinear operator I on $\mathbb{C}^l = \mathbb{R}^l + i\mathbb{R}^l$: for $u = x + iy \in \mathbb{C}^l$,

$$Iu = I(x + iy) = x - iy,\ x, y \in \mathbb{R}^l, \text{ so that } I\mu u = \overline{\mu}Iu,\ I^2 = \mathbf{1}_{\mathbb{C}^l},\ I\widetilde{\tau}I \overset{(6.5)}{=} \widetilde{\tau}. \tag{6.41}$$

Theorem 6.13 *Let* $G = D \ltimes N$ *and* (6.40) *hold. Let* β: $\mathbb{R}^l \to B(\mathfrak{H}, \mathbb{C}e)$ *be a linear map.*

(i) β *is* $(\rho\iota, U)$*-cocyclic if and only if, for* $u \in \mathbb{C}^l$ *and some operator* $A \in B(\mathbb{C}^l, \mathfrak{H})$,

$$\widetilde{\beta}(u) = AIu \otimes e \text{ and } A\widetilde{\tau}_d = \overline{\delta_0(d)}U(d)A \text{ for } d \in D. \tag{6.42}$$

(ii) *If* U^D *has no finite-dimensional invariant subspaces, then only* $\beta = 0$ *is* $(\rho\iota, U)$*-cocyclic.*

(iii) *Let the representation* $\widetilde{\tau}$ *of* D *on* $\mathbb{C}^l$ *be similar to unitary.*

1) *If the character* δ_0 *on* D *is not unitary then* $\beta = 0$ *is the only* $(\rho\iota, U)$*-cocyclic map.*

2) *If* δ_0 *is unitary and* $\widetilde{\tau}$ *irreducible, then* β *is* $(\rho\iota, U)$*-cocyclic if and only if* $\widetilde{\beta}(u) = tVIu \otimes e$ *and* $\overline{\delta_0(d)}U(d)V = V\widetilde{\tau}_d$ *for all* $u \in \mathbb{C}^l$, $d \in D$, *and some isometry* V: $\mathbb{C}^l \to \mathfrak{H}$ *and* $t > 0$.

Proof. (i) Each operator in $B(\mathfrak{H},\mathbb{C}e)$ has form $x\otimes e$ for some $x\in\mathfrak{H}$ (see (2.7)). Let $\widetilde{\beta}$ be the complexification of β. Then $\widetilde{\beta}(u)=f(u)\otimes e$ for a map f: $u\in\mathbb{C}^l\to\mathfrak{H}$. As $\widetilde{\beta}$ is linear then, by (2.8), $f(\mu u_1+u_2)=\overline{\mu}f(u_1)+f(u_2)$ for $\mu\in\mathbb{C}$, $u_1,u_2\in\mathbb{C}^l$. So $f(u)=AIu$ for $A\in B(\mathbb{C}^l,\mathfrak{H})$. Thus

$$\widetilde{\beta}(u)=AIu\otimes e,\ \text{so that}\ \widetilde{\beta}(u)x\overset{(2.8)}{=}\langle x,AIu\rangle_{\mathfrak{H}}\,e\ \text{for}\ x\in\mathfrak{H},\tag{6.43}$$

$$\lambda(d)\widetilde{\beta}(u)\overset{(6.43)}{=}\delta_0(d)(AIu\otimes e)\overset{(2.8)}{=}(\overline{\delta_0(d)}AIu)\otimes e,$$

$$\widetilde{\beta}(\widetilde{\tau}_du)U(d)\overset{(6.43)}{=}(AI\widetilde{\tau}_du\otimes e)U(d)\overset{(2.8)}{=}(U^*(d)AI\widetilde{\tau}_du)\otimes e.$$

By (6.11), β is $(\rho\iota,U)$-cocyclic if and only if $\overline{\delta_0(d)}AI=U^*(d)AI\widetilde{\tau}_d$, $d\in D$. As U^D is unitary,

$$\overline{\delta_0(d)}U(d)A=U(d)(U^*(d)AI\widetilde{\tau}_d)I=AI\widetilde{\tau}_dI\overset{(6.41)}{=}A\widetilde{\tau}_d\ \text{for}\ d\in D.\tag{6.44}$$

Part (ii) follows from (6.42) as $\widetilde{\tau}$ is finite-dimensional.

(iii) Let $S^1=\{z\in\mathbb{C}\colon\ |z|=1\}$. Then $\mathrm{Sp}(U(d))$ and $\mathrm{Sp}(\widetilde{\tau}_d)$ lie in S^1 for $d\in D$.

1) If δ_0 is not unitary, $|\delta_0(d_1)|\neq1$ for some $d_1\in D$, by (2.4). Hence $\mathrm{Sp}(\overline{\delta_0(d_1)}U(d_1))\cap\mathrm{Sp}(\widetilde{\tau}_{d_1})=\varnothing$. So $A=0$ in (6.42) by Lemma 2.1. Thus all $(\rho\iota,U)$-cocyclic maps are zero.

2) Let δ_0 be unitary, $\widetilde{\tau}$ be irreducible and $A=VT$ be the polar form of A, $0<T\in B(\mathbb{C}^l)$. Then $\overline{\delta_0(d)}U(d)VT\overset{(6.42)}{=}VT\widetilde{\tau}_d$. As $\widetilde{\tau}$ is irreducible, $\ker T=\{0\}$. So V is an isometry from $\mathbb{C}^l$ to $\mathfrak{H}$. Set $R(d)=V^*\overline{\delta_0(d)}U(d)V$ for $d\in D$. Then R is a unitary representation of D on $\mathbb{C}^l$ and $R(d)T=T\widetilde{\tau}_d$. So

$$\widetilde{\tau}_d^*TT\widetilde{\tau}_d=TR(d)^*R(d)T=T^2\ \text{for}\ d\in D.$$

Thus $\widetilde{\tau}_dT^2=T^2\widetilde{\tau}_d$. As $\widetilde{\tau}$ is irreducible, $T^2=t^2\mathbf{1}_{\mathbb{C}^l}$. As $T>0$, $T=t\mathbf{1}_{\mathbb{C}^l}$ for some $t>0$. ■

Let D be an Engel group. By Theorem 6.8, a map β: $\mathbb{R}^l\to B(\mathfrak{H},\mathbb{C}e)$ is $(\rho\iota,U)$-cocyclic if and only if there are linear maps b_ν: $Z_\nu\to B(\mathfrak{H}_{\delta_0/\nu},\mathbb{C}e)$, $\nu\in\Omega_{\delta_0}$, such that

$$\widetilde{\beta}(u)=\sum\nolimits_{\nu\in\Omega_{\delta_0}}b_\nu(Q_\nu u)P_{\delta_0/\nu}\ \text{and}\ \nu(d)b_\nu(z)=b_\nu(\widetilde{\tau}_dz),\tag{6.45}$$

for $u\in\mathbb{C}^l$, $z\in Z_\nu$, $d\in D$. For $\nu\in D^*$, set $\overline{\nu}(d)=\overline{\nu(d)}$ for $d\in D$. By (6.41), $\nu\to\overline{\nu}$ is an isomorphism of $\mathrm{sign}(\widetilde{\tau})$, $IZ_\nu=Z_{\overline{\nu}}$ and $IQ_\nu=Q_{\overline{\nu}}I$.

Corollary 6.14 *Let D be Engel. A linear map β: $\mathbb{R}^l\to B(\mathfrak{H},\mathbb{C}e)$ is $(\rho\iota,U)$-cocyclic if and only if*

$$\widetilde{\beta}(u)=\sum\nolimits_{\nu\in\Omega_{\delta_0}}R_\nu Q_{\overline{\nu}}Iu\otimes e\ \textit{for}\ u\in\mathbb{C}^l,\tag{6.46}$$

where the operators $R_\nu\in B(Z_{\overline{\nu}},\mathfrak{H}_{\delta_0/\nu})$ satisfy $R_\nu\widetilde{\tau}_dQ_{\overline{\nu}}=\overline{\nu}(d)R_\nu Q_{\overline{\nu}}$ for $d\in D$.

If $\widetilde{\tau}|_{Z_\nu}=\nu\mathbf{1}_{Z_\nu}$ for some $\nu\in\Omega_{\delta_0}$, then R_ν can be any operator in $B(Z_{\overline{\nu}},\mathfrak{H}_{\delta_0/\nu})$.

Proof. Let b_ν: $Z_\nu\to B(\mathfrak{H}_{\delta_0/\nu},\mathbb{C}e)$ be a linear map. Since $IZ_\nu=Z_{\overline{\nu}}$, we have, as in (6.43), that $b_\nu(z)=R_\nu Iz\otimes e$ for all $z\in Z_\nu$ and some $R_\nu\in B(Z_{\overline{\nu}},\mathfrak{H}_{\delta_0/\nu})$. As $P_{\delta_0/\nu}R_\nu=R_\nu$ and $IQ_\nu=Q_{\overline{\nu}}I$,

$$b_\nu(Q_\nu u)=(R_\nu IQ_\nu u\otimes e)P_{\delta_0/\nu}\overset{(2.8)}{=}\left(P_{\delta_0/\nu}R_\nu IQ_\nu u\right)\otimes e=(R_\nu Q_{\overline{\nu}}Iu)\otimes e,$$

$$\nu(d)b_\nu(Q_\nu u)\overset{(2.8)}{=}\left(\overline{\nu(d)}R_\nu IQ_\nu u\right)\otimes e\ \text{and}$$

$$b_\nu(\widetilde{\tau}_dQ_\nu u)=(R_\nu I\widetilde{\tau}_dQ_\nu u)\otimes e\overset{(6.41)}{=}(R_\nu\widetilde{\tau}_dIQ_\nu u)\otimes e\ \text{for}\ u\in\mathbb{C}^l.$$

So (6.45) implies that $\widetilde{\beta}$ has form as in (6.46). If the second condition in (6.45) holds then $R_\nu\widetilde{\tau}_d I Q_\nu = \overline{\nu}(d) R_\nu I Q_\nu$. As $IQ_\nu = Q_{\overline{\nu}} I$ and $I^2 = \mathbf{1}_{\mathbb{C}^l}$, this implies that $R_\nu\widetilde{\tau}_d Q_{\overline{\nu}} = \overline{\nu}(d) R_\nu Q_{\overline{\nu}}$.

If $\widetilde{\tau}|_{Z_\nu} = \nu\mathbf{1}_{Z_\nu}$, condition $R_\nu\widetilde{\tau}_d Q_{\overline{\nu}} = \overline{\nu}(d) R_\nu Q_{\overline{\nu}}$ holds automatically. Indeed, as $IQ_\nu = Q_{\overline{\nu}}I$,

$$\widetilde{\tau}_d Q_{\overline{\nu}} = \widetilde{\tau}_d I Q_\nu I \overset{(6.41)}{=} I\widetilde{\tau}_d Q_\nu I = I\nu(d)Q_\nu I \overset{(6.41)}{=} \overline{\nu(d)} I Q_\nu I = \overline{\nu(d)} Q_{\overline{\nu}}.$$

So R_ν can be any operator in $B(Z_{\overline{\nu}}, \mathfrak{H}_{\delta_0/\nu})$. ■

As $\mathrm{sign}(\lambda^D) = \{\delta_0\}$, we have from (6.32) that $\Phi = \Phi_{\delta_0} = \{\phi = \delta_0/\nu\colon \nu \in \mathrm{sign}(\widetilde{\tau}), \mathfrak{H}_\phi \neq \{0\}\}$,

$$\Theta = \{\delta_0\} \text{ if } \mathfrak{H}_{\delta_0} \neq \{0\}, \text{ or } \Theta = \varnothing,\ \mathfrak{H}_{\Phi\cup\Theta} = \oplus_{\phi\in\Phi\cup\Theta}\mathfrak{H}_\phi \text{ and } \mathfrak{H} = \mathfrak{H}_{\Phi\cup\Theta}\oplus\mathfrak{H}_\aleph.$$

If $\xi = (\xi|_{\mathfrak{H}_{\Phi\cup\Theta}}, \xi|_{\mathfrak{H}_\aleph})$ is a $(\rho\iota, U)$-cocycle, then $\xi|_{\mathfrak{H}_{\Phi\cup\Theta}} = (\xi^{\delta_0,\phi})_{\phi\in\Phi\cup\Theta}$ and (see (6.34)) there are a $(\rho\iota, U)$-cocyclic map $\beta\colon \mathbb{R}^l \to B(\mathfrak{H}, \mathbb{C}e)$ and a $(\delta_0\iota^D, U^D)$-cocycle η such that

$$\xi^{\delta_0,\phi}(dh) = \xi(dh)P_\phi = \lambda(d)\beta^{\delta_0,\phi}(\omega_N(h)) + \eta^{\delta_0,\phi}(d), \text{ where } \beta^{\delta_0,\phi} = \beta P_\phi,\ \eta^{\delta_0,\phi} = \eta P_\phi.$$

Let $\phi_0 = \delta_0/\nu_0$ for $\nu_0 \in \Omega_{\delta_0}$. As $R_\nu \in B(Z_{\overline{\nu}}, \mathfrak{H}_{\delta_0/\nu})$, $P_{\phi_0}R_{\nu_0} = R_{\nu_0}$ and $P_{\phi_0}R_\nu = 0$ if $\nu \neq \nu_0$. So

$$\beta^{\delta_0,\phi}(t) = \beta(t)P_{\phi_0} \overset{(6.46)}{=} \sum\nolimits_{\nu\in\Omega_{\delta_0}} P_{\phi_0}R_\nu Q_{\overline{\nu}} t \otimes e = R_{\nu_0}Q_{\overline{\nu_0}} t \otimes e \text{ for } t \in \mathbb{R}^l. \tag{6.47}$$

If $\Theta = \{\delta_0\}$ then (see (6.37)) there are a continuous epimorphism $\omega_D\colon D \to \mathbb{R}^{l_D}$ and a linear map $\beta_{\delta_0}\colon \mathbb{R}^{l_D} \to B(\mathfrak{H}_{\delta_0}, \mathbb{C}e)$ such that $\eta^{\delta_0,\delta_0}(d) = \eta(d)P_{\delta_0} = \delta_0(d)\,\beta_{\delta_0}(\omega_D(d))$ for $d \in D$. As β_{δ_0} is linear, $\beta_{\delta_0}(r) = Ar \otimes e$ for some $A \in B(\mathbb{R}^{l_D}, \mathfrak{H}_{\delta_0})$ and all $r \in \mathbb{R}^{l_D}$. So

$$\eta^{\delta_0,\delta_0}(d) = \delta_0(d)\,\beta_{\delta_0}(\omega_D(d)) = \delta_0(d)\,(A\omega_D(d) \otimes e) \text{ for } d \in D. \tag{6.48}$$

Combining (6.47), (6.48) and Corollary 6.10 yield

Corollary 6.15 (i) *If $\delta_0 \notin \Phi$ then $\beta^{\delta_0,\delta_0} = 0$. If $\phi \in \Phi$ then*

$$\beta^{\delta_0,\phi}(t) = R_\nu Q_{\overline{\nu}} t \otimes e, t \in \mathbb{R}^l, \text{ where } \nu = \delta_0/\phi$$

and $R_\nu \in B(Z_{\overline{\nu}}, \mathfrak{H}_{\delta_0/\nu})$ is an arbitrary operator satisfying $R_\nu\widetilde{\tau}_d Q_{\overline{\nu}} = \overline{\nu}(d)R_\nu Q_{\overline{\nu}}$ for $d \in D$.

If $\widetilde{\tau}|_{Z_\nu} = \nu\mathbf{1}_{Z_\nu}$ then R_ν is an arbitrary operator in $B(Z_{\overline{\nu}}, \mathfrak{H}_{\delta_0/\nu})$.

(ii) *Let $\phi \in \Phi \cup \Theta$. If $\phi \neq \delta_0$ then η^ϕ is a $(\delta_0\iota^D, \phi\mathbf{1}_{\mathfrak{H}_\phi})$-coboundary. If $\phi = \delta_0$ then*

$$\eta^{\delta_0}(d) = \delta_0(d)\,(A\omega_D(d) \otimes e),\ d \in D, \text{ and } A \text{ is an arbitrary operator in } B(\mathbb{R}^{l_D}, \mathfrak{H}_{\delta_0}).$$

(iii) *$\xi(dh)|_{\mathfrak{H}_\aleph} = \eta(d)|_{\mathfrak{H}_\aleph}$, where $\eta|_{\mathfrak{H}_\aleph}$ is a $(\delta_0\iota^D, U^D|_{\mathfrak{H}_\aleph})$-cocycle.*

(iv) *If D is nilpotent, $\eta|_{\mathfrak{H}_\aleph}$ is either a coboundary, or $\mathfrak{H}_\aleph = \oplus_{i=1}^\infty \mathfrak{H}_i$, where all $\mathfrak{H}_i$ are U-invariant and all maps $\eta|_{\mathfrak{H}_i}\colon D \to B(\mathfrak{H}_i, L)$ are $(\delta_0\iota^D, U^D|_{\mathfrak{H}_i})$-coboundaries. So $\eta|_{\mathfrak{H}_\aleph}$ has form given in* (1.5).

Remark 6.16 (i) Let δ_0 and U^D be spectrally disjoint (for example, δ_0 be non-unitary). Then each $(\delta_0\iota^D, U^D)$-cocycle η is a coboundary by Theorem 2.13: $\eta(d) = \delta_0(d)T - TU(d)$ for some $T \in B(\mathfrak{H}, \mathbb{C}e)$. Since $T = y \otimes e$ for some $y \in \mathfrak{H}$, we get from (2.8) and (6.40) that

$$\eta(d) = (\overline{\delta_0(d)}y - U^*(d)y) \otimes e \text{ and } \xi_\eta(dh) = \eta(d) = (\overline{\rho(g)}y - U^*(g)y) \otimes e. \tag{6.49}$$

(ii) If D is compact then δ_0 is unitary. By Corollary III.2.1 [6], η is a coboundary.

7 (λ, U)-cocycles on subgroups of $GL(n,\mathbb{R}) \ltimes \mathbb{R}^n$

Let λ, U be N-trivial representations of G on L and $\mathfrak{H}$, and ξ be a (λ, U)-cocycle. By (6.32), $\mathfrak{H} = \mathfrak{H}_{\Phi\cup\Theta} \oplus \mathfrak{H}_{\aleph}$, so that $\xi = (\xi|_{\mathfrak{H}_{\Phi\cup\Theta}}, \xi|_{\mathfrak{H}_{\aleph}})$. By (6.33) and (6.34),

$$\xi = \xi_\beta + \xi_\eta, \text{ where } \xi_\beta(dh) = \lambda(d)\beta(\omega_N(h)),\ \xi_\eta(dh) = \eta(d) \text{ are } (\lambda, U)\text{-cocycles,}$$
$$\xi|_{\mathfrak{H}_{\Phi\cup\Theta}} = (\xi^{\delta,\phi})_{\delta\in\mathrm{sign}(\lambda^D),\phi\in\Phi\cup\Theta}, \text{ where } \xi^{\delta,\phi}(dh) = \lambda_\delta(d)\beta^{\delta,\phi}(\omega_N(h)) + \eta^{\delta,\phi}(d).$$

We will consider various groups $G = D \ltimes N$, where $N = \mathbb{R}^n$, $n \geq 2$, and D is a connected Engel subgroup of the group $GL(n,\mathbb{R})$. For these groups we describe the component $\xi^{\delta,\phi}$ of $\xi|_{\mathfrak{H}_{\Phi\cup\Theta}}$.

Let $\{e_i\}_{i=1}^n$ be a basis in $\mathbb{R}^n$ and denote by $d\cdot h$, $d \in D$, $h \in \mathbb{R}^n$, the action of d on $\mathbb{R}^n$. Then

$$G = D \ltimes \mathbb{R}^n = \{(d,h)\colon\ d \in D,\ h \in \mathbb{R}^n\} \text{ with } (d,h)(c,f) = (dc, d\cdot f + h).$$

The group G is also the semidirect product of its subgroups $\widehat{D}$ and $\widetilde{\mathbb{R}^n}$: $G = \widehat{D} \ltimes \widetilde{\mathbb{R}^n}$, where $\widehat{D} = \{\widehat{d} = (d,0)\colon\ d \in D\}$ and $\widetilde{\mathbb{R}^n} = \{\widetilde{h} = (e,h)\colon\ h \in \mathbb{R}^n\}$. For $d \in D$ and $h \in \mathbb{R}^n$,

$$\begin{aligned}
&(d,h) = \widehat{d}\,\widetilde{(d^{-1}\cdot h)} = (d,0)(e,d^{-1}\cdot h) = \widetilde{h}\widehat{d},\quad \widehat{d}\,\widetilde{h} = (d, d\cdot h),\\
&\widehat{d}\,\widetilde{h}\widehat{d}^{-1} = \widetilde{d\cdot h} \in \widetilde{\mathbb{R}^n},\ (d,h)^{-1} = (d^{-1}, -d^{-1}h),\ \widehat{d}\,\widetilde{h}\widehat{d}^{-1}\widetilde{h}^{-1} = \widetilde{d\cdot h - h} \in \widetilde{\mathbb{R}^n}.
\end{aligned} \tag{7.1}$$

Then $\omega_N(\widetilde{h}) = h \in \mathbb{R}^n$, $l_N = n$ (see (4.1)). So $\tau_d h \overset{(6.4)}{=} \omega_N(\widehat{d}\,\widetilde{h}\widehat{d}^{-1}) = \omega_N(\widetilde{d\cdot h}) = d\cdot h$. Thus

$$\widetilde{\tau}_d u \overset{(6.5)}{=} \tau_d x + i\tau_d y = d\cdot x + id\cdot y = d\cdot u \text{ for } u = x + iy \in \mathbb{C}^n,\ x, y \in \mathbb{R}^n. \tag{7.2}$$

7.1 D is a subgroup of the group of all positive diagonal matrices.

Denote by $\mathcal{D} = \{d = (d_{ij}) \in GL(n,\mathbb{R})\colon\ d_{ij} = 0 \text{ if } i \neq j, \text{ and } d_{ii} > 0\}$ the commutative, connected group of all positive diagonal matrices. Let $\mathcal{D}_k = \{d = (d_{ij}) \in \mathcal{D}\colon\ d_{11} = ... = d_{kk}\}$, $1 \leq k \leq n$, be subgroups of $\mathcal{D}$ and $D = \mathcal{D}_k$ for some k. Then $G = D \ltimes \mathbb{R}^n$ is solvable. It follows from (7.2) that

$$\begin{aligned}
&\mathrm{sign}(\widetilde{\tau}) = \{\nu_1, \nu_{k+1}, ..., \nu_n\}, \text{ where } \nu_i(d) = d_{ii} \text{ are real characters on } D = \mathcal{D}_k,\\
&Z_{\nu_1} = \mathbb{C}e_1 \oplus ... \oplus \mathbb{C}e_k \text{ and } Z_{\nu_i} = \mathbb{C}e_i,\ i \geq k+1, \text{ where } \{e_i\}_{i=1}^n \text{ is a basis in } \mathbb{R}^n.
\end{aligned} \tag{7.3}$$

As all $\nu \in \mathrm{sign}(\widetilde{\tau})$ are real characters, (6.20) and (6.32) imply that, for each $\delta \in \mathrm{sign}(\lambda^D)$, either $\Omega_\delta = \Phi_\delta = \varnothing$, or $\Omega_\delta = \{\nu_j\}$ and $\Phi_\delta = \{\delta/\nu_j\}$ for some $\nu_j \in \mathrm{sign}(\widetilde{\tau})$. In this case δ is non-unitary and there is not more than one $\nu_j \in \mathrm{sign}(\widetilde{\tau})$ such that δ/ν_j is unitary.

Let $\delta \in \mathrm{sign}(\lambda^D)$, $\phi \in \Phi \cup \Theta$ in (6.32) and $t = (t_i)_{i=1}^n \in \mathbb{R}^n$. By Corollary 6.10 and (7.3),

$$\begin{aligned}
&\beta^{\delta,\phi} = 0 \text{ if either } \Phi_\delta = \varnothing, \text{ or if } \Phi_\delta = \{\delta/\nu_j\} \text{ and } \phi \neq \delta/\nu_j,\\
&\beta^{\delta,\delta/\nu_1}(t) = (t_1 b_{\nu_1}(e_1) + ... + t_k b_{\nu_1}(e_k))P_{\delta/\nu_1}, \text{ if } \Phi_\delta = \{\delta/\nu_1\},\\
&\beta^{\delta,\delta/\nu_j}(t) = t_j b_{\nu_j}(e_j)P_{\delta/\nu_j}, \text{ if } \Phi_\delta = \{\delta/\nu_j\} \text{ and } j \geq k+1,
\end{aligned} \tag{7.4}$$

where $b_{\nu_1}(e_i)$ can be any operators from $B(\mathfrak{H}_{\delta/\nu_1}, L^1_\delta)$ and $b_{\nu_j}(e_j)$ any operators from $B(\mathfrak{H}_{\delta/\nu_j}, L^1_\delta)$.

If $\phi \neq \delta$ then $\eta^{\delta,\phi}$ is a $(\lambda_\delta, \phi\mathbf{1}_{\mathfrak{H}_\phi})$-coboundary and has form given in (6.36).

Let $\phi = \delta \in \Theta$. If $\lambda_\delta \neq \delta\mathbf{1}_{L_\delta}$ then $\eta^{\delta,\delta}$ is a $(\lambda_\delta, \delta\mathbf{1}_{\mathfrak{H}_\delta})$-cocycle. As D is commutative,

$$\eta^{\delta,\delta}(cd) = \lambda_\delta(c)\eta^{\delta,\delta}(d) + \delta(d)\eta^{\delta,\delta}(c) = \eta^{\delta,\delta}(dc) = \lambda_\delta(d)\eta^{\delta,\delta}(c) + \delta(c)\eta^{\delta,\delta}(d),$$

so that $\eta^{\delta,\delta}\colon D \to B(\mathfrak{H}_\delta, L_\delta)$ can be any map satisfying the condition

$$(\lambda_\delta(c) - \delta(c)\mathbf{1}_{L_\delta})\eta^{\delta,\delta}(d) = (\lambda_\delta(d) - \delta(d)\mathbf{1}_{L_\delta})\eta^{\delta,\delta}(c) \text{ for } c, d \in D. \tag{7.5}$$

If $\lambda_\delta = \delta\mathbf{1}_{L_\delta}$ then $D/D^{[1]} \approx \mathcal{D}_k \approx \mathbb{R}^{n-k+1}$ and the epimorphism $\omega_D\colon D \to \mathbb{R}^{n-k+1}$ is given by

$$\omega_D(d) = (\ln d_1)\, f_1 + \sum\nolimits_{i=2}^{k} (\ln d_{i+k-1})\, f_i \text{ in a basis } \{f_j\}_{j=1}^{n-k+1} \text{ in } \mathbb{R}^{n-k+1}.$$

By (6.37), $\eta^{\delta,\delta}(d) = \delta\,(d)\,\beta_\delta(\omega_D(d))$ for a linear map $\beta_\delta\colon \mathbb{R}^{n-k+1} \to B(\mathfrak{H}_\delta, L_\delta)$. So

$$\eta^{\delta,\delta}(d) = \delta(d)[(\ln d_1)\,\beta_\delta(f_1) + (\ln d_{k+1})\,\beta_\delta(f_2) + ... + (\ln d_n)\,\beta_\delta(f_{n-k+1})], \tag{7.6}$$

and all $\beta_\delta(f_i)$ can be any operators from $B(\mathfrak{H}_\delta, L_\delta)$. This gives a description of all $\xi^{\delta,\phi}$.

7.2 $D = SO(n)-$ the group of all orthogonal matrices in $GL(n, \mathbb{R})$.

Let D be the compact group $SO(n)$, $n \geq 2$. Then $G_n = D \ltimes \mathbb{R}^n$ is the Euclidean motion group.

Lemma 7.1 *The representation $\widetilde{\tau}$ (see (7.2)) of D on $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$, $n \geq 3$, is irreducible.*

Proof. Let L be a $\widetilde{\tau}$-invariant $\mathbb{C}$-subspace in $\mathbb{C}^n$. Let $\mathbb{R}^n \cap L = \{0\}$ and $z = x + iy \in L$. Then $0 \neq y \notin \mathbb{R}x$. So $y = tx \oplus u$ for $t \in \mathbb{R}$ and $0 \neq u \perp x$. As $n \geq 3$, there is $d_0 \in D$ such that $d_0 \cdot x = x$ and $d_0 \cdot u \neq u$. Hence $0 \neq i(z - d_0 z) = d_0 u - u \in L \cap \mathbb{R}^n$, a contradiction. So $\mathbb{R}^n \cap L \neq \{0\}$. Therefore, as $SO(n)$ acts transitively on the sphere S^{n-1} in $\mathbb{R}^n$, $\mathbb{R}^n \subset L$. Thus $L = \mathbb{C}^n$. ∎

Combining Lemma 7.1, Theorems 6.4 and 6.13 and Remark 6.16 yield

Corollary 7.2 *Let U be a unitary N-trivial representation of $G = D \ltimes \mathbb{R}^n$, $n \geq 3$, on $\mathfrak{H}$.*
A linear map β: $\mathbb{R}^l \to B(\mathfrak{H}, \mathbb{C}e)$ is (ι, U)-cocyclic if and only if for some $t > 0$,

$$\widetilde{\beta}(u) = tVIu \otimes e,\ u \in \mathbb{C}^n,\ \textit{for some isometry } V\colon\ \mathbb{C}^n \to \mathfrak{H} \textit{ satisfying } U(d)V = Vd.$$

As $\omega_N(h) = h$, a map ξ: $G \to B(\mathfrak{H}, \mathbb{C}e)$ is a (ι, U)-cocycle if and only if

$$\xi = \xi_\eta + \xi_\beta,\ \textit{where } \xi_\beta(\widehat{d}\,\widetilde{h}) = \beta(h) \textit{ for } h \in \mathbb{R}^n,\ \textit{and}$$
$$\xi_\eta(\widehat{d}\,\widetilde{h}) = \eta(\widehat{d}) = (y - U^*(\widehat{d})y) \otimes e \textit{ for all } d \in D \textit{ and some } y \in \mathfrak{H} \ (\textit{cf. } (6.49)).$$

The cohomology group $\mathcal{H}(G, \iota, U)$ was considered in [4, Example 2] without the assumption $U(dh) = U(d)$.

The group $D = SO(2) = \left\{ d = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix} : s \in [0, 2\pi) \right\}$ is commutative and $G = D \ltimes \mathbb{R}^2$ is solvable. All characters on D are unitary. Let $e_1 = (1, 0)$, $e_2 = (0, 1)$ be a basis in $\mathbb{R}^2$. By (7.2),

$$\text{sign}(\widetilde{\tau}) = \{\nu_-, \nu_+\},\ \text{where } \nu_\pm(d) = \exp(\pm is) \text{ are unitary characters on } D,$$
$$\mathbb{C}^2 = Z_{\nu_+} \oplus Z_{\nu_-},\ \text{where } Z_{\nu_\pm} = \mathbb{C}f_\pm \text{ and } f_\pm = (e_1 \pm ie_2)/\sqrt{2}.$$

Denote by $Q_{\nu_\pm}$ the projections on $Z_{\nu_\pm}$. Then $Q_{\nu_\pm} e_1 = f_\pm/\sqrt{2}$ and $Q_{\nu_\pm} e_2 = \pm f_\pm / i\sqrt{2}$.

As D is compact, each $\delta \in \operatorname{sign}(\lambda^D)$ is unitary and the representation λ_δ is similar to unitary. So $\lambda_\delta = \delta \mathbf{1}_{L_\delta}$ and $L^1_\delta = L_\delta$. The set Ω_δ can be any subset of $\{\nu_-, \nu_+\}$. So

$$\text{either } \Phi_\delta = \varnothing, \text{ or } \Phi_\delta = \{\delta/\nu_+\}, \text{ or } \Phi_\delta = \{\delta/\nu_-\}, \text{ or } \Phi_\delta = \{\delta/\nu_+, \delta/\nu_-\}.$$

By Corollary 6.10, if $\phi = \delta/\nu \in \Phi_\delta$ then $\beta^{\delta,\delta/\nu}(t) = b^\delta_\nu(Q_\nu t)P_\phi$, $t \in \mathbb{R}^2$, where b^δ_ν: $Z_\nu \to B(\mathfrak{H}_{\delta/\nu}, L^1_\delta)$ can be any linear map. Otherwise, $\beta^{\delta,\phi} = 0$. Thus, for $t = t_1 e_1 + t_2 e_2 \in \mathbb{R}^2$,

$$\beta^{\delta,\phi}(t) = b^\delta_{\nu_+}(Q_{\nu_+} t)P_\phi = 2^{-1/2}(t_1 - it_2) b^\delta_{\nu_+}(f_+)P_\phi, \text{ if } \phi = \delta/\nu_+ \in \Phi_\delta,$$
$$\beta^{\delta,\phi}(t) = b^\delta_{\nu_-}(Q_{\nu_-} t)P_\phi = 2^{-1/2}(t_1 + it_2) b^\delta_{\nu_-}(f_-)P_\phi, \text{ if } \phi = \delta/\nu_- \in \Phi_\delta,$$

where $b^\delta_{\nu_\pm}(f_\pm)$ can be any operators from $B(\mathfrak{H}_{\delta/\nu_\pm}, L^1_\delta)$.

If $\phi \neq \delta$ then $\eta^{\delta,\phi}$ is a $(\lambda_\delta, \phi \mathbf{1}_{\mathfrak{H}_\phi})$-coboundary. By (6.36), $\eta^{\delta,\phi} = (\delta - \phi)T$ for some $T \in B(\mathfrak{H}_\phi, L_\delta)$.

If $\phi = \delta$ then $\eta^{\delta,\delta}$ is a $(\delta \mathbf{1}_{L_\delta}, \delta \mathbf{1}_{\mathfrak{H}_\delta})$-cocycle. As D is compact, we get from (6.37) that $\eta^{\delta,\delta} = 0$.

7.3 D is the group of all upper triangular matrices with 1 on the diagonal

In this section D is the nilpotent group of all upper triangular $n \times n$ matrices with 1 on the diagonal

$$D := T_n = \{d = (d_{ij}) \in GL(n, \mathbb{R})\colon d_{11} = \dots = d_{nn} = 1,\ d_{ij} = 0 \text{ if } i > j\}.$$

Since $\widetilde{\tau}_d u = d \cdot u$ for $d \in T_n$, $u \in \mathbb{C}^n$, we have $\operatorname{sign}(\widetilde{\tau}) = \{\chi_e\}$ and $Z_{\chi_e} = \mathbb{C}^n$. Thus (see (6.32))

$$\Theta = \Phi = \{\delta \in \operatorname{sign}(\lambda^D)\colon \mathfrak{H}_\delta \neq \{0\}\} \text{ and } \Phi_\delta = \{\delta\} \text{ if } \delta \in \Theta,\ \Phi_\delta = \varnothing \text{ if } \delta \notin \Theta.$$

If $\phi \neq \delta$ then $\eta^{\delta,\phi}$ is a $(\lambda_\delta, \phi \mathbf{1}_{\mathfrak{H}_\phi})$-coboundary and has form given in (6.36).

Let $\phi = \delta \in \Theta$. If $\lambda_\delta \neq \delta \mathbf{1}_{L_\delta}$, $\eta^{\delta,\delta}$ is a $(\lambda_\delta, \delta \mathbf{1}_{\mathfrak{H}_\delta})$-cocycle (it may have complicated structure).

Let $\lambda_\delta = \delta \mathbf{1}_{L_\delta}$. As $D/D^{[1]} \approx \mathbb{R}^{n-1}$, the epimorphism ω_D: $D \to \mathbb{R}^{n-1}$ defined as in (4.2) has form

$$\omega_D(d) = \sum\nolimits_{i=1}^{n-1} d_{i,i+1} g_i \text{ for } d = (d_{ij}) \in T_n \text{ in some basis } \{g_j\}_{j=1}^{n-1} \text{ in } \mathbb{R}^{n-1}.$$

We have from (6.37) that $\eta^{\delta,\delta}(d) = \delta(d) \beta_\delta(\omega_D(d))$ for a linear map β_δ: $\mathbb{R}^{n-1} \to B(\mathfrak{H}_\delta, L_\delta)$. So

$$\eta^{\delta,\delta}(d) = \delta(d) \sum\nolimits_{i=1}^{n-1} d_{i,i+1} \beta_\delta(g_i), \text{ where } \beta_\delta(g_i) \text{ can be any operators from } B(\mathfrak{H}_\delta, L_\delta).$$

Consider now $\beta = (\beta^{\delta,\phi})_{\delta \in \operatorname{sign}(\lambda^D), \phi \in \Phi}$. Let $\{e_i\}_{i=1}^n$ be a basis in $\mathbb{R}^n$. As before, for $\delta \in \operatorname{sign}(\lambda^D)$, let

$$\mathfrak{H}^T_\delta = \{x \in \mathfrak{H}\colon U(d)x = \delta(d)x \text{ for } d \in T_n\} \text{ and } P_\delta \text{ be the projection on } \mathfrak{H}^T_\delta. \tag{7.7}$$

Let $\phi \in \Phi$, $t \in \mathbb{R}^n$, $u \in \mathbb{C}^n$ and $d \in T_n$. From (6.22) and (6.35) it follows that

$$\beta^{\delta,\phi} = 0 \text{ if } \delta \notin \Phi, \text{ or if } \phi \neq \delta \in \Phi; \text{ and}$$
$$\beta^{\delta,\delta}(t) = b^\delta(t)P_\delta \text{ if } \delta \in \Phi, \text{ where } \lambda_\delta(d) b^\delta(u) = \delta(d) b^\delta(d \cdot u) \text{ and}$$

b^δ: $\mathbb{C}^n \to B(\mathfrak{H}^T_\delta, L_\delta)$ is a linear map. Set $\lambda^\delta_{\chi_e} = \lambda_\delta/\delta$. Then $\lambda^\delta_{\chi_e}$ is a χ_e-representation and

$$\lambda^\delta_{\chi_e}(d) b^\delta(u) = b^\delta(d \cdot u),\ d \in D,\ u \in \mathbb{C}^n. \tag{7.8}$$

We will describe the map b^δ satisfying (7.8) in two particular cases.

Proposition 7.3 (i) *If $\lambda^\delta_{\chi_e} = \mathbf{1}_{L_\delta}$ in (7.8) then $b^\delta(u) = u_n b^\delta(e_n)$, for $u = \sum_{i=1}^n u_i e_i \in \mathbb{C}^n$, where $b^\delta(e_n)$ can be any operator in $B(\mathfrak{H}^T_\delta, L_\delta)$.*

(ii) *Let* $\mathrm{sign}(\lambda^D) = \{\delta\}$, $L = L_\delta \approx \mathbb{C}^n$ *and, in some basis* $\{f_i\}_{i=1}^n$ *in* L,

$$\lambda^\delta_{\chi_e}(d)v = d \cdot v, \ \ d \in T_n, \ \ v \in L. \tag{7.9}$$

Then a linear map $b^\delta \colon \mathbb{C}^n \to B(\mathfrak{H}^T_\delta, L)$ *satisfies* (7.8) *if and only if, for some (any)* $x, y \in \mathfrak{H}^T_\delta$,

$$b^\delta(e_i) = x \otimes f_i \text{ for } i = 1, ..., n-1, \text{ and } b^\delta(e_n) = y \otimes f_1 + x \otimes f_n. \tag{7.10}$$

Proof. (i) If $\lambda^\delta_{\chi_e} = \mathbf{1}_L$ then $b^\delta(u) \overset{(7.8)}{=} b^\delta(d \cdot u)$, $d \in T_n$, $u \in \mathbb{C}^n$. So $b^\delta((d - \mathbf{1}_{\mathbb{C}^n}) \cdot u) = 0$. Choose $\{d_j\}_{j=1}^{n-1}$ such that $(d_j - \mathbf{1}_{\mathbb{C}^n}) \cdot e_{j+1} = e_j$. Thus $b^\delta(e_j) = 0$ for $j = 1, ..., n-1$, and $b^\delta(e_n)$ can be any operator in $B(\mathfrak{H}^T_\delta, L_\delta)$.

(ii) Each operator $A \in B(\mathfrak{H}^T_\delta, L)$ has form (see (2.8))

$$A = \sum\nolimits_{i=1}^n y_i \otimes f_i \text{ for some } y_i \in \mathfrak{H}^T_\delta, \text{ and } RA = R\sum\nolimits_{i=1}^n y_i \otimes f_i \overset{(2.8)}{=} \sum\nolimits_{i=1}^n y_i \otimes Rf_i \tag{7.11}$$

for $R \in B(L)$. Since $b^\delta(e_j) \in B(\mathfrak{H}^T_\delta, L)$, we have from (7.11) that

$$b^\delta(e_j) = x_{j1} \otimes f_1 + ... + x_{jn} \otimes f_n \text{ for some } x_{ji} \in \mathfrak{H}^T_\delta. \tag{7.12}$$

Set $\lambda = \lambda^\delta_{\chi_e}$ and $b = b^\delta$. Then

$$\begin{aligned}(\lambda(d) - \mathbf{1}_L)b^\delta(e_j) &= (\lambda(d) - \mathbf{1}_L)\sum\nolimits_{i=1}^n x_{ji} \otimes f_i \overset{(7.11)}{=} \sum\nolimits_{i=1}^n x_{ji} \otimes (\lambda(d) - \mathbf{1}_L)f_i \\ &\overset{(7.9)}{=} \sum_{i=1}^n x_{ji} \otimes (d - \mathbf{1}_L) \cdot f_i.\end{aligned}$$

Thus (7.8) is equivalent to the condition

$$\sum\nolimits_{i=1}^n x_{ji} \otimes (d - \mathbf{1}_L) \cdot f_i = b^\delta((d - \mathbf{1}_{\mathbb{C}^n}) \cdot e_j) \text{ for } d \in T_n \text{ and } j = 1, ..., n. \tag{7.13}$$

Let $\widehat{d}_{km} = (d_{ij})$ be such that $d_{km} = 1$ and all other $d_{ij} = 0$. Then (7.13) is equivalent to

$$\sum\nolimits_{i=1}^n x_{ji} \otimes \widehat{d}_{km} \cdot f_i = b^\delta(\widehat{d}_{km} \cdot e_j) \text{ for } j = 1, ..., n \text{ and } 1 \leq k < m \leq n. \tag{7.14}$$

As $\widehat{d}_{km} \cdot f_i = \delta_{mi} f_k$ and $\widehat{d}_{km} \cdot e_i = \delta_{mi} e_k$, where $\delta_{mi} = 0$ if $m \neq i$ and $\delta_{mm} = 1$, we have

$$\sum\nolimits_{i=1}^n x_{ji} \otimes \delta_{mi} f_k = x_{jm} \otimes f_k \overset{(7.14)}{=} b^\delta(\delta_{mj} e_k) \overset{(7.12)}{=} \delta_{mj} \sum\nolimits_{i=1}^n x_{ki} \otimes f_i. \tag{7.15}$$

Hence

$$x_{jm} = 0 \text{ if } j \neq m \text{ for } m = 2, ..., n. \tag{7.16}$$

If $2 \leq j = m \leq n$ then

$$\begin{gathered} x_{mm} \otimes f_1 \overset{(7.15)}{=} \sum\nolimits_{i=1}^n x_{1i} \otimes f_i \overset{(7.16)}{=} x_{11} \otimes f_1, \text{ so that } x_{11} = x_{22} = ... = x_{nn}, \\ x_{mm} \otimes f_k \overset{(7.15)}{=} \sum\nolimits_{i=1}^n x_{ki} \otimes f_i \overset{(7.16)}{=} x_{k1} \otimes f_1 + x_{kk} \otimes f_k \text{ for } 1 < k < m \leq n. \end{gathered}$$

Thus $x_{k1} = 0$ for $k = 2, ..., n-1$. Set $x = x_{11}$ and $y = x_{n1}$. Then (7.10) follows from (7.12). ■

7.4 D is the group of all upper triangular matrices with positive diagonal entries

In this section D is the solvable connected group of upper triangular matrices

$$D := S_n = \{s = (s_{ij})\colon\ s_{ij} \in \mathbb{R},\ s_{ij} = 0 \text{ if } i > j, \text{ and } s_{ii} > 0\} = \mathcal{D} \ltimes T_n, \tag{7.17}$$

where $\mathcal{D}$ is the group of all diagonal matrices with positive diagonal entries. The group $G_S = S_n \ltimes \mathbb{R}^n$ is connected and solvable.

As D is not nilpotent, Corollary 6.10(iv) cannot be applied, so we have no information about (λ^D, U^D)-cocycles η. Thus we only consider the structure of the (λ, U)-cocyclic maps β: $\mathbb{R}^n \to B(\mathfrak{H}, L)$.

Let representations λ, U of G_S on L and $\mathfrak{H}$ satisfy (6.15). As S_n is solvable, matrices $\lambda(s)$, $s \in S_n$, are upper triangular in a basis in L with characters $\{\delta_j\}$ on the diagonal: $\lambda_{jj}(s) = \delta_j(s)$. They can be also treated as characters on the subgroup $\mathcal{D}$. Some of them may coincide. Let

$$\text{sign}(\lambda^{\mathcal{D}}) = \text{ sign}(\lambda^S) = \{\delta_1, ..., \delta_p\} \text{ be the set of all } \textbf{distinct} \text{ diagonal characters.}$$

Recall that a map β: $\mathbb{R}^n \to B(\mathfrak{H}, L)$ is (λ, U)-cocyclic if $\widetilde{\beta}$ satisfies (6.11):

$$\lambda(d)\widetilde{\beta}(u) = \widetilde{\beta}(\widetilde{\tau}_d u)U(d) \text{ for } d \in D \text{ and } u \in \mathbb{C}^n. \tag{7.18}$$

Clearly, (7.18) holds if and only if it holds for $\mathcal{D}$ and T_n. Let P be the projection on $\mathfrak{H}^T_{\chi_e}$ (see (7.7)).

Lemma 7.4 *A linear map β: $\mathbb{R}^n \to B(\mathfrak{H}, L)$ is (λ, U)-cocyclic on G_S if and only if*

$$\widetilde{\beta}(u) = b(u)P, \tag{7.19}$$

where b: $\mathbb{C}^n \to B(\mathfrak{H}^T_{\chi_e}, L)$ is a linear map that satisfies the following conditions:

$$\lambda(t)b(u) = b(t \cdot u) \text{ for } t \in T_n \text{ and } u \in \mathbb{C}^n, \tag{7.20}$$

$$\lambda(d)b(u)|_{\mathfrak{H}^T_{\chi_e}} = b(d \cdot u)U(d)|_{\mathfrak{H}^T_{\chi_e}} \text{ for } d \in \mathcal{D},\ u \in \mathbb{C}^n. \tag{7.21}$$

Proof. As $T_n = S_n^{[1]}$, all matrices $\lambda(t)$, $t \in T_n$, have 1 on the diagonal. So $\text{sign}(\lambda^T) = \{\chi_e\}$ and $\text{sign}(\widetilde{\tau}^T) = \{\chi_e\}$. It follows from Theorem 6.8 that (7.18) holds for $t \in T_n$ if and only if $\widetilde{\beta}(u) = b(u)P$, $u \in \mathbb{C}^n$, where b: $\mathbb{C}^n \to B(\mathfrak{H}^T_{\chi_e}, L)$ is a linear map that satisfies (7.20).

As T_n is a normal subgroup of S_n, $\mathfrak{H}^T_{\chi_e}$ is U-invariant. Thus U and P commute. So, for $d \in \mathcal{D}$,

$$\lambda(d)\widetilde{\beta}(u) \overset{(7.19)}{=} \lambda(d)b(u)P \text{ and } \widetilde{\beta}(\widetilde{\tau}_d u)U(d)|_{\mathfrak{H}^T_{\chi_e}} \overset{(7.19)}{=} b(\widetilde{\tau}_d u)U(d)P \overset{(7.2)}{=} b(d \cdot u)U(d)P.$$

Hence (7.18) holds for $\mathcal{D}$ if and only if (7.21) holds which completes the proof. ■

By Lemma 7.4, without loss of generality, we can assume that $\mathfrak{H} = \mathfrak{H}^T_{\chi_e}$. As $\theta|_{T_n} = 1$ for $\theta \in S_n^*$,

$$\mathfrak{H}^{\mathcal{D}}_\theta := \{x \in \mathfrak{H}\colon\ U(d)x = \theta(d)x \text{ for } d \in \mathcal{D}\} = \{x \in \mathfrak{H}\colon\ U(s)x = \theta(s)x \text{ for } s \in S_n\}. \tag{7.22}$$

Let $P^{\mathcal{D}}_\theta$ be the projection on $\mathfrak{H}^{\mathcal{D}}_\theta$. As $\mathcal{D}$ is Engel, it follows from (6.16) that

$$L = \oplus_{\delta \in \text{sign}(\lambda^{\mathcal{D}})} L_\delta \text{ and } \lambda^{\mathcal{D}} = \oplus_{\delta \in \text{sign}(\lambda^{\mathcal{D}})} \lambda^{\mathcal{D}}_\delta,$$

where L_δ are $\lambda^{\mathcal{D}}$-invariant subspaces, $\lambda^{\mathcal{D}}_\delta = \lambda^{\mathcal{D}}|_{L_\delta}$ are δ-representations of $\mathcal{D}$. Then

$$b = (b^\delta)_{\delta\in\operatorname{sign}(\lambda^{\mathcal{D}})} \text{ and } \lambda_\delta(d)b^\delta(u) = b^\delta(\widetilde{\tau}_d u)U(d) \text{ for } u\in\mathbb{C}^l,\ d\in\mathcal{D}, \tag{7.23}$$

where b^δ: $\mathbb{C}^n \to B(\mathfrak{H}, L_\delta)$ are $(\lambda^{\mathcal{D}}_\delta, U^{\mathcal{D}})$-cocyclic maps. As in (7.3) for $k = 1$,

$$\operatorname{sign}(\widetilde{\tau}^{\mathcal{D}}) = \{\nu_i\}_{i=1}^n, \text{ where } \nu_i(d) = d_{ii}, \text{ are real characters and } Z_{\nu_i} = \mathbb{C}e_i \text{ for all } i. \tag{7.24}$$

Thus (see (6.20)), for $\delta\in\operatorname{sign}(\lambda^{\mathcal{D}})$, either $\Omega_\delta = \varnothing$, or $\Omega_\delta = \{\nu_{i_\delta}\}$ for some $1\le i_\delta\le n$. Set

$$\Delta = \{\delta\in \operatorname{sign}(\lambda^{\mathcal{D}})\colon \Omega_\delta = \{\nu_{i_\delta}\}\}, \text{ i.e., } \mathfrak{H}^{\mathcal{D}}_{\delta/\nu_{i_\delta}} \neq \{0\}. \tag{7.25}$$

Using Corollary 6.9 and Lemma 7.4, we obtain th following description of (λ, U)-cocyclic maps.

Theorem 7.5 *A linear map* β: $\mathbb{R}^n \to B(\mathfrak{H}, L)$ *is* (λ, U)*-cocyclic on* G_S *if and only if*

$$\widetilde{\beta}(u) = b(u)P \text{ and } b(u) = (b^\delta(u))_{\delta\in\operatorname{sign}(\lambda^{\mathcal{D}})} \text{ for } u = (u_i)_{i=1}^n\in\mathbb{C}^n, \text{ where}$$
$$b^\delta = 0 \text{ if } \delta\notin\Delta, \text{ and } b^\delta(u) = u_{i_\delta} b^\delta_{\nu_{i_\delta}}(e_{i_\delta})P^{\mathcal{D}}_{\delta/\nu_{i_\delta}} \text{ if } \delta\in\Delta,$$

$P^{\mathcal{D}}_{\delta/\nu_{i_\delta}}$ *are the projections on* $\mathfrak{H}^{\mathcal{D}}_{\delta/\nu_{i_\delta}}$ *and* $b^\delta_{\nu_{i_\delta}}(e_{i_\delta})$ *are operators in* $B(\mathfrak{H}^{\mathcal{D}}_{\delta/\nu_{i_\delta}}, L^1_\delta)$ *such that*

$$\lambda(t)b(u) = b(t\cdot u) \text{ for } t\in T_n.$$

In other words, for $u = (u_i)_{i=1}^n\in\mathbb{C}^n$ and $d\in T_n$,

$$\widetilde{\beta}(u) = b(u)P = \sum\nolimits_{\delta\in\Delta} u_{i_\delta} b^\delta_{\nu_{i_\delta}}(e_{i_\delta})P^{\mathcal{D}}_{\delta/\nu_{i_\delta}} \text{ and } \lambda(d)b(u) = b(d\cdot u). \tag{7.26}$$

As in Proposition 7.3, we will find the map $\widetilde{\beta}$ in a particular case when $L\approx\mathbb{C}^n$ and

$$\lambda(s)v = s\cdot v \text{ for } v\in L \text{ and } s\in S_n \text{ in some basis } \{f_i\}_{i=1}^n \text{ in } L. \tag{7.27}$$

Proposition 7.6 *Let (7.27) hold. If* $\mathfrak{H}^{\mathcal{D}}_{\chi_e} = \{0\}$ *then* $\beta = 0$ *is the only* (λ, U)*-cocyclic map.*
Let $\mathfrak{H}^{\mathcal{D}}_{\chi_e}\neq\{0\}$*. A linear map* β: $\mathbb{R}^n\to B(\mathfrak{H}, L)$ *is* (λ, U)*-cocyclic if and only if*

$$\widetilde{\beta}(u) = x\otimes\sum\nolimits_{i=1}^n u_i f_i \text{ for an arbitrary } x\in\mathfrak{H}^{\mathcal{D}}_{\chi_e} \text{ and all } u = (u_i)_{i=1}^n\in\mathbb{C}^l. \tag{7.28}$$

Proof. As $\lambda(d)v = d\cdot v$ for $v\in L$, $d\in\mathcal{D}$, we have $\operatorname{sign}(\lambda^{\mathcal{D}}) = \{\delta_i\}_{i=1}^n$, where $\delta_i(d) = d_{ii}$, and $L = \oplus_{i=1}^n L_{\delta_i}$, with $L_{\delta_i} = L^1_{\delta_i} = \mathbb{C}f_i$ for all i.
If $\mathfrak{H}^{\mathcal{D}}_{\chi_e} = \{0\}$, it follows from (7.24) and (7.25) that $\Delta = \varnothing$. So $\beta = 0$ by (7.26).
Let $\mathfrak{H}^{\mathcal{D}}_{\chi_e}\neq\{0\}$ and $P^{\mathcal{D}}_{\chi_e}$ be the projection on $\mathfrak{H}^{\mathcal{D}}_{\chi_e}$. It follows from (7.24) and (7.25) that

$$\Delta = \operatorname{sign}(\lambda^{\mathcal{D}}) \text{ and } \Omega_{\delta_i} = \{\nu_i = \delta_i\} \text{ for all } 1\le i\le n, \text{ as } \delta_i/\nu_i = \chi_e \text{ and } \mathfrak{H}^{\mathcal{D}}_{\chi_e}\neq\{0\}.$$

Let $u = \sum_{i=1}^n u_i e_i\in\mathbb{C}^n$. By (7.26), a map β is (λ, U)-cocyclic if and only if it has form

$$\widetilde{\beta}(u) = \sum\nolimits_{i=1}^n u_i b^{\delta_i}_{\nu_i}(e_i)P^{\mathcal{D}}_{\delta_i/\nu_i} = \sum\nolimits_{i=1}^n u_i b^{\delta_i}_{\nu_i}(e_i)P^{\mathcal{D}}_{\chi_e}, \tag{7.29}$$

where $b^{\delta_i}_{\nu_i}(e_i)$ are operators in $B(\mathfrak{H}^{\mathcal{D}}_{\chi_e}, L^1_{\delta_i})$ such that $\lambda(t)b(u) = b(t\cdot u)$ for $t\in T_n$.

As $L^1_{\delta_i} = \mathbb{C}f_i$ for all i, $b^{\delta_i}_{\nu_i}(e_i)P^{\mathcal{D}}_{\chi_e} = x_i \otimes f_i$ for some $x_i \in \mathfrak{H}^{\mathcal{D}}_{\chi_e}$ by (7.11). By (7.11), (7.27),

$$\widetilde{\beta}(u) \overset{(7.29)}{=} u_1(x_1 \otimes f_1) + ... + u_n(x_n \otimes f_n), \text{ so that } \widetilde{\beta}(e_i) = x_i \otimes f_i, \tag{7.30}$$
$$\text{and } \lambda(t)\widetilde{\beta}(e_n) \overset{(7.27)}{=} t(x_n \otimes f_n) \overset{(7.11)}{=} x_n \otimes t \cdot f_n, \text{ for } t \in T_n.$$

Let $t_{i,n} \in T_n$ be a matrix with (i,n)-th and all diagonal entries 1, and all other are 0. Then

$$\lambda(t_{i,n})\widetilde{\beta}(e_n) = x_n \otimes t_{i,n} \cdot f_n = x_n \otimes f_i + x_n \otimes f_n$$
$$\overset{(7.26)}{=} \widetilde{\beta}(t_{i,n} \cdot e_n) = \widetilde{\beta}(e_i) + \widetilde{\beta}(e_n) \overset{(7.30)}{=} x_i \otimes f_i + x_n \otimes f_n.$$

Hence $x_i = x_n$ for $1 \leq i \leq n-1$. Combining this with (7.30), we complete the proof. ■

8 Cocyclic maps on the group $S_n = \mathcal{D} \ltimes T_n$.

The group $S_n = \mathcal{D} \ltimes T_n$ in (7.17) is the semidirect product of $D = \mathcal{D}$ and $N = T_n$. It is solvable and not Engel. Each $h \in N$ has form $h = (\widehat{h}_0, \widehat{h}_1, .., \widehat{h}_{n-1})$, where $\widehat{h}_0 = (1, ..., 1)$ is the main diagonal and $\widehat{h}_i = (h_{11+i}, h_{22+i}, .., h_{n-i,n})$, $1 \leq i \leq n-1$, is the i-th upper diagonal of h.

Then $N^{[1]} = \{h \in N\colon \widehat{h}_1 = 0\}$ and $N/N^{[1]} \approx \mathbb{R}^{n-1}$. So $l = l_{_N} = n-1$ (see (4.1)) and

$$\omega_{_N}(h) = \widehat{h}_1 = (h_{12}, ..., h_{n-1,n}) = h_{12}e_1 + ... + h_{n-1,n}e_{n-1} \in \mathbb{R}^{n-1} \tag{8.1}$$

is the epimorphism from N to $\mathbb{R}^{n-1}$. Let $d = (d_{11}, ..., d_{nn}) \in D$. Set $d_i = d_{ii}$. By (6.4),

$$\tau_d\omega_{_N}(h) = \omega_{_N}(h_d) \overset{(8.1)}{=} (\widehat{h_d})_1 = (\nu_1(d)h_{12}, ..., \nu_{n-1}(d)h_{n-1,n}), \text{ where } \nu_i(d) = d_i/d_{i+1}$$

for $1 \leq i \leq n-1$, are distinct real characters on D. Hence, for $\delta \in \operatorname{sign}(\lambda^D)$,

$$\operatorname{sign}(\widetilde{\tau}) = \{\nu_i\}_{i=1}^{n-1}, \ \mathbb{C}^{n-1} = \oplus_{i=1}^{n-1} Z_{\nu_i}, \text{ where all } Z_{\nu_i} = \mathbb{C}e_i, \text{ and}$$
$$\text{either } \Omega_\delta = \Phi_\delta = \varnothing, \text{ or } \Omega_\delta = \{\nu_i\} \text{ and } \Phi_\delta = \{\delta/\nu_i\} \text{ for some } \nu_i \in \operatorname{sign}(\widetilde{\tau}), \tag{8.2}$$

The results in this case are similar to the results for $\mathcal{D}_1 \ltimes \mathbb{R}^n$ in Section 7.1.

Let $\phi \in \Phi \cup \Theta$ and $t = (t_i)_{i=1}^{n-1} \in \mathbb{R}^{n-1}$ (see (6.32)). From Corollary 6.10 and (8.2) we have

$$\beta^{\delta,\phi} = 0 \text{ if either } \Phi_\delta = \varnothing, \text{ or if } \Phi_\delta = \{\delta/\nu_i\} \text{ and } \phi \neq \delta/\nu_i;$$
$$\beta^{\delta,\phi}(t) = t_j b_{\nu_j}(e_j)P_{\delta/\nu_j}, \text{ if } \Phi_\delta = \{\delta/\nu_j\} \text{ and } \phi = \delta/\nu_j \text{ for some } j \geq 1,$$

where $b_{\nu_j}(e_j)$ can be any operators from $B(\mathfrak{H}_{\delta/\nu_j}, L^1_\delta)$.

If $\phi \neq \delta$ then $\eta^{\delta,\phi}$ is a $(\lambda_\delta, \phi\mathbf{1}_{\mathfrak{H}_\phi})$-coboundary and has form given in (6.36).

Let $\phi = \delta \in \Theta$. If $\lambda_\delta \neq \delta\mathbf{1}_{L_\delta}$ then $\eta^{\delta,\delta}$ is a $(\lambda_\delta, \delta\mathbf{1}_{\mathfrak{H}_\delta})$-cocycle. As D is commutative, we have, as in (7.5), that $\eta^{\delta,\delta}\colon D \to B(\mathfrak{H}_\delta, L_\delta)$ can be any map satisfying the condition

$$(\lambda_\delta(c) - \delta(c)\mathbf{1}_{L_\delta})\eta^{\delta,\delta}(d) = (\lambda_\delta(d) - \delta(d)\mathbf{1}_{L_\delta})\eta^{\delta,\delta}(c) \text{ for } c, d \in D.$$

Let $\lambda_\delta = \delta\mathbf{1}_{L_\delta}$. The map $\omega_{_D}\colon D \to \mathbb{R}^n$ defined by $\omega_{_D}(d) = \sum_{i=1}^n (\ln d_i)\, e_i$ is an epimorphism. By (6.37), $\eta^{\delta,\delta}(d) = \delta\,(d)\,\beta_\delta(\omega_{_D}(d))$ for some linear map $\beta_\delta\colon \mathbb{R}^n \to B(\mathfrak{H}_\delta, L_\delta)$. So

$$\eta^{\delta,\delta}(d) = \delta(d)\sum_{i=1}^n (\ln d_i)\, \beta_\delta(e_i) \text{ and } \beta_\delta(e_i) \text{ can be any operators in } B(\mathfrak{H}_\delta, L_\delta).$$

References

[1] Araki H., Indecomposable representations with invariant inner product, Commun. Math. Phys., 97(1985), 149-159.

[2] Delorme P., 1-cohomologie des representations unitaires des groupes de Lie semisimples et resolubles, Bull. Soc. Math. France, 105(1977), 281-336.

[3] Dubin D. A. and Tarski J., J. Math. Phys., Indefinite metric resulting from regularization in the infrared region, 7, 574 (1966).

[4] Guichardet A., Sur la cohomologie des groupes topologiques, II, Bull. Soc. Math., 96(1972), 305-332.

[5] Guichardet A., Cohomology of topological groups and positive definite functions, J. Multivariate Anal., 3 (1973), 249-261.

[6] Guichardet A., Cohomologie des groupes topologiques et des algebres de Lie, Cedic/Fernand Nathan, Paris, 1980.

[7] Halmos P.R., "A Hilbert space problem book", D. van Nostrand Co. Inc., 1967.

[8] Hochschild G., Serre J.P., Cohomology of group extensions, Trans. Amer. Math. Soc., 74(1953), p. 110-134.

[9] Ismagilov R.S., Unitary representations of the Lorentz group in spaces with indefinite metric, Izv. Akad. Nauk SSSR, 30, No 3(1966), 497-522.

[10] Ismagilov R.S., On irreducible representations of the discrete group SL$(2, P)$which are unitary with respect to an indefinite metric, Izv. Akad. Nauk SSSR, 30, No 4(1966), 923-950.

[11] Ismagilov R.S., On the problem of extension of representations, Matem. Zametki, 35, No 1(1984), 99-105.

[12] Kissin E. and Shulman V.S., Representations on Krein Spaces and Derivations of C*-algebras, Addison Wesley Longman, London, 1997.

[13] Kissin E. and Shulman V. S., Non-unitary representations of nilpotent groups, I: cohomologies, extensions and neutral cocycles, J. Func. Anal., 269(2015), 2564-2610.

[14] Kissin E. and Shulman V. S., Representations of nilpotent groups on spaces with indefinite metric, J. Int. Eq. Oper. Theory, 87, No 1(2017), 81-116, DOI 10.1007/s00020-016-2337-7.

[15] Kissin E. and Shulman V.S., On representations of groups and algebras in spaces with indefinite metric (in Russian), Contemporary Mathematics, Fundamental directions., 67(2021), 2, pp. 295-315. ISSN 2413-3639. DOI 10.22363/2413-3639-2021-67-2-29.

[16] Kissin E. and Shulman V.S., On cohomologies and representations of groups with normal Engel subgroups, Eurasian Mathem. J., 13(2022), No 2, 70-81.

[17] Kissin E., Symmetric operator extensions of unbounded derivations of C*-algebras, Journal of Funct. Anal., 81(1988), 38-53.

[18] Kuratowski K., "Topology", 1–2 , Acad. Press, 1966–1968.

[19] Morchio G., Pierotti D. and Strocchi F. , Infrared and Vacuum Structure in Two-Dimensional Local Quantum Field Theory, J. Math. Phys. 31 (1990), no. 6, 1467-1477.

[20] Morris S. A., Pontryagin duality and the structure of locally compact Abelian groups, London Math. Soc., Lecture Notes Series 29, Cambridge University Press, 1977.

[21] Naimark M.A., On unitary group representations in spaces with indefinite metric, Acta Sci. Szeged, 26(1965), 3-4, 201-209.

[22] Naimark M.A., On unitary representations of solvable groups in spaces with an indefinite metric, Izv. Akad. Nauk SSSR, 27(1963), 1181-1185 (AMS Transl. 49(1966), 86-91).

[23] Naimark M.A. and Ismagilov R.S., Representations of groups and algebras in spaces with indefinite metric, Mat. Anal., 1968, Moscow 1969, 73-105.

[24] Radjavi R. and Rosenthal P., "Invariant Subspaces", Springer-Verlag, Berlin, 1973.

[25] Sakai K., On J-unitary representations of amenable groups, Sci. Rep. Kagoshima Univ., 26(1977), 33-41.

[26] Sakai K., Some remarks on unitary representations of the Euclidean motion group in Π_m-spaces, Sci. Rep. Kagoshima Univ., 29(1980), 13-26.

[27] Schmidt A. U., Infinite infrared regularization and a state space for the Heisenberg algebra .J. Math. Phys. 43, 243 (2002).

[28] Strocchi F., Selected Topics on the General Properties of Quantum Field Theory, Lecture Notes in Physics, vol. 51, World Scientific, Singapore, London, Hong Kong, 1993.

[29] Suprunenko D.A. and Garaščuk M.S., Linear groups with Engel's condition, Dokl. Akad. Nauk BSSR, 6(1962), 277-280.

e-mail: e.kissin@londonmet.ac.uk

STORM, London Metropolitan University
166-220 Holloway Road,
London N7 8DB, Great Britain.

e-mail: shulman.victor80@gmail.com

Department of Mathematics,
Vologda State University,
Vologda, Russia.