ON GLOBAL DYNAMICS OF TYPE-K COMPETITIVE KOLMOGOROV DIFFERENTIAL SYSTEMS

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Abstract. This paper deals with global asymptotic behaviour of the dynamics for N-dimensional type-K competitive Kolmogorov systems of differential equations defined in the first orthant. It is known that the backward dynamics of such systems is type-K monotone. Assuming the system is dissipative and the origin is a repeller, it is proved that there exists a compact invariant set Σ which separates the basin of repulsion of the origin and the basin of repulsion of infinity and attracts all the non-trivial orbits. There are two closed sets S_H and S_V, their restriction to the interior of the first orthant are $(N-1)$-dimensional hypersurfaces, such that the asymptotic dynamics of the type-K system in the first orthant can be described by a system on either S_H or S_V: each trajectory in the interior of the first orthant is asymptotic to one in S_H and one in S_V. Geometric and asymptotic features of the global attractor Σ are investigated. It is proved that the partition $\Sigma = \Sigma_H \cup \Sigma_0 \cup \Sigma_V$ holds such that $\Sigma_H \cup \Sigma_0 \subset S_H$ and $\Sigma_V \cup \Sigma_0 \subset S_V$. Thus, Σ_0 contains all the ω-limit sets for all interior trajectories of any type-K subsystems and the closure $\overline{\Sigma_H \cup \Sigma_V}$ as a subset of Σ is invariant and the upper boundary of the basin of repulsion of the origin. This Σ has the same asymptotic feature as the modified carrying simplex for a competitive system: every nontrivial trajectory below Σ is asymptotic to one in Σ and the ω-limit set is in Σ for every other nontrivial trajectory.

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1. Introduction

In this paper, we are concerned with the global dynamical behaviour of the flow φ_t generated by the N-dimensional autonomous Kolmogorov differential system of equations

$$\dot{x}_i = x_i f_i(x) = F_i(x), x \in C, i \in I_N = \{1, 2, \dots, N\}, \tag{1}$$

where $C = \mathbb{R}^N_+ = \{x \in \mathbb{R}^N : \forall i \in I_N, x_i \geq 0\}$ and $f \in C^1(C, \mathbb{R}^N)$. System (1) is a typical mathematical model for the population dynamics of a community of N species, where each $x_i(t)$ represents the population size or density at time t and the function $f_i(x)$ denotes the *per capita* growth rate of the ith species. For this reason, system (1) has the following classification in terms of the off-diagonal entries of the Jacobian $Df(x)$ for all $x \in C$.

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(a) The system is called *cooperative* if $\frac{\partial f_i(x)}{\partial x_j} \geq 0$ for all $i, j \in I_N$ with $i \neq j$ because increase of the jth population boosts the per capita growth rate of the ith species.

(b) The system is called *competitive* if $\frac{\partial f_i(x)}{\partial x_j} \leq 0$ for all $i, j \in I_N$ with $i \neq j$ as increase of the jth population reduces the per capita growth rate of the ith species. A competitive system is called *totally competitive* if $\frac{\partial f_i(x)}{\partial x_j} < 0$ for all $i, j \in I_N$. A competitive system is said to have *strong internal competition* if $\frac{\partial f_i(x)}{\partial x_i} < 0$ for all $i \in I_N$.

(c) The system models the population dynamics of two communities: community A consists of k species with $x_1(t), \ldots, x_k(t)$ as their populations and community B consists of $N-k$ species with $x_{k+1}(t), \ldots, x_N(t)$ $(1 \leq k < N)$ as their populations. The system is called *type-K monotone* if each community as a subsystem is cooperative but there are competitions between the two communities, that is,

$$Df(x) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{2}$$

where the A_{ij} are matrices with A_{11} being $k \times k$, all the off-diagonal entries of A_{11} and A_{22} are nonnegative and all the entries of A_{12} and A_{21} are nonpositive.

(d) The system models the population dynamics of two communities A and B. It is called *type-K competitive* if each community as a subsystem is competitive but they are cooperative between the two communities, that is, all the off-diagonal entries of A_{11} and A_{22} in (2) are nonpositive and all the entries of A_{12} and A_{21} are nonnegative.

System (1) is called *dissipative* if there is a compact invariant set $A \subset C$ that uniformly attracts each compact set of initial values.

System (1) and its various particular instances, such as the classic Lotka-Volterra systems, have attracted huge interests from researchers spanning in the last half century. Consequently, numerous research results for such systems can be found in literature. We have no intention to give a detailed survey here but just pick up a few closely related examples. Hirsch [10]–[14] investigated the dynamical behaviour of cooperative and competitive systems (1) in line with development of the theory on monotone dynamical systems. Smith [39] investigated asymptotic behaviour of cooperative systems. For type-K monotone systems, Smith [40] proved a generalised Kamke theorem and Liang and Jiang [32] classified 3-dimensional Lotka-Volterra systems. Liang and Jiang [33] investigated the dynamical behaviour of type-K competitive systems and applied the theory to 3-dimensional Lotka-Volterra systems.

It is known [41] that while the flow $\varphi_t(x)$ generated by a cooperative system is monotone, i.e., with the order generated by the positive cone C, $x \leq y$ in C implies $\varphi_t(x) \leq \varphi_t(y)$ for $t \geq 0$, the backward flow generated by a competitive system is monotone, i.e., $x \leq y$ in C implies $\varphi_t(x) \leq \varphi_t(y)$ for $t \leq 0$. With an order generated by the cone $K = \{x \in \mathbb{R}^N :$

$\forall i \in I_k, x_i \geq 0, \forall j \in I_N \setminus I_k, x_j \leq 0\}$ (see section 2 for definition), the flow generated by a type-K monotone system is type-K monotone [40], i.e., $x \leq_K y$ implies $\varphi_t(x) \leq_K \varphi_t(y)$ for $t \geq 0$, whereas the backward flow generated by a type-K competitive system is type-K monotone [33], i.e., $x \leq_K y$ implies $\varphi_t(x) \leq_K \varphi_t(y)$ for $t \leq 0$.

One of the important results in literature is the carrying simplex theory for totally competitive systems. When system (1) is dissipative, the infinity ∞ can be viewed as a repeller and its basin of repulsion is denoted by $\mathcal{R}(\infty)$. If the origin O is also a repeller with $\mathcal{R}(O)$ as its basin of repulsion in C, we see that both $\mathcal{R}(\infty)$ and $\mathcal{R}(O)$ are open sets of C. So the set $\Sigma = C \setminus (\mathcal{R}(\infty) \cup \mathcal{R}(O))$ as the separator of the two basins of repulsion is a compact invariant set attracting all the nontrivial points in C. Under certain conditions, the set is an $(N-1)$-dimensional manifold with interesting geometric and dynamical features. The following is one of the theorems given in [12].

Theorem 1.1. *Assume that (1) satisfies the following conditions:*

(i) *The system is competitive.*

(ii) *The system is dissipative.*

(iii) *The Jacobian matrix $Df(x)$ is irreducible in C.*

(iv) *The origin O is a repeller.*

(v) *At each nonzero equilibrium, every entry of Df is negative.*

Then (1) has a compact invariant submanifold $\Sigma \subset C$ homeomorphic to $\Delta^{N-1} = \{x \in C : x_1 + \cdots + x_N = 1\}$ by radial projection such that every nontrivial trajectory in C is asymptotic to one in Σ.

This theorem is outstanding, phenomenal and has a big impact to later researches on competitive systems due to the important and interesting features of the set Σ: compact, invariant, unordered ($p \leq q$ implies $p = q$ for $p, q \in \Sigma$), homeomorphic to Δ^{N-1} by radial projection, and attracting all the points in $C \setminus \{O\}$. This theorem proves that the dynamical behaviour of the system in C is essentially described by that of the system on this co-dimension 1 surface Σ. Zeeman [50] called Σ the *carrying simplex* and it has been known as carrying simplex since then.

There is a wide range of applications of this theorem with a huge number of citations in literature. For example, Zeeman and Zeeman [48], Tineo [42], Baigent [3] and [4] investigated convexity of Σ; Mierczyński [34]–[37] studied the smoothness of Σ. Zeeman and Zeeman [49], Hou and Baigent [22], [24] and [7], and Hou [15] investigated global stability and repulsion of a fixed point; Hou and Baigent [23] studied the existence and global stability of heteroclinic limit cycles; Zeeman [50], Jiang and Niu [26]–[28] and Jiang et al. [31] investigated the dynamical behaviour on carrying simplex. Some research projects were inspired by Hirsch's theorem although they could not use theorem 1.1 because some of its conditions were not met. For example, Hou [16]–[18] investigated permanence and stability without assuming the existence of a carrying simplex; Liang and Jiang [33] studied the dynamical behaviour of type-K competitive systems; Mierczyński and Schreiber [38] established permanence conditions; Tu and Jiang [43] found coexistence conditions for systems with limited competition; Yu et al. [47] gave a criterion for global stability of three-dimensional system. The concept of carrying simplex has been extended to discrete competitive dynamical systems. Research in this area has been flourishing in the last two decades. Typical examples are [13], [44], [45], [25], [29], [30], [9], [5], [6] and [19].

We note that total competition implies the fulfilment of conditions (iii) and (v) of theorem 1.1. For convenience, almost all users of theorem 1.1 assume total competition of the system. But total competition is very restrictive and costly. For this reason, Hou [20] and [21] provided an updated version of the theorem requiring strong internal competition only. (The notation $x \gg y$ in C means $x_i > y_i$ for all $i \in I_N$.)

Theorem 1.2. *Assume that system (1) satisfies the following conditions:*

(i) $f(O) \gg O$.

(ii) *There exists a vector* $r \gg O$ *such that*

$$\forall i \in I_N, \forall x \in C \text{ with } x_i \geq r_i, f_i(x) < 0.$$

(iii) $f \in C^1([O,r], \mathbb{R}^N)$ *and*

$$\forall x \in [O,r], \forall i,j \in I_N, \frac{\partial f_i(x)}{\partial x_i} < 0 \text{ and } \frac{\partial f_i(x)}{\partial x_j} \leq 0.$$

Then (1) has a compact invariant submanifold $\Sigma \subset C$ *homeomorphic to* $\Delta^{N-1} = \{x \in C : x_1 + \cdots + x_N = 1\}$ *by radial projection such that every nontrivial trajectory in* C *below* Σ *is asymptotic to one in* Σ *and every point* $x \in C$ *above* Σ *satisfies* $\omega(x) \subset \Sigma$.

So far the geometric and dynamic features of the set $C \setminus (\mathcal{R}(\infty) \cup \mathcal{R}(O))$ is clear for competitive systems, but not much is known for other types of systems. For type-K competitive systems, Liang and Jiang [33] obtained the following global attractivity result.

Theorem 1.3. *Assume that system (1) satisfies the following conditions:*

(i) *The system is dissipative.*

(ii) *There exists* $\varepsilon > 0$ *such that* $\forall x \in C$,

$$\forall i,j \in I_k \text{ or } \forall i,j \in I_N \setminus I_k, \frac{\partial f_i(x)}{\partial x_j} < -\varepsilon;$$

$$\forall i \in I_k, \forall j \in I_N \setminus I_k \text{ or } \forall i \in I_N \setminus I_k, \forall j \in I_k, \frac{\partial f_i(x)}{\partial x_j} > \varepsilon.$$

Then there are two $(N-1)$-dimensional surfaces S_1 and S_2 in C that are homeomorphic to $\mathbb{R}^{N-1}$. Moreover, for each interior point x of C, there are $x' \in \overline{S_1}$ (the closure of S_1) and $x'' \in \overline{S_2}$ such that

$$\lim_{t\to\infty} \|\varphi_t(x) - \varphi_t(x')\| = \lim_{t\to\infty} \|\varphi_t(x) - \varphi_t(x'')\| = 0.$$

This theorem proves the asymptotic dynamic behaviour of system (1) in terms of two $(N-1)$-dimensional surfaces. But the condition (ii) is unnecessarily strong and restrictive. Besides, the feature of the dynamics on the invariant set $C \setminus (\mathcal{R}(\infty) \cup \mathcal{R}(O))$ has not been investigated so far when the origin is a repeller.

The aim of this paper is to prove under appropriate conditions that the globally attracting invariant set $\Sigma = C \setminus (\mathcal{R}(\infty) \cup \mathcal{R}(O))$ is formed by two closed sets, each of which is homeomorphic to a closed set of an $(N-1)$-dimensional hyperplane by projection, with the shape of a two-sloped roof of a hut plus some decorative thing on top of the ridge. We also aim at investigating various features of the dynamics on Σ.

We shall describe the notation and provide some preliminary results in section 2, present our main results in section 3 and prove them in section 6. In section 4 we deal with two-dimensional type-K competitive systems. In section 5, we try to find some possible configurations of Σ for three-dimensional type-K competitive systems. Concrete examples are given in sections 4 and 5. We then conclude the paper in section 7 with a few open problems.

2. Notation and preliminary results

For $C = \mathbb{R}^N_+$ we let $\dot{C} = \{x \in C : \forall i \in I_N, x_i > 0\}$ and $\partial C = C \setminus \dot{C}$. Then $\dot{C}$ is the interior of C and ∂C is the boundary of C. For any $x \in C \setminus \{O\}$, by the support of x we mean the set $I = \{i \in I_N : x_i > 0\}$. Let

$$\pi_i = \{x \in C : x_i = 0\}, i \in I_N. \tag{3}$$

Then π_i is the part of ∂C restricted to the ith coordinate plane. Denote the ith standard unit vector by e_i, i.e. the ith component of e_i is 1 and others are 0. For any nonempty subset $I \subset I_N$, define

$$C_I = \{x \in C : \forall j \in I_N \setminus I, x_j = 0\}, \tag{4}$$

$$\dot{C}_I = \{x \in C_I : \forall i \in I, x_i > 0\}. \tag{5}$$

Parallel to C and $\dot{C}$, for some fixed integer $1 \le k < N$ with $H = \{1, \dots, k\}$ (for "horizontal") and $V = \{k+1, \dots, N\}$ (for "vertical") we let

$$K = \{x \in \mathbb{R}^N : \forall i \in H, x_i \ge 0; \forall j \in V, x_j \le 0\}, \tag{6}$$

$$\dot{K} = \{x \in \mathbb{R}^N : \forall i \in H, x_i > 0; \forall j \in V, x_j < 0\}. \tag{7}$$

For any $x, y \in \mathbb{R}^N$, we define

$$x \le y : y - x \in C,\ x < y : x \le y\ (x \ne y),\ x \ll y : y - x \in \dot{C}. \tag{8}$$

We also use $y \geq x, y > x, y \gg x$ for $x \leq y, x < y, x \ll y$ respectively. Similarly,

$$x \leq_K y : y - x \in K,\ x <_K y : x \leq_K y\ (x \neq y),\ x \ll_K y : y - x \in \dot{K}. \tag{9}$$

For any $a, b \in \mathbb{R}^N$ and $c \in C$, we let

$$[a,b] = \{x \in \mathbb{R}^N : a \leq x \leq b\}\ if\ a \leq b, \tag{10}$$
$$(a,b) = \{x \in \mathbb{R}^N : a \ll x \ll b\}\ if\ a \ll b, \tag{11}$$

$$[a,b) = \{x \in \mathbb{R}^N : a \leq x \ll b\}\ if\ a \ll b, \tag{12}$$
$$(a,b] = \{x \in \mathbb{R}^N : a \ll x \leq b\}\ if\ a \ll b, \tag{13}$$

$$[c,+\infty) = \{x \in C : x \geq c\}, \tag{14}$$
$$(-\infty,c] = \{x \in C : x \leq c\}, \tag{15}$$

$$(c,+\infty) = \{x \in C : x \gg c\}, \tag{16}$$
$$(-\infty,c) = \{x \in C : x \ll c\}\ if\ c \gg O. \tag{17}$$

We also define $[a,b]_K, (a,b)_K, [a,b)_K, (a,b]_K, [c,+\infty)_K, (-\infty,c]_K, (c,+\infty)_K, (-\infty,c)_K$ by (10)–(17) with the replacement of $\leq, \ll, \geq, \gg$ by $\leq_K, \ll_K, \geq_K, \gg_K$ respectively. We may also apply any one of these notations to C_I instead of C if no confusion arises. For example, for $I = \{2, \dots, N\}$ and $c \in \dot{C}_I$, $(-\infty,c)_K = \emptyset$ in C but we use $(-\infty,c)_K \cap C_I$ to represent the set of points $x \ll_K c$ in C_I.

Note that the sets $[c,+\infty), (-\infty,c], (c,+\infty), (-\infty,c), [c,+\infty)_K, (-\infty,c]_K, (c,+\infty)_K$ and $(-\infty,c)_K$ are all subsets of C. Although ∞ appears in each of these sets, not all are unbounded. Indeed, $(-\infty,c] = [O,c]$ and $(-\infty,c) = [O,c)$ are bounded and the rest are unbounded. (See figure 1 for illustration of these sets.)

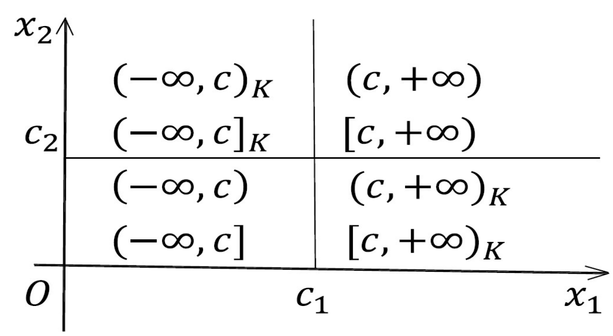

FIGURE 1. Illustration of $[c,+\infty), (-\infty,c], (c,+\infty), (-\infty,c), [c,+\infty)_K, (-\infty,c]_K, (c,+\infty)_K$ and $(-\infty,c)_K$: a set with (without) $c]$ or $[c$ is the region including (excluding) the boundary lines $x_1 = c_1$ and $x_2 = c_2$. If c is on x_1-axis (x_2-axis) then $(c,+\infty)_K = \emptyset$ ($(-\infty,c)_K = \emptyset$).

General assumptions From now onward we always assume the following for (1):

(A1) The system is type-K competitive.

(A2) The function f satisfies $f(O) \gg O$ so the origin O is a repeller.

(A3) The system is dissipative.

Any conditions that ensure (A3) are not trivial nor obvious. So a couple of sufficient conditions for (A3) will be provided at the end of this section. From (A1) we know that for any ordered pair $x, y \in C$ with $x \leq_K y$ and for all $t \leq 0$ in the maximum existence common interval for both $\varphi_t(x)$ and $\varphi_t(y)$, we have $\varphi_t(x) \leq_K \varphi_t(y)$. From (A3) we can view ∞ as a repeller with basin of repulsion $\mathcal{R}(\infty) \subset C$ such that $C \setminus \mathcal{R}(\infty)$ is a compact invariant set attracting every compact set of initial values uniformly as $t \to +\infty$. Since O is a repeller with basin of repulsion $\mathcal{R}(O) \subset C$ from (A2), we must have $\mathcal{R}(\infty) \cap \mathcal{R}(O) = \emptyset$ so $\mathcal{R}(O) \subset (C \setminus \mathcal{R}(\infty))$. Let

$$\Sigma = C \setminus (\mathcal{R}(\infty) \cup \mathcal{R}(O)). \tag{18}$$

As both $\mathcal{R}(\infty)$ and $\mathcal{R}(O)$ are open sets of C, Σ is a compact set separating $\mathcal{R}(\infty)$ and $\mathcal{R}(O)$ and $\omega(x) \subset \Sigma$ for all $x \in C \setminus \{O\}$. We are going to explore the asymptotic feature of (1) in terms of Σ.

For each $x \in \mathcal{R}(O) \cap \dot{C}$, since each half line in C from x to infinity intersects $\mathcal{R}(\infty)$ and Σ, from (14) and (15) and figure 1 we see that $[x, +\infty)_K \cap \Sigma \neq \emptyset$, $[x, +\infty)_K \cap \mathcal{R}(\infty) \neq \emptyset$, $(-\infty, x]_K \cap \Sigma \neq \emptyset$ and $(-\infty, x]_K \cap \mathcal{R}(\infty) \neq \emptyset$. Thus, for any $y \in [x, +\infty)_K \cap \Sigma$, as $x \leq_K y$ implies $\varphi_t(x) \leq_K \varphi_t(y)$ for all $t < 0$ and $x \in \mathcal{R}(O)$ implies $\lim_{t \to -\infty} \varphi_t(x) = O$, we must have $\varphi_t(x)_j \geq \varphi_t(y)_j$ for all $j \in V$ and $t < 0$. It follows from this that $\lim_{t \to -\infty} \varphi_t(y)_j = 0$ for all $j \in V$ so $\alpha(y) \subset \cap_{j \in V} \pi_j = C_H$. For any $z \in [x, +\infty)_K \cap \mathcal{R}(\infty)$, if $(t_z, +\infty)$ is the maximum existence interval of $\varphi_t(z)$, by the same reason as above, $\varphi_t(z)_j$ is bounded for all $t \in (t_z, +\infty)$ and $j \in V$ but there is at least one $i \in H$ such that $\varphi_t(z)_i \to +\infty$ as $t \to t_z$. Similarly, for each $y \in (-\infty, x]_K \cap \Sigma$, $\alpha(y) \subset \cap_{i \in H} \pi_i = C_V$; for each $z \in (-\infty, x]_K \cap \mathcal{R}(\infty)$, $\varphi_t(z)_i$ is bounded for all $t \in (t_z, +\infty)$ and $i \in H$ but there is at least one $j \in V$ such that $\varphi_t(z)_j \to +\infty$ as $t \to t_z$. Now let

$$\Sigma_H = \{x \in \Sigma : \alpha(x) \subset C_H\}, \ \Sigma_V = \{x \in \Sigma : \alpha(x) \subset C_V\} \tag{19}$$

and $\Sigma_0 = \Sigma \setminus (\Sigma_H \cup \Sigma_V)$. Denoting the open ball in $\mathbb{R}^N$ centred at a with radius $\delta > 0$ by $\mathcal{B}(a, \delta)$ and the closure of any set $S \subset \mathbb{R}^N$ by $\overline{S}$, we let

$$\mathcal{R}_H(\infty) = \{x \in \mathcal{R}(\infty) : \forall j \in V, \sup_{t \in (t_x, +\infty)} |\varphi_t(x)_j| < \infty\}, \tag{20}$$

$$\mathcal{R}_V(\infty) = \{x \in \mathcal{R}(\infty) : \forall i \in H, \sup_{t \in (t_x, +\infty)} |\varphi_t(x)_i| < \infty\}, \tag{21}$$

$$\dot{\mathcal{R}}_H(\infty) = \{x \in \mathcal{R}_H(\infty) : \exists \delta > 0, (\mathcal{B}(x, \delta) \cap C) \subset \mathcal{R}_H(\infty)\}, \tag{22}$$

$$\dot{\mathcal{R}}_V(\infty) = \{x \in \mathcal{R}_V(\infty) : \exists \delta > 0, (\mathcal{B}(x, \delta) \cap C) \subset \mathcal{R}_V(\infty)\}, \tag{23}$$

$$S_H = \overline{\mathcal{R}_H(\infty)} \setminus \dot{\mathcal{R}}_H(\infty), \ \ S_V = \overline{\mathcal{R}_V(\infty)} \setminus \dot{\mathcal{R}}_V(\infty). \tag{24}$$

It is clear that all of these sets defined by (19)–(24) are nonempty, Σ_H and Σ_V are mutually exclusive, and $\mathcal{R}_H(\infty)$ and $\mathcal{R}_V(\infty)$ are mutually exclusive. We shall see later that $\Sigma_0 \neq \emptyset$.

We say that a set B_1 separates B into B_2 and B_3 if B is the union of mutually exclusive nonempty sets B_1, B_2, B_3, for any points $x \in B_2$, $y \in B_3$ and any curve $\ell \subset B$ joining x to y, $\ell \cap B_1 \neq \emptyset$. If $B = B_1 \cup B_2 \cup B_3$ where B_1, B_2, B_3 are nonempty mutually exclusive, B_1 is closed and B_2, B_3 are open, then the finite open covering theorem for compact sets ensures that B_1 separates B into B_2 and B_3.

Proposition 1. *Under the assumptions (A1)–(A3), the following conclusions hold:*

(i) *Each of $\mathcal{R}_H(\infty)$, $\mathcal{R}_V(\infty)$, $\dot{\mathcal{R}}_H(\infty)$, $\dot{\mathcal{R}}_V(\infty)$, $\overline{\mathcal{R}_H(\infty)}$, $\overline{\mathcal{R}_V(\infty)}$, S_H and S_V is invariant.*

(ii) *$x \in \mathcal{R}_H(\infty)$ implies $[x, +\infty)_K \subset \mathcal{R}_H(\infty)$ and $x \in \mathcal{R}_V(\infty)$ implies $(-\infty, x]_K \subset \mathcal{R}_V(\infty)$.*

(iii) *$\dot{\mathcal{R}}_H(\infty)$ and $\dot{\mathcal{R}}_V(\infty)$ are open sets of C.*

(iv) *S_H and S_V are closed.*

(v) *For each $x \in S_H$, $[x, +\infty)_K \subset \overline{\mathcal{R}_H(\infty)}$; if $(x, +\infty)_K \neq \emptyset$ then $(x, +\infty)_K \subset \mathcal{R}_H(\infty)$; if $(-\infty, x)_K \neq \emptyset$ then $(-\infty, x)_K \cap \overline{\mathcal{R}_H(\infty)} = \emptyset$.*

(vi) *For each $x \in S_V$, $(-\infty, x]_K \subset \overline{\mathcal{R}_V(\infty)}$; if $(-\infty, x)_K \neq \emptyset$ then $(-\infty, x)_K \subset \mathcal{R}_V(\infty)$; if $(x, +\infty)_K \neq \emptyset$ then $(x, +\infty)_K \cap \overline{\mathcal{R}_V(\infty)} = \emptyset$.*

(vii) *The sets S_H and S_V are unordered in $\ll_K$, i.e. no two points in S_H (S_V) are related by $\ll_K$.*

(viii) *$S_H \cap \dot{C}$ is homeomorphic to $\mathbb{R}^{N-1}$, so it is an $(N-1)$-dimensional surface separating $\dot{C}$ into two mutually exclusive simply connected open sets: $\dot{\mathcal{R}}_H(\infty) \cap \dot{C}$ on one side and $\dot{C} \setminus \overline{\mathcal{R}_H(\infty)}$ on the other side of $S_H \cap \dot{C}$. The set S_H is homeomorphic to a closed set of an $(N-1)$-dimensional plane by projection and it separates C into two mutually exclusive open sets of C: $\dot{\mathcal{R}}_H(\infty)$ and $C \setminus \overline{\mathcal{R}_H(\infty)}$.*

(ix) *$S_V \cap \dot{C}$ is homeomorphic to $\mathbb{R}^{N-1}$, so it is an $(N-1)$-dimensional surface separating $\dot{C}$ into two mutually exclusive simply connected open sets: $\dot{\mathcal{R}}_V(\infty) \cap \dot{C}$ on one side and $\dot{C} \setminus \overline{\mathcal{R}_V(\infty)}$ on the other side of $S_V \cap \dot{C}$. The set S_V is homeomorphic to a closed set of an $(N-1)$-dimensional plane by projection and it separates C into two mutually exclusive open sets of C: $\dot{\mathcal{R}}_V(\infty)$ and $C \setminus \overline{\mathcal{R}_V(\infty)}$.*

(x) *$\dot{\mathcal{R}}_H(\infty) \subset (\overline{\mathcal{R}_H(\infty)} \setminus (S_H \cap S_V)) \subset (C \setminus \overline{\mathcal{R}_V(\infty)})$ and $\dot{\mathcal{R}}_V(\infty) \subset (\overline{\mathcal{R}_V(\infty)} \setminus (S_H \cap S_V)) \subset (C \setminus \overline{\mathcal{R}_H(\infty)})$.*

Proof. (i) The invariance of $\mathcal{R}_H(\infty)$ and $\mathcal{R}_V(\infty)$ follows from each set consisting of certain whole orbits. The rest follows from the definition of each set and continuous dependence on initial values.

(ii) It is obvious from the definition of $[x,+\infty)_K, (-\infty,x]_K$, (20), (21) and the backward monotone property of the flow for (1).

(iii) It is obvious from (22), (23).

(iv) That S_H and S_V are closed follows from (iii) and (24).

(v) For each $x \in S_H$, if $x \in \mathcal{R}_H(\infty)$ then $[x,+\infty)_K \subset \mathcal{R}_H(\infty)$ from (ii). So $[x,+\infty)_K \subset \overline{\mathcal{R}_H(\infty)}$ and, if $(x,+\infty)_K \neq \emptyset$, $(x,+\infty)_K \subset [x,+\infty)_K \subset \mathcal{R}_H(\infty)$. If $x \in \overline{\mathcal{R}_H(\infty)} \setminus \mathcal{R}_H(\infty)$, there is a sequence $\{x^{(n)}\} \subset \mathcal{R}_H(\infty)$ such that $\lim_{n\to\infty} x^{(n)} = x$. As $[x^{(n)},+\infty)_K \subset \mathcal{R}_H(\infty)$ from (ii), for each $y \geq_K x$ we have $x^{(n)} + (y-x) \geq_K x^{(n)}$ and $\lim_{n\to\infty} x^{(n)} + (y-x) = y$ so $[x,+\infty)_K \subset \overline{\mathcal{R}_H(\infty)}$. If $(x,+\infty)_K \neq \emptyset$, for any $y \in (x,+\infty)_K$, $y \gg_K x$ implies $y \gg_K x^{(n)}$ for large enough n so that $y \in [x^{(n)},+\infty)$ and $(x,+\infty)_K \subset \mathcal{R}_H(\infty)$.

Suppose $(-\infty,x)_K \neq \emptyset$ for some $x \in S_H$. Since $(-\infty,x)_K$ is an open set of C, if $(-\infty,x)_K \cap \overline{\mathcal{R}_H(\infty)} \neq \emptyset$ we would have a $y \in (-\infty,x)_K \cap \mathcal{R}_H(\infty)$. Then $[y,+\infty)_K \subset \mathcal{R}_H(\infty)$ by (ii). As $y \ll_K x$, with $\delta = \min\{\frac{1}{2}|y_i - x_i| : i \in I_N\} > 0$ we have $(\mathcal{B}(x,\delta) \cap C) \subset [y,+\infty)_K \subset \mathcal{R}_H(\infty)$ so $x \in \dot{\mathcal{R}}_H(\infty)$, a contradiction to $x \in S_H$. Hence, $(-\infty,x)_K \cap \overline{\mathcal{R}_H(\infty)} = \emptyset$.

(vi) It is similar to the proof of (v).

(vii) This follows from (v) and (vi).

(viii) Define $M : \dot{C} \to \mathbb{R}^N$ by

$$M(x) = (\ln(x_1), \ln(x_2), \ldots, \ln(x_N)),\ x \in \dot{C}.$$

Then M is a homeomorphism between $\dot{C}$ and $\mathbb{R}^N$. As $\ln(x_i)$ is strictly increasing in x_i, M maintains the orders in $\leq_K$, $<_K$ and $\ll_K$. Thus, for any $x \in \dot{C}$,

$$\begin{aligned} M([x,+\infty)_K \cap \dot{C}) &= \{y \in \mathbb{R}^N : y \geq_K M(x)\}, \\ M((x,+\infty)_K \cap \dot{C}) &= \{y \in \mathbb{R}^N : y \gg_K M(x)\}, \\ M((-\infty,x]_K \cap \dot{C}) &= \{y \in \mathbb{R}^N : y \leq_K M(x)\}, \\ M((-\infty,x)_K \cap \dot{C}) &= \{y \in \mathbb{R}^N : y \ll_K M(x)\}. \end{aligned}$$

Moreover, from (v) and (vii), $M(S_H \cap \dot{C})$ is unordered in $\ll_K$ and, for each $y \in M(S_H \cap \dot{C})$, $\{x \in \mathbb{R}^N : x \gg_K y\} \subset M(\mathcal{R}_H(\infty) \cap \dot{C})$ but $\{x \in \mathbb{R}^N : x \ll_K y\} \cap M(\mathcal{R}_H(\infty) \cap \dot{C}) = \emptyset$. Further, $M(\dot{\mathcal{R}}_H(\infty) \cap \dot{C})$ is identical to

$$\{x \in M(\mathcal{R}_H(\infty) \cap \dot{C}) : \exists \delta > 0, \mathcal{B}(x,\delta) \subset M(\mathcal{R}_H(\infty) \cap \dot{C})\}.$$

Let $u \in \mathbb{R}^N$ such that $u \gg_K O$ with $\langle u,u \rangle = u_1^2 + \cdots + u_N^2 = 1$ and let $E = \{x \in \mathbb{R}^N : \langle u,x \rangle = 0\}$. Then E is the orthogonal hyperplane of the unit vector u. The projection $P_E : \mathbb{R}^N \to E$ is given by $P_E(x) = x - \langle u,x \rangle\, u$ for $x \in \mathbb{R}^N$. Now define $m : M(S_H \cap \dot{C}) \to E$

by the restriction of P_E to $M(S_H \cap \dot{C})$ so $m(x) = x - \langle u, x \rangle\, u$ for $x \in M(S_H \cap \dot{C})$. We claim that m is a homeomorphism. As E is homeomorphic to $\mathbb{R}^{N-1}$ and $S_H \cap \dot{C}$ through M and m is homeomorphic to E, we have shown that $S_H \cap \dot{C}$ is homeomorphic to $\mathbb{R}^{N-1}$. Note that E separates $\mathbb{R}^N$ into two mutually exclusive simply connected open sets with one on each side of E. As $M(S_H \cap \dot{C})$ is homeomorphic to E through m, $M(S_H \cap \dot{C})$ is an $(N-1)$-dimensional surface separating $\mathbb{R}^N$ into two mutually exclusive simply connected open sets with one on each side of $M(S_H \cap \dot{C})$. Since $\dot{C}$ and $\mathbb{R}^N$ are homeomorphic through M and M maps $S_H \cap \dot{C}$ onto $M(S_H \cap \dot{C})$, $S_H \cap \dot{C}$ is an $(N-1)$-dimensional surface separating $\dot{C}$ into two mutually exclusive simply connected open sets, $\dot{\mathcal{R}}_H(\infty) \cap \dot{C}$ and $\dot{C} \setminus \overline{\mathcal{R}_H(\infty)}$, with one on each side of $S_H \cap \dot{C}$. Clearly, S_H, $\dot{\mathcal{R}}_H(\infty)$ and $C \setminus \overline{\mathcal{R}_H(\infty)}$ are nonempty mutually exclusive and $C = S_H \cup \dot{\mathcal{R}}_H(\infty) \cup (C \setminus \overline{\mathcal{R}_H(\infty)})$. As S_H is closed and $\dot{\mathcal{R}}_H(\infty)$, $C \setminus \overline{\mathcal{R}_H(\infty)}$ are open, S_H separates C into $\dot{\mathcal{R}}_H(\infty)$ and $C \setminus \overline{\mathcal{R}_H(\infty)}$.

Next we prove that m is a homeomorphism. Clearly, m is continuous and one-one by (vii). For each $y \in E$, the line through y parallel to u consists of the points $y + \lambda u$ for all $\lambda \in \mathbb{R}$. Clearly, $y + \lambda_1 u \gg_K y + \lambda_2 u$ if and only if $\lambda_1 > \lambda_2$. For any $z \in M(S_H \cap \dot{C})$, it follows from (v) that

$$\begin{aligned} \{x \in \mathbb{R}^N : x \gg_K z\} &\subset M(\mathcal{R}_H(\infty) \cap \dot{C}), \\ \{x \in \mathbb{R}^N : x \ll_K z\} &\cap M(\mathcal{R}_H(\infty) \cap \dot{C}) = \emptyset. \end{aligned}$$

Since $y + \lambda u \gg_K z$ so $y + \lambda u \in M(\mathcal{R}_H(\infty) \cap \dot{C})$ for sufficiently large λ, and $y + \lambda u \ll_K z$ so $y + \lambda u \notin M(\mathcal{R}_H(\infty) \cap \dot{C})$ for sufficiently large $-\lambda$, by letting

$$\lambda(y) = \inf\{\lambda \in \mathbb{R} : y + \lambda u \in M(\mathcal{R}_H(\infty) \cap \dot{C})\},$$

we have $y + \lambda u \in M(\mathcal{R}_H(\infty) \cap \dot{C})$ for $\lambda > \lambda(y)$ but $y + \lambda u \notin M(\mathcal{R}_H(\infty) \cap \dot{C})$ for $\lambda < \lambda(y)$. Thus, $m^{-1}(y) = y + \lambda(y)u \in M(S_H \cap \dot{C})$. Then m is a homeomorphism if $m^{-1} : E \to M(S_H \cap \dot{C})$ is continuous.

For continuity of m^{-1} we need only show that $\lambda : E \to \mathbb{R}$ is continuous. For any bounded set $B \subset E$, there is a $\lambda_1 > 0$ such that $y + \lambda u \gg_K z$ for all $y \in B$ and $\lambda \geq \lambda_1$ and $y + \lambda u \ll_K z$ for all $y \in B$ and $\lambda \leq -\lambda_1$. By the definition of $\lambda(y)$, we have $|\lambda(y)| \leq \lambda_1$ for all $y \in B$. Thus, $\lambda(y)$ is bounded for y in any bounded set of E. Now suppose $\lambda : E \to \mathbb{R}$ is not continuous. Then, for any convergent sequence $\{y^{(n)}\} \subset E$ such that $\lim_{n\to\infty} y^{(n)} = y^{(0)}$, the boundedness of $\{\lambda(y^{(n)})\}$ implies that it has a convergent subsequence. Without loss of generality, we assume that $\lim_{n\to\infty} \lambda(y^{(n)}) = \lambda_0 \neq \lambda(y^{(0)})$. It follows from this that

$$\lim_{n\to\infty} m^{-1}(y^{(n)}) = y^{(0)} + \lambda_0 u \neq y^{(0)} + \lambda(y^{(0)})u = m^{-1}(y^{(0)}).$$

As $m^{-1}(y^{(0)}) \in M(S_H \cap \dot{C})$, we must have $x \in M(\mathcal{R}_H(\infty) \cap \dot{C})$ for all $x \gg_K m^{-1}(y^{(0)})$ and $x \notin M(\mathcal{R}_H(\infty) \cap \dot{C})$ for all $x \ll_K m^{-1}(y^{(0)})$. If $\lambda_0 > \lambda(y^{(0)})$ then $y^{(0)} + \lambda_0 u \gg_K m^{-1}(y^{(0)})$, so there is a small $\delta > 0$ such that every $p \in \mathcal{B}(y^{(0)} + \lambda_0 u, \delta)$ satisfies $p \gg_K m^{-1}(y^{(0)})$, i.e. $\mathcal{B}(y^{(0)} + \lambda_0 u, \delta) \subset M(\mathcal{R}_H(\infty) \cap \dot{C})$. Hence, $\mathcal{B}(m^{-1}(y^{(n)}), \frac{1}{2}\delta) \subset \mathcal{B}(y^{(0)} + \lambda_0 u, \delta) \subset M(\mathcal{R}_H(\infty) \cap \dot{C})$ so that $m^{-1}(y^{(n)}) \in M(\dot{\mathcal{R}}_H(\infty) \cap \dot{C})$ for large enough n. This contradicts

$m^{-1}(y^{(n)}) \in M(S_H \cap \dot{C})$. If $\lambda_0 < \lambda(y^{(0)})$ then $y^{(0)} + \lambda_0 u \ll_K m^{-1}(y^{(0)})$, so there is a small $\delta > 0$ such that every $p \in \mathcal{B}(m^{-1}(y^{(0)}), \delta)$ satisfies $p \gg_K m^{-1}(y^{(n)}$ for sufficiently large n. Thus, $\mathcal{B}(m^{-1}(y^{(0)}), \delta) \subset M(\mathcal{R}_H(\infty) \cap \dot{C})$ so that $m^{-1}(y^{(0)}) \in M(\dot{\mathcal{R}}_H(\infty) \cap \dot{C})$, a contradiction to $m^{-1}(y^{(0)}) \in M(S_H \cap \dot{C})$. Therefore, we must have $\lambda_0 = \lambda(y^{(0)})$ so m^{-1} is continuous.

Finally, we consider the map $g : S_H \to P_E(S_H)$ defined by $g(x) = x - \langle u, x \rangle u$ for $x \in S_H$, the restriction of the projection P_E to S_H. From (v) and (vii) we can see the existence of $g^{-1} : P_E(S_H) \to S_H$ and $g^{-1}(y) = y + \lambda(y)u$. By the same technique as above we can show that $\lambda(y)$ is continuous, so g^{-1} is continuous and g is a homeomorphism. (Actually, the continuity of g^{-1} can follow from that it is Lipschitz shown for corollary 3 in next section.) As S_H is closed, $P_E(S_H)$ is closed in E.

(ix) Similar to the proof of (viii).

(x) These set inclusions follow from (20)–(24), (viii) and (ix). □

Remark 2.1 The idea of the proof of proposition 1 (viii) is the same as that of proposition 3.3 in [33] though the notation there is different from ours.

Remark 2.2 For system (1) on C, the sets $\mathcal{R}(\infty)$, $\mathcal{R}(O)$, $\mathcal{R}_H(\infty)$, $\dot{\mathcal{R}}_H(\infty)$, S_H, $\mathcal{R}_V(\infty)$, $\dot{\mathcal{R}}_V(\infty)$, S_V, Σ, Σ_H, Σ_V and Σ_0 are defined as subsets of C. For any proper subset $I \subset I_N$, since C_I is invariant, for the subsystem of (1) restricted to C_I we can define $\mathcal{R}^I(\infty)$, $\mathcal{R}^I(O)$, $\mathcal{R}^I_H(\infty)$, $\dot{\mathcal{R}}^I_H(\infty)$, S^I_H, $\mathcal{R}^I_V(\infty)$, $\dot{\mathcal{R}}^I_V(\infty)$, S^I_V, Σ^I, Σ^I_H, Σ^I_V and Σ^I_0 as subsets of C_I correspondingly. Then it is clear from the definitions that

$$\begin{aligned} \mathcal{R}^I(\infty) &= \mathcal{R}(\infty) \cap C_I, & \mathcal{R}^I(O) &= \mathcal{R}(O) \cap C_I, \\ \mathcal{R}^I_H(\infty) &= \mathcal{R}_H(\infty) \cap C_I, & \mathcal{R}^I_V(\infty) &= \mathcal{R}_V(\infty) \cap C_I, \\ \Sigma^I = \Sigma \cap C_I, \Sigma^I_H &= \Sigma_H \cap C_I, & \Sigma^I_V &= \Sigma_V \cap C_I, \Sigma^I_0 = \Sigma_0 \cap C_I. \end{aligned}$$

However, since $(\dot{\mathcal{R}}_H(\infty) \cap C_I) \subset \dot{\mathcal{R}}^I_H(\infty)$ and $\overline{\mathcal{R}^I_H(\infty)} \subset (\overline{\mathcal{R}_H(\infty)} \cap C_I)$, we have

$$S^I_H = (\overline{\mathcal{R}^I_H(\infty)} \setminus \dot{\mathcal{R}}^I_H(\infty)) \subset ((\overline{\mathcal{R}_H(\infty)} \cap C_I) \setminus (\dot{\mathcal{R}}_H(\infty) \cap C_I)) = S_H \cap C_I. \tag{25}$$

Similarly, we also have

$$S^I_V = (\overline{\mathcal{R}^I_V(\infty)} \setminus \dot{\mathcal{R}}^I_V(\infty)) \subset ((\overline{\mathcal{R}_V(\infty)} \cap C_I) \setminus (\dot{\mathcal{R}}_V(\infty) \cap C_I)) = S_V \cap C_I. \tag{26}$$

If $I = H$ then $\dot{\mathcal{R}}^I_H(\infty) = \mathcal{R}^I_H(\infty) = \mathcal{R}^I(\infty)$, $\mathcal{R}^I_V(\infty) = \emptyset$, and $S^I_H = \Sigma^I$ is the modified carrying simplex of the competitive subsystem on C_H. If $I = V$ then $\dot{\mathcal{R}}^I_V(\infty) = \mathcal{R}^I_V(\infty) = \mathcal{R}^I(\infty)$, $\mathcal{R}^I_H(\infty) = \emptyset$, and $S^I_V = \Sigma^I$ is the modified carrying simplex of the competitive subsystem on C_V. If $I \cap H \neq \emptyset$ and $I \cap V \neq \emptyset$, proposition 1 can be applied to these subsets of C_I. In particular, $S^I_H \cap \dot{C}_I$ ($S^I_V \cap \dot{C}_I$) as a subset of $S_H \cap \dot{C}_I$ ($S_V \cap \dot{C}_I$) is homeomorphic to $\mathbb{R}^{|I|-1}$, where $|I|$ denotes the cardinality of I.

Remark 2.3 Are the equalities

$$S_H^I = S_H \cap C_I \, (if \; S_H^I \; exists), S_V^I = S_V \cap C_I \, (if \; S_V^I \; exists) \tag{27}$$

true for all proper subset $I \subset I_N$? If the answer is YES, then S_H and S_V can be decomposed as

$$\begin{aligned} S_H &= (S_H \cap \dot{C}) \cup S_H^H \cup (\cup_I S_H^I \cap \dot{C}_I), \\ S_V &= (S_V \cap \dot{C}) \cup S_V^V \cup (\cup_I S_V^I \cap \dot{C}_I), \end{aligned}$$

where $\cup_I$ is the union over all proper subsets $I \subset I_N$ with $I \cap H \neq \emptyset$ and $I \cap V \neq \emptyset$. As Σ^H (Σ^V) is an $(|H|-1)$-dimensional ($(|V|-1)$-dimensional) manifold by theorem 1.2 and $S_H^I \cap \dot{C}_I$ ($S_V^I \cap \dot{C}_I$) is homeomorphic to $\mathbb{R}^{|I|-1}$, we can say that S_H (S_V) is an $(N-1)$-dimensional surface with a finite number of lower dimensional manifolds on its edges in ∂C. We shall see in next section (remark 3.2) that

$$(S_H^I \setminus \mathcal{R}(\infty)) = (S_H \setminus \mathcal{R}(\infty)) \cap C_I \;\; ((S_V^I \setminus \mathcal{R}(\infty)) = (S_V \setminus \mathcal{R}(\infty)) \cap C_I) \tag{28}$$

holds for all proper subset $I \subset I_N$ if S_H^I (S_V^I) exists. For the proof or disproof of general (27), we leave it as an open problem.

We note that the assumption (A3) is actually posing a challenging problem in practice: find a easily checkable sufficient condition for dissipation. Any sufficient condition for (1) to be dissipative is not trivial. For convenience, we provide two related results below. For any matrix A, we denote its ith row vector by $(A)_i$. We say that A is *stable* if every eigenvalue of A has a negative real part.

Proposition 2. *Assume the existence of a stable matrix A with negative diagonal entries and nonnegative off-diagonal entries. If there is an $r \gg O$ such that*

$$\forall x \in C \setminus [O, r], \;\; Df(x) \leq A,$$

then system (1) is dissipative.

Proof. Since A is stable, $-A$ is an M-matrix so there is a vector $v \gg O$ such that $\mu = Av \ll O$ (see [1] p.6 or [2]). Then, for each $i \in I_N$, $f_i(r_i e_i) - (A)_i r_i e_i + k\mu_i < 0$ holds for large enough $k > 0$. Let $k_0 > 0$ such that

$$k_0 v \geq r, \;\; \forall i \in I_N, f_i(r_i e_i) - (A)_i r_i e_i + k_0 \mu_i < 0. \tag{29}$$

We now show that for each $k \geq k_0$ and $i \in I_N$, $f_i(x) < 0$ for all $x \in [O, kv]$ with $x_i = kv_i$ so that $[O, kv]$ is forward invariant and for each $x \in [O, kv]$ with $x_j = kv_j$ for some $j \in I_N$, $\varphi_t(x) \in [O, kv)$ for all $t > 0$. It then follows that $[O, k_0 v)$ is forward invariant and every orbit in C will enter into $[O, k_0 v)$, so (1) is dissipative.

For each $k \geq k_0$, $i \in I_N$ and $x \in [O, kv]$ with $x_i = kv_i$, we have $x_i \geq k_0 v_i \geq r_i$ so $x - r_i e_i \geq 0$. By the assumption,

$$\begin{aligned} f_i(x) &= f_i(r_ie_i) + \left(\int_0^1 (Df(r_ie_i + s(x - r_ie_i)))_i ds\right)(x - r_ie_i) \\ &\leq f_i(r_ie_i) + (A)_i(x - r_ie_i) \\ &\leq f_i(r_ie_i) + (A)_i(kv - r_ie_i) \\ &= f_i(r_ie_i) - (A)_i r_ie_i + k\mu_i \\ &\leq f_i(r_ie_i) - (A)_i r_ie_i + k_0\mu_i < 0. \end{aligned}$$

□

Note that proposition 2 also shows that $[O, k_0v)$ is forward invariant and globally attractive for $v \gg O$ and $k_0 > 0$ satisfying $Av \ll O$ and (29). Moreover, this proposition does not require (1) to be a type-K system.

For any $x \in C$, we can write it as $x = x_H + x_V$ where $x_H \in C_H$ and $x_V \in C_V$.

Proposition 3. *Assume the existence of a continuous curve $u : \mathbb{R}_+ \to \dot{C}$ such that $u(s_1) \geq u(s_2) \gg O$ for $s_1 \geq s_2 \geq 0$ and for all $i \in I_N$, $u_i(s) \to +\infty$ as $s \to +\infty$. Moreover, for each $s \geq 0$, $i \in H$ and $j \in V$,*

$$f_i(u_i(s)e_i + u(s)_V) < 0, \quad f_j(u_j(s)e_j + u(s)_H) < 0.$$

Then the type-K competitive system (1) is dissipative with $[O, u(0))$ forward invariant and globally attractive.

Proof. For each $s \geq 0$, we show that

$$\forall i \in I_N, \forall x \in [O, u(s)] \text{ with } x_i = u_i(s), \quad f_i(x) < 0, \tag{30}$$

so that $[O, u(s)]$ is forward invariant and $\omega(x) \subset [O, u(s))$ for every $x \in [O, u(s)]$. As $u(s)$ is monotone and $u_i(s) \to +\infty$ as $s \to +\infty$ for all $i \in I_N$, $[O, u(0))$ is forward invariant and globally attractive.

For each $i \in H$, since (1) is type-K competitive, $f_i(x)$ is nonincreasing in x_j and non-decreasing in x_k for $j \in H \setminus \{i\}$ and $k \in V$. Thus, for fixed $s \geq 0$, $x \in [O, u(s)]$ with $x_i = u_i(s)$, we have

$$f_i(x) \leq f_i(u_i(s)e_i + x_V) \leq f_i(u_i(s)e_i + u(s)_V).$$

From the assumption $f_i(u_i(s)e_i + u(s)_V) < 0$ we have $f_i(x) < 0$ for all $i \in H$. Similarly, for each $j \in V$, $f_j(x)$ is nondecreasing in x_i and nonincreasing in x_k for $i \in H$ and $k \in V \setminus \{j\}$ so for fixed $s \geq 0$, $x \in [O, u(s)]$ with $x_j = u_j(s)$, we have

$$f_j(x) \leq f_j(x_H + u_j(s)e_j) \leq f_j(u(s)_H + u_j(s)e_j).$$

From the assumption $f_j(u_j(s)e_j + u(s)_H) < 0$ we obtain $f_j(x) < 0$ for all $j \in V$. Then (30) follows. □

3. The main results

With the preparation in section 2 we are now able to present our main results. From proposition 1 we know that $S_H \cap \dot{C}$ and $S_V \cap \dot{C}$ are two $(N-1)$-dimensional invariant surfaces each separating $\dot{C}$ into two mutually exclusive regions. Also, each of S_H and S_V is homeomorphic to a closed set in an $(N-1)$-dimensional plane orthogonal to any vector $u \gg_K O$. Under the assumptions (A1)–(A3) and further conditions we shall see that the asymptotic dynamics of (1) on C can be described by the system restricted to S_H or S_V.

We shall use the norm $\|x\| = \max\{|x_i| : i \in I_N\}$ for $x \in \mathbb{R}^N$. Then, for $x \in C$, we have $\|x_H\| = \max\{x_i : i \in H\}$ and $\|x_V\| = \max\{x_j : j \in V\}$.

Theorem 3.1. *Under the assumptions (A1)–(A3) we further assume that the following conditions hold.*

(i) *There is an $r \gg O$ such that $C \setminus \mathcal{R}(\infty) \subset [O, r)$.*

(ii) *Each diagonal entry of $Df(x)$ is negative for all $x \in [O, r]$.*

(iii) *If $N \geq 3$, for each $I \subset I_N$ with $I \cap H \neq \emptyset$, $I \cap V \neq \emptyset$ and the cardinality $|I| \geq 3$,*

$$\forall i \in H \cap I, \exists j \in V \cap I, \qquad \forall x \in [O, r] \cap C_I, \frac{\partial f_j(x)}{\partial x_i} > 0,$$
$$\forall j \in V \cap I, \exists i \in H \cap I, \qquad \forall x \in [O, r] \cap C_I, \frac{\partial f_i(x)}{\partial x_j} > 0.$$

(iv) *If $N \geq 3$, there is a $\rho_0 \geq \|r\|$ such that for each $i \in H$ and every $j \in V$,*

$$f_i(\rho_0 e_i + r_V) < 0, \qquad f_j(\rho_0 e_i + r_V) > 0,$$
$$f_i(\rho_0 e_j + r_H) > 0, \qquad f_j(\rho_0 e_j + r_H) < 0.$$

Then, for each $I \subset I_N$ satisfying $I \cap H \neq \emptyset$ and $I \cap V \neq \emptyset$ and every $x \in \dot{C}_I$, there are $T \geq 0$, $x' \in S_H \cap C_I \cap [O, r]$ and $x'' \in S_V \cap C_I \cap [O, r]$ such that

$$\lim_{t \to +\infty} \|\varphi_{t+T}(x) - \varphi_t(x')\| = \lim_{t \to +\infty} \|\varphi_{t+T}(x) - \varphi_t(x'')\| = 0. \tag{31}$$

Remark 3.1 For N-dimensional systems with $N \geq 3$, a condition stronger than condition (iii) is

$$\forall i \in H, \forall j \in V, \forall x \in [O, r], \ \frac{\partial f_i(x)}{\partial x_j} > 0, \ \frac{\partial f_j(x)}{\partial x_i} > 0. \tag{32}$$

For two-dimensional systems, conditions (iii) and (iv) in theorem 3.1 are redundant and $H = \{1\}$ and $V = \{2\}$. Writing S_H and S_V simply as S_1 and S_2, we simplify theorem 3.1 as the corollary below.

Corollary 1. *Assume that system (1) with* $N = 2$ *meets the assumptions (A1)–(A3) and the conditions (i) and (ii) of theorem 3.1. Then, for each* $x \in \dot{C}$*, there is an equilibrium* $p \in S_1 \cap S_2 \cap (O, r)$ *such that* $\lim_{t\to+\infty} \varphi_t(x) = p$.

The proof of this corollary will be covered by that of theorem 4.1 in the next section so that the conclusion of theorem 3.1 for $I \subset I_N$ with $|I| = 2$ is proved.

Since S_H and S_V are closed and invariant, for $x' \in S_H$ and $x'' \in S_V$ we have $\omega(x') \subset S_H$ and $\omega(x'') \subset S_V$. But from theorem 3.1, for $x \in \dot{C}_I$ satisfying (31) we must have $\omega(x) = \omega(x') = \omega(x'')$. Thus, $S_H \cap S_V \neq \emptyset$ and, for each $I \subset I_N$ satisfying $I \cap H \neq \emptyset$ and $I \cap V \neq \emptyset$ and every $x \in \dot{C}_I$, $\omega(x) \subset S_H \cap S_V \cap [O, r)$.

But what if $x \in C_H$ or $x \in C_V$? The answer to this question will be clear from the following observation: The subsystem of (1) on C_H is competitive. The conditions of theorem 3.1 guarantee that theorem 1.2 can be applied to this subsystem. Thus, this subsystem has a modified carrying simplex Σ^H, which separates the basin of repulsion of ∞ from the basin of repulsion of the origin in C_H, such that $\omega(x) \subset \Sigma^H$ for each $x \in C_H \setminus \{O\}$. Moreover, if $x \neq O$ is below Σ^H then there is $x' \in \Sigma^H$ such that $\lim_{t\to\infty} \|\varphi_t(x) - \varphi_t(x')\| = 0$. We shall see from remark 3.2 that $\Sigma^H = \Sigma \cap C_H = S_H \cap C_H$. Therefore, the asymptotic dynamics of the subsystem of (1) on C_H can be described by that of the system restricted to $S_H \cap C_H$. In the same way as above, the subsystem of (1) on C_V is also competitive with a modified carrying simplex $\Sigma^V = \Sigma \cap C_V = S_V \cap C_V$ and the asymptotic dynamics of the subsystem on C_V can be described by that of the system on $S_V \cap C_V$. From theorem 3.1 and this observation we obtain the corollary below.

Corollary 2. *Under the conditions of theorem 3.1, the asymptotic dynamics of system (1) restricted to* $C \setminus C_H$ *can be described by the system restricted to* $S_V \cap [O, r]$*; the asymptotic dynamics of system (1) restricted to* $C \setminus C_V$ *can be described by the system restricted to* $S_H \cap [O, r]$*. In other words, for each* $x \in C \setminus C_H$*, if* $x \notin C_V \cap \dot{\mathcal{R}}_V(\infty)$ *then* $\varphi_t(x)$ *for* $t \geq 0$ *is asymptotic to one in* $S_V \cap [O, r]$*, otherwise* $\omega(x) \subset S_V \cap C_V$*; for each* $x \in C \setminus C_V$*, if* $x \notin C_H \cap \dot{\mathcal{R}}_H(\infty)$ *then* $\varphi_t(x)$ *for* $t \geq 0$ *is asymptotic to one in* $S_H \cap [O, r]$*, otherwise* $\omega(x) \subset S_H \cap C_H$.

Let $E = \{x \in \mathbb{R}^N : \langle x, u\rangle = 0\}$ be the hyperplane in $\mathbb{R}^N$ for any given unit vector $u \gg_K O$. From proposition 1 (viii) we know that each of S_H and S_V is homeomorphic to a closed set in E by projection. Let $P_E : \mathbb{R}^N \to E$ be the projection and $g : S_H \to P_E(S_H)$ $(g : S_V \to P_E(S_V))$ be the homeomorphism, the restriction of P_E to S_H (S_V). Then $g(x) = x - \langle u, x\rangle\, u$ for $x \in S_H\,(S_V)$ and $g^{-1}(y) = y + \lambda(y)u$ for $y \in P_E(S_H)\,(P_E(S_V))$, where $\lambda(y)$ is continuous (see the proof of proposition 1 (viii)). It is obvious that g is Lipschitz. We now show that g^{-1} is also Lipschitz (by the same technique used in the proof of proposition 2.6 in [12]). Let $S_E \subset E$ be the set of all unit vectors in E. Then S_E is compact, so there is a $\mu > 0$ such that $x + \rho u \gg_K O$ for all $\rho \geq \mu$ and $x \in S_E$ and $x + \rho u \ll_K O$ for all $\rho \leq -\mu$ and $x \in S_E$. Hence, if some $p \in S_E$ and $\rho \in \mathbb{R}$ satisfy neither $p + \rho u \gg_K O$ nor $p + \rho u \ll_K O$ then we must have $|\rho| < \mu$. Now for any distinct $x, y \in P_E(S_H)\,(P_E(S_V))$,

$$\frac{g^{-1}(x) - g^{-1}(y)}{\|x-y\|} = \frac{x-y}{\|x-y\|} + \frac{(\lambda(x) - \lambda(y))}{\|x-y\|}u.$$

As both $g^{-1}(x)$ and $g^{-1}(y)$ are in $S_H\,(S_V)$, which is unordered in $\ll_K$, we have neither $\frac{g^{-1}(x)-g^{-1}(y)}{\|x-y\|} \gg_K O$ nor $\frac{g^{-1}(x)-g^{-1}(y)}{\|x-y\|} \ll_K O$. With $p = \frac{x-y}{\|x-y\|}$ and $\rho = \frac{(\lambda(x)-\lambda(y))}{\|x-y\|}$, we see that $p \in S_E$, $\rho \in \mathbb{R}$ and neither $p + \rho u \gg_K O$ nor $p + \rho u \ll_K O$. Thus, $|\rho| = \frac{|\lambda(x)-\lambda(y)|}{\|x-y\|} < \mu$ and

$$\|g^{-1}(x) - g^{-1}(y)\| \leq (1+\mu)\|x-y\|.$$

Then we obtain the following corollary.

Corollary 3. *Under the conditions of theorem 3.1, the flow φ_t on S_H (S_V) is conjugate, via the Lipschitz homeomorphism g, to the flow θ_t of a Lipschitz vector field on a closed set of the plane $E = \{x \in \mathbb{R}^N : \langle x, u\rangle = 0\}$ for any given unit vector $u \gg_K O$, that is, with $\theta_t = g \circ \varphi_t \circ g^{-1}$ and $G(y) = P_E \circ F \circ g^{-1}(y)$,*

$$\forall x \in g(S_H) \subset E \ (\forall x \in g(S_V) \subset E), \frac{d}{dt}(\theta_t(x)) = G(\theta_t(x)).$$

Recall that

$$\Sigma = C \setminus (\mathcal{R}(\infty) \cup \mathcal{R}(O)) = \Sigma_H \cup \Sigma_0 \cup \Sigma_V,$$

where Σ_H and Σ_V are defined by (19). Clearly, S_H and S_V are closed and unbounded and part of each is in the open set $\mathcal{R}(\infty)$. Then $S_H \setminus \mathcal{R}(\infty)$ and $S_V \setminus \mathcal{R}(\infty)$ are compact sets. What are the relationships between $S_H \setminus \mathcal{R}(\infty)$ and $\Sigma_H, \Sigma_0, \Sigma_V$ and between $S_V \setminus \mathcal{R}(\infty)$ and $\Sigma_H, \Sigma_0, \Sigma_V$? Our next theorem answers this question.

Theorem 3.2. *The following conclusions hold under the conditions of theorem 3.1.*

(a) *For each $x \in (\mathcal{R}(O) \setminus \{O\})$, let $I \subset I_N$ be its support. If $I \subset H$ then there is an $x' \in \Sigma_H \cap C_I$ such that*

$$\lim_{t\to+\infty} \|\varphi_t(x) - \varphi_t(x')\| = 0. \tag{33}$$

If $I \subset V$ then there is an $x'' \in \Sigma_V \cap C_I$ such that

$$\lim_{t\to+\infty} \|\varphi_t(x) - \varphi_t(x'')\| = 0. \tag{34}$$

If $I \cap H \neq \emptyset$ and $I \cap V \neq \emptyset$, then there are $x' \in \Sigma_H \cap C_I$ and $x'' \in \Sigma_V \cap C_I$ such that both (33) and (34) hold.

(b) $(S_H \setminus \mathcal{R}(\infty)) = (\Sigma_H \cup \Sigma_0)$ *and* $(S_V \setminus \mathcal{R}(\infty)) = (\Sigma_V \cup \Sigma_0)$.

(c) $(\overline{\mathcal{R}(\infty)} \cap \overline{\mathcal{R}(O)}) = (\overline{\mathcal{R}(O)} \setminus \mathcal{R}(O)) = (\overline{\Sigma_H \cup \Sigma_V}) = (\overline{\Sigma_H} \cup \Sigma_V) = (\Sigma_H \cup \overline{\Sigma_V}) \subset \Sigma$ *and* $(\overline{\mathcal{R}(\infty)} \setminus \mathcal{R}(\infty)) = \Sigma$.

Remark 3.2 From theorem 3.2 (b) we can see that (28) in remark 2.3 holds for each proper $I \subset I_N$ with $S_H \cap C_I \neq \emptyset$ ($S_V \cap C_I \neq \emptyset$). Indeed, S_H^I (S_V^I) exists for each such I. Applying theorem 3.2 to (1) on C and to the subsystem of (1) on C_I, we have

$$\begin{aligned} S_H^I \setminus \mathcal{R}(\infty) &= S_H^I \setminus \mathcal{R}^I(\infty) = \Sigma_H^I \cup \Sigma_0^I \\ &= (\Sigma_H \cup \Sigma_0) \cap C_I = (S_H \setminus \mathcal{R}(\infty)) \cap C_I, \\ S_V^I \setminus \mathcal{R}(\infty) &= S_V^I \setminus \mathcal{R}^I(\infty) = \Sigma_V^I \cup \Sigma_0^I \\ &= (\Sigma_V \cup \Sigma_0) \cap C_I = (S_V \setminus \mathcal{R}(\infty)) \cap C_I. \end{aligned}$$

The main features of Σ are described by the following theorem, where for any $\varepsilon > 0$ and any nonempty set $S \subset \mathbb{R}^N$, $\mathcal{B}(S, \varepsilon) = \cup_{x \in S} \mathcal{B}(x, \varepsilon)$.

Theorem 3.3. *Assume that the conditions of theorem 3.1 hold. Then the following conclusions hold about $\Sigma = \Sigma_H \cup \Sigma_0 \cup \Sigma_V$ given by (18) and (19).*

(a) *Each of $\Sigma_H, \Sigma_0, \Sigma_V, \overline{\Sigma_H}, \overline{\Sigma_V}$ and any set obtained from these with $\cup, \cap, \setminus$ is invariant.*

(b) *Each of $\overline{\Sigma_H}, \overline{\Sigma_V}, \Sigma_0, \Sigma_H \cup \Sigma_0, \Sigma_V \cup \Sigma_0, \overline{\Sigma_H \cup \Sigma_V}$ and any set obtained from these with $\cup, \cap$ is compact.*

(c) *Σ is the global attractor of (1) on $C \setminus \{O\}$, that is, for any compact set $Q \subset C \setminus \{O\}$ and any $\varepsilon > 0$, there is a $T > 0$ such that $\varphi_t(Q) \subset \mathcal{B}(\Sigma, \varepsilon)$ for all $t \geq T$.*

(d) *Each of $\Sigma_H \cap \dot{C}$ and $\Sigma_V \cap \dot{C}$ is an $(N-1)$-dimensional invariant manifold homeomorphic to an open set of the hyperplane $E = \{x \in \mathbb{R}^N : \langle x, u \rangle = 0\}$, where $u \gg_K O$ is a unit vector, by projection. Each of $\overline{\Sigma_H}$ and $\overline{\Sigma_V}$ is homeomorphic to a closed set of E by projection. The set $\overline{\Sigma_H \cup \Sigma_V}$ separates C into two mutually exclusive connected open sets $\mathcal{R}(O)$ and $C \setminus \overline{\mathcal{R}(O)}$.*

(e) *The set $\Sigma_H \setminus C_H$ does not contain any limit point and for each $x \in \Sigma_H \setminus C_H$, $\alpha(x) \subset \Sigma_H \cap C_H$ and $\omega(x) \subset \Sigma_0$.*

(f) *The set $\Sigma_V \setminus C_V$ does not contain any limit point and for each $x \in \Sigma_V \setminus C_V$, $\alpha(x) \subset \Sigma_V \cap C_V$ and $\omega(x) \subset \Sigma_0$.*

(g) *The set Σ_0 is the global attractor of (1) on $C \setminus (C_H \cup C_V)$, that is, for any compact set $Q \subset C \setminus (C_H \cup C_V)$ and any $\varepsilon > 0$, there is a $T \geq 0$ such that $\varphi_t(Q) \subset \mathcal{B}(\Sigma_0, \varepsilon)$ for all $t \geq T$.*

(h) *The set $\Sigma \cap C_H = \Sigma_H \cap C_H$ is the global attractor of the subsystem of (1) on $C_H \setminus \{O\}$ but is the global repeller of (1) on $\overline{\Sigma_H}$ with its dual attractor $\overline{\Sigma_H} \setminus \Sigma_H$.*

(i) *The set $\Sigma \cap C_V = \Sigma_V \cap C_V$ is the global attractor of the subsystem of (1) on $C_V \setminus \{O\}$ but is the global repeller of (1) on $\overline{\Sigma_V}$ with its dual attractor $\overline{\Sigma_V} \setminus \Sigma_V$.*

Remark 3.3 From theorem 3.3 we see that the three mutually exclusive compact invariant sets $\Sigma_H \cap C_H$, $\Sigma_V \cap C_V$ and Σ_0 contain all the nontrivial limit sets of (1), $\Sigma_H \setminus C_H$ consists of trajectories linking $\Sigma_H \cap C_H$ to Σ_0 and $\Sigma_V \setminus C_V$ consists of trajectories linking $\Sigma_V \cap C_V$ to Σ_0. Intuitively, $\Sigma_H \cap C_H$ can be viewed as a "saddle point" with a "stable manifold" $C_H \setminus \{O\}$ and "unstable manifold" $\Sigma_H \setminus C_H$, $\Sigma_V \cap C_V$ can be viewed as a "saddle point" with a "stable manifold" $C_V \setminus \{O\}$ and "unstable manifold" $\Sigma_V \setminus C_V$, and Σ_0 can be viewed as a "stable node". This supports the geometric description of Σ as a two-sloped roof of a hut with a decorative thing on top of the ridge (see figure 2).

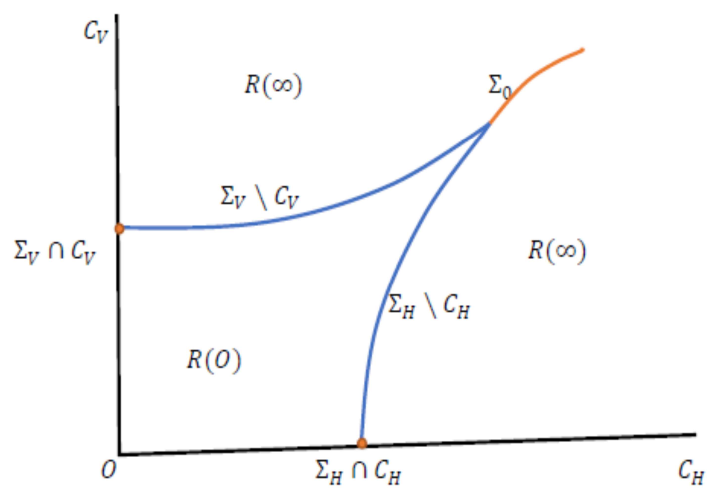

FIGURE 2. Illustration of the global attractor Σ: $\overline{\Sigma_H \cup \Sigma_V}$ (the upper boundary of $\mathcal{R}(O)$) plus some possible extra in Σ_0.

Remark 3.4 Note that C_H is k-dimensional and a part of the boundary ∂C and C_V is $(N-k)$-dimensional and a part of of the boundary of C. From theorem 3.3 (e)–(g) we see that the global behaviour of (1) in $\dot{C}$, and in $C \setminus (C_H \cup C_V)$ in general, is concentrated on Σ_0. For a competitive system with carrying simplex, it is known that the global behaviour of the system on C can be described by that of the system on the carrying simplex. It is also known that any complicated behaviour can be embedded into the carrying simplex of a competitive system. Is this also true for Σ_0 of type-K competitive system (1)? Also, we don't know much about the geometric feature of Σ_0. It is possible for Σ_0 to be an $(N-1)$-dimensional manifold homeomorphic to a closed region in E by projection. It is equally possible for Σ_0 to consists of a finite number of lower dimensional invariant manifolds. There is a huge space for further exploration of Σ_0, its composition and the behaviour on it. The next two sections serve as examples of such exploration.

Remark 3.5 From theorem 3.3 (d) and remarks 2.2, 2.3 and 3.2 we see that Σ_H (Σ_V) can be viewed as an $(N-1)$-dimensional invariant manifold with a finite number of lower dimensional manifolds on its edges in ∂C. It is strongly believed that the following conjecture is true:

- **Conjecture** The set $(\overline{\Sigma_H} \cap \overline{\Sigma_V}) \cap \dot{C}$ is an $(N-2)$-dimensional invariant manifold and $(\overline{\Sigma_H} \cap \overline{\Sigma_V}) \cap \partial C$ consists of a finite number of lower dimensional manifolds.

If this is true, then $\overline{\Sigma_H} \cap \overline{\Sigma_V}$ serves as a common edge of Σ_H and Σ_V glueing the two pieces together so that $\overline{\Sigma_H \cup \Sigma_V}$ can be viewed as a two sloped $(N-1)$-dimensional manifold with a finite number of lower dimensional manifolds on its edges. In particular, when $\Sigma_0 = \overline{\Sigma_H} \cap \overline{\Sigma_V}$, the whole global attractor $\Sigma = \overline{\Sigma_H \cup \Sigma_V}$ on $C \setminus \{O\}$ has this geometric feature. We shall see from theorem 4.1 in next section that this conjecture is true for $N = 2$: $(\overline{\Sigma_H} \cap \overline{\Sigma_V}) \cap \dot{C}$ consists of a single equilibrium, a 0-dimensional manifold. For $N > 2$ this conjecture is left as an open problem.

4. Two-dimensional type-K competitive systems

In this section, we look into Σ for more details for two-dimensional type-K competitive systems (1) which can be written as

$$x_1' = x_1 f_1(x_1, x_2), x_2' = x_2 f_2(x_1, x_2),\ x = (x_1, x_2) \in \mathbb{R}^2_+ = C. \tag{35}$$

By definition, (35) is type-K competitive if the two-species system is cooperative (as its two subsystems are one-dimensional with no off-diagonal elements of the Jacobian matrices). Since the flow of cooperative systems is monotone for positive time, the flow of (35) is monotone in $\leq$ for forward time and monotone in $\leq_K$ for backward time. Conditions (iii) and (iv) in theorem 3.1 require $N \geq 3$ so they are redundant for (35). The following theorem for (35) includes corollary 1 as part of it. A curve ℓ is said to be *ordered* in $\leq$ ($\ll$) if every pair of distinct points on ℓ are related by $\leq$ ($\ll$).

Theorem 4.1. *Assume that f in (35) satisfies (A1)–(A3) and the conditions (i) and (ii) of theorem 3.1. Then the conclusions below hold for (35).*

(a) *The system has one equilibrium Q_i on each positive x_i-axis, which is a saddle point with the positive x_i-axis as its stable manifold $W^s(Q_i)$ and a trajectory as its unstable manifold $W^u(Q_i) \subset \dot{C}$.*

(b) *The system has at least one interior equilibrium $p_0 \gg O$. If p_0 is the only one interior equilibrium then it is globally asymptotically stable in $\dot{C}$. Otherwise, there are two equilibria $p_1 \gg p_0 \gg O$ such that $\varphi_t(x)$ converges to an equilibrium $p \in [p_0, p_1]$ for all $x \in \dot{C}$.*

(c) *The set $\Sigma = C \setminus (\mathcal{R}(\infty) \cup \mathcal{R}(O))$ is the global attractor of (35) on $C \setminus \{O\}$ and it has the decomposition $\Sigma = \Sigma_H \cup \Sigma_0 \cup \Sigma_V$, where Σ_H consists of Q_1 and its unstable manifold $W^u(Q_1)$, a trajectory linking Q_1 to p_0, forming an ordered curve in $\leq$ and Σ_V consists of Q_2 and its unstable manifold $W^u(Q_2)$, a trajectory linking Q_2*

to p_0, forming an ordered curve in $\leq$. If p_0 is the unique equilibrium in $\dot{C}$ then $\Sigma_0 = \{p_0\}$; otherwise, Σ_0 is an ordered curve in $\ll$ from p_0 to p_1 consisting of heteroclinic trajectories and equilibria.

(d) *$S_H \setminus \mathcal{R}(\infty) = \Sigma_H \cup \Sigma_0$ and $S_V \setminus \mathcal{R}(\infty) = \Sigma_V \cup \Sigma_0$ (see figure 3 below).*

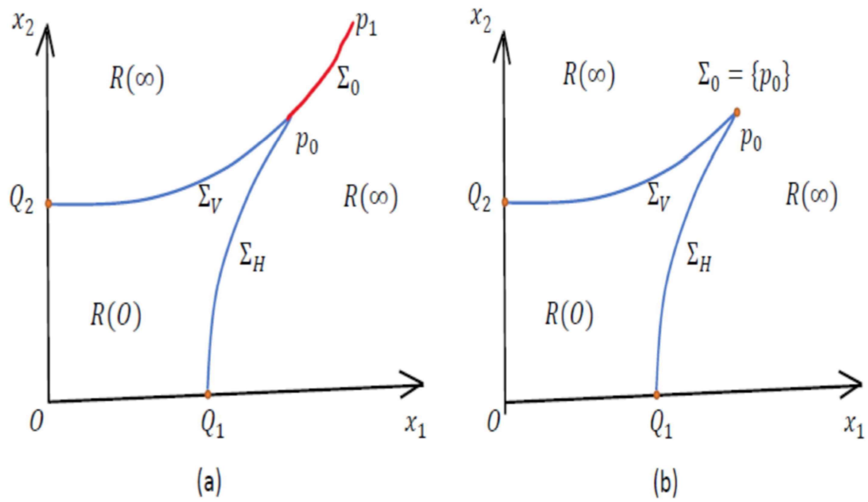

FIGURE 3. Illustration of $\Sigma = \Sigma_H \cup \Sigma_0 \cup \Sigma_V$ when $N = 2$: (a) Σ_0 is an ordered curve, (b) $\Sigma_0 = \{p_0\}$.

Remark 4.1 An alternative condition for (A3) and (i) in theorem 4.1 is the following:

- There is a continuous $g : \mathbb{R}_+ \to \mathbb{R}^2_+$ such that $g(s_2) \geq g(s_1) \gg O$ for all $s_2 \geq s_1 \geq 0$, $g_i(s) \to +\infty$ as $s \to +\infty$ for $i = 1, 2$, and $f(g(s)) \ll O$ for all $s \geq 0$.

Indeed, under this condition, by proposition 3 (35) is dissipative with $[0, g(0))$ forward invariant and globally attractive.

Example 4.1 Consider the planar system

$$
\begin{aligned}
x_1' &= x_1(x_2 - 1 - a(x_1 - 1) - (x_1 - 1)^3), \\
x_2' &= x_2(x_1 - 1 - a(x_2 - 1) - (x_2 - 1)^3),
\end{aligned}
\tag{36}
$$

where $a > 0$ is a constant and $(x_1, x_2) \in C$. Viewing (36) as (35) we have $f(O) = (a, a) \gg O$. With $g(s) = (s + 2, s + 2)$ for $s \geq 0$,

$$f(g(s)) = -(s + 1)(a + s(s + 2))(1, 1) \ll O.$$

By remark 4.1, (36) is dissipative with $C \setminus \mathcal{R}(\infty) \subset [O, r)$ for $r = (2, 2)$. Then (A1)–(A3) and conditions (i)–(ii) of theorem 3.1 are all satisfied. Let s_0 be the unique solution of the equation $1 + a(s - 1) + (s - 1)^3 = 0$. Then $s_0 \in (0, 1)$ and $Q_1 = (s_0, 0)$, $Q_2 = (0, s_0)$ are the two axial equilibria. Since $f_1(x_2, x_1) = f_2(x_1, x_2)$, the dynamics of (36) is symmetric about the line $x_2 = x_1$ and the line is an invariant manifold. The dynamics on this manifold can be described by the scalar equation

$$u' = u(u-1)(1-a-(u-1)^2). \tag{37}$$

For $0 < a < 1$, (37) has four equilibria $0, 1-\sqrt{1-a}, 1, 1+\sqrt{1-a}$ with repellers 0 and 1 and attractors $1 \pm \sqrt{1-a}$. For $a \geq 1$, (37) has only two equilibria 0 (a repeller) and 1 (an attractor). Then $P_1 = (1-\sqrt{1-a}, 1-\sqrt{1-a})$, $P_2 = (1,1)$ and $P_3 = (1+\sqrt{1-a}, 1+\sqrt{1-a})$ are the only equilibria of (36) in $\dot{C}$ for $0 < a < 1$ and P_2 is the only one for $a \geq 1$. The global dynamics is clear from the phase portraits below (see figure 4).

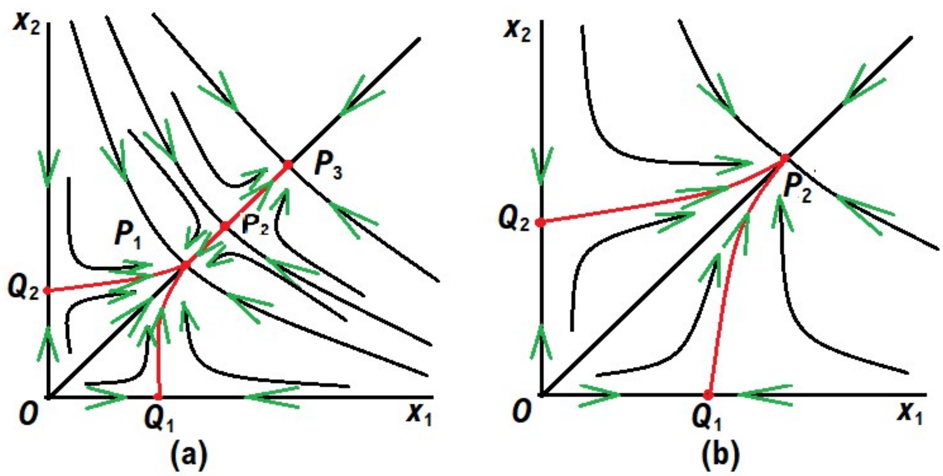

FIGURE 4. Phase portraits of (36) with the red colour for the global attractor Σ: (a) $0 < a < 1$, (b) $a \geq 1$.

Proof of theorem 4.1: Since (35) is cooperative as well as type-K competitive, the dynamics of (35) is monotone in $\leq$ for forward time and in $\leq_K$ for backward time. As O is a repeller, the compact set $\Sigma = C \setminus (\mathcal{R}(\infty) \cup \mathcal{R}(O))$ attracts all the nontrivial points in C.

(a) By (A2), (A3) and conditions (i) and (ii) of theorem 3.1, $f_1(O) > 0$, $f_1(x_1, 0)$ is strictly decreasing for $x_1 \in [0, r_1]$, and $[0, r_1)$ attracts all the points on positive x_1-axis. Thus, $f_1(r_1, 0) < 0$ and there is a unique equilibrium Q_1 on the positive x_1-axis such that $f_1(x_1, 0) > 0$ (< 0) for x_1 below (above) Q_1 within $[0, r_1]$ so that Q_1 is globally attracting on the positive x_1-axis. By (A1) and (ii), $f_2(Q_1) \geq f_2(O) > 0$ and $\frac{\partial f_1(Q_1)}{\partial x_1} < 0$. So the

Jacobian $DF(Q_1)$ has two eigenvalues with opposite signs. Hence, Q_1 is a saddle point with the positive half x_1-axis as its $W^s(Q_1)$ and a trajectory in $\dot{C}$ from Q_1 as its $W^u(Q_1)$.

Replacing the subscript 1 by 2 in the above paragraph we obtain a unique equilibrium Q_2 on the positive x_2-axis within $[0, r_2)$ such that Q_2 is a saddle point with the positive half x_2-axis as its $W^s(Q_2)$ and a trajectory in $\dot{C}$ from Q_2 as its $W^u(Q_2)$.

(b) It is known that every bounded trajectory converges to an equilibrium for two-dimensional cooperative systems (see theorem 2.2 in [41] Chapter 3). Since the origin O is a repeller, for any $x \in (O, r)$ close enough to the origin, $f(x) \gg O$ so $\varphi_t(x) \gg x$ for small $t > 0$. By the forward monotone property of cooperative systems, $\varphi_t(x)$ is monotone and converges to an equilibrium $p_0 \gg O$. Then, for any $y \gg O$, since $\lim_{t\to-\infty} \varphi_t(x) = O$ and $\lim_{t\to+\infty} \varphi_t(x) = p_0$, there is $t_1 \in \mathbb{R}$ such that $\varphi_{t_1}(x) \ll y$ so that $\varphi_{t+t_1}(x) \leq \varphi_t(y)$ for all $t \geq 0$. This shows that $\lim_{t\to+\infty} \varphi_t(y) = p$ for some equilibrium $p \geq p_0$. In particular, there is an equilibrium $p_1 \in [p_0, r)$ such that $\lim_{t\to+\infty} \varphi_t(r) = p_1$.

By (i), for each $y \gg O$, there is $t_2 > 0$ such that $\varphi_{t_2}(y) \leq r$ so that $\varphi_{t+t_2}(y) \leq \varphi_t(r)$ for all $t \geq 0$. This shows that $\lim_{t\to+\infty} \varphi_t(y) = p$ for some equilibrium $p \leq p_1$. Combining this with the previous paragraph we conclude that $\varphi_t(y)$ converges to an equilibrium $p \in [p_0, p_1]$ for all $y \in \dot{C}$. If $p_0 = p_1$ then the forward monotone property of cooperative systems and the global attraction of p_0 in $\dot{C}$ imply the global stability of p_0 in $\dot{C}$.

(c) The proof that Σ is a global attractor of (35) in $C \setminus \{O\}$ will be covered by the proof of theorem 3.3 (c) given in section 6.

Since $f_i(p_0) = 0$ and $f_i(Q_i) = 0$, by (A1) $f_1(p_{0H}) \leq f_1(p_0) = 0$ and $f_2(p_{0V}) \leq f_2(p_0) = 0$. By (ii), $Q_1 \leq p_{0H} < p_0$ and $Q_2 \leq p_{0V} < p_0$. So, for any $x \in W^u(Q_i)$ close to Q_i, $x < p_0$. Therefore, by forward monotone property and conclusion (b), $\lim_{t\to+\infty} \varphi_t(x) = p_0$. Hence, $W^u(Q_i)$ is a heteroclinic trajectory from Q_i to p_0.

Then, from the saddle nature of Q_1 and Q_2 and the repelling nature of the origin, it is clear from a simple phase portrait that $\{Q_1, p_0, Q_2\} \cup W^u(Q_1) \cup W^u(Q_2)$ is the upper boundary of $\mathcal{R}(O)$, i.e.

$$\overline{\mathcal{R}(O)} \setminus \mathcal{R}(O) = \{Q_1, p_0, Q_2\} \cup W^u(Q_1) \cup W^u(Q_2).$$

For each fixed $x \in W^u(Q_1)$ and every $y \in C$ satisfying $y \gg_K x$ close enough to x, by backward monotone property of type-K systems and the saddle nature of Q_1, $y \in \mathcal{R}_H(\infty) \subset \mathcal{R}(\infty)$. This shows that

$$W^u(Q_1) \subset \overline{\mathcal{R}(O)} \cap \overline{\mathcal{R}_H(\infty)} \cap \overline{\mathcal{R}(\infty)}. \tag{38}$$

Thus, by (19), $\Sigma_H = \{Q_1\} \cup W^u(Q_1)$. We claim that Σ_H is an ordered curve in $\leq$. Indeed, if there were unordered pair $p, q \in \Sigma_H$, i.e. $p \nleq q \nleq p$ then we must have either $p \ll_K q$ or $q \ll_K p$. But the former implies $q \in \mathcal{R}(\infty)$ and the latter implies $p \in \mathcal{R}(\infty)$, contradictions to $p, q \in \Sigma_H$.

In parallel to the above, we also have

$$W^u(Q_2) \subset \overline{\mathcal{R}(O)} \cap \overline{\mathcal{R}_V(\infty)} \cap \overline{\mathcal{R}(\infty)}. \tag{39}$$

and $\Sigma_V = \{Q_2\} \cup W^u(Q_2)$. Moreover, Σ_V is an ordered curve from Q_2 to p_0.

If p_0 is the only one equilibrium in $\dot{C}$ then $\Sigma_0 = \{p_0\}$ and the proof is complete. Now suppose (35) has more than one interior equilibria. By part (b) all interior equilibria are in $[p_0, p_1]$. Let us consider the curves

$$\ell_1 = \{x \in [O, r] : f_1(x) = 0\}, \quad \ell_2 = \{x \in [O, r] : f_2(x) = 0\}.$$

Then by (A1) and (ii) there are continuously differentiable functions $x_1 = g(x_2)$ and $x_2 = h(x_1)$ such that ℓ_1 is the graph of g and ℓ_2 is the graph of h. As

$$g'(x_2) = -\frac{\partial f_1}{\partial x_2} / \frac{\partial f_1}{\partial x_1} \geq 0, \ h'(x_1) = -\frac{\partial f_2}{\partial x_1} / \frac{\partial f_2}{\partial x_2} \geq 0,$$

both g and h are nondecreasing. As each equilibrium p in $\dot{C}$ satisfies $f_i(p) = 0$, p is an intersection point of ℓ_1 and ℓ_2. Thus, the monotone nature of g and h determines that $p_0 \ll p_1$ and all equilibria in $[p_0, p_1]$ are ordered in $\ll$. Now suppose $p, q \in [p_0, p_1]$ are two equilibria such that $p \ll q$ and no other equilibrium in (p, q). Then between p and q either ℓ_1 is above ℓ_2 or ℓ_1 is below ℓ_2 (see figure 5). In either case, the closed region bounded by ℓ_1 and ℓ_2 between p and q is forward invariant. For x in the interior of this region, $x_1' x_2' = x_1 x_2 f_1(x) f_2(x) > 0$. So $\lim_{t \to +\infty} \varphi_t(x) = p$ in the first case and $\lim_{t \to +\infty} \varphi_t(x) = q$ in the second case.

We prove that there is at least one heteroclinic trajectory γ linking the two equilibria. Since (35) is a planar system, instead of using the results and methods used in [8] and [46] we prove this intuitively as follows. Take two points $P \in \ell_1$ and $Q \in \ell_2$ between p and q (see figure 5 (a)). Then the line segment $\overline{PQ}$ is homeomorphic to the interval $[a_{10}, a_{11}] = [0, 1]$ through $h(s) = P + s(Q - P)$. Suppose $\lim_{t \to +\infty} \varphi_t(h(s)) = p$ for all $s \in [0, 1]$. Then, for each $s \in [0, 1]$, either (ia) $\varphi_t(h(s))$ leaves the region bounded by ℓ_1 and ℓ_2 crossing ℓ_1 in a finite negative time or (ib) $\varphi_t(h(s))$ leaves the bounded region crossing ℓ_2 in a finite negative time or (ic) $\varphi_t(h(s))$ stays in the region forever and $\lim_{t \to -\infty} \varphi_t(h(s)) = q$. Clearly, $s = a_{10}$ is in case (ia) and $s = a_{11}$ in case (ib). If for some $s' \in (0, 1)$ such that $s = s'$ is in case (ia) then, by uniqueness and continuous dependence on initial values, there is a sufficiently small $\delta > 0$ such that s is also in case (ia) for all $s \in [0, s' + \delta)$. Similarly, if $s = s'$ is in case (ib) for some $s' \in (0, 1)$ then for a small $\delta > 0$, s is also in case (ib) for all $s \in (s' - \delta, 1]$. Now test $s_1 = \frac{1}{2}(a_{10} + a_{11})$. If $s = s_1$ is in case (ic) then the trajectory $\varphi_t(h(s_1))$ for $t \in \mathbb{R}$ is a heteroclinic trajectory linking p to q so the proof is complete. If $s = s_1$ is in case (ia) we let $[a_{20}, a_{21}] = [s_1, a_{11}]$; if $s = s_1$ is in case (ib) we let $[a_{20}, a_{21}] = [a_{10}, s_1]$; and in either case we test $s_2 = \frac{1}{2}(a_{20} + a_{21})$. If $s = s_2$ is in case (ic) then the proof is complete. Otherwise, repeat the above process. If we obtain a heteroclinic trajectory linking p to q after repeating n times then the proof is complete.

Otherwise, we obtain a series $\{[a_{n0}, a_{n1}]\}$ of intervals satisfying

$$[a_{(n+1)0}, a_{(n+1)1}] \subset [a_{n0}, a_{n1}], a_{(n+1)1} - a_{(n+1)0} = \frac{1}{2}(a_{n1} - a_{n0}) = \frac{1}{2^n}$$

and a sequence $s_n = \frac{1}{2}(a_{n0} + a_{n1})$ for all $n \geq 1$. Clearly, there is a unique $s_0 \in (a_{n0}, a_{n1})$ for all $n \geq 1$ and, as $n \to \infty$, all of a_{n0}, a_{n1} and s_n converge to s_0. From the above process we see that $s = a_{n0}$ is in case (ia) and $s = a_{n1}$ is in case (ib) for all $n \geq 1$. Therefore, s is in case (ia) for all $s \in [0, s_0)$ and s is in case (ib) for all $s \in (s_0, 1]$. From the δ-features of s' in either (ia) or (ib) we see that $s = s_0$ must be in case (ic) and $\gamma = \{\varphi_t(h(s_0)) : t \in \mathbb{R}\}$ is the required heteroclinic trajectory.

Clearly, γ is an ordered curve in $\ll$. We show in the next paragraph that γ is the unique heteroclinic trajectory linking p and q. Then Σ_0 is a ordered curve from p_0 to p_1 consisting of all equilibria in $\dot{C}$ and heteroclinic trajectories.

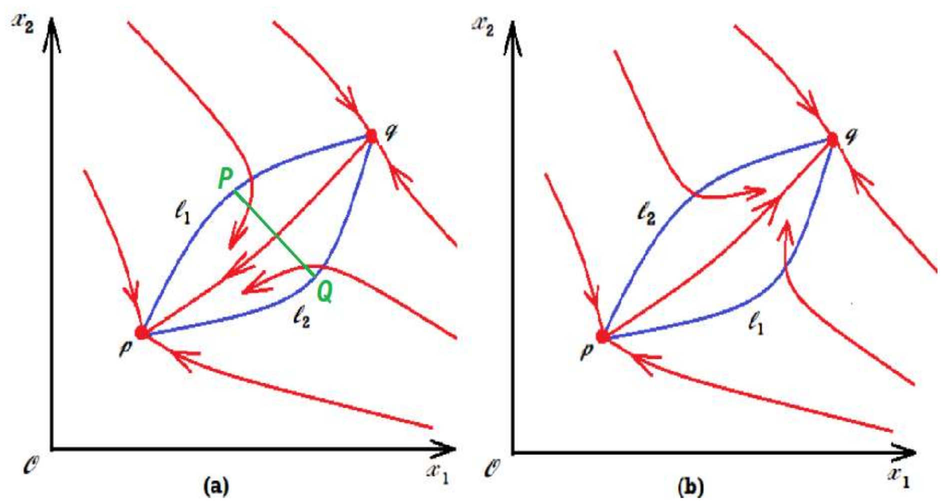

FIGURE 5. Local phase portraits near two neighbouring interior equilibria $p \ll q$: (a) p attracts all trajectories entering into the region bounded by ℓ_1 and ℓ_2, (b) q attracts all such trajectories.

If there is another heteroclinic trajectory γ_1 between p and q distinct from γ, then there are two points $x \ll_K y$, x on one of γ and γ_1 and y on the other. By the backward monotone property, $\varphi_t(x) \leq_K \varphi_t(y)$ for all $t \leq 0$. From (35) we have

$$\left(\frac{\varphi_t(y)_1}{\varphi_t(x)_1}\right)' = \frac{\varphi_t(y)_1}{\varphi_t(x)_1}[f_1(\varphi_t(y)) - f_1(\varphi_t(x))].$$

From (A1) and (ii), $(\frac{\partial f_1}{\partial x_1}(u), \frac{\partial f_1}{\partial x_2}(u)) \leq_K O$ for $u \in [O, r]$. This and $\varphi_t(y) - \varphi_t(x) \geq_K O$ ensure

$$f_1(\varphi_t(y)) - f_1(\varphi_t(x)) = \left(\int_0^1 (\frac{\partial f_1}{\partial x_1}, \frac{\partial f_1}{\partial x_2}) ds\right)(\varphi_t(y) - \varphi_t(x)) \leq 0,$$

where the argument of $\frac{\partial f_1}{\partial x_1}$ and $\frac{\partial f_1}{\partial x_2}$ is $\varphi_t(x) + s(\varphi_t(y) - \varphi_t(x)) \in [O, r]$. It then follows that $\frac{\varphi_t(y)_1}{\varphi_t(x)_1}$ is nondecreasing when t decreases. Hence, $\frac{\varphi_t(y)_1}{\varphi_t(x)_1} \geq \frac{y_1}{x_1} > 1$ for $t \leq 0$, a contradiction to $\lim_{t\to-\infty} \frac{\varphi_t(y)_1}{\varphi_t(x)_1} = 1$ as $\lim_{t\to-\infty} \varphi_t(x) = \lim_{t\to-\infty} \varphi_t(y)$. Therefore, γ is the unique heteroclinic trajectory between p and q.

(d) From (38) and $\Sigma_H = \{Q_1\} \cup W^u(Q_1)$ we see that $\Sigma_H \subset \overline{\mathcal{R}_H(\infty)}$. Since $\Sigma_H \cap \mathcal{R}_H(\infty) = \emptyset$, by (24) we must have $\Sigma_H \subset S_H$. Similarly, from (39) and $\Sigma_V = \{Q_2\} \cup W^u(Q_2)$ we see that $\Sigma_V \subset \overline{\mathcal{R}_V(\infty)}$. Then $\Sigma_V \subset S_V$ follows from $\Sigma_V \cap \mathcal{R}_V(\infty) = \emptyset$ and (24). For each $x \in \Sigma_0$ and all $y, z \in C$ satisfying $y \ll_K x \ll_K z$, from $\varphi_t(y) \leq_K \varphi_t(x) \leq_K \varphi_t(z)$ for $t < 0$ in their common existence interval and the boundedness of $\varphi_t(x)$ we know that $\varphi_t(y)_1$ and $\varphi_t(z)_2$ are bounded. From (c) we see that $y, z \in \mathcal{R}(\infty)$. Thus, $y \in \mathcal{R}_V(\infty)$ and $z \in \mathcal{R}_H(\infty)$. This shows that $\Sigma_0 \subset \overline{\mathcal{R}_H(\infty)} \cap \overline{\mathcal{R}_H(\infty)}$. By $\Sigma_0 \cap \mathcal{R}(\infty) = \emptyset$ we obtain $\Sigma_0 \subset S_H \cap S_V$. Therefore, $S_H \setminus \mathcal{R}(\infty) = \Sigma_H \cup \Sigma_0$ and $S_V \setminus \mathcal{R}(\infty) = \Sigma_V \cup \Sigma_0$. □

5. Configuration of Σ for three-dimensional type-K competitive systems

In this section, assuming the truth of the conjecture in remark 3.5 for $N = 3$, i.e. $\overline{\Sigma_H} \cap \overline{\Sigma_V}$ is a curve, we look at possible configurations of the global attractor Σ under the conditions of theorem 3.1 for system (1) when $N = 3$. Here the cone K given by (6) could have $H = \{1, 2\}$ and $V = \{3\}$ or $H = \{1\}$ and $V = \{2, 3\}$. Since the configurations for the two cases are essentially the same, we assume $H = \{1, 2\}$ and $V = \{3\}$. Then, by theorem 3.3 (d) and remark 3.5, $\overline{\Sigma_H \cup \Sigma_V}$ is a two-dimensional surface, both $\overline{\Sigma_H}$ and $\overline{\Sigma_V}$ are unordered in $\ll_K$ and $\overline{\Sigma_H} \cap \overline{\Sigma_V}$ is the common boundary curve shared by Σ_H and Σ_V joining the two equilibria P_1 and P_2, P_1 is an interior equilibrium of the subsystem for (x_2, x_3) and P_2 is an interior equilibrium of the subsystem for (x_1, x_3) (see figure 6). There is a trajectory joining the axial equilibrium Q_3 to P_1 and a trajectory joining Q_3 to P_2. Then the surface $\overline{\Sigma_V}$ is bounded by these two trajectories and the curve $\overline{\Sigma_H} \cap \overline{\Sigma_V}$. There is a curve joining the two axial equilibria Q_1 and Q_2, which is the modified carrying simplex $\Sigma_H \cap C_H$ for the competitive subsystem for (x_1, x_2). There is a trajectory joining Q_1 to P_2 and a trajectory joining Q_2 to P_1. Then the surface $\overline{\Sigma_H}$ is bounded by these two trajectories and the curves $\Sigma_H \cap C_H$ and $\overline{\Sigma_H} \cap \overline{\Sigma_V}$.

From remark 3.4 we see that not much of the geometric and dynamical features of Σ_0 is known. A thorough classification with rigour for configurations of Σ_0 when $N = 3$ is beyond the scope of this paper. However, based on theorem 4.1 for subsystems for (x_1, x_3) and (x_2, x_3), we may have the following possible configurations of Σ_0 (see figure 7). Case 1: P_1 is the unique interior equilibrium of the subsystem for (x_2, x_3) and P_2 is the unique interior equilibrium of the subsystem for (x_1, x_3). If P_1 or P_2 or an equilibrium in $\dot{C}$ is globally attracting then configuration (1) happens and $\Sigma = \overline{\Sigma_H \cup \Sigma_V}$. If there are at least

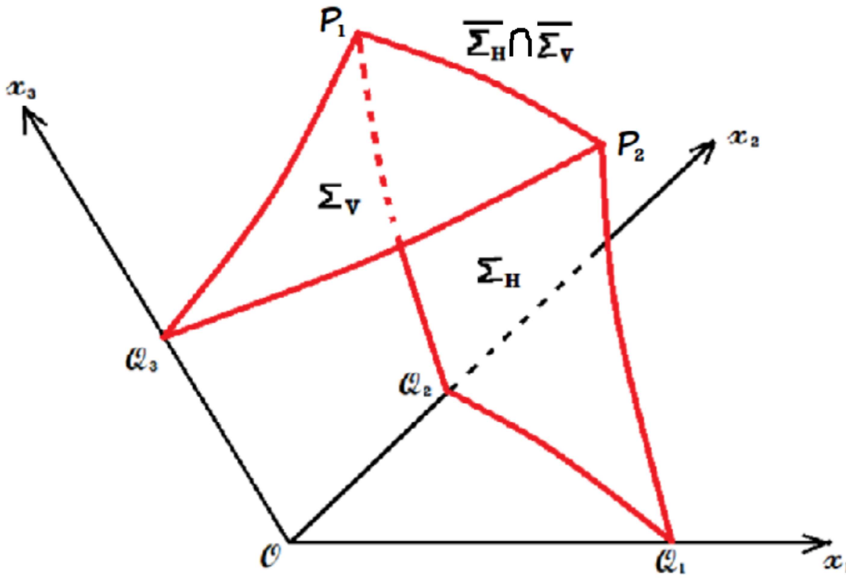

FIGURE 6. Illustration of $\overline{\Sigma_H \cup \Sigma_V}$ for three-dimensional type-K competitive systems.

two equilibria in $\dot{C}$, then (2) or (3) may happen. Case 2: one of P_1 and P_2 is the unique interior equilibrium for that subsystem and the other is not unique. Then (4) or (5) or (6) or (7) may happen. Case 3: none of P_1 and P_2 is the unique interior equilibrium for that subsystem. Then one of (8)–(13) may happen. It is possible that some of (1)–(13) may never happen. It is also possible that there are many other configurations not included in (1)–(13). These problems are left for future investigation.

The configurations of Σ are obtained by simply sitting each configuration of Σ_0 on top of the curve $\overline{\Sigma_H} \cap \overline{\Sigma_V}$ in figure 6.

If the conjecture in remark 3.5 is not true for $N = 3$, then $\overline{\Sigma_H} \cap \overline{\Sigma_V}$ may not be a curve and the configurations (1)–(13) may not be applicable.

Example 5.1 consider the 3-dimensional system

$$\begin{aligned} x_1' &= x_1(x_3 - 1 - (a + 0.25)x_2 - a(x_1 - 1) - (x_1 - 1)^3), \\ x_2' &= x_2(x_3 - 1 - (a + 0.25)x_1 - a(x_2 - 1) - (x_2 - 1)^3), \\ x_3' &= x_3(x_1 + x_2 - 1 - a(x_3 - 1) - (x_3 - 1)^3), \end{aligned} \tag{40}$$

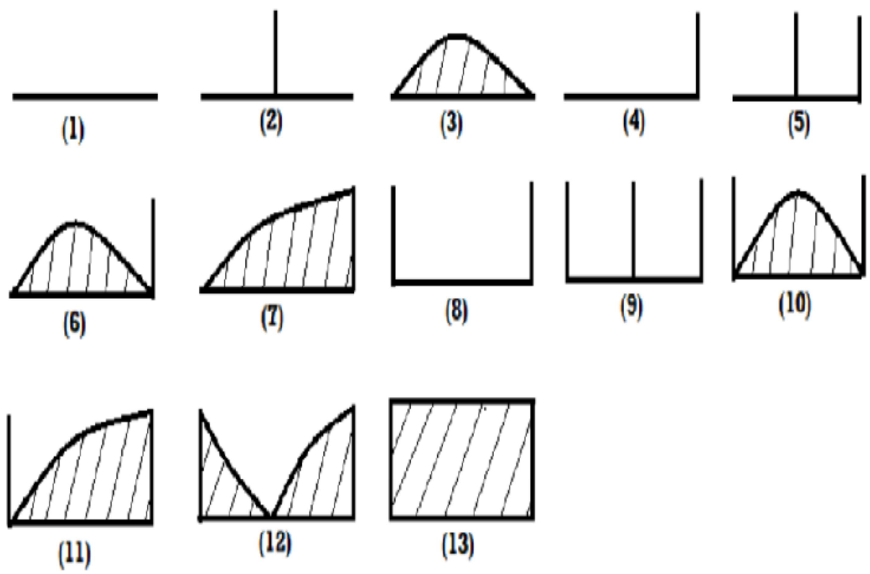

FIGURE 7. Some possible configurations of Σ_0 for three-dimensional type-K competitive systems under the assumption that $\overline{\Sigma_H} \cap \overline{\Sigma_V}$ is a curve.

where $a > 0$ is a constant. Viewing (40) as (1) with $N = 3$, we have

$$Df(x) = \begin{pmatrix} -a - 3(x_1 - 1)^2 & -(a + 0.25) & 1 \\ -(a + 0.25) & -a - 3(x_2 - 1)^2 & 1 \\ 1 & 1 & -a - 3(x_3 - 1)^2 \end{pmatrix}.$$

From this it is clear that (40) is type-K competitive with $H = \{1, 2\}$ and $V = \{3\}$. Also, $f(O) = (a, a, a) \gg O$. With $u(s) = (s + 3, s + 3, s + 3)$ for all $s \geq 0$, we see that

$$\begin{aligned} f_1(u_1(s)e_1 + u(s)_V) &= (s + 2)(1 - a - (s + 2)^2) \leq -6 < 0, \\ f_2(u_2(s)e_2 + u(s)_V) &= (s + 2)(1 - a - (s + 2)^2) \leq -6 < 0, \\ f_3(u_3(s)e_3 + u(s)_H) &= 1 + (s + 2)(2 - a - (s + 2)^2) \leq -3 < 0. \end{aligned}$$

By proposition 2.3, (40) is dissipative. Thus, (40) satisfies (A1)–(A3). Moreover, with $r = u(0) = (3, 3, 3) \gg O$ and from $Df(x)$ and proposition 2.3, the conditions (i)–(iii) of theorem 3.1 are met. For $\rho_0 = 10 + 5a$ and $i = 1, 2$,

$$\begin{aligned} f_i(\rho_0 e_i + r_V) &= 2 - (\rho_0 - 1)(a + (\rho_0 - 1)^2) < 0, \\ f_i(\rho_0 e_3 + r_H) &= \rho_0 - 9 - 3(a + 0.25) - 2a = 0.25 > 0, \\ f_3(\rho_0 e_i + r_V) &= \rho_0 - 9 - 2a = 1 + 3a > 0, \\ f_3(\rho_0 e_3 + r_H) &= 5 - a(\rho_0 - 1) - (\rho_0 - 1)^3 < 5 - 9^3 < 0. \end{aligned}$$

Thus, (iv) of theorem 3.1 is also met. Then all theorems, corollaries and remarks in section 3 can be applied to (40).

The subsystems for (x_1, x_3) when $x_2 = 0$ and (x_2, x_3) when $x_1 = 0$ are the same as (36) in example 4.1. Note that $f_1(x_2, x_1, x_3) = f_2(x_1, x_2, x_3)$ and $f_3(x_2, x_1, x_3) = f_3(x_1, x_2, x_3)$. This indicates that the dynamics of (40) in C is symmetric about the plane $x_1 = x_2$ and this plane is an invariant manifold. The dynamics of (40) on this manifold is described by the planar system

$$(41) \qquad \begin{aligned} x' &= x(y - 1 + a - (2a + 0.25)x - (x - 1)^3), \\ y' &= y(2x - 1 - a(y - 1) - (y - 1)^3) \end{aligned}$$

with $x_1 = x_2 = x, x_3 = y$. Viewing this as $x' = xg_1(x, y), y' = yg_2(x, y)$, the Jacobian matrix is

$$Dg(x, y) = \begin{pmatrix} -(2a + 0.25) - 3(x - 1)^2 & 1 \\ 2 & -a - 3(y - 1)^2 \end{pmatrix}.$$

By a routine check we see that (41) satisfies all the conditions of theorem 4.1. The system has at least one interior equilibrium $p_0 = (0.5, 1)$, which is the inflection point of the cubic curve $g_2(x, y) = 0$ with maximum gradient $2a^{-1}$ at p_0. The curve $g_1(x, y) = 0$ is also cubic with inflection point $(1, a + 1.25)$. The gradient of $g_1 = 0$ at p_0 is $2a + 1$. If $0 < a < (\sqrt{17} - 1)/4$ then $2a + 1 < 2a^{-1}$ so the two curves $g_1 = 0$ and $g_2 = 0$ have at least three intersection points including p_0, one in (O, p_0) and at least one in $(p_0, +\infty)$. By theorem 4.1, the set $\mathbb{R}^2_+ \setminus (\mathcal{R}(O) \cup \mathcal{R}(\infty))$ is shown by figure 3 (a). If $a \geq (\sqrt{16^2 + 1} - 1)/16$, the gradient of $g_1 = 0$ has minimum $2a + 0.25$ at $(1, a + 1.25)$ and $2a + 0.25 \geq 2a^{-1}$. Hence, p_0 is the unique interior equilibrium of (41) and the set $\mathbb{R}^2_+ \setminus (\mathcal{R}(O) \cup \mathcal{R}(\infty))$ is shown by figure 3 (b).

Now inputting the information of (41) into the system (40) on $x_1 = x_2$ and the results of example 4.1 into the subsystem with $x_2 = 0$ and the subsystem with $x_1 = 0$, we obtain the following conclusions for (40):

If $0 < a < (\sqrt{17} - 1)/4$ then Σ_0 contains three equilibria in the interior of π_1:

$$(0, 1 - \sqrt{1 - a}, 1 - \sqrt{1 - a}) \leq (0, 1, 1) \leq (0, 1 + \sqrt{1 - a}, 1 + \sqrt{1 - a});$$

three equilibria in the interior of π_2:

$$(1 - \sqrt{1 - a}, 0, 1 - \sqrt{1 - a}) \leq (1, 0, 1) \leq (1 + \sqrt{1 - a}, 0, 1 + \sqrt{1 - a});$$

at least three equilibria in $\dot{C} : O \ll R_1 \ll (0.5, 0.5, 1) \ll R_2$, where R_1 and R_2 correspond to the one of (41) in (O, p_0) and the one in $(p_0, +\infty)$ respectively. It is clear that $(0, 1 -$

$\sqrt{1-a}, 1-\sqrt{1-a})$, R_1 and $(1-\sqrt{1-a}, 0, 1-\sqrt{1-a})$ are all in $\overline{\Sigma_H} \cap \overline{\Sigma_V}$ but others are in $\Sigma_0 \setminus (\overline{\Sigma_H} \cap \overline{\Sigma_V})$. Thus, $\overline{\Sigma_H} \cap \overline{\Sigma_V}$ is a proper subset of Σ_0. Further investigation is needed to confirm whether Σ_0 has one of the configurations (9)–(11) and (13) in figure 7.

If $(\sqrt{16^2+1}-1)/16 \leq a < 1$ then Σ_0 contains the same six equilibria in ∂C as in previous case but $(0.5, 0.5, 1) \in \Sigma_0$ is the unique equilibrium on $x_1 = x_2$, and there may be more equilibria in $\dot{C}$. In this case, Σ_0 may have the configuration (8) or (12) in figure 7.

If $a \geq 1$, Σ_0 contains $(0, 1, 1) \in \pi_1$, $(1, 0, 1) \in \pi_2$, $(0.5, 0.5, 1)$ on $x_1 = x_2$, and maybe more equilibria in $\dot{C}$. In this case, Σ_0 may have configuration (1) or outside of (1)–(13) in figure 7.

6. Proof of the main results

Lemma 6.1. *Assume that (A1)–(A3) and the conditions (i)–(iii) of theorem 3.1 hold. Let $I \subset I_N$ such that $I \cap H \neq \emptyset$, $I \cap V \neq \emptyset$ and $N \geq |I| \geq 3$. If for some $x \in \dot{C}_I$ there exists $x' \in C_I$ ($x'' \in C_I$) satisfying $\varphi_t(x) \geq_K \varphi_t(x')$ ($\varphi_t(x'') \geq_K \varphi_t(x)$) for all $t \geq 0$, then*

$$\forall i \in I, \lim_{t\to+\infty}[\varphi_t(x)_i - \varphi_t(x')_i] = 0 \quad (\lim_{t\to+\infty}[\varphi_t(x)_i - \varphi_t(x'')_i] = 0). \tag{42}$$

Proof. By condition (i) of theorem 3.1, $\omega(x), \omega(x')$ and $\omega(x'')$ are all subsets of $[O, r)$. Without loss of generality we may assume that $\varphi_t(x), \varphi_t(x'), \varphi_t(x'') \in [O, r]$ for all $t \geq 0$. Let

$$\varepsilon = \min\left\{-\frac{\partial f_k(x)}{\partial x_k} : x \in [O, r], k \in I_N\right\}.$$

Then $\varepsilon > 0$ by condition (ii) of theorem 3.1. Clearly, $x \in \dot{C}_I$ and $x', x'' \in C_I$ imply $\varphi_t(x) \in \dot{C}_I$ and $\varphi_t(x'), \varphi_t(x'') \in C_I$ for all $t \geq 0$. Then, for each $i \in I \cap H$, $j \in I \cap V$ and all $t \geq 0$,

$$\varphi_t(x'')_i \geq \varphi_t(x)_i \geq \varphi_t(x')_i \geq 0, \;\; 0 \leq \varphi_t(x'')_j \leq \varphi_t(x)_j \leq \varphi_t(x')_j.$$

So, $\frac{\varphi_t(x'')_i}{\varphi_t(x)_i} \geq 1$ and

$$\left(\frac{\varphi_t(x'')_i}{\varphi_t(x)_i}\right)' = \frac{\varphi_t(x'')_i}{\varphi_t(x)_i}[f_i(\varphi_t(x'')) - f_i(\varphi_t(x))], \; t \geq 0. \tag{43}$$

Note that

$$\begin{aligned} f_i(\varphi_t(x'')) - f_i(\varphi_t(x)) &= \left(\int_0^1 [Df]_i ds\right)[\varphi_t(x'') - \varphi_t(x)] \\ &\leq \left(\int_0^1 \frac{\partial f_i}{\partial x_i} ds\right)[\varphi_t(x'')_i - \varphi_t(x)_i] \\ &\leq -\varepsilon[\varphi_t(x'')_i - \varphi_t(x)_i] \leq 0, \end{aligned}$$

where the argument of Df and $\frac{\partial f_i}{\partial x_i}$ is $\varphi_t(x) + s(\varphi_t(x'') - \varphi_t(x))$. So $\frac{\varphi_t(x'')_i}{\varphi_t(x)_i}$ is nonincreasing. If $\frac{x''_i}{x_i} = 1$ then $\frac{\varphi_t(x'')_i}{\varphi_t(x)_i} \equiv 1$ and the second equality in (42) is true for this i. Now suppose

$\frac{x''_i}{x_i} > 1$. Then, from (43) and the above inequality we have

$$\frac{\varphi_t(x'')_i}{\varphi_t(x)_i} \le \frac{x''_i}{x_i} \exp\left(-\varepsilon \int_0^t [\varphi_s(x'')_i - \varphi_s(x)_i] ds\right),\ t \ge 0.$$

Suppose $\lim_{t\to+\infty}[\varphi_t(x'')_i - \varphi_t(x)_i] = 0$ is not true. Since $0 \le \varphi_t(x'')_i - \varphi_t(x)_i \le r_i$, we must have $\limsup_{t\to+\infty}[\varphi_t(x'')_i - \varphi_t(x)_i] > 0$. Then there is a positive increasing sequence $t_n \to +\infty$ as $n \to \infty$ such that $\varphi_{t_n}(x'')_i - \varphi_{t_n}(x)_i \ge \delta > 0$ for all $n > 0$. Let $M = \max\{|x_i f_i(x)| : x \in [O, r], i \in I_N\}$ and define $g : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\forall n > 0, \forall t \in [t_n, t_{n+1}], g(t) = \max\{0, \delta - 2M|t - t_n|, \delta - 2M|t - t_{n+1}|\}$$

and $g(t) = \max\{0, \delta - 2M|t - t_1|\}$ for $t \in [0, t_1]$. Then, for $t \in [t_n - \frac{\delta}{2M}, t_n + \frac{\delta}{2M}]$,

$$\begin{aligned} \varphi_t(x'')_i - \varphi_t(x)_i &= \varphi_{t_n}(x'')_i - \varphi_{t_n}(x)_i \\ &\quad + \int_{t_n}^t [\varphi_s(x'')_i f_i(\varphi_s(x'')) - \varphi_s(x)_i f_i(\varphi_s(x))] ds \\ &\ge \delta - 2M|t - t_n|. \end{aligned}$$

Combining this with

$$\varphi_t(x'')_i - \varphi_t(x)_i \ge \delta - 2M|t - t_{n+1}|$$

for $t \in [t_{n+1} - \frac{\delta}{2M}, t_{n+1} + \frac{\delta}{2M}]$, we obtain $\varphi_t(x'')_i - \varphi_t(x)_i \ge g(t)$ for $t \ge 0$. As $\int_0^{+\infty} g(t)dt$ diverges, we have

$$\frac{\varphi_t(x'')_i}{\varphi_t(x)_i} \le \frac{x''_i}{x_i} \exp\left(-\varepsilon \int_0^t g(s) ds\right) \to 0\ (t \to +\infty),$$

a contradiction to $\frac{\varphi_t(x'')_i}{\varphi_t(x)_i} \ge 1$. This contradiction shows the truth of $\lim_{t\to+\infty}[\varphi_t(x'')_i - \varphi_t(x)_i] = 0$ for all $i \in I \cap H$.

Replacing $\frac{\varphi_t(x'')_i}{\varphi_t(x)_i}$ by $\frac{\varphi_t(x')_j}{\varphi_t(x)_j}$ in the above process we also shows the truth of $\lim_{t\to+\infty}[\varphi_t(x')_j - \varphi_t(x)_j] = 0$ for all $j \in I \cap V$.

Next, we show that $\lim_{t\to+\infty}[\varphi_t(x)_j - \varphi_t(x'')_j] = 0$ for each $j \in I \cap V$ so that the second equality in (42) follows. By condition (iii) in theorem 3.1, for this j there is $i \in I \cap H$ such that $\frac{\partial f_i(x)}{\partial x_j} > 0$ for all $x \in [O, r] \cap C_I$. Then

$$\varepsilon_0 = \min\left\{\frac{\partial f_i(x)}{\partial x_j} : x \in [O, r] \cap C_I\right\} > 0.$$

Replacing $\frac{\partial f_i(\varphi_t(x) + s(\varphi_t(x'') - \varphi_t(x)))}{\partial x_i}$ in the inequalities below (43) by $\frac{\partial f_i(\varphi_t(x) + s(\varphi_t(x'') - \varphi_t(x)))}{\partial x_j}$ and utilising ε_0, we obtain

$$f_i(\varphi_t(x'')) - f_i(\varphi_t(x)) \le -\varepsilon_0[\varphi_t(x)_j - \varphi_t(x'')_j] \le 0.$$

It follows from (43) that

$$\frac{\varphi_t(x'')_i}{\varphi_t(x)_i} \le \frac{x''_i}{x_i} \exp\left(-\varepsilon_0 \int_0^t [\varphi_s(x)_j - \varphi_s(x'')_j] ds\right),\ t \ge 0.$$

Since $0 \le \varphi_t(x)_j - \varphi_t(x'')_j \le r_j$, by the same reasoning as above we would derive a contradiction from this if $\lim_{t\to+\infty}[\varphi_t(x)_j - \varphi_t(x'')_j] = 0$ is not true. Therefore, we must have $\lim_{t\to+\infty}[\varphi_t(x)_j - \varphi_t(x'')_j] = 0$ and the second in (42) follows.

In a similar manner we can show that $\lim_{t\to+\infty}[\varphi_t(x)_i - \varphi_t(x')_i] = 0$ for each $i \in I \cap H$ so that the first equality in (42) holds. □

Lemma 6.2. *Under the assumptions (A1)–(A3) and the conditions (i) and (ii) of theorem 3.1, if there are two points $x, y \in C \setminus (\mathcal{R}(\infty) \cup \{O\})$ such that $x <_K y$, then $\alpha(x) \subset \pi_i$ for each $i \in H$ with $x_i < y_i$ and $\alpha(y) \subset \pi_j$ for each $j \in V$ with $x_j > y_j$.*

Proof. Clearly, for any two points $x, y \in C \setminus (\mathcal{R}(\infty) \cup \{O\})$ with $x <_K y$, both $\varphi_t(x)$ and $\varphi_t(y)$ exist for $t \in \mathbb{R}$. So $\varphi_t(x) \le_K \varphi_t(y)$ for all $t \le 0$. Suppose $x_i < y_i$ for some $i \in H$. If $x_i = 0$ then $\varphi_t(x)_i \equiv 0$ for $t \in \mathbb{R}$ so $\alpha(x) \subset \pi_i$. If $x_i > 0$ then $\varphi_t(y)_i \ge \varphi_t(x)_i > 0$, so

$$\frac{\varphi_t(y)_i}{\varphi_t(x)_i} \ge 1, \quad \left(\frac{\varphi_t(y)_i}{\varphi_t(x)_i}\right)' = \frac{\varphi_t(y)_i}{\varphi_t(x)_i}[f_i(\varphi_t(y)) - f_i(\varphi_t(x))],$$

and

$$\frac{\varphi_t(y)_i}{\varphi_t(x)_i} = \frac{y_i}{x_i} \exp\left[-\left(\int_t^0 [f_i(\varphi_\ell(y)) - f_i(\varphi_\ell(x))]d\ell\right)\right] \tag{44}$$

for all $t \le 0$. As $f_i(\varphi_t(y)) - f_i(\varphi_t(x))$ can be written as

$$\left(\int_0^1 [Df(\varphi_t(x) + s(\varphi_t(y) - \varphi_t(x)))]_i ds\right)(\varphi_t(y) - \varphi_t(x)),$$

by $\varphi_t(y) \ge_K \varphi_t(x)$, assumption (A1) and condition (ii) we have $f_i(\varphi_t(y)) - f_i(\varphi_t(x)) \le 0$. From this and (44) we see that $\frac{\varphi_t(y)_i}{\varphi_t(x)_i}$ is nondecreasing when t decreases to $-\infty$. If $\lim_{t\to-\infty} \varphi_t(x)_i \ne 0$, then there is a sequence $\{t_n\}$ satisfying $t_{n+1} < t_n - 1 < -1$ and $\varphi_{t_n}(x)_i \ge \delta > 0$ for all $n \ge 0$. Then, for

$$\varepsilon = \min\left\{-\frac{\partial f_k(x)}{\partial x_k} : x \in [O, r], k \in I_N\right\}$$

(same as in the proof of lemma 6.1), we have $\varepsilon > 0$ and

$$f_i(\varphi_t(y)) - f_i(\varphi_t(x)) \le -\varepsilon[\varphi_t(y)_i - \varphi_t(x)_i].$$

From $\varphi_t(y)_i - \varphi_t(x)_i = \varphi_t(x)_i \left[\frac{\varphi_t(y)_i}{\varphi_t(x)_i} - 1\right] \ge \left[\frac{y_i}{x_i} - 1\right] \varphi_t(x)_i$ we obtain

$$f_i(\varphi_t(y)) - f_i(\varphi_t(x)) \le -\varepsilon \left[\frac{y_i}{x_i} - 1\right] \varphi_t(x)_i.$$

For $M = \max\{|x_i f_i(x)| : x \in [O, r], i \in I_N\}$ (same as in the proof of lemma 6.1) we have $M > 0$ and

$$|\varphi_t(x)_i - \varphi_{t_n}(x)_i| = \left|\int_{t_n}^t \varphi_s(x)_i f_i(\varphi_s(x))ds\right| \le M|t - t_n|,$$

so

$$\varphi_t(x)_i \geq \varphi_{t_n}(x)_i - M|t-t_n| \geq \delta - M|t-t_n|,$$

for $t_n - \frac{\delta}{M} \leq t \leq t_n + \frac{\delta}{M}$. Without loss of generality, we may assume that $\frac{\delta}{M} < \frac{1}{2}$ as we can always make $\delta > 0$ smaller. Now define $g(t) = \delta - M|t - t_n|$ for each $n \geq 0$ and all t satisfying $t_n - \frac{\delta}{M} \leq t \leq t_n + \frac{\delta}{M}$ and $g(t) = 0$ for other values of $t \in (-\infty, 0]$. Then

$$-\left(\int_t^0 [f_i(\varphi_\ell(y)) - f_i(\varphi_\ell(x))]d\ell\right) \geq \varepsilon \left[\frac{y_i}{x_i} - 1\right] \left(\int_t^0 g(s)ds\right) \to +\infty$$

as $t \to -\infty$. It follows from this and (44) that $\frac{\varphi_t(y)_i}{\varphi_t(x)_i} \to +\infty$ as $t \to -\infty$, a contradiction to the boundedness of $\frac{\varphi_{t_n}(y)_i}{\varphi_{t_n}(x)_i} \leq \frac{\varphi_{t_n}(y)_i}{\delta}$. Therefore, we must have $\lim_{t\to-\infty} \varphi_t(x)_i = 0$ so $\alpha(x) \subset \pi_i$.

If $x_j > y_j$ for some $j \in V$, by the same reasoning as above we also have $\alpha(y) \subset \pi_j$. □

Consider the following subsets of C:

$$\Omega_H = \{y \in C : y_V \leq r_V, \|y_H\| \geq \rho_0\}, \tag{45}$$

$$\Omega_V = \{y \in C : y_H \leq r_H, \|y_V\| \geq \rho_0\}, \tag{46}$$

$$\dot{\Omega}_H = \{y \in C : \forall j \in V, y_j < r_j, \|y_H\| > \rho_0\}, \tag{47}$$

$$\dot{\Omega}_V = \{y \in C : \forall i \in H, y_i < r_i, \|y_V\| > \rho_0\}, \tag{48}$$

where ρ_0 is given in theorem 3.1.

Lemma 6.3. *Under the assumptions of theorem 3.1 with $N \geq 3$ we have*

$$\Omega_H \subset \mathcal{R}_H(\infty),\ \dot{\Omega}_H \subset \dot{\mathcal{R}}_H(\infty),\ \Omega_V \subset \mathcal{R}_V(\infty),\ \dot{\Omega}_V \subset \dot{\mathcal{R}}_V(\infty). \tag{49}$$

Proof. For each $y \in \Omega_V$ with $\|y_V\| = \rho \geq \rho_0$, $y_j = \rho$ for some $j \in V$ so $y \leq_K \rho e_j + r_H$. By (A1) and conditions (ii) and (iv) of theorem 3.1,

$$\forall i \in H,\ f_i(y) \geq f_i(\rho_0 e_j + r_H) > 0,\ f_j(y) \leq f_j(\rho_0 e_j + r_H) < 0.$$

These ensure that $\varphi_t(y)_i < y_i$ but $\varphi_t(y)_j > y_j = \rho$, so $\varphi_t(y)_H \leq y_H \leq r_H$ and $\|\varphi_t(y)_V\| > \|y_V\| \geq \rho_0$, for small $-t > 0$. Hence, $\varphi_t(y) \in \Omega_V$ for all $t \leq 0$ in its existence interval and Ω_V is backward invariant. As $\Omega_V \subset C \setminus [O, r) \subset \mathcal{R}(\infty)$ by (i) and $\varphi_t(y)_H \leq r_H$ for each $y \in \Omega_V$ and all $t \leq 0$ in its existence interval, we must have $\Omega_V \subset \mathcal{R}_V(\infty)$.

Obviously, $\dot{\Omega}_V \subset \Omega_V$ and $\dot{\Omega}_V$ is an open set of C. By the definition of $\dot{\mathcal{R}}_V(\infty)$, we have $\dot{\Omega}_V \subset \dot{\mathcal{R}}_V(\infty)$.

That $\Omega_H \subset \mathcal{R}_H(\infty)$ and $\dot{\Omega}_H \subset \dot{\mathcal{R}}_H(\infty)$ follow from the above proof by a simple swapping of H with V. □

Lemma 6.4. *Let $x \in (O, r)$. Then the following statements are true and $\|y\| \leq \rho_0$ for all y in each of the sets.*

(a) *The set* $(-\infty, x]_K \cap S_H$ *is nonempty and compact for* $x \in \dot{\mathcal{R}}_H(\infty)$.

(b) *The set* $[x, +\infty)_K \cap S_H$ *is nonempty and compact for* $x \in C \setminus \overline{\mathcal{R}_H(\infty)}$.

(c) *The set* $[x, +\infty)_K \cap S_V$ *is nonempty and compact for* $x \in \dot{\mathcal{R}}_V(\infty)$.

(d) *The set* $(-\infty, x]_K \cap S_V$ *is nonempty and compact for* $x \in C \setminus \overline{\mathcal{R}_V(\infty)}$.

(e) *The sets in (50) and (51) are nonempty and compact for* $x \in \mathcal{R}(O)$ *and the equalities (50) and (51) hold:*

$$([x, +\infty)_K \cap S_H) = ([x, +\infty)_K \cap \Sigma_H) = ([x, +\infty)_K \cap \Sigma), \tag{50}$$

$$((-\infty, x]_K \cap S_V) = ((-\infty, x]_K \cap \Sigma_V) = ((-\infty, x]_K \cap \Sigma). \tag{51}$$

Proof. (a) For each $y \in (-\infty, x]_K$ with $\|y_V\| > \rho_0$, as $x \ll r$ and $y_H \leq x_H$, we have $y \in \dot{\Omega}_V$. By lemma 6.3, $y \in \dot{\mathcal{R}}_V(\infty)$. Since $\dot{\mathcal{R}}_V(\infty)$ and $\dot{\mathcal{R}}_H(\infty)$ are mutually exclusive open sets, we must have $\dot{\mathcal{R}}_V(\infty) \cap \overline{\mathcal{R}_H(\infty)} = \emptyset$ so $y \notin \overline{\mathcal{R}_H(\infty)}$. Since $x \in \dot{\mathcal{R}}_H(\infty) \cap (O, r)$, by proposition 1 (viii), the line segment from x to y must intersect S_H. Therefore, $(-\infty, x]_K \cap S_H \neq \emptyset$ and

$$((-\infty, x]_K \cap S_H) = (\{y \in C : y \leq_K x, \|y_V\| \leq \rho_0\} \cap S_H). \tag{52}$$

As the set $\{y \in C : y \leq_K x, \|y_V\| \leq \rho_0\}$ is compact in C and S_H is closed, their intersection is compact. Thus, $(-\infty, x]_K \cap S_H$ is compact and $\|y\| \leq \rho_0$ for all y in it.

(b) For each $y \in [x, +\infty)_K$ with $\|y_H\| > \rho_0$, as $y_j \leq x_j < r_j$ for all $j \in V$, we have $y \in \dot{\Omega}_H$. By lemma 6.3, $y \in \dot{\mathcal{R}}_H(\infty)$. By proposition 1 (viii), the line segment from x to y intersects S_H so $[x, +\infty)_K \cap S_H \neq \emptyset$. The compactness of $[x, +\infty)_K \cap S_H$ follows from

$$([x, +\infty)_K \cap S_H) = (\{y \in C : y \geq_K x, \|y_H\| \leq \rho_0\} \cap S_H)$$

and the compactness of $\{y \in C : y \geq_K x, \|y_H\| \leq \rho_0\}$. It is clear from the equality that $\|y\| \leq \rho_0$ for all $y \in [x, +\infty)_K \cap S_H$.

(c) The proof is similar to that of (a).

(d) The proof is similar to that of (b).

(e) As $\mathcal{R}(O)$ and $\dot{\mathcal{R}}_H(\infty)$ are mutually exclusive open sets of C, $\mathcal{R}(O) \cap \overline{\mathcal{R}_H(\infty)} = \emptyset$ so $x \in \mathcal{R}(O) \subset (C \setminus \overline{\mathcal{R}_H(\infty)})$. By (b), $[x, +\infty)_K \cap S_H$ is nonempty and compact. Since Σ separates C into mutually exclusive open sets $\mathcal{R}(O)$ and $\mathcal{R}(\infty)$, from figure 1 we see that $[x, +\infty)_K \cap \mathcal{R}(\infty) \neq \emptyset$ so $[x, +\infty)_K \cap \Sigma$ is not empty. It is compact since Σ is compact and $[x, +\infty)_K$ is closed. For each $y \in [x, +\infty)_K \cap \Sigma$, $x \leq_K y$ implies that $\varphi_t(x) \leq_K \varphi_t(y)$, so $\varphi_t(y)_V \leq \varphi_t(x)$, for all $t < 0$. It follows from this and $x \in \mathcal{R}(O)$ that $y \in \Sigma_H$. Therefore, $([x, +\infty)_K \cap \Sigma) \subset ([x, +\infty)_K \cap \Sigma_H)$. But since $\Sigma_H \subset \Sigma$, we have

$$([x, +\infty)_K \cap \Sigma) = ([x, +\infty)_K \cap \Sigma_H) \tag{53}$$

so $[x, +\infty)_K \cap \Sigma_H$ is also nonempty and compact. Next, we prove that

$$([x, +\infty)_K \cap \Sigma) = ([x, +\infty)_K \cap S_H). \tag{54}$$

For each $y \in [x, +\infty)_K \cap \mathcal{R}(\infty)$, we have $\varphi_t(x) \leq_K \varphi_t(y)$ and $\varphi_t(y)_V \leq \varphi_t(x)$ for all $t \in (t_y, 0]$. As $\varphi_t(x)$ is bounded for all $t \in \mathbb{R}$, $\varphi_t(y)_V$ is bounded so $y \in \mathcal{R}_H(\infty)$. By $\mathcal{R}_H(\infty) \subset \mathcal{R}(\infty)$, we must have

$$([x, +\infty)_K \cap \mathcal{R}(\infty)) = ([x, +\infty)_K \cap \mathcal{R}_H(\infty)).$$

Now fix a point $x' \in (-\infty, x)_K \cap \mathcal{R}(O)$ so that $[x', +\infty)_K \cap \mathcal{R}(\infty) = [x', +\infty)_K \cap \mathcal{R}_H(\infty)$. As $[x, +\infty)_K \subset (x', +\infty)_K \subset [x', +\infty)_K$, we have

$$(x', +\infty)_K \cap \mathcal{R}(\infty) = (x', +\infty)_K \cap \mathcal{R}_H(\infty).$$

Since $(x', +\infty)_K \cap \mathcal{R}(\infty)$ is open, for each $y \in (x', +\infty)_K \cap \mathcal{R}_H(\infty)$, there is a $\delta > 0$ such that $(\mathcal{B}(y, \delta) \cap C) \subset ((x', +\infty)_K \cap \mathcal{R}_H(\infty))$ so $y \in \dot{\mathcal{R}}_H(\infty)$ by (22). Thus, $(x', +\infty)_K \cap \mathcal{R}(\infty) = (x', +\infty)_K \cap \dot{\mathcal{R}}_H(\infty)$. Then

$$([x, +\infty)_K \cap \mathcal{R}(\infty)) \subset ((x', +\infty)_K \cap \mathcal{R}(\infty)) = ((x', +\infty)_K \cap \dot{\mathcal{R}}_H(\infty)).$$

From this, $\dot{\mathcal{R}}_H(\infty) \subset \mathcal{R}_H(\infty)$ and the equality $([x, +\infty)_K \cap \mathcal{R}(\infty)) = ([x, +\infty)_K \cap \mathcal{R}_H(\infty))$ it follows that

$$([x, +\infty)_K \cap \mathcal{R}(\infty)) = ([x, +\infty)_K \cap \dot{\mathcal{R}}_H(\infty)).$$

For each $y \in ([x, +\infty)_K \cap S_H)$, if $y \in \mathcal{R}(\infty)$ then the above equality implies that $y \in \dot{\mathcal{R}}_H(\infty)$. This contradiction to $y \in S_H$ shows that

$$([x, +\infty)_K \cap S_H) = ([x, +\infty)_K \cap (S_H \setminus \mathcal{R}(\infty))).$$

Then, for each $y \in ([x, +\infty)_K \cap S_H)$ and every $t \leq 0$, $\varphi_t(x) \leq_K \varphi_t(y)$ and $\varphi_t(y)_V \leq_K \varphi_t(x)$. By $x \in \mathcal{R}(O)$ we obtain $\alpha(y) \subset C_H$ so $y \in \Sigma_H$. Therefore, from (53),

$$([x, +\infty)_K \cap S_H) \subset ([x, +\infty)_K \cap \Sigma_H) = ([x, +\infty)_K \cap \Sigma).$$

Suppose there is a $y \in ([x, +\infty)_K \cap \Sigma)$ such that $y \notin S_H$. Then y is in $[x, +\infty)_K \setminus \overline{\mathcal{R}_H(\infty)}$, an open set of $[x, +\infty)_K$. Let $I \subset I_N$ be the support of y so $y \in \dot{C}_I$. Then we can choose a $z \in \dot{C}_I \cap ([x, +\infty)_K \setminus \overline{\mathcal{R}_H(\infty)})$ such that $y \leq_K z$ in C and $y \ll_K z$ in C_I. Choose an $i \in I \cap H$ such that the half line defined by $w(\lambda) = z + \lambda e_i$ for all $\lambda \geq 0$ is in $[x, +\infty)_K$. As $w(0) \notin \overline{\mathcal{R}_H(\infty)}$ and $w(\lambda) \in \dot{\mathcal{R}}_H(\infty)$ for large λ, by proposition 1 (viii) there is a $\lambda_0 > 0$ such that $w(\lambda_0) \in ([x, +\infty)_K \cap S_H) \subset ([x, +\infty)_K \cap \Sigma)$. Then both $y, w(\lambda_0) \in \dot{C}_I \cap \Sigma$ and $y \ll_K w(\lambda_0)$ in C_I. By lemma 6.2, $\alpha(y) \subset C_V$ so $y \in \Sigma_V$, a contradiction to $y \in \Sigma_H$ by (53). This proves (54). Then (50) follows from (53) and (54). From these equalities we have $\|y\| \leq \|r\| \leq \rho_0$ for all y in the sets (50).

The proof for compactness of the sets in (51) and the equalities (51) is similar to the above. □

Lemma 6.5. *Assume that the conditions of theorem 3.1 are met. Let $I \subset I_N$ be such that $I \cap H \neq \emptyset$, $I \cap V \neq \emptyset$ and $N \geq |I| \geq 3$. Then the following conclusions hold.*

(a) *For each $x \in \dot{\mathcal{R}}_H(\infty) \cap \dot{C}_I$, if $0 < \varphi_t(x)_i < r_i$ for all $i \in I$ and $t \geq 0$ then there is an $x' \in S_H \cap C_I$ such that $\varphi_t(x') \leq_K \varphi_t(x)$ for all $t \geq 0$.*

(b) *For each* $x \in \dot{\mathcal{R}}_V(\infty) \cap \dot{C}_I$, *if* $0 < \varphi_t(x)_i < r_i$ *for all* $i \in I$ *and* $t \geq 0$ *then there is an* $x'' \in S_V \cap C_I$ *such that* $\varphi_t(x) \leq_K \varphi_t(x'')$ *for all* $t \geq 0$.

Proof. (a) We first prove the statement for the case of $I = I_N$ with $C_I = C$. We need only prove the conclusion for each $x \in \dot{\mathcal{R}}_H(\infty) \cap (O, r)$ with $\varphi_t(x) \in (O, r)$ for all $t \geq 0$. By lemma 6.4 (a), $(-\infty, x]_K \cap S_H$ is a compact set. Then, by the invariance of $\dot{\mathcal{R}}_H(\infty)$ from proposition 1 (i), $\varphi_t(x) \in \dot{\mathcal{R}}_H(\infty) \cap (O, r)$. By lemma 6.4 (a) again, $(-\infty, \varphi_t(x)]_K \cap S_H$ is a nonempty compact set for all $t \geq 0$.

Next, we show that

$$\forall t \geq s \geq 0,\ \varphi_{-t}((-\infty, \varphi_t(x)]_K \cap S_H) \subset \varphi_{-s}((-\infty, \varphi_s(x)]_K \cap S_H). \tag{55}$$

For each $y \in (-\infty, \varphi_t(x)]_K \cap S_H$, $y \leq_K \varphi_t(x)$ so $\varphi_u(y) \leq_K \varphi_{t+u}(x)$ for $u \leq 0$ in the existence interval of $\varphi_u(y)$. We first claim that $u = -t$ is in this interval. For if not, there is a $u_0 \in (-t, 0)$ such that $\varphi_u(y)_H \leq \varphi_{t+u}(x)_H \leq r_H$ for $u \in (u_0, 0]$ but $\|\varphi_u(y)_V\| \to \infty$ as $u \to u_0+$. On the other hand, however, S_H is invariant so $\varphi_u(y) \in (-\infty, \varphi_{t+u}(x)]_K \cap S_H$ for $u \in (u_0, 0]$. By lemma 6.4, $\|\varphi_u(y)\| \leq \rho_0$. This contradiction shows the existence of $\varphi_{-t}(y)$ and $\varphi_{-t}(y) \in (-\infty, x]_K \cap S_H$. Therefore,

$$\varphi_{-t}((-\infty, \varphi_t(x)]_K \cap S_H) \subset (-\infty, x]_K \cap S_H.$$

Now replacing x by $\varphi_s(x)$ for any $s \geq 0$ in the above, we obtain

$$\varphi_{-t}((-\infty, \varphi_{t+s}(x)]_K \cap S_H) \subset (-\infty, \varphi_s(x)]_K \cap S_H.$$

Application of φ_{-s} to both sides of this inclusion leads us to

$$\varphi_{-(t+s)}((-\infty, \varphi_{t+s}(x)]_K \cap S_H) \subset \varphi_{-s}((-\infty, \varphi_s(x)]_K \cap S_H).$$

Then (55) follows.

Now that $\varphi_{-t}((-\infty, \varphi_t(x)]_K \cap S_H)$ is compact, from (55) we have

$$\bigcap_{t \geq 0} \varphi_{-t}((-\infty, \varphi_t(x)]_K \cap S_H) \neq \emptyset.$$

Take any $x' \in \cap_{t \geq 0} \varphi_{-t}((-\infty, \varphi_t(x)]_K \cap S_H)$. Then, $x' \in (-\infty, x]_K \cap S_H$ and for all $t \geq 0$, $x' \in \varphi_{-t}((-\infty, \varphi_t(x)]_K \cap S_H)$ implies that $\varphi_t(x') \in (-\infty, \varphi_t(x)]_K \cap S_H$ so $\varphi_t(x') \leq_K \varphi_t(x)$ and $\varphi_t(x') \in S_H$.

Next, suppose I is a proper subset of I_N and consider the subsystem of (1) on C_I. Then, for $x \in (\dot{\mathcal{R}}^I_H(\infty) \cap \dot{C}_I)$, applying the above proved result to the subsystem on C_I, we obtain $x' \in S^I_H \subset C_I$ such that $\varphi_t(x') \leq_K \varphi_t(x)$ for all $t \geq 0$. By remark 2.2, $S^I_H \subset (S_H \cap C_I)$ and $(\dot{\mathcal{R}}_H(\infty) \cap C_I) \subset \dot{\mathcal{R}}^I_H(\infty)$. The proof of part (a) is complete.

(b) The proof is similar to the above with appropriate changes of $(-\infty, x]_K$ to $[x, +\infty)_K$ and swapping H with V. □

Lemma 6.6. *Assume that the conditions of theorem 3.1 are met. Let* $I \subset I_N$ *be such that* $I \cap H \neq \emptyset$, $I \cap V \neq \emptyset$ *and* $N \geq |I| \geq 3$. *Then the following conclusions hold.*

(a) *For each* $x \in \dot{C}_I \setminus \overline{\mathcal{R}_H(\infty)}$, *if* $0 < \varphi_t(x)_i < r_i$ *for all* $i \in I$ *and* $t \geq 0$ *then there is an* $x' \in S_H \cap C_I$ *such that* $\varphi_t(x') \geq_K \varphi_t(x)$ *for all* $t \geq 0$.

(b) *For each* $x \in \dot{C}_I \setminus \overline{\mathcal{R}_V(\infty)}$, *if* $0 < \varphi_t(x)_i < r_i$ *for all* $i \in I$ *and* $t \geq 0$ *then there is an* $x'' \in S_V \cap C_I$ *such that* $\varphi_t(x'') \leq_K \varphi_t(x)$ *for all* $t \geq 0$.

Proof. (a) We first prove the statement for the case of $I = I_N$ with $C_I = C$. We need only prove the conclusion for $x \in (O, r) \setminus \overline{\mathcal{R}_H(\infty)}$ with $\varphi_t(x) \in (O, r)$ for all $t \geq 0$. As $\overline{\mathcal{R}_H(\infty)}$ is invariant, $\varphi_t(x) \in (O, r) \setminus \overline{\mathcal{R}_H(\infty)}$ for all $t \geq 0$. By lemma 6.4 (b), $[\varphi_t(x), +\infty)_K \cap S_H$ is nonempty and compact.

We next show that

$$\forall t \geq s \geq 0,\ \varphi_{-t}([\varphi_t(x), +\infty)_K \cap S_H) \subset \varphi_{-s}(([\varphi_s(x), +\infty)_K \cap S_H). \tag{56}$$

For each $y \in [\varphi_t(x), +\infty)_K \cap S_H$, $y \geq_K \varphi_t(x)$ so $\varphi_u(y) \geq_K \varphi_{t+u}(x)$ for $u \leq 0$ in the existence interval of $\varphi_u(y)$. By the same reasoning as that used in the proof of (55) we know the existence of $\varphi_{-t}(y)$ and $\varphi_{-t}(y) \in [x, +\infty)_K \cap S_H$. Therefore,

$$\varphi_{-t}([\varphi_t(x), +\infty)_K \cap S_H) \subset [x, +\infty)_K \cap S_H.$$

Now replacing x by $\varphi_s(x)$ for any $s \geq 0$ in the above, we obtain

$$\varphi_{-t}([\varphi_{t+s}(x), +\infty)_K \cap S_H) \subset [\varphi_s(x), +\infty)_K \cap S_H.$$

Applying φ_{-s} to both sides we further obtain

$$\varphi_{-(t+s)}([\varphi_{t+s}(x), +\infty)_K \cap S_H) \subset \varphi_{-s}([\varphi_s(x), +\infty)_K \cap S_H).$$

Then (56) follows.

As $\varphi_{-t}([\varphi_t(x), +\infty)_K \cap S_H)$ is compact, from (56) we have

$$\bigcap_{t \geq 0} \varphi_{-t}([\varphi_t(x), +\infty)_K \cap S_H) \neq \emptyset.$$

Take any $x' \in \cap_{t \geq 0} \varphi_{-t}([\varphi_t(x), +\infty)_K \cap S_H)$. Then, $x' \in [x, +\infty)_K \cap S_H$ and for all $t \geq 0$, $x' \in \varphi_{-t}([\varphi_t(x), +\infty)_K \cap S_H)$ implies that $\varphi_t(x') \in [\varphi_t(x), +\infty)_K \cap S_H$ so $\varphi_t(x') \geq_K \varphi_t(x)$ and $\varphi_t(x') \in S_H$.

Next, suppose I is a proper subset of I_N and consider the subsystem of (1) on C_I. Then, for $x \in (\dot{C}_I \setminus \overline{\mathcal{R}_H^I(\infty)})$, applying the above proved result to the subsystem on C_I, we obtain $x' \in S_H^I \subset C_I$ such that $\varphi_t(x') \geq_K \varphi_t(x)$ for all $t \geq 0$. By remark 2.2, $S_H^I \subset (S_H \cap C_I)$ and $\overline{\mathcal{R}_H^I(\infty)} \subset (\overline{\mathcal{R}_H(\infty)} \cap C_I)$. The proof of part (a) is complete.

(b) The proof is similar to the above with appropriate changes of $[x, +\infty)_K$ to $(-\infty, x]_K$ and swapping H with V. □

Armed with the lemmas 6.1–6.6 we are now able to prove theorems 3.1, 3.2 and 3.3.

Proof of theorem 3.1: The case of $N = 2$ or $N > |I| = 2$ is covered by theorem 4.1. So we only consider the case of $I \subset I_N$ with $I \cap H \neq \emptyset$, $I \cap V \neq \emptyset$ and $N \geq |I| \geq 3$. For each $x \in \dot{C}_I$, by condition (i) there is $T \geq 0$ such that $0 < \varphi_t(x)_i < r_i$ for all $t \geq T$ and $i \in I$. By lemmas 6.5 and 6.6, for $\varphi_T(x)$ there are $x' \in S_H \cap C_I$ and $x'' \in S_V \cap C_I$ such that

$$\forall t \geq 0, \varphi_{t+T}(x) \leq_K \varphi_t(x') \quad \textit{or} \quad \forall t \geq 0, \varphi_{t+T}(x) \geq_K \varphi_t(x')$$

and

$$\forall t \geq 0, \varphi_{t+T}(x) \leq_K \varphi_t(x'') \quad \textit{or} \quad \forall t \geq 0, \varphi_{t+T}(x) \geq_K \varphi_t(x'').$$

By (i) we may assume that $\varphi_t(x'), \varphi_t(x'') \in [O, r]$ for all $t \geq 0$. Then (31) follows from lemma 6.1. $\square$

Proof of theorem 3.2: (a) Let $x \in (\mathcal{R}(O) \setminus \{O\})$ with support $I \subset I_N$ so $x \in \dot{C}_I$. If $I \subset H$ then the subsystem of (1) on C_I is competitive. By theorem 1.2, $\Sigma \cap C_I = \Sigma_H \cap C_I$ is its modified carrying simplex and, as $x \in \mathcal{R}(O) \cap \dot{C}_I$, there is a point $x' \in \Sigma_H \cap C_I$ satisfying (33). In parallel to the above, if $I \subset V$ then the subsystem of (1) on C_I is also competitive. By theorem 1.2 again, there is an $x'' \in \Sigma_V \cap C_I$ satisfying (34).

Now suppose $I \cap H \neq \emptyset$ and $I \cap V \neq \emptyset$. If $N \geq |I| = 2$ then the conclusion follows from theorem 4.1. So we assume $N \geq |I| \geq 3$. Since $\varphi_t(x) \in \dot{C}_I \cap \mathcal{R}(O)$ for all $t \in \mathbb{R}$, by lemma 6.4 (e) for the subsystem of (1) on C_I and remark 2.2, the sets $[\varphi_t(x), +\infty)_K \cap C_I \cap \Sigma_H$ and $(-\infty, \varphi_t(x)]_K \cap C_I \cap \Sigma_V$ are compact. Moreover, for each $y \in [\varphi_t(x), +\infty)_K \cap C_I \cap \Sigma_H$, $y \geq_K \varphi_t(x)$ so $\varphi_{-t}(y) \geq_K x$ and $\varphi_{-t}(y) \in [x, +\infty)_K \cap C_I \cap \Sigma_H$. Hence,

$$\varphi_{-t}([\varphi_t(x), +\infty)_K \cap C_I \cap \Sigma_H) \subset ([x, +\infty)_K \cap C_I \cap \Sigma_H).$$

In parallel,

$$\varphi_{-t}((-\infty, \varphi_t(x)]_K \cap C_I \cap \Sigma_V) \subset ((-\infty, x]_K \cap C_I \cap \Sigma_V).$$

Replacing x by $\varphi_s(x)$ for any $s \geq 0$, we obtain

$$\varphi_{-t}([\varphi_{t+s}(x), +\infty)_K \cap C_I \cap \Sigma_H) \subset ([\varphi_s(x), +\infty)_K \cap C_I \cap \Sigma_H)$$

and

$$\varphi_{-t}((-\infty, \varphi_{t+s}(x)]_K \cap C_I \cap \Sigma_V) \subset ((-\infty, \varphi_s(x)]_K \cap C_I \cap \Sigma_V).$$

Then it follows from these two inclusions that

$$\varphi_{-(t+s)}([\varphi_{t+s}(x), +\infty)_K \cap C_I \cap \Sigma_H) \subset \varphi_{-s}([\varphi_s(x), +\infty)_K \cap C_I \cap \Sigma_H)$$

and

$$\varphi_{-(t+s)}((-\infty, \varphi_{t+s}(x)]_K \cap C_I \cap \Sigma_V) \subset \varphi_{-s}((-\infty, \varphi_s(x)]_K \cap C_I \cap \Sigma_V).$$

Therefore,

$$\bigcap_{s \geq 0} \varphi_{-s}([\varphi_s(x), +\infty)_K \cap C_I \cap \Sigma_H) \neq \emptyset$$

and

$$\bigcap_{s \geq 0} \varphi_{-s}((-\infty, \varphi_s(x)]_K \cap C_I \cap \Sigma_V) \neq \emptyset.$$

Now take $x' \in \cap_{s\geq 0}\varphi_{-s}([\varphi_s(x),+\infty)_K \cap C_I \cap \Sigma_H)$ and $x'' \in \cap_{s\geq 0}\varphi_{-s}((-\infty,\varphi_s(x)]_K \cap C_I \cap \Sigma_V)$. Then $\varphi_t(x'') \leq_K \varphi_t(x) \leq_K \varphi_t(x')$ for all $t \geq 0$. By lemma 6.1, we obtain (33) and (34).

(b) As $\mathcal{R}(O)$ and $\mathcal{R}_H(\infty)$ are mutually exclusive and $\mathcal{R}(O)$ is open, from (24) we know that $S_H \cap \mathcal{R}(O) = \emptyset$. Thus, $S_H \setminus \mathcal{R}(\infty) \subset \Sigma$. Similar reasoning gives $S_V \cap \mathcal{R}(O) = \emptyset$ so $S_V \setminus \mathcal{R}(\infty) \subset \Sigma$.

Since $\Sigma_H \cap \Sigma_V = \emptyset$, from part (a) we know that $\omega(x) \subset \overline{\Sigma_H} \cap \overline{\Sigma_V} \subset \Sigma_0$ for all $x \in \mathcal{R}(O) \setminus (C_H \cup C_V)$. Thus $\Sigma_0 \neq \emptyset$. For any nonempty set $I \subset I_N$, suppose there are two distinct points $x, y \in \Sigma \cap C_I$ such that $x_i < y_i$ for all $i \in H \cap I$ but $x_j > y_j$ for all $j \in V \cap I$. By lemma 6.2, $\lim_{t\to-\infty}\varphi_t(x)_H = O$ and $\lim_{t\to-\infty}\varphi_t(y)_V = O$. Thus, $x \in \Sigma_V$ and $y \in \Sigma_H$. This shows that both $(\Sigma \setminus \Sigma_H) \cap C_I$ and $(\Sigma \setminus \Sigma_V) \cap C_I$ are unordered in $\ll_K$.

Next, we show that $S_H \setminus \mathcal{R}(\infty) \subset \Sigma \setminus \Sigma_V$. By $S_H \setminus \mathcal{R}(\infty) \subset \Sigma$ and (19), we need only show that $(S_H \setminus \mathcal{R}(\infty)) \cap \Sigma_V = \emptyset$. For if this is not true, then there are $x \in (S_H \setminus \mathcal{R}(\infty)) \cap \Sigma_V$ and $I \subset I_N$ such that $x \in \dot{C}_I$. Since $x \in \Sigma_V \cap \dot{C}_I$ implies that $\alpha(x) \subset C_V \cap C_I$, for each $y \in \alpha(x)$ we have $O \in [y, +\infty)_K$. This shows that $(\varphi_t(x), +\infty)_K \cap C_I \cap \mathcal{R}(O) \neq \emptyset$ for large enough $-t$. But by the invariance of S_H and $\mathcal{R}(\infty)$, $x \in (S_H \setminus \mathcal{R}(\infty)) \cap \dot{C}_I$ implies that $\varphi_t(x) \in (S_H \setminus \mathcal{R}(\infty)) \cap \dot{C}_I$ for all $t \in \mathbb{R}$. By proposition 1 (v) applied to the subsystem of (1) on C_I and remark 2.2, $((\varphi_t(x), +\infty)_K \cap C_I) \subset \mathcal{R}^I_H(\infty) = (\mathcal{R}_H(\infty) \cap C_I) \subset \mathcal{R}_H(\infty)$, a contradiction. Therefore, $(S_H \setminus \mathcal{R}(\infty)) \cap \Sigma_V = \emptyset$ and $S_H \setminus \mathcal{R}(\infty) \subset \Sigma \setminus \Sigma_V$. Similarly, we also have $(S_V \setminus \mathcal{R}(\infty)) \cap \Sigma_H = \emptyset$ so that $S_V \setminus \mathcal{R}(\infty) \subset \Sigma \setminus \Sigma_H$.

Finally, we show that $\Sigma \setminus \Sigma_V \subset S_H \setminus \mathcal{R}(\infty)$. For each $x \in \Sigma \setminus \Sigma_V$ and $I \subset I_N$ with $x \in \dot{C}_I$, if there is a $y \in \Sigma \cup \mathcal{R}(O)$ satisfying $y \gg_K x$ in C_I, then lemma 6.2 ensures that $\alpha(x) \subset C_V$ and $x \in \Sigma_V$, a contradiction to $x \in \Sigma \setminus \Sigma_V$. Thus, for each y satisfying $y \gg_K x$ in C_I we must have $y \in \mathcal{R}(\infty)$, i.e. $((x, +\infty)_K \cap \dot{C}_I) \subset \mathcal{R}(\infty)$. For each $y \in ((x, +\infty)_K \cap \dot{C}_I)$, by the boundedness of $\varphi_t(x)$ for all $t \in \mathbb{R}$ and $\varphi_t(y) \geq_K \varphi_t(x)$ for $t \leq 0$ in the existence interval of $\varphi_t(y)$, $y \in \mathcal{R}_H(\infty)$ so $x \in \overline{\mathcal{R}_H(\infty)}$. But since $x \notin \mathcal{R}_H(\infty)$, we must have $x \in (\overline{\mathcal{R}_H(\infty)} \setminus \mathcal{R}_H(\infty)) \subset S_H$. Thus, $\Sigma \setminus \Sigma_V \subset S_H \setminus \mathcal{R}(\infty)$. Similarly, $\Sigma \setminus \Sigma_H \subset S_V \setminus \mathcal{R}(\infty)$.

Combining the conclusions obtained in the above two paragraphs, we obtain

$$\Sigma_H \cup \Sigma_0 = \Sigma \setminus \Sigma_V = S_H \setminus \mathcal{R}(\infty),\ \Sigma_V \cup \Sigma_0 = \Sigma \setminus \Sigma_H = S_V \setminus \mathcal{R}(\infty).$$

(c) From (b) we see that $\Sigma_H \subset S_H$, $\Sigma_V \subset S_V$ and $\Sigma_0 \subset S_H \cap S_V$. As $(S_H \cup S_V) \subset \overline{\mathcal{R}(\infty)}$, we have $\Sigma \subset \overline{\mathcal{R}(\infty)}$. This and $\Sigma \cap \mathcal{R}(\infty) = \emptyset$ imply that $\Sigma \subset (\overline{\mathcal{R}(\infty)} \setminus \mathcal{R}(\infty))$. On the other hand, however, $(\overline{\mathcal{R}(\infty)} \setminus \mathcal{R}(\infty)) \subset (C \setminus \mathcal{R}(\infty)) = (\Sigma \cup \mathcal{R}(O))$ but $(\overline{\mathcal{R}(\infty)} \setminus \mathcal{R}(\infty)) \cap \mathcal{R}(O) = \emptyset$. So $(\overline{\mathcal{R}(\infty)} \setminus \mathcal{R}(\infty)) \subset \Sigma$. Therefore, $(\overline{\mathcal{R}(\infty)} \setminus \mathcal{R}(\infty)) = \Sigma$.

Clearly, $(\overline{\mathcal{R}(O)} \setminus \mathcal{R}(O)) \subset \Sigma$. For each $x \in (\overline{\mathcal{R}(O)} \setminus \mathcal{R}(O))$, we have either $x \in \Sigma_H$ or $x \in \Sigma_V$ or $x \in \Sigma_0$. We shall show that, for a fixed $x \in (\overline{\mathcal{R}(O)} \setminus \mathcal{R}(O))$,

$$x \in \Sigma_0 \Longrightarrow x \in (\overline{\Sigma_H} \setminus \Sigma_H), x \in (\overline{\Sigma_V} \setminus \Sigma_V), x \in (\overline{\Sigma_H \cup \Sigma_V} \setminus (\Sigma_H \cup \Sigma_V)), \tag{57}$$

so that $\overline{\mathcal{R}(O)} \setminus \mathcal{R}(O)$ is a subset of $\overline{\Sigma_H} \cup \Sigma_V$, $\Sigma_H \cup \overline{\Sigma_V}$ and $\overline{\Sigma_H \cup \Sigma_V}$.

We now prove (57). Let $I \subset I_N$ such that $x \in \dot{C}_I$. By $x \in \Sigma_0 \cap \dot{C}_I$ and remark 2.2, $x \in \Sigma_0^I \subset S_H^I \cap S_V^I \subset (S_H \cap S_V \cap C_I)$. Applying proposition 1 (v) and (vi) to the subsystem of (1) on C_I, we have $((-\infty, x)_K \cap C_I) \subset \mathcal{R}_V^I(\infty) = (\mathcal{R}_V(\infty) \cap C_I)$ and $((x, +\infty)_K \cap C_I) \subset \mathcal{R}_H^I(\infty) = (\mathcal{R}_H(\infty) \cap C_I)$. Since $x \in \overline{\mathcal{R}(O)} \setminus \mathcal{R}(O)$ and $\mathcal{R}(O)$ is open, there is a sequence $\{x_n\} \subset \dot{C} \cap \mathcal{R}(O)$ converging to x as $n \to \infty$. For each $n > 0$, by lemma 6.4 (e), $(-\infty, x_n]_K \cap S_V$ is a nonempty compact subset of Σ_V, $[x_n, +\infty)_K \cap S_H$ is a nonempty compact subset of Σ_H, and (50) and (51) hold. Suppose $x \notin \overline{\Sigma_H}$. Then there is a $\delta > 0$ such that $\overline{\Sigma_H} \cap \overline{\mathcal{B}(x, \delta)} = \emptyset$. By $\lim_{n\to\infty} x_n = x$, we have $\overline{\mathcal{B}(x_n, \frac{1}{2}\delta)} \subset \overline{\mathcal{B}(x, \delta)}$, so $\overline{\mathcal{B}(x_n, \frac{1}{2}\delta)} \cap [x_n, +\infty)_K \cap \Sigma = \emptyset$ by (50), for large enough n. By proposition 1 (viii), $S_H \cap [x_n, +\infty)_K$ separates $[x_n, +\infty)_K$ into two sets $[x_n, +\infty)_K \cap \dot{\mathcal{R}}_H(\infty)$ and $[x_n, +\infty)_K \setminus \overline{\mathcal{R}_H(\infty)}$. Since (50) shows that S_H, Σ, Σ_H restricted to $[x_n, +\infty)_K$ are identical, $[x_n, +\infty)_K \cap \dot{\mathcal{R}}_H(\infty)$ is on one side of Σ and $[x_n, +\infty)_K \setminus \overline{\mathcal{R}_H(\infty)}$ is on the other side of Σ. As Σ separates C into $\mathcal{R}(O)$ and $\mathcal{R}(\infty)$, one on each side of Σ, by $x_n \in \mathcal{R}(O)$ we see that $([x_n, +\infty)_K \setminus \overline{\mathcal{R}_H(\infty)}) \subset \mathcal{R}(O)$. Since $\overline{\mathcal{B}(x_n, \frac{1}{2}\delta)} \cap [x_n, +\infty)_K$ is a connected subset of $[x_n, +\infty)_K$ with no intersection point with Σ, we must have

$$(\overline{\mathcal{B}(x_n, \delta/2)} \cap [x_n, +\infty)_K) \subset ([x_n, +\infty)_K \setminus \overline{\mathcal{R}_H(\infty)}) \subset \mathcal{R}(O)$$

for large enough n. Letting $n \to \infty$, we obtain $(\overline{\mathcal{B}(x, \frac{1}{2}\delta)} \cap [x, +\infty)_K) \subset \overline{\mathcal{R}(O)}$. Thus, $\overline{\mathcal{B}(x, \frac{1}{2}\delta)} \cap [x, +\infty)_K \cap \mathcal{R}_H(\infty) = \emptyset$. But this contradicts $((x, +\infty)_K \cap C_I) \subset \mathcal{R}_H(\infty)$ as $\overline{\mathcal{B}(x, \frac{1}{2}\delta)} \cap (x, +\infty)_K \cap C_I$ is a nonempty subset of $\overline{\mathcal{B}(x, \frac{1}{2}\delta)} \cap [x, +\infty)_K$. The contradiction shows that we must have $x \in \overline{\Sigma_H}$. In a similar manner, we also have $x \in \overline{\Sigma_V}$ and, hence, $x \in \overline{\Sigma_H \cup \Sigma_V}$. Then (57) follows from $x \notin (\Sigma_H \cup \Sigma_V)$.

If we can show that

$$(\Sigma_H \cup \Sigma_V) \subset (\overline{\mathcal{R}(O)} \setminus \mathcal{R}(O)), \tag{58}$$

as $\overline{\mathcal{R}(O)} \setminus \mathcal{R}(O)$ is closed, we can conclude from (58) that $\overline{\Sigma_H} \cup \Sigma_V$, $\Sigma_H \cup \overline{\Sigma_V}$ and $\overline{\Sigma_H \cup \Sigma_V}$ are all subsets of $\overline{\mathcal{R}(O)} \setminus \mathcal{R}(O)$. From this and the line below (57) we obtain

$$(\overline{\mathcal{R}(O)} \setminus \mathcal{R}(O)) = (\overline{\Sigma_H} \cup \Sigma_V) = (\Sigma_H \cup \overline{\Sigma_V}) = (\overline{\Sigma_H \cup \Sigma_V}).$$

That $\overline{\mathcal{R}(\infty)} \cap \overline{\mathcal{R}(O)} = \overline{\mathcal{R}(O)} \setminus \mathcal{R}(O)$ follows from $\overline{\mathcal{R}(\infty)} \setminus \mathcal{R}(\infty) = \Sigma$, $\overline{\mathcal{R}(O)} \setminus \mathcal{R}(O) = \overline{\Sigma_H \cup \Sigma_V} \subset \Sigma$ and $\overline{\mathcal{R}(\infty)} \cap \overline{\mathcal{R}(O)} = (\overline{\mathcal{R}(\infty)} \setminus \mathcal{R}(\infty)) \cap (\overline{\mathcal{R}(O)} \setminus \mathcal{R}(O))$.

For the proof of (58) we take $x \in \Sigma_H \subset S_H$ with $I \subset I_N$ such that $x \in \dot{C}_I$. For any $y \in (-\infty, x)_K \cap C_I$, we have $\varphi_t(y) \leq_K \varphi_t(x)$ and $0 \leq \varphi_t(y)_H \leq \varphi_t(x)$ for $t \in (t_y, 0)$. The boundedness of $\varphi_t(x)$ follows from that of Σ_H and $\varphi_t(x) \in \Sigma_H$ for all $t \in \mathbb{R}$. Thus, if $y \in \mathcal{R}(\infty)$ then $y \in \mathcal{R}_V(\infty)$. Hence, if $((-\infty, x)_K \cap C_I) \subset \mathcal{R}(\infty)$ then $((-\infty, x)_K \cap C_I) \subset \mathcal{R}_V(\infty)$ so $x \in \overline{\mathcal{R}_V(\infty)}$. By $x \notin \mathcal{R}_V(\infty)$ we have $x \in S_V$. Thus, $x \in (\Sigma \cap S_H \cap S_V) = \Sigma_0$, a contradiction to $x \in \Sigma_H$. This shows the existence of $y \in (C_I \cap (\Sigma \cup \mathcal{R}(O)))$ such that

$y \ll_K x$ in C_I. We claim that

$$\forall s \in (0,1),\ z_s \equiv y + s(x-y) \in \mathcal{R}(O). \tag{59}$$

Indeed, from lemma 6.2 we know that $(\Sigma_H \cup \Sigma_0) \cap C_I$ is unordered in $\ll_K$ in C_I. So we have either $y \in \Sigma_V \cap C_I$ or $y \in \mathcal{R}(O) \cap C_I$. For each $s \in (0,1)$, $y \ll_K z_s \ll_K x$ in C_I so $\varphi_t(y) \leq_K \varphi_t(z_s) \leq_K \varphi_t(x)$ for all $t \leq 0$. Thus, $\varphi_t(z_s)$ is bounded and $z_s \in (C_I \setminus (\mathcal{R}(\infty) \cup \{O\}))$. If $z_s \notin \mathcal{R}(O)$, by lemma 6.2 and $x \in \Sigma_H$ we have $z_s \in \Sigma_V$. As $y \ll_K z_s$ in C_I and $(\Sigma_V \cup \Sigma_0) \cap C_I$ is unordered in $\ll_K$ in C_I, we must have $y \notin \Sigma_V$ so $y \in \mathcal{R}(O)$. This implies that $\lim_{t\to-\infty} \varphi_t(z_s)_V \leq \lim_{t\to-\infty} \varphi_t(y)_V = O$. As $z_s \in \Sigma_V$ means that $\lim_{t\to-\infty} \varphi_t(z_s)_H = O$, combination of the two equalities gives $z_s \in \mathcal{R}(O)$, a contradiction. Therefore, (59) holds. It then follows from (59) that $\Sigma_H \subset (\overline{\mathcal{R}(O)} \setminus \mathcal{R}(O))$. By the same reasoning as above, we also obtain $\Sigma_V \subset (\overline{\mathcal{R}(O)} \setminus \mathcal{R}(O))$. Then (58) follows. □

Proof of theorem 3.3: (a) The invariance of Σ_H (Σ_V) follows from $x \in \Sigma_H$ (Σ_V) implying $\varphi_t(x) \in \Sigma_H$ (Σ_V) for all $t \in \mathbb{R}$ by (19). The invariance of Σ_0 follows from that of $C, \mathcal{R}(\infty), \mathcal{R}(O), \Sigma_H, \Sigma_V$ and $\Sigma_0 = C \setminus (\mathcal{R}(\infty) \cup \mathcal{R}(O) \cup \Sigma_H \cup \Sigma_V)$. The invariance of $\overline{\Sigma_H}$ ($\overline{\Sigma_V}$) follows from that of Σ_H (Σ_V) and continuous dependence on initial values. The rest is obvious.

(b) The sets $\overline{\Sigma_H}, \overline{\Sigma_V}$ and $\overline{\Sigma_H \cup \Sigma_V}$ are compact since they are closed and bounded. Since S_H and S_V are closed and $\mathcal{R}(\infty)$ is open, by theorem 3.2 (b) $\Sigma_H \cup \Sigma_0$ and $\Sigma_V \cup \Sigma_0$ are closed, and so is $\Sigma_0 = (S_H \cap S_V) \setminus \mathcal{R}(\infty)$. Since $\Sigma_H \cup \Sigma_0, \Sigma_V \cup \Sigma_0, \Sigma_0$ are subsets of Σ, they are bounded and compact.

(c) Since (1) is dissipative, its fundamental attractor $(C \setminus \mathcal{R}(\infty)) = (\Sigma \cup \mathcal{R}(O))$ attracts each compact set of initial values in C uniformly. By assumption (A2), for $\delta > 0$ small enough the compact set $(\Sigma \cup \mathcal{R}(O)) \setminus \mathcal{B}(O,\delta)$ is positively invariant. So, by the invariance of Σ, $\Sigma \subset \cap_{t\geq 0}\varphi_t((\Sigma \cup \mathcal{R}(O)) \setminus \mathcal{B}(O,\delta))$. Now, if there is $x \in \cap_{t\geq 0}\varphi_t((\Sigma \cup \mathcal{R}(O)) \setminus \mathcal{B}(O,\delta))$ but $x \notin \Sigma$, then its whole backward semiorbit is in $\mathcal{R}(O) \setminus \mathcal{B}(O,\delta)$, which is impossible. This shows that

$$\Sigma = \cap_{t\geq 0}\varphi_t((\Sigma \cup \mathcal{R}(O)) \setminus \mathcal{B}(O,\delta)).$$

Thus, Σ attracts each compact set contained in $(\Sigma \cup \mathcal{R}(O)) \setminus \{O\}$. Then, for any compact set $U \subset (C \setminus \{O\})$, for any $\varepsilon > 0$, there is a $T \geq 0$ such that $\varphi_t(U) \subset \mathcal{B}(\Sigma \cup \mathcal{R}(O), \varepsilon)$ for all $t \geq T$. If $(U \cap \mathcal{R}(O)) = \emptyset$ then $(\varphi_t(U) \cap \mathcal{R}(O)) = \emptyset$ for all $t \geq 0$ so $\varphi_t(U) \subset \mathcal{B}(\Sigma, \varepsilon)$ for all $t \geq T$. If $(U \cap \mathcal{R}(O)) \neq \emptyset$, since Σ attracts the compact set $U \cap (\Sigma \cup \mathcal{R}(O))$ uniformly, there is $T_1 \geq T$ such that $\varphi_t(U) \subset \mathcal{B}(\Sigma, \varepsilon)$ for all $t \geq T_1$. Hence, Σ attracts U uniformly.

(d) From theorem 3.2 (b) and (c) we see that $\overline{\Sigma_H}$ as a subset of S_H ($\overline{\Sigma_V}$ as a subset of S_V) is part of the common boundary of $\mathcal{R}(\infty)$ and $\mathcal{R}(O)$. From proposition 1 (viii) and (ix) we know that $S_H \cap \dot{C}$ and $S_V \cap \dot{C}$ are $(N-1)$-dimensional unordered surfaces in $\ll_K$. For each $x \in \Sigma_H \cap \dot{C}$, as $x \in S_H$ but $x \notin S_V$, there is a small $\delta > 0$ such that $\mathcal{B}(x,\delta) \subset \dot{C}$, $\mathcal{B}(x,\delta) \cap S_V = \emptyset$ and $\mathcal{B}(x,\delta) = B_1 \cup B_2 \cup B_3$ with $B_1 \subset \mathcal{R}_H(\infty)$, $B_2 = S_H \cap \mathcal{B}(x,\delta) = \Sigma_H \cap \mathcal{B}(x,\delta)$ and $B_3 \subset \mathcal{R}(O)$. Thus, for any unit vector $u \gg_K O$,

B_2 is homeomorphic to an open disc in $E = \{y \in \mathbb{R}^N : \langle y, u\rangle = 0\}$ and $\Sigma_H \cap \dot{C}$ is homeomorphic to an open set of E that is the union of open discs. The invariance of $\Sigma_H \cap \dot{C}$ is obvious. So $\Sigma_H \cap \dot{C}$ is an $(N-1)$-dimensional invariant manifold. By the same reasoning as above, $\Sigma_V \cap \dot{C}$ is an $(N-1)$-dimensional invariant manifold. That each of $\overline{\Sigma_H}$ and $\overline{\Sigma_V}$ is homeomorphic to a closed set in E follows from proposition 1 and theorem 3.2 (b). As $\overline{\Sigma_H \cup \Sigma_V} = \overline{\mathcal{R}(O)} \setminus \mathcal{R}(O)$ from theorem 3.2 (c), it is on the boundary of $\mathcal{R}(O)$ and separates C into two mutually exclusive simply connected open sets $\mathcal{R}(O)$ and $C \setminus \overline{\mathcal{R}(O)}$.

(e) For each point $x \in C \setminus \{O\}$, $\omega(x) \subset \Sigma_H \cap C_H$ if $x \in C_H$, $\omega(x) \subset \Sigma_V \cap C_V$ if $x \in C_V$, and by theorems 3.1 and 3.2, $\omega(x) \subset \Sigma_0$ if $x \notin C_H \cup C_V$; $\alpha(x)$ does not exist if $x \in \mathcal{R}(\infty)$, $\alpha(x) = \{O\}$ if $x \in \mathcal{R}(O)$, $\alpha(x) \subset \Sigma_H \cap C_H$ if $x \in \Sigma_H$, $\alpha(x) \subset \Sigma_V \cap C_V$ if $x \in \Sigma_V$, and $\alpha(x) \subset \Sigma_0$ if $x \in \Sigma_0$. Then the conclusion follows.

(f) The proof for (e) is also valid here.

(g) Let $A = \cup_{t\geq 0}\varphi_t([O, r])$. Since $\Sigma \cup \mathcal{R}(O)$ is the fundamental global attractor of (1) on C and it is a subset of $[O, r)$, for the compact set $[O, r]$ there is a $t_0 > 0$ such that $\varphi_t([O, r]) \subset [O, r]$ for all $t \geq t_0$ so $A = \cup_{0\leq t\leq t_0}\varphi_t([O, r])$. It can be checked that A is a forward invariant compact set. By $f(O) \gg O$ there is a small $\delta_0 > 0$ such that $\mathcal{B}(O, \delta_0) \subset \varphi_t(\mathcal{B}(O, \delta_0))$ for all $t \geq 0$. Then $A \setminus \mathcal{B}(O, \delta_0)$ is a forward invariant compact set. By (c), Σ is the global attractor of (1) on $C \setminus \{O\}$, so we must have $\cap_{t\geq 0}\varphi_t(A \setminus \mathcal{B}(O, \delta_0)) = \Sigma$.

Now consider the compact set $B_1 = (A \setminus \mathcal{B}(O, \delta_0)) \cap C_H$. For each $x \in B_1$ and every $j \in V$, we have $f_j(x) \geq f_j(O) > 0$. By continuity of f and the compactness of B_1, for each sufficiently small $\varepsilon > 0$ there is a $\delta_1 > 0$ such that

$$\forall x \in \mathcal{B}(B_1, \delta_1) \cap C,\ \forall j \in V,\ f_j(x) \geq \varepsilon > 0.$$

Hence,

$$\forall x \in (\mathcal{B}(B_1, \delta_1) \cap C) \setminus C_H,\ \forall j \in V,\ \forall t \geq 0,\ \varphi_t(x)_j \geq x_j e^{\varepsilon t}$$

provided $\varphi_t(x)$ is still in $\mathcal{B}(B_1, \delta_1)$. This shows that B_1 repels any compact set of initial points in $(A\setminus\mathcal{B}(O, \delta_0))\setminus B_1$ away from B_1 uniformly and $(A\setminus\mathcal{B}(O, \delta_0))\setminus\mathcal{B}(B_1, \delta_1)$ is a forward invariant compact set. In parallel to the above, for the compact set $B_2 = (A\setminus\mathcal{B}(O, \delta_0))\cap C_V$ and small enough $\delta_1 > 0$, B_2 repels any compact set of initial points in $(A \setminus \mathcal{B}(O, \delta_0)) \setminus B_2$ away from B_2 uniformly and $(A \setminus \mathcal{B}(O, \delta_0)) \setminus (\mathcal{B}(B_1, \delta_1) \cup \mathcal{B}(B_2, \delta_1))$ is a forward invariant compact set. We claim that

$$B_3 \equiv \cap_{t\geq 0}\varphi_t((A \setminus \mathcal{B}(O, \delta_0)) \setminus (\mathcal{B}(B_1, \delta_1) \cup \mathcal{B}(B_2, \delta_1))) = \Sigma_0.$$

In fact, since $\Sigma_0 \subset (A \setminus \mathcal{B}(O, \delta_0)) \setminus (\mathcal{B}(B_1, \delta_1) \cup \mathcal{B}(B_2, \delta_1))$ for sufficiently small δ_1, the invariance of Σ_0 implies that $\Sigma_0 \subset B_3$. If there is a point $x \in B_3 \setminus \Sigma_0$ then we must have $x \in \Sigma \setminus \Sigma_0$ so that $\alpha(x) \subset (B_1 \cup B_2)$, a contradiction to the invariance of B_3. This proves the claim $B_3 = \Sigma_0$.

Now for any compact set $D \subset (C \setminus (C_H \cup C_V))$, since Σ is the global attractor of (1) on $C \setminus \{O\}$ and $A \setminus \mathcal{B}(O, \delta_0)$ contains a small neighbourhood of Σ restricted to C, there is a

$t_1 > 0$ such that $\varphi_t(D) \subset (A \setminus \mathcal{B}(O, \delta_0))$ for all $t \geq t_1$. As $D \subset (C \setminus (C_H \cup C_V))$ implies $\varphi_{t_1}(D) \subset (C \setminus (C_H \cup C_V))$, $\varphi_{t_1}(D)$ is a compact set in $A \setminus (\mathcal{B}(O, \delta_0) \cup B_1 \cup B_2)$. Thus, for $\delta_1 > 0$ small enough, $\varphi_{t_1}(D) \subset (A \setminus \mathcal{B}(O, \delta_0)) \setminus (\mathcal{B}(B_1, \delta_1) \cup \mathcal{B}(B_2, \delta_1))$. By $B_3 = \Sigma_0$, Σ_0 attracts D uniformly.

(h) The fundamental attractor of the subsystem of (1) on C_H is $(\Sigma \cup \mathcal{R}(O)) \cap C_H$. Since O repels, by the same reasoning as that used in (c) above, $(\Sigma \cap C_H) = (\Sigma_H \cap C_H)$ attracts each compact set in $C_H \setminus \{O\}$ uniformly.

For any compact set $S \subset (\overline{\Sigma_H} \setminus C_H)$, from the proof of (g) above we see that B_1 repels S away from B_1 to Σ_0. But since $\overline{\Sigma_H}$ is invariant, $(\Sigma_H \cap C_H) = (\Sigma_H \cap B_1)$, and $(\overline{\Sigma_H} \cap \Sigma_0) = (\overline{\Sigma_H} \setminus \Sigma_H)$, $\Sigma_H \cap C_H$ repels S away from $\Sigma_H \cap C_H$ to $\overline{\Sigma_H} \setminus \Sigma_H$. Thus, restricting (1) to $\overline{\Sigma_H}$, $\Sigma_H \cap C_H$ is the global repeller with its dual attractor $\overline{\Sigma_H} \setminus \Sigma_H$.

(i) Similar to (h) above. □

7. Conclusion

For autonomous type-K competitive system (1) with dissipation and the origin O as a repeller, we have found two closed invariant sets S_H and S_V, the former is the boundary of the set of unbounded trajectories $\varphi_t(x)$ with bounded $\varphi_t(x)_V$ and the latter is the boundary of the set of unbounded trajectories $\varphi_t(x)$ with bounded $\varphi_t(x)_H$. We have shown that $S_H \cap \dot{C}_I$ and $S_V \cap \dot{C}_I$ are $(N-1)$-dimensional surfaces homeomorphic to $\mathbb{R}^{N-1}$ and that each of S_H and S_V is unordered in $\ll_K$ and homeomorphic to a closed set in an $(N-1)$-dimensional plane by projection.

We then have decomposed the compact invariant set $\Sigma = C \setminus (\mathcal{R}(\infty) \cup \mathcal{R}(O))$ into $\Sigma = \Sigma_H \cup \Sigma_0 \cup \Sigma_V$, where Σ_0 is compact invariant, each trajectory in $\Sigma_H \setminus C_H$ has its α-limit set in $\Sigma_H \cap C_H$ and ω-limit set in $\overline{\Sigma_H} \setminus \Sigma_H \subset \Sigma_0$, and each trajectory in $\Sigma_V \setminus C_V$ has its α-limit set in $\Sigma_V \cap C_V$ and ω-limit set in $\overline{\Sigma_V} \setminus \Sigma_V \subset \Sigma_0$.

Under certain conditions weaker than those appeared in the literature, we have established the asymptotic behaviour: each trajectory in $C \setminus (C_H \cup C_V)$ is eventually asymptotic to one in $S_H \cap [O, r]$ and one in $S_V \cap [O, r]$ for a given $r \gg 0$.

Under the same conditions we have proved that each trajectory in $\mathcal{R}(O) \setminus C_H$ is asymptotic to one in Σ_V, each trajectory in $\mathcal{R}(O) \setminus C_V$ is asymptotic to one in Σ_H, and hence each trajectory in $\mathcal{R}(O) \setminus (C_H \cup C_V)$ is asymptotic to one in Σ_V and one in Σ_H.

We then have established the relationships between S_H, S_V and Σ_H, Σ_V and Σ_0: $\Sigma_H \subset S_H$, $\Sigma_V \subset S_V$ and $\Sigma_0 \subset (S_H \cap S_V)$.

We have further established the relationships between $\overline{\mathcal{R}(\infty)}$, $\overline{\mathcal{R}(O)}$ and Σ, $\overline{\Sigma_H}$, $\overline{\Sigma_V}$ and $\overline{\Sigma_H \cup \Sigma_V}$: $\Sigma = \overline{\mathcal{R}(\infty)} \setminus \mathcal{R}(\infty)$ but $\overline{\Sigma_H \cup \Sigma_V} = \overline{\mathcal{R}(O)} \setminus \mathcal{R}(O)$, which is identical to $\overline{\Sigma_H} \cup \Sigma_V$, $\Sigma_H \cup \overline{\Sigma_V}$ and $\Sigma_H \cup (\overline{\Sigma_H} \cap \overline{\Sigma_V}) \cup \Sigma_V$.

We then have concluded that the global attractor Σ for (1) in $C \setminus \{O\}$ consists of $\overline{\Sigma_H \cup \Sigma_V}$, the closure of $(N-1)$-dimensional invariant manifolds $\Sigma_H \cap \dot{C}_I$ and $\Sigma_V \cap \dot{C}_I$, which has a closed invariant subset $\overline{\Sigma_H} \cap \overline{\Sigma_V}$, and a compact invariant subset Σ_0 of $S_H \cap S_V$.

It is clear that Σ for type-K competitive system (1) has the same geometric and asymptotic features as those of modified carrying simplex for a competitive system: each nontrivial trajectory below Σ is asymptotic to one in Σ and every trajectory above Σ has its ω-limit set in Σ. In particular, when $\Sigma = \overline{\Sigma_H \cup \Sigma_V}$, i.e. $\Sigma_0 = \overline{\Sigma_H} \cap \overline{\Sigma_V}$, Σ is the closure of $(N-1)$-dimensional invariant manifolds $\Sigma_H \cap \dot{C}_I$ and $\Sigma_V \cap \dot{C}_I$ and has more similarities to a modified carrying simplex for a competitive system.

In general, however, $\overline{\Sigma_H} \cap \overline{\Sigma_V}$ is a proper subset of Σ_0 as demonstrated in sections 4–5 by examples 4.1 and 5.1. So far not much is known about geometric and dynamical feature of Σ_0. It is hoped that the results obtained here lay the foundation for further investigation. We end this paper with the following open problems and hope any further investigations will enrich the existing theory.

Open Problems:

1. Explore conditions for $\Sigma_0 = \overline{\Sigma_H} \cap \overline{\Sigma_V}$ to hold.
2. Explore conditions for Σ_0 to be an $(N-1)$-dimensional invariant manifold.
3. Classify geometric configurations of Σ_0 supported by conditions, examples and proofs, and investigate the dynamical behaviour on Σ_0, for three-dimensional type-K competitive systems.
4. It is known that any complicated behaviour can be embedded into the carrying simplex of a competitive system. Investigate the possibility of such embedding of complicated behaviour into Σ_0 for type-K competitive systems.
5. Conduct further investigation of Σ_0 in whichever aspects for general N-dimensional type-K systems.
6. Prove or disprove (27) in remark 2.3.
7. Prove or disprove the conjecture for $N \geq 3$ given in remark 3.5.

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