THE ASYMPTOTIC AND OSCILLATORY BEHAVIOUR OF DIFFERENCE AND DIFFERENTIAL EQUATIONS

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ABSTRACT

This thesis deals with the asymptotic and oscillatory behaviour of the solutions of certain differential and difference equations. It mainly consists of three parts. The first part is to study the asymptotic behaviour of certain differential equations. The second part is to look for oscillatory criteria for certain nonlinear neutral differential equations. And, the third part is to establish new criteria for a class of nonlinear neutral difference equations of any order with continuous variable and another type of higher even order nonlinear neutral difference equations to be oscillatory.

At first, we are concerned with the first order differential system of the form

$$x_i'(t) = b_i(t)x_i(t)\left(1 - \sum_{j=1}^{n} a_{ij}(t)x_j(t)\right), \quad i \in N(1,n),$$

where the functions $a_{ij}(t)$ and $b_i(t)$ are continuous on R and bounded above and below by strictly positive numbers. Sufficient conditions are established for the solutions to be stable.

Secondly, we consider the oscillation of second order nonlinear neutral differential equations of the form

$$(a(t)(x(t) + \delta p(t)x(t-\tau))')' + f(t, x(t-\sigma)) - g(t, x(t-\rho)) = 0,$$

where $\delta = \pm 1$, $t \geq t_0$, $\tau, \sigma, \rho \in [0, \infty)$ are constants, $a(t)$ is a continuously differentiable function and $p(t)$ is a bounded continuous function with $a(t) > 0$ and $p(t) \geq 0$, and $f(t,u)$ and $g(t,v)$ are continuous functions. We obtain some crite-

ria for bounded oscillation, bounded almost oscillation and almost oscillation for these equations.

Thirdly, we consider the mth order nonlinear neutral difference equations of the form

$$\Delta_\tau^m(x(t)-px(t-r))+f(t,x(g(t)))=0,$$

where $p \geq 0, m \geq 2$, τ and r are positive constants, $\Delta_\tau x(t)=x(t+\tau)-x(t)$, $0 < g(t) < t$, $g \in C^1([t_0,\infty),R^+)$ and $g'(t)>0$, and $f \in C([t_0,\infty)\times R,R)$. Oscillatory criteria are obtained for the second, third, fourth, higher even order, and higher odd order equation.

In addition, we consider the even order nonlinear neutral difference equation

$$\Delta^{m-1}(a_n\Delta(x_n+\varphi(n,x_{\tau_n})))+q_nf(x_{g_n})=0,$$

where m is an even positive integer, $n \geq n_0$, $\{\tau_n\}$ and $\{g_n\}$ are sequences of nondecreasing nonnegative integers with $\tau_n \leq n$, $g_n \leq n$, $\tau_n \to \infty$ and $g_n \to \infty$ as $n \to \infty$, $\{a_n\}$ and $\{q_n\}$ are sequences of real numbers with $a_n > 0$, $q_n \geq 0$ and $q_n \not\equiv 0$. By mainly using Riccati's technique, we obtain some oscillation criteria for the above equation.

Contents

Chapter 1

INTRODUCTION

The aim of this chapter is to lay the fundamental basis for the thesis and confine the terms to the objective of the thesis. In this respect, we define some terminologies at the very beginning, which will appear later on throughout the thesis. In section 1.2, we state some notations and basic theorems that will be needed in later sections and chapters. These notations are commonly used and can be found in some monographs. Then a brief survey of the development and current state of stability and oscillation of solutions of differential and difference equations is given in section 1.3. In section 1.4, we present the reasons why this thesis has taken place. Finally, we close this chapter with the outline of the work presented in the thesis.

1.1 TERMINOLOGY

We begin this section with some terms commonly used in the literature which will appear throughout the thesis.

A functional differential equation is a differential equation involving the values of the unknown functions at present, as well as at past or future time. The word "time" here stands for the independent variable. In the thesis, the concept of a functional differential equation is confined to ordinary differential equations although it suits partial ones as well.

Functional differential equations can be classified into four types according to their deviations: retarded, advanced, neutral and mixed.

A neutral equation is one in which derivative of functionals of the past history and the present state are involved but no future states occur in the equation.

The order of a differential equation is the order of the highest derivative of the unknown function.

A difference equation is a specific type of recurrence relation, which is an equation that defines a sequence recursively: each term of the sequence is defined as a function of the preceding terms. On the other hand, difference equations can be thought of as the discrete analogue of the corresponding differential equations.

By analogy with differential equations, difference equations also can be classified into four types: delay, advanced, neutral, and mixed.

The order of a difference equation is the difference between the largest and the smallest values of the integer variable explicitly involved in the difference equation.

1.2 PRELIMINARIES

We begin this section with the definition of notations, which will be used throughout this thesis later. Let $R = (-\infty, +\infty)$, $R_0 = [0, +\infty)$, $R^+ = (0, +\infty)$ and $R^- = (-\infty, 0)$ be the usual sets of real, nonnegative, positive, and negative numbers, respectively. Let R^* denote the extended real line, i.e., $R^* = R \cup \{-\infty, +\infty\}$. Let R^n be an n-dimensional real linear vector space with norm $||\cdot||$, and $C([a,b], R^n)$ be the Banach space of continuous functions from

$[a, b]$ to R^n. Let

$$L^1[t_0, \infty) = \left\{ x(t) \,\middle|\, \int_{t_0}^{\infty} |x(s)| ds < \infty \right\}$$

be the Banach space of functions from $[t_0, \infty)$ to R with topology of uniform convergence. Put $N = \{1, 2, 3, \cdots\}$ and $\bar{N} = \{0, 1, 2, \cdots\}$. For integers $a \geq 0$ and $b > a$, we denote the discrete intervals by $N(a, b) = \{a, a+1, \cdots, b\}$ and $N(b) = \{b, b+1, b+2, \cdots\}$.

Let Δ be the forward difference operator: $\Delta x_n = x_{n+1} - x_n$, $\Delta^m x_n = \Delta(\Delta^{m-1} x_n)$, $\Delta_\tau x(t) = x(t+\tau) - x(t)$, $\Delta_\tau^m x(t) = \Delta_\tau(\Delta_\tau^{m-1} x(t))$. In addition, let x' and x'' denote the first and second order derivatives of x and let $x^{(m)}$ denote the mth order derivative of x. Then

$$x^{(n)} = \frac{dx^{(n-1)}}{dt} = \frac{d}{dt}\left(\frac{d}{dt} \cdots \frac{dx}{dt}\right) \quad \text{if} \quad n \geq 2.$$

In order to give the definition of the solution of differential equations, we at first give the following general nth order functional differential equation

$$F(t, x, x', x'', \cdots, x^{(n)}) = 0, \tag{1.2.1}$$

where $F(t, \cdot)$ is a functional involving the value of the $x^{(k)}$ on an interval.

Definition 1.2.1 By a solution of (1.2.1) we mean a function $x(t)$, $t \in [t_x, \infty) \subset R$ which is n times continuously differentiable and satisfies (1.2.1) on the interval $[t_x, \infty)$. A solution x of equation (1.2.1) is called oscillatory if x is neither eventually positive nor negative, in other words, x has an unbounded set of zeros in $[t_x, \infty)$. If all solutions x of (1.2.1) are oscillatory, then differential equation (1.2.1) is called oscillatory. If every bounded solution of (1.2.1) is oscillatory, then (1.2.1) is called bounded oscillatory. If every solution of (1.2.1) not in the class of $o(1)$ as $t \to \infty$ is oscillatory, then (1.2.1) is called almost oscillatory. If every bounded solution of (1.2.1) which is not in the class of $o(1)$ as $t \to \infty$ is oscillatory, then (1.2.1) is called bounded almost oscillatory.

At the same time, to define the solution of difference equations we give the following corresponding general difference equation

$$F(n, x_n, x_{n+1}, \cdots, x_{n+k}) = 0, \qquad k \in N, \tag{1.2.2}$$

where $F(n, \cdot)$ is a given function of the independent variable n and the dependent variable of $x(n)$ at $n \in N$.

Definition 1.2.2 By a solution of equation (1.2.2) we means a sequence $\{x_n\}$ of points $x_n \in R$ for $n \in N$, which satisfies equation (1.2.2). A sequence $\{x_n\}$ of real numbers is said to be oscillatory if the terms x_n are neither eventually positive nor eventually negative. If all the solutions $\{x_n\}$ of (1.2.2) are oscillatory, then equation (1.2.2) is said to be oscillatory. If every bounded solution $\{x_n\}$ of (1.2.2) is oscillatory, then (1.2.2) is called bounded oscillatory. If for every solution $\{x_n\}$ of (1.2.2), either $\{x_n\}$ or $\{\Delta^{k-1}x_n\}$ is oscillatory, then (1.2.2) is called almost oscillatory. If for every bounded solution $\{x_n\}$ of (1.2.2), either $\{x_n\}$ or $\{\Delta^{k-1}x_n\}$ is oscillatory, then (1.2.2) is called bounded almost oscillatory.

For later use, we present the following well-known Gronwall's inequality.

Lemma 1.2.1 Let $I = [t_0, T) \subset R$ and suppose that

$$u(t) \leq c + \int_{t_0}^{t} q(s)u(s)ds \;\text{ for }\; t \in I, \tag{1.2.3}$$

where c is a nonnegative constant and $u, q \in C(I, R^+)$. Then,

$$u(t) \leq c \exp\left(\int_{t_0}^{t} q(s)ds\right) \quad \text{for} \quad t \in I. \tag{1.2.4}$$

Lemma 1.2.2 (Taylor's Formula) Suppose that $f(x)$ satisfies the following two conditions:

(i) $f^{(n)}(x)$ is continuous on $[a, b]$,

(ii) $f^{(n+1)}(x)$ exists in the open interval (a, b).

Then for all $x \in (a, b)$, there is at least one $\xi \in (a, x)$ such that

$$\begin{aligned} f(x) \quad = \quad & f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \\ & \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}. \end{aligned}$$

In particular, the above conclusion holds if $f^{(n+1)} \in C[a, b]$.

1.3 BACKGROUND AND HISTORY REVIEW

We would like to point out here that in this section only a general and basic background and history review will be given, more specific and recent background and review will be given in each chapter later.

It is well known that differential and difference equations have played an important role in applicable analysis for recent few decades. Due to the importance of equations in application with an increasing number of interesting mathematical problems involved, see [46], [52], [42] and [20] for example, the subject has been developed very fast and attracted a huge number of researchers and some basic theories of differential and difference equations have been established. We refer to [15], [28], [29], [2], [8] for a few examples of such theories.

Since then there have appeared a good deal of results reflecting various interests. For example, Gopalsamy's book [21] includes most of the recent results on stability and oscillation of delay differential equations of population dynamics while Györi and Ladas' book [27] devotes to the recent results in the oscillation theory of functional differential equations.

For differential equations, miscellaneous problems have been broadly investigated for various classes of particular equations, such as the initial value problem, existence, uniqueness, stability, oscillation, and so on. Here are some examples mentioned.

Johnson and Karlsson [31] studied the equation

$$x''(t) + \frac{a}{r}x'(t) + \frac{b}{r}\sin x(t-r) = 0,$$

which is often referred to as the sunflower equation because of its origin in the circummutation of plants. Oscillation of certain special differential equations were studied, for instance,

$$y'(t) = \sum_{i=1}^{n} p_i y(t+\tau_i), \qquad p_i > 0 \text{ and } \tau_i > 0,\ i = 1, 2, \cdots, n$$

by Ladas and Stavroulakis [35] and

$$y^{(n)}(t) = p(t)y(g(t)) + q(t)y(h(t)), \qquad g(t) < t \text{ and } h(t) > t$$

by Kusano [34].

It is worth pointing out that with the importance in application, such as in Physics, Economics, and Control Systems, stability of differential equations with time delays has attracted a vast number of researchers. It has been investigated since 1960s by Razumikhin [48], Ezeilo [18] and many others. Since then some books and many papers have been published dealing with functional differential equations (for example, see [28], [38] and [70]). Generally, there are two of the main ideas to consider stability of time delay systems based on Lyapunov's direct method. The first one is to construct Lyapunov functionals to obtain the criteria for stability. The second one is the Lyapunov-Razumikhin approach, which is to construct Lyapunov functions rather than functionals and was first introduced by Razumikhin [48]. Since functions are much simpler to use, it is natural to explore the possibility of using functions to determine sufficient conditions for stability. By this method, Hale gave sufficient conditions of stability and boundedness of the first order and second order delay differential equations [28] and Razumikhin [48] dealt with third order equations. Ezeilo [18], Abou-El-Ela [1], Yu and Chen [60] and Okoronkwo [45] considered stability of a certain fourth order differential equations. There are few results on higher order functional differential equations up to date because it is very difficult to construct a Lyapunov functional or function. For example, Sadek [50] considered the following third order differential equations

$$x^{(3)}(t) + ax''(t) + bx'(t) + f(x(t-r)) = p(t)$$

and

$$x^{(3)}(t) + ax''(t) + \phi(x'(t-r)) + f(x(t)) = p(t).$$

Some sufficient conditions were given for the stability and boundedness of solutions. Further, Sadek [51] also investigated the fourth order equations

$$x^{(4)}(t) + \alpha_1 x^{(3)}(t) + \alpha_2 x''(t) + \alpha_3 x(t) + f(x(t-r)) = 0$$

and

$$x^{(4)}(t) + \alpha_1 x^{(3)}(t) + \alpha_2 x''(t) + \phi(x(t-r)) + f(x(t)) = 0$$

and obtained sufficient conditions for the zero solution to be asymptotically stable.

As a special class, neutral differential equations have been studied broadly and some basic problems, such as the initial value problems, stability, oscillation, have been solved for certain particular classes of equations. We refer to Ladde, Lakshmikantham and Zhang [36], Bainov and Misher [11], Grammatikopoulos [25] and the references therein for more details.

At the same time, the theory of difference equation has grown at a faster pace in past decade and has occupied an important position in analysis. It is no doubt that difference equations will carry on to play an important role in Mathematics as a whole. The basic theories have been established, see [8], [2], [32], for a few examples of such theories. Since then there are many results about the qualitative properties of solutions of difference equations, such as [4], [3] and the references therein for more details.

Although many researchers have engaged in the study of the qualitative behaviour of differential and difference equations, there is no general theory available for certain particular classes of equations. It is worthy to investigate the qualitative properties of solutions of these equations.

1.4 MOTIVATION FOR THE THESIS

In both theory and application, there is an increasing need to investigate the properties of the solutions of differential and difference equations. It is worth the effort to investigate a broader class of equations and to establish a theory for some fundamental problems such as asymptotical behaviour at infinity, oscillatory properties and oscillatory criteria. The main reason is that the solutions of most differential and difference equations cannot be formulated explicitly, and in some cases are troublesome even numerically. Thus it is very important that one can obtain the criteria for the behaviour of the solutions even in the higher dimension without knowing the solutions themselves. Therefore, this thesis will only concentrate on theoretical investigations of the behaviour of the solutions of certain types of differential and difference equations.

1.5 OUTLINE OF THE THESIS

A brief description of the organization of the thesis is as follows. Chapter 1 introduces the terminologies and notations appeared throughout the thesis and gives the background of the thesis.

Chapter 2 studies the stability of one class of differential equations. N-

dimensional non-autonomous first order differential systems are investigated of the form

$$x'(t) = b_i(t)x_i(t)\left(1 - \sum_{j=1}^{n} a_{ij}(t)x_j(t)\right), \qquad i \in N.$$

Sufficient conditions are established for the solutions to be stable.

Chapter 3 deals with the oscillation of the second order nonlinear neutral differential equations of the form

$$(a(t)(x(t) + \delta p(t)x(t-\tau))')' + f(t, x(t-\sigma)) - g(t, x(t-\rho)) = 0,$$

where $\delta = 1$ or $\delta = -1$. In this chapter, we obtain criteria for bounded oscillation, bounded almost oscillation and almost oscillation of the solutions. Furthermore, examples are given to illustrate the criteria in each case, respectively.

Chapters 4 and 5 are the main part of this thesis. In these two chapters, we investigate the oscillation of the nonlinear neutral difference equations with continuous variable

$$\Delta_\tau^m(x(t) - px(t-r)) + f(t, x(g(t))) = 0,$$

where $m \geq 2$. The second, fourth, higher even order, third, and higher odd order of the above equation have been discussed, respectively. Chapter 4 contains the even orders $m \geq 2$ and Chapter 5 is devoted to the odd orders $m \geq 3$. In Chapter 4, mainly using an integral transformation, the Riccati transformation and iteration, oscillation criteria are established for the second order, fourth order and higher even order, respectively. Furthermore, examples are given to illustrate the applications of the results in each case. In Chapter 5, the above difference equations are converted to the corresponding differential equations or inequalities by using an integral transformation. Based on the results of differential equations or inequalities, sufficient conditions are obtained for the bounded solutions to be oscillatory. Moreover, examples are given to illustrate the results. We should point out that the results on the second order have been published in a peer-viewed journal (see [56]).

Chapter 6 deals with the following higher even order nonlinear neutral difference equations of the form

$$\Delta^{m-1}\left(a_n \Delta(x_n + \varphi(n, x_{\tau_n}))\right) + q_n f(x_{g_n}) = 0.$$

By using generalized Riccati technique and Riccati type difference inequalities, oscillation criteria are achieved for the solutions. In addition, examples are given to illustrate the results. The results have been published in a peer-viewed journal (see [40]).

We draw conclusions in Chapter 7.

Chapter 2

STABILITY OF DIFFERENTIAL EQUATIONS

2.1 INTRODUCTION

In this chapter, we are concerned with the nonautonomous Lotka-Volterra system with the form

$$x_i'(t) = b_i(t)x_i(t)\left(1 - \sum_{j=1}^{n} a_{ij}(t)x_j(t)\right), \quad i \in N(1,n), \tag{2.1.1}$$

where the functions $a_{ij}(t)$ and $b_i(t)$ are continuous on R and bounded above and below by strictly positive numbers. Throughout this chapter the following condition

$$\frac{b_i(t)}{b_1(t)} \to \bar{b}_i \quad \text{as} \quad t \to \infty, \quad i \in N(1,n) \tag{2.1.2}$$

will be assumed. Since (2.1.1) is well known as a model of a community of n mutually competing species, x_i denoting the population size of the ith species at time t, we adopt the tradition of restricting attention to the closed positive cone denoted by R^n_+ and the open positive cone is denoted by $\dot{R}^n_+$.

We are interested in the existence of a global attractor $x^* \in R^n_+$ with $x_1^* \in \dot{R}_+$, which means that all but one of the species will go extinct while the only one species will stabilise at x_1^*. Oca and Zeeman [44] considered the following Lotka-Volterra system

$$x_i'(t) = x_i(t)\left(b_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t)\right), \quad i \in N(1,n), \tag{2.1.3}$$

and established a criterion which guarantees that all but one of the species is driven to extinction. Ahmad and Lazer [10] considered the above system, where a_{ij} and b_j are periodic or almost periodic, and found conditions under which there exists a unique solution that attracts all other solutions with positive components.

Oca and Zeeman [43] generalized results given in [44] and [10] to a situation when $n-r$ components vanish, whilst the remaining r components approach a canonical solution of an r-dimensional subsystem. Zeeman [62] considered the autonomous system in the form

$$x_i'(t) = x_i(t)\left(b_i - \sum_{j=1}^{n} a_{ij}x_j(t)\right), \quad i \in N(1,n)$$

and exhibited a criterion which guarantees that all but one of the species are driven to extinction, whilst the one remaining population stabilities at its own carrying capacity. Ahmad [9] considered the two-dimensional system

$$\begin{cases} u'(t) & = u(t)\left(a(t) - b(t)u(t) - c(t)v(t)\right) \\ v'(t) & = v(t)\left(d(t) - e(t)u(t) - f(t)v(t)\right) \end{cases}$$

and showed under certain conditions that one of its components vanishes while the other approaches a certain solution of a logistic equation.

The purpose of this chapter is to find new conditions that are less restrictive for the existence of a global attractor $x^* \in R_+^n$. We shall establish some new criteria for a particular solution to be stable. The main result will be presented in section 2.2 and its proof will be given in section 2.3. At last we draw a conclusion in section 2.4.

2.2 MAIN RESULT

Together with (2.1.1), we also consider the nonautonomous logistic equation

$$x_i'(t) = b_i(t)x_i(t)(1 - a_{ii}(t)x_i(t)), \quad i \in N(1,n). \tag{2.2.4}$$

The following lemmas 2.2.1 and 2.2.2 are applied to (2.2.4). They have showed that in the nonautonomous logistics equation the role of the globally attracting capacity of the nonautonomous systems is played by a well defined canonical solution $x_i^*(t)$ to which all other solutions converge. Both lemmas can be found in [44].

Lemma 2.2.1 Equation (2.2.4) has a unique solution $x_i^*(t)$ which is bounded above and below by strictly positive numbers for all $t \in R$.

Lemma 2.2.2 If $u(t)$, $v(t)$ are any two solutions of (2.2.4), then

$$u(t) - v(t) \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty.$$

We call $x_i^*(t)$ the canonical solution of (2.2.4). Note that any solution $x_i(t)$ of (2.2.4) is bounded above and below by strictly positive numbers and $x_i(t) - x_i^*(t) \rightarrow 0$ as $t \rightarrow \infty$.

Theorem 2.2.1 Assume that for all $k > 1$ there exists an $i_k < k$ such that

$$a_{kj}(t) \geq a_{i_k j}(t) + \varepsilon_j, \qquad \text{for all } j \leq k, \tag{2.2.5}$$

where ε_j are any positive constants. Then every trajectory with initial condition in $\dot{R}^n_+$ is asymptotic to $(x_1^*, 0, ..., 0)$.

In other words, for strictly positive initial values, the species x_j $(j \in N(2,n))$ are driven to extinction while species x_1 stabilizes at the unique bounded solution x_1^* of the logistic equation (2.2.4) for $i = 1$.

2.3 THE PROOF

First we prove the following lemmas then derive the proof of the theorem immediately. For a given function $g(t)$ defined on R, let

$$g_L = \inf_R g(t), \qquad g_M = \sup_R g(t).$$

Lemma 2.3.1 If $x(t)$ is a solution of (2.1.1) with initial condition in $\dot{R}^n_+$, then there exist some $\delta > 0$, $M > 0$, and $T \in R$ such that

$$\delta \leq \sum_{i=1}^{n} x_i(t) \leq M \quad \text{and} \quad 0 < x_i(t) \leq M, \quad \text{for all } \; i \in N(1,n) \tag{2.3.6}$$

for all $t > T$.

Proof It is obvious that the open and closed positive cones are invariant under (2.1.1). Now let

$$M = 2\max\left\{\frac{1}{a_{ijL}}: \quad i,j \in N(1,n)\right\},$$
$$\delta = \frac{1}{2}\min\left\{\frac{1}{a_{ijM}}: \quad i,j \in N(1,n)\right\}$$

and

$$S = \left\{x \in R^n_+: \quad \delta \leq \sum_{i=1}^{n} x_i \leq M\right\}.$$

We will show that S is a globally attracting positively invariant compact set in $R^n_+\backslash\{0\}$. Then if $x(t)$ is a solution of (2.1.1) with initial condition in $\dot{R}^n_+$, there exists a $T \in R$ such that $x(t) \in S$ for all $t \geq T$. And the conclusion follows immediately. From (2.1.1), we have

$$\begin{aligned} x_i'(t) &= b_i(t)x_i(t)\left(1-\sum_{j=1}^{n} a_{ij}(t)x_j\right) \\ &\le b_i(t)x_i(t)\left(1-\sum_{j=1}^{n} a_{ijL}x_j\right) \\ &\le b_i(t)x_i(t)\left(1-\alpha\sum_{j=1}^{n} x_j\right), \end{aligned}$$

where $\alpha = \min_{i,j}\{a_{ijL}\}$. Then we have

$$\left(\sum_{i=1}^{n} x_i(t)\right)' \le \sum_{i=1}^{n} b_i(t)x_i(t)\left(1-\alpha\sum_{j=1}^{n} x_j\right).$$

If $\sum_{j=1}^{n} x_j > 1/\alpha$, then $(\sum_{i=1}^{n} x_i)' < 0$, i.e., $\sum_{i=1}^{n} x_i$ is decreasing. So there must be a $T_1 \in R$ such that $\sum_{i=1}^{n} x_i < M = 2/\alpha$ for $t \ge T_1$.

By the similar procedure, we obtain

$$\begin{aligned} x_i'(t) &= b_i(t)x_i(t)\left(1-\sum_{j=1}^{n} a_{ij}(t)x_j\right) \\ &\ge b_i(t)x_i(t)\left(1-\sum_{j=1}^{n} a_{ijM}x_j\right) \\ &\ge b_i(t)x_i(t)\left(1-\beta\sum_{j=1}^{n} x_j\right), \end{aligned}$$

where $\beta = \max_{i,j}\{a_{ijM}\}$. So

$$\left(\sum_{i=1}^{n} x_i\right)' \ge \sum_{i=1}^{n} b_i(t)x_i(t)\left(1-\beta\sum_{j=1}^{n} x_j\right).$$

If $\sum_{j=1}^{n} x_j < 1/\beta$, then $(\sum_{i=1}^{n} x_i)' > 0$, i.e., $\sum_{i=1}^{n} x_i$ is increasing. So there must be a $T \ge T_1$ such that $\sum_{i=1}^{n} x_i > \delta = 1/(2\beta)$ for $t \ge T$. Therefore, S is a positively invariant compact set in $R_+^n\backslash\{0\}$. And for each $x(t_0) > 0$, $\sum_{i=1}^{n} x_i(t)$ belongs to S as $t \ge T$. Thus, S is a compact attracting set for (2.1.1) on $R_+^n\backslash\{0\}$.

Remark 2.3.1 Although there are results similar to Lemma 2.3.1 available in the literature for system (2.1.3), we put this lemma here for convenience.

Lemma 2.3.2 If (2.1.1) satisfies (2.2.5) and $x(t)$ is a solution of (2.1.1) with $x(t_0) \in \dot{R}^n_+$ for some t_0, then for all $i \in N(2, n)$

$$x_i(t) \leq K_i e^{\sigma_i t}, \qquad t \geq t_0,$$

where $K_i > 0$ and $\sigma_i < 0$.

Proof Let $x(t)$ be a solution of (2.1.1) with $x(t_0) \in \dot{R}^n_+$ for some t_0. By Lemma 2.3.1, $x(t) \in S$ for all $t \geq t_0$. We will prove the conclusion for $x_n(t)$ at first. Let i_n be given in (2.2.5) by i. From (2.2.5), it follows that

$$\begin{aligned}
\frac{x_n'(t)}{x_n(t)} &= b_n(t)\left(1 - \sum_{j=1}^{n} a_{nj}(t)x_j\right) \\
&= \frac{b_n(t)}{b_i(t)} b_i(t)\left(1 - \sum_{j=1}^{n} a_{ij}(t)x_j + \sum_{j=1}^{n} a_{ij}(t)x_j - \sum_{j=1}^{n} a_{nj}(t)x_j\right) \\
&= \frac{b_n(t)}{b_i(t)} b_i(t)\left(1 - \sum_{j=1}^{n} a_{ij}(t)x_j\right) + \frac{b_n(t)}{b_i(t)} b_i(t)\sum_{j=1}^{n}(a_{ij}(t) - a_{nj}(t))x_j \\
&= \frac{\bar{b}_n}{\bar{b}_i}\frac{x_i'(t)}{x_i(t)} + \frac{\bar{b}_n}{\bar{b}_i} b_i(t)\sum_{j=1}^{n}\left(a_{ij}(t) - a_{nj}(t)\right)x_j + o(1)
\end{aligned}$$

as $t \to \infty$. Then, from (2.2.5) again, we have

$$\begin{aligned}
\left[\ln\left(\frac{x_n^{\frac{\bar{b}_i}{\bar{b}_n}}(t)}{x_i(t)}\right)\right]' &= \frac{\bar{b}_i}{\bar{b}_n}\frac{x_n'(t)}{x_n(t)} - \frac{x_i'(t)}{x_i(t)} \\
&= b_i(t)\sum_{j=1}^{n}(a_{ij}(t) - a_{nj}(t))x_j + o(1) \\
&\leq b_i(t)\sum_{j=1}^{n}(-\varepsilon_j x_j) + o(1) \\
&\leq -b_{iL}\min_j\{\varepsilon_j\}\delta + o(1)
\end{aligned}$$

as $t \to \infty$. Hence, there exists a $t_1 > t_0$ such that for all $t \geq t_1$,

$$\frac{d}{dt} \ln \left(x_n^{\frac{\bar{b}_i}{\bar{b}_n}}(t) x_i^{-1}(t) \right) \leq -b_{iL} \min_j \{\varepsilon_j\} \delta / 2 = M_n.$$

Integrating both sides, we obtain

$$\ln \left(\frac{x_n^{\frac{\bar{b}_i}{\bar{b}_n}}(t)}{x_i(t)} \right) \Bigg|_{t_1}^{t} \leq M_n(t - t_1)$$

and

$$\begin{aligned} x_n^{\frac{\bar{b}_i}{\bar{b}_n}}(t) &\leq x_i(t) \frac{x_n^{\frac{\bar{b}_i}{\bar{b}_n}}(t_1)}{x_i(t_1)} e^{-M_n t_1} e^{M_n t} \\ &< M \frac{x_n^{\frac{\bar{b}_i}{\bar{b}_n}}(t_1)}{x_i(t_1)} e^{-M_n t_1} e^{M_n t} \\ &= C e^{M_n t} \end{aligned}$$

for $t \geq t_1$, where $C = M \frac{x_n^{\frac{\bar{b}_i}{\bar{b}_n}}(t_1)}{x_i(t_1)} e^{-M_n t_1}$. Thus,

$$x_n(t) \leq K_n e^{\sigma_n t}$$

for some $K_n > 0$, $\sigma_n < 0$, and all $t \geq t_0$.

We now prove that $x_{n-1}(t) \leq K_{n-1} e^{\sigma_{n-1} t}$ using the result $x_n \leq K_n e^{\sigma_n t}$ for $t \geq t_0$. This method is essentially the same as that used above for x_n. Now let $i_{(n-1)}$ be given in (2.1.1) by i. From (2.2.5) and $\lim_{t \to \infty} x_n(t) = 0$, it follows that

$$\begin{aligned} &\frac{x'_{n-1}(t)}{x_{n-1}(t)} \\ &= b_{n-1}(t) \left(1 - \sum_{j=1}^{n} a_{(n-1)j}(t) x_j \right) \\ &= \frac{b_{n-1}(t)}{b_i(t)} b_i(t) \left(1 - \sum_{j=1}^{n} a_{ij}(t) x_j + \sum_{j=1}^{n} a_{ij}(t) x_j - \sum_{j=1}^{n} a_{(n-1)j}(t) x_j \right) \\ &= \frac{\bar{b}_{n-1}}{\bar{b}_i} b_i(t) \left(1 - \sum_{j=1}^{n} a_{ij}(t) x_j \right) + \frac{\bar{b}_{n-1}}{\bar{b}_i} b_i(t) \sum_{j=1}^{n} (a_{ij}(t) - a_{(n-1)j}(t)) x_j + o(1) \\ &= \frac{\bar{b}_{n-1}}{\bar{b}_i} \frac{x'_i(t)}{x_i(t)} + \frac{\bar{b}_{n-1}}{\bar{b}_i} b_i(t) \sum_{j=1}^{n-1} (a_{ij}(t) - a_{(n-1)j}(t)) x_j + o(1). \end{aligned}$$

By (2.2.5) again, there is a $T_n > t_0$ such that

$$\frac{x'_{n-1}(t)}{x_{n-1}(t)} \le \frac{\bar{b}_{n-1}}{\bar{b}_i}\frac{x'_i(t)}{x_i(t)} - \frac{\bar{b}_{n-1}}{\bar{b}_i} b_{iL}\frac{1}{2}\min_j\{\epsilon_j\}\delta$$

for $t \ge T_n$, i.e.,

$$\frac{d}{dt}\left(\ln\left(x_{n-1}^{\frac{\bar{b}_i}{\bar{b}_{(n-1)}}}(t)x_i^{-1}(t)\right)\right) \le -b_{iL}\frac{1}{2}\min_j\{\varepsilon_j\}\delta = M_{n-1}.$$

Integrating the above inequality, we have

$$\ln\left(x_{n-1}^{\frac{\bar{b}_i}{\bar{b}_{(n-1)}}}(t)x_i^{-1}(t)\right)\Bigg|_{T_n}^{t} \le M_{n-1}(t - T_n)$$

and so

$$x_{n-1}(t) \le K_{n-1}e^{\sigma_{n-1}t}$$

for some $K_{n-1} > 0$, $\sigma_{n-1} < 0$ and all $t \ge t_0$. Repeating the above procedure $n-1$ times, we have shown the conclusion holds for all $i \in N(2,n)$.

Lemma 2.3.3 If (2.1.1) satisfies (2.2.5) and $x(t)$ is a solution of (2.2.5) with $x(t_0) \in \dot{R}^n_+$ for some t_0, then $x_1(t) - x_1^*(t) \to 0$ as $t \to \infty$.

Proof By Lemma 2.3.1, we assume that $x(t_0) \in S \cap \dot{R}^n_+$ then $x(t) \in S$ for all $t \ge t_0$. From Lemma 2.3.2, $\lim_{t\to\infty} x_i(t) = 0$ when $i > 1$, so $x_1(t)$ is bounded above and below by positive constants.

Let $u_1(t)$ be a solution of the nonautonomous logistic equation (2.2.4) such that $u_1(t_0) \ge x_1(t_0)$. Then $u_1(t)$ is bounded by Lemmas 2.2.1-2.2.2. We claim that $u_1(t) > x_1(t)$ for all $t > t_0$. Indeed, if there exists a number $t_1 > t_0$ such that $u_1(t_1) - x_1(t_1) = 0$ and $u_1(t) > x_1(t)$ for $t \in (t_0, t_1)$, then this would imply that $u_1'(t_1) - x_1'(t_1) \leq 0$. But this contracts the fact that

$$u_1'(t_1) - x_1'(t_1) = b_1(t_1)x_1(t_1)\sum_{j=2}^{n} a_{1j}(t_1)x_j(t_1) > 0.$$

So we have

$$\begin{aligned}
\frac{d}{dt}\left(\ln\frac{x_1(t)}{u_1(t)}\right) &= \frac{x_1'(t)}{x_1(t)} - \frac{u_1'(t)}{u_1(t)} \\
&= b_1(t)\left(1 - \sum_{j=1}^{n} a_{1j}(t)x_j(t)\right) - b_1(t)(1 - a_{11}(t)u_1(t)) \\
&= b_1(t)a_{11}(t)(u_1(t) - x_1(t)) - b_1(t)\sum_{j=2}^{n} a_{1j}(t)x_j(t)
\end{aligned}$$

and

$$\begin{aligned}
u_1(t) - x_1(t) &= \frac{1}{a_{11}(t)b_1(t)}\left(\frac{d}{dt}\left(\ln\frac{x_1(t)}{u_1(t)}\right) + b_1(t)\sum_{j=2}^{n} a_{1j}(t)x_j(t)\right) \\
&\leq \frac{1}{a_{11L}b_{1L}}\left(\frac{d}{dt}\left(\ln\frac{x_1(t)}{u_1(t)}\right) + b_1(t)\sum_{j=2}^{n} a_{1j}(t)x_j(t)\right).
\end{aligned}$$

Integrating this inequality we have

$$\begin{aligned}
\int_{t_0}^{t}(u_1(s) - x_1(s))ds &\leq \frac{1}{a_{11L}b_{11L}}\left(\ln\frac{x_1(t)}{u_1(t)}\Bigg|_{t_0}^{t} + \int_{t_0}^{t} b_1(s)\sum_{j=2}^{n} a_{1j}(s)x_j(s)ds\right) \\
&\leq \frac{1}{a_{11L}b_{11L}}\left(\ln\left(\frac{x_1(t)u_1(t_0)}{u_1(t)x_1(t_0)}\right) + b_{1M}\sum_{j=2}^{n} a_{1jM}\int_{t_0}^{t} x_j(s)ds\right) \\
&< K < \infty,
\end{aligned}$$

where K is a constant independent of t. Suppose $\limsup_{t\to\infty}(u_1(t) - x_1(t)) = \gamma > 0$. Then there is a sequence $\{t_n\}$ such that $t_n \to \infty$ and $u_1(t_n) - x_1(t_n) \to \gamma$ as $n \to \infty$, $t_n > t_0$ and $u_1(t_n) - x_1(t_n) > \gamma/2$ for $n \geq 1$. Since $u_1'(t) - x_1'(t)$ is bounded, $u_1(t) - x_1(t)$ is uniformly continuous on $[t_0, \infty)$. Thus, for $\gamma/4 > 0$, there is an $l > 0$ such that

$$|(u_1(t) - x_1(t)) - (u_1(t_n) - x_1(t_n))| < \gamma/4$$

for each $n \geq 1$ and all $t \in [t_n - l, t_n + l]$. Then

$$\int_{t_n-l}^{t_n+l} (u_1(t) - x_1(t))dt \geq \int_{t_n-l}^{t_n+l} \left(u_1(t_n) - x_1(t_n) - \frac{\gamma}{4}\right) dt \geq \frac{1}{2}\gamma l > 0.$$

Since we can always choose a subsequence of $\{t_n\}$ to replace $\{t_n\}$ if necessary, without loss of generality, we assume that

$$t_0 \leq t_1 - l \leq \cdots \leq t_n + l \leq t_{n+1} - l \leq \cdots .$$

Then

$$\int_{t_0}^{t_n+l} (u_1(t) - x_1(t))dt \geq \sum_{i=1}^{n} \int_{t_i-l}^{t_i+l} (u_1(t) - x_1(t))dt \geq \frac{1}{2}\gamma n l \to \infty$$

as $n \to \infty$. This contradiction to the convergence of $\int_{t_0}^{\infty}(u_1(t) - x_1(t))dt$ shows that $\limsup_{t\to\infty}(u_1(t) - x_1(t)) = 0$. Note that $u_1(t) - x_1(t) > 0$. Hence $u_1(t) - x_1(t) \to 0$ as $t \to \infty$. Moreover, by Lemma 2.2.2

$$u_1(t) - x_1^*(t) \to 0 \text{ as } t \to \infty$$

and hence

$$x_1(t) \to x_1^*(t) \quad \text{as} \quad t \to \infty.$$

Proof of Theorem 2.2.1 Suppose that $x(t)$ is a solution of (2.1.1) with initial condition in $\dot{R}_+^n$. By Lemma 2.3.1, we have

$$\delta \leq \sum_{i=1}^{n} x_i(t) \leq M \quad \text{and} \quad 0 < x_i(t) \leq M, \quad \text{for all } \ i \in N(1,n)$$

for all $t > T$, where $\delta > 0$, $M > 0$, and $T \in R$. Then from Lemma 2.3.2, we can see that $x_i(t) \to 0$ as $t \to \infty$ for $2 \leq i \leq n$ and $x_1(t)$ is bounded above and below by positive constants. According to Lemma 2.3.3, $x_1(t) \to x^*(t)$ as $t \to$

∞. Therefore, every trajectory with initial condition in $\dot{R}^n_+$ is asymptotic to $(x_1^*, 0, \dots, 0)$.

2.4 CONCLUSION

The aim of this chapter is to study the stability of a particular class of solutions of the first order nonautonomous system (2.1.1). In the meantime, (2.1.1) is a well-known population model of n mutually competing species. In this chapter, we are interested in the existence of a global attractor $x^* = (x_1^*, 0, \cdots, 0)$. From the previous studies, we know that logistic equation (2.2.4) has a canonical solution x_i^*, which is bounded above and below by positive numbers. Based on this, we have obtained a weaker condition (2.2.5) and improved the result. Under condition (2.2.5), all species $x_i (i \geq 2)$ but x_1 with strict positive initial conditions are driven to extinction.

Chapter 3

OSCILLATION OF NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS

3.1 INTRODUCTION

In this chapter, we consider the oscillation of second order nonlinear neutral differential equations of the form

$$(a(t)(x(t)+\delta p(t)x(t-\tau))')' + f(t,x(t-\sigma)) - g(t,x(t-\rho)) = 0, \qquad (3.1.1)$$

where $\delta = +1$ or -1, $t \geq t_0$, $a(t)$ is a continuously differentiable function, $p(t)$ is a continuous bounded function with $a(t) > 0$, $p(t) \geq 0$, $f(t,u)$ and $g(t,v)$ are continuous functions, the constants τ, σ, $\rho \in [0,\infty)$. Denote $\lambda = \max\{\tau,\sigma,\rho\}$, $t_1 = t_0 + \lambda$, $L^1[t_0,\infty) = \left\{x(t) \,\middle|\, \int_{t_0}^{\infty} |x(s)|ds < \infty\right\}$.

Some of the following conditions will be assumed later:

(H_1) $\int_t^{\infty} \frac{1}{a(s)} ds = \infty$ for all $t \geq t_0$,

(H_2) $\frac{f(t,u)}{u} \geq q(t-\sigma) > 0$ for $u \neq 0$ and $0 < \frac{g(t,v)}{v} \leq r(t-\rho)$ for $v \neq 0$,

(H_3) $0 < \frac{f(t,u)}{u} \leq q(t-\sigma)$ for $u \neq 0$ and $\frac{g(t,v)}{v} \geq r(t-\rho) > 0$ for $v \neq 0$,

(H_4) $\frac{1}{r(t)-q(t)}$ is bounded, where $q, r \in C([t_0,\infty), R^+)$.

In other words, this equation is a second order neutral differential equations with positive and negative terms. In this chapter, we shall first obtain some criteria for bounded oscillation, bounded almost oscillation and almost oscillation for equation (3.1.1) when $\delta = +1$ in section 3.2 and then for equation (3.1.1) when $\delta = -1$ in section 3.4. Following the theorems, some examples will be given in section 3.5 to illustrate the criteria. Finally, we will finish this chapter with a conclusion in section 3.6.

In this chapter, we always assume that $x(t)$ is a nontrivial solution of equation (3.1.1). The investigation of oscillatory behaviour of solutions of various types of differential equations done by many researchers is motivated by many application problems in Physics [17], Biology [41], Ecology [55], and the study of infectious diseases [12]. For some contributions made to the oscillation theory, we refer to the articles [22], [57], [25], [26], [19], [30], [33], [47], [49], [14] and [13] and the books by Györi and Ladas [27], Agarwal, Grace and O'Regan [5], Erbe, Kong and Zhang [15], Ladde, Lakshmikantham and Zhang [36], and the references therein. There are a great number of papers devoted to various particular cases of (3.1.1) such as the linear equation (without delays)

$$x''(t) + q(t)x(t) = 0,$$

the more general linear equation (without delays)

$$(a(t)x'(t))' + q(t)x(t) = 0,$$

the nonlinear equation (without delays)

$$(a(t)x'(t))' + q(t)f(x(t)) = 0,$$

and the more general nonlinear equation

$$(a(t)x'(t))' + p(t)x'(t) + q(t)f(x(t)) = 0$$

when $\tau = \sigma = \rho = 0$. See [22], [37], [57], and [14] for example. Grammatikopoulos, Ladas and Meimaridou [26] considered

$$(x(t) + p(t)x(t-\tau))'' + q(t)x(t-\sigma) = 0,$$

where $p(t) \equiv p$ and $q(t) \equiv q > 0$ and obtained some sufficient conditions for the above equation to be oscillatory. Gai, Shi and Zhang [19] considered

$$(x(t) + p(t)x(\sigma(t)))'' + q(t)f(x(\tau(t)))g(x'(t)) = 0$$

and

$$(x(t) + p(t)x(\sigma(t)))'' + q(t)f(x(t), x(\tau(t)))g(x'(t)) = 0$$

and established the criteria for the solution to be oscillatory. These equations are more general when $0 \leq p(t) < 1$ than (3.1.1). Ruan [49] considered

$$(a(t)(x(t) + p(t)x(t - \tau))')' + q(t)f(x(t - \sigma)) = 0$$

and obtained some oscillation criteria. Again it requires that $0 \leq p(t) \leq 1$. The differential equations of the form

$$x^{(n)}(t) + f(t, x(t), x'(g_1(t)), ..., x^{(n-1)}(g_n(t))) = 0,$$

where $g_i(t) \leq t$, $i \in N(1, n)$, were considered in [30], [33], [14] and [13]. Note that the highest derivative does not involve delays.

3.2 MAIN RESULTS WHEN $\delta = +1$

In this section, we consider equation (3.1.1) with $\delta = +1$. We rewrite equation (3.1.1) as

$$(a(t)(x(t) + p(t)x(t - \tau))')' + f(t, x(t - \sigma)) - g(t, x(t - \rho)) = 0. \qquad (3.2.2)$$

Four oscillatory criteria will be presented here for equation (3.2.2) to be bounded oscillatory, almost oscillatory and bounded almost oscillatory, respectively.

Theorem 3.2.1 Suppose conditions (H_1), (H_2) and (H_4) hold, $q(t) > r(t)$, $r(t)$ is bounded and $\sigma \geq \rho$. Then (3.2.2) is bounded oscillatory.

Proof Let $x(t)$ be a bounded non-oscillatory solution. Suppose $x(t)$ is an eventually positive solution. Then there exists a $t_2 \geq t_1$ such that $x(t) > 0$ and $x(t-\lambda) > 0$ for $t \geq t_2$. Let

$$z(t) = a(t)(x(t) + p(t)x(t-\tau))' - \int_{t-\sigma}^{t-\rho} r(s)x(s)ds. \tag{3.2.3}$$

From (3.2.2) and (H_2), it follows that

$$z'(t) \leq (r(t-\sigma) - q(t-\sigma))x(t-\sigma) < 0, \quad t \geq t_2. \tag{3.2.4}$$

So $z(t)$ is decreasing, and

$$-\infty \leq \lim_{t\to\infty} z(t) = c < \infty.$$

If $c = -\infty$, from (3.2.3) and the boundedness of $x(t)$ and $r(t)$, we have

$$\lim_{t\to\infty} a(t)(x(t) + p(t)x(t-\tau))' = -\infty.$$

Then there exist an $l_1 > 0$ and a $t_3 \geq t_2$ such that

$$(x(t) + p(t)x(t-\tau))' \leq -\frac{l_1}{a(t)}, \quad t \geq t_3.$$

Integrating both sides of the above inequality, according to (H_1), we obtain

$$\lim_{t\to\infty} (x(t) + p(t)x(t-\tau)) = -\infty,$$

which contradicts the boundedness of $x(t)$ and $p(t)$. This contradiction shows that $|c| < \infty$, i.e., $z(t)$ is bounded.

From (3.2.4) it follows that

$$x(t-\sigma) \leq \frac{1}{r(t-\sigma)-q(t-\sigma)} z'(t). \tag{3.2.5}$$

So $x \in L^1[t_0,\infty)$ by (H_4).

(i) If $c > 0$, from (3.2.3) we have

$$z(t) \leq a(t)(x(t)+p(t)x(t-\tau))', \quad t \geq t_2.$$

Therefore, since $z(t) \to c$ as $t \to \infty$,

$$(x(t)+p(t)x(t-\tau))' \geq \frac{c}{a(t)}, \quad t \geq t_2.$$

From (H_1) we have $\lim_{t\to\infty}(x(t)+p(t)x(t-\tau)) = \infty$, which contradicts the boundedness of $x(t)$.

(ii) If $c < 0$, in view of $x \in L^1[t_0,\infty)$, we have

$$\lim_{t\to\infty} \int_{t-\sigma}^{t-\rho} r(s)x(s)ds = 0.$$

Then, since $z(t) \to c$ as $t \to \infty$, there exist an $\epsilon \in (0,-c)$ and a $t_4 \geq t_2$ such that

$$a(t)(x(t)+p(t)x(t-\tau))' \leq c+\varepsilon < 0, \quad t \geq t_4.$$

Hence, by (H_1) again, we obtain

$$\lim_{t\to\infty}(x(t)+p(t)x(t-\tau)) = -\infty,$$

a contradiction to the boundedness of $x(t)$ and $p(t)$.

(iii) If $c = 0$, in view of $z'(t) < 0$, we have $z(t) > 0$. So

$$a(t)(x(t)+p(t)x(t-\tau))' > \int_{t-\sigma}^{t-\rho} r(s)x(s)ds > 0, \quad t \geq t_2.$$

Since $x(t)+p(t)x(t-\tau)$ is positive and increasing, the integral

$$\int_{t_0}^{\infty}(x(t)+p(t)x(t-\tau))dt$$

is divergent, a contradiction to $x \in L^1[t_0,\infty)$. The contradictions obtained in the above three cases show that (3.2.2) has no bounded eventually positive solution. Now suppose $x(t)$ is a bounded eventually negative solution. Then $x(t-\lambda) < 0$ for some $t_2 > t_1$ and all $t \geq t_2$. From (3.2.2), (3.2.3) and (H_2), we have

$$z'(t) \geq (r(t-\sigma)-q(t-\sigma))x(t-\sigma) > 0, \quad t \geq t_2. \tag{3.2.4$'$}$$

So $z(t)$ is increasing and $-\infty < \lim_{t\to\infty} z(t) = c \leq \infty$. Then an argument parallel to the above also leads to contradictions. Therefore, every bounded solution of (3.2.2) is oscillatory.

Theorem 3.2.2 Suppose conditions (H_1), (H_2) and (H_4) hold, $q(t) > r(t)$, $q(t)$, $1/a(t)$ are bounded and $\sigma < \rho$. Then (3.2.2) is almost oscillatory.

Proof Without loss of generality, suppose that $x(t)$ is an eventually positive solution. Take $t_2 \geq t_1$ such that $x(t-\lambda) > 0$ for all $t \geq t_2$. Let

$$z(t) = a(t)(x(t)+p(t)x(t-\tau))' + \int_{t-\rho}^{t-\sigma} q(s)x(s)ds. \tag{3.2.6}$$

From (3.2.2) it follows that

$$z'(t) \leq (r(t-\rho)-q(t-\rho))x(t-\rho) < 0, \quad t \geq t_2. \tag{3.2.7}$$

So $z(t)$ is decreasing and

$$-\infty \leq \lim_{t\to\infty} z(t) = c < \infty.$$

If $c = -\infty$, then

$$\lim_{t\to\infty} a(t)(x(t) + p(t)x(t-\tau))' = -\infty.$$

By (H_1), we obtain $\lim_{t\to\infty}(x(t) + p(t)x(t-\tau)) = -\infty$, which contradicts $x(t) + p(t)x(t-\tau) > 0$. Therefore $|c| < \infty$ so $z(t)$ is bounded.

From (3.2.7), we have

$$x(t-\rho) \leq \frac{1}{r(t-\rho) - q(t-\rho)} z'(t), \quad t \geq t_2 \tag{3.2.8}$$

so, by (H_4), $x \in L^1[t_0, \infty)$ and $\lim_{t\to\infty} \int_{t-\rho}^{t-\sigma} q(s)x(s)ds = 0$. Since $1/a(t)$ is bounded, by (3.2.6), $(x(t) + p(t)x(t-\tau))'$ is bounded. This implies that $x(t) + p(t)x(t-\tau)$ is uniformly continuous on $[t_1, \infty)$. Note that the property $x \in L^1[t_0, \infty)$ and the boundedness of $p(t)$ imply that $x(t) + p(t)x(t-\tau) \in L^1[t_0, \infty)$. Hence $\lim_{t\to\infty}(x(t) + p(t)x(t-\tau)) = 0$ so $\lim_{t\to\infty} x(t) = 0$. Therefore, every solution x of (3.2.2) which is not in the class of $o(1)$ as $t \to \infty$ is oscillatory.

Theorem 3.2.3 Suppose conditions (H_1), (H_3) and (H_4) hold, $q(t) < r(t)$, $r(t)$, $1/a(t)$ are bounded and $\sigma \geq \rho$. Then (3.2.2) is bounded almost oscillatory.

Proof Without loss of generality, assume that $x(t)$ is a bounded eventually positive solution and $z(t)$ is defined by (3.2.3). Take $t_2 \geq t_1$ such that $x(t-\lambda) > 0$ for $t \geq t_2$. From (3.2.2) and condition (H_3), we have

$$z'(t) \geq (r(t-\sigma) - q(t-\sigma))x(t-\sigma) > 0, \quad t \geq t_2. \tag{3.2.9}$$

So $z(t)$ is increasing. Then

$$-\infty < \lim_{t\to\infty} z(t) = d \leq \infty.$$

If $\lim_{t\to\infty} z(t) = \infty$, then from (3.2.3) and the boundedness of $x(t)$ and $r(t)$, we obtain

$$\lim_{t\to\infty} a(t)(x(t) + p(t)x(t-\tau))' = \infty.$$

Then there exist an $l_2 > 0$ and a $t_3 \geq t_2$ such that

$$a(t)(x(t) + p(t)x(t-\tau))' \geq l_2, \quad t \geq t_3.$$

From (H_1) it follows that

$$\lim_{t\to\infty} (x(t) + p(t)x(t-\tau)) = \infty,$$

a contradiction to the boundedness of $x(t)$ and $p(t)$. So $|d| < \infty$ and $z(t)$ is bounded. From (3.2.9) we have

$$x(t-\sigma) \leq \frac{1}{r(t-\sigma) - q(t-\sigma)} z'(t), \quad t \geq t_2.$$

Therefore, by (H_4), $x \in L^1[t_0, \infty)$. By the same reasoning as that used in the proof of Theorem 3.2.2, we have $\lim_{t\to\infty} x(t) = 0$. Therefore, every bounded solution x of (3.2.2) which is not in the class of $o(1)$ as $t \to \infty$ must be oscillatory.

Theorem 3.2.4 Suppose conditions (H_1), (H_3) and (H_4) hold, $q(t) < r(t)$, $q(t)$ is bounded and $\sigma < \rho$. Then (3.2.2) is bounded almost oscillatory.

Proof Without loss of generality, suppose that $x(t)$ is a bounded eventually positive solution. Let $z(t)$ be defined by (3.2.6). Take $t_2 \geq t_1$ such that $x(t-\lambda) > 0$ for $t \geq t_2$. From (3.2.2) and (H_3), we have

$$z'(t) \geq (r(t-\rho) - q(t-\rho))x(t-\rho) > 0, \quad t \geq t_2. \tag{3.2.10}$$

Hence $z(t)$ is increasing and

$$-\infty < \lim_{t\to\infty} z(t) = d \leq \infty.$$

By using the method similar to that used in the proof of Theorem 3.2.3, we have $-\infty < d < \infty$. Therefore $z(t)$ is bounded. From (3.2.10) it follows that

$$x(t-\rho) \leq \frac{1}{r(t-\rho) - q(t-\rho)} z'(t), \quad t \geq t_2.$$

Thus, by (H_4), $x \in L^1[t_0, \infty)$ and $\lim_{t\to\infty} \int_{t-\rho}^{t-\sigma} q(s)x(s)ds = 0$. Then it follows from (3.2.6) that

$$\lim_{t\to\infty} a(t)(x(t) + p(t)x(t-\tau))' = d.$$

(i) If $d > 0$, then there exists a $t_5 \geq t_2$ such that

$$a(t)(x(t) + p(t)x(t-\tau))' \geq \frac{d}{2}, \quad t \geq t_5.$$

From (H_1) we have

$$\lim_{t\to\infty} (x(t) + p(t)x(t-\tau)) = \infty,$$

which contradicts the boundedness of $x(t)$ and $p(t)$.

(ii) If $d < 0$, similar to the case (i), we have

$$\lim_{t\to\infty} (x(t) + p(t)x(t-\tau)) = -\infty,$$

a contradiction to the boundedness of $x(t)$ and $p(t)$ again. Hence $d = 0$, i.e., $\lim_{t\to\infty} z(t) = 0$. On the other hand, from (3.2.10) and $\lim_{t\to\infty} z(t) = 0$, we have $z(t) < 0$. In view of (3.2.6), $(x(t) + p(t)x(t-\tau))' < 0$ which implies that $x(t) + p(t)x(t-\tau)$ is decreasing. From $x(t) + p(t)x(t-\tau) \in L^1[t_0, \infty)$, we have $\lim_{t\to\infty}(x(t)+p(t)x(t-\tau)) = 0$. Thus $\lim_{t\to\infty} x(t) = 0$. Therefore, every bounded solution x of (3.2.2) which is not in the class of $o(1)$ as $t \to \infty$ must be oscillatory.

3.3 EXAMPLES FOR (3.2.2)

We will give three examples here to illustrate the results obtained in the last section.

Example 3.3.1 Consider the differential equation

$$\left(\left(1+\frac{1}{t}\right)(x(t)+2x(t-2\pi))'\right)' + 3\left(1+\frac{1}{t}\right)x(t-2\pi)$$
$$+3\left(1+\frac{1}{t}\right)x^5(t-2\pi) - \frac{3}{t^2}x\left(t-\frac{\pi}{2}\right) = 0. \qquad (3.3.11)$$

Viewing (3.3.11) as (3.2.2), we have $a(t) = 1 + (1/t)$, $p(t) = 2 > 0$, and

$$q(t) = 3\left(1+\frac{1}{t+2\pi}\right) > r(t) = \frac{3}{\left(t+\frac{\pi}{2}\right)^2}.$$

Moreover, $\tau = 2\pi$, $\sigma = 2\pi > \rho = \pi/2$, and $r(t)$ is bounded for $t \geq 2\pi$. Note that conditions $(H_1), (H_2)$ and (H_4) are satisfied and by Theorem 3.2.1, equation (3.3.11) is bounded oscillatory.

Example 3.3.2 Consider the differential equation

$$(t(x(t)+x(t-\pi))')' + \frac{t\pi}{t-2\pi}x(t-2\pi) - \frac{\pi}{t-\frac{7}{2}\pi}x\left(t-\frac{7}{2}\pi\right) = 0. \qquad (3.3.12)$$

Viewing (3.3.12) as (3.2.2), we have $a(t) = t$, $p(t) = 1 > 0$, and

$$q(t) = \frac{(t+2\pi)\pi}{t} > r(t) = \frac{\pi}{t}.$$

Moreover, $\tau = \pi$, $\sigma = 2\pi < \rho = 7\pi/2$, $q(t)$ is bounded for $t \geq 4\pi$. Note that conditions (H_1), (H_2) and (H_4) are satisfied. By Theorem 3.2.2, then equation (3.3.12) is almost oscillatory. Indeed, $x(t) = t\sin t$ is a unbounded oscillatory solution of equation (3.3.12).

Example 3.3.3 Consider the differential equation

$$\begin{aligned}\left(\frac{t+1-3\pi}{t-3\pi}\left(x(t)+\frac{t-\pi}{t(t+1-\pi)}x(t-\pi)\right)'\right)' &+ \\ \frac{2t+2-3\pi}{(t-3\pi)(2t-3\pi)}x\left(t-\frac{3\pi}{2}\right)-x(t-3\pi) &= 0. \qquad (3.3.13)\end{aligned}$$

Viewing (3.3.13) as (3.2.2), we have

$$\begin{aligned} a(t) &= \frac{t+1-3\pi}{t-3\pi}, \\ p(t) &= \frac{t-\pi}{t(t+1-\pi)} > 0, \\ q(t) &= \frac{2t+2}{t(2t-3\pi)} < r(t) = 1. \end{aligned}$$

Also,

$$\tau = \pi, \ \sigma = \frac{3\pi}{2} < \rho = 3\pi \text{ and } q(t) \text{ is bounded for } t \geq 3\pi.$$

We note that conditions $(H_1), (H_3)$ and (H_4) are satisfied and by Theorem 3.2.4, equation (3.3.13) is bounded almost oscillatory. In fact, $x(t) = (1+(1/t))\sin t$ is a bounded oscillatory solution of equation (3.3.13).

3.4 MAIN RESULTS WHEN $\delta = -1$

In this section, we consider equation (3.1.1) when $\delta = -1$. We rewrite equation (3.1.1) as

$$(a(t)(x(t) - p(t)x(t-\tau))')' + f(t, x(t-\sigma)) - g(t, x(t-\rho)) = 0. \qquad (3.4.14)$$

Four theorems will be given for equation (3.4.14) to be bounded oscillatory and bounded almost oscillatory.

Theorem 3.4.1 Suppose conditions (H_1), (H_3) and (H_4) hold, $p(t) \geq 1$, $q(t) < r(t)$, $\sigma \leq \rho$ and $r(t)$ is bounded. Then (3.4.14) is bounded oscillatory.

Proof Suppose $x(t)$ is a bounded non-oscillatory solution. Without loss of generality, we assume that $x(t)$ is an eventually positive solution. Let

$$z(t) = a(t)(x(t) - p(t)x(t-\tau))' + \int_{t-\rho}^{t-\sigma} q(s)x(s)ds. \qquad (3.4.15)$$

From a proof similar to that of Theorem 3.2.3, we obtain

$$z'(t) > 0, \ \lim_{t\to\infty} z(t) = c, \ |c| < \infty, \ \text{and } x \in L^1[t_0, \infty).$$

(i) If $c > 0$, from (3.4.15) it follows that

$$\lim_{t\to\infty} a(t)(x(t) - p(t)x(t-\tau))' = c > \frac{c}{2}.$$

So, for large enough t,

$$(x(t) - p(t)x(t-\tau))' \geq \frac{c}{2a(t)}.$$

Hence $\lim_{t\to\infty}(x(t) - p(t)x(t-\tau)) = \infty$ by (H_1), which contradicts the boundedness of $x(t)$ and $p(t)$.

(ii) If $c < 0$, in view of $\lim_{t\to\infty} z(t) = c$ and $x \in L^1[t_0, \infty)$, there exists a $t_6 \geq t_1$ such that

$$a(t)(x(t) - p(t)x(t-\tau))' \leq \frac{c}{2} < 0, \quad t \geq t_6.$$

Hence $\lim_{t\to\infty}(x(t) - p(t)x(t-\tau)) = -\infty$ by (H_1), a contradiction to the boundedness of $x(t)$ and $p(t)$ again.

(iii) If $c = 0$, in view of $z'(t) > 0$, we have $z(t) < 0$. Further,

$$(x(t) - p(t)x(t-\tau))' < 0.$$

We show that $x(t) - p(t)x(t-\tau) > 0$. In fact, if there exists a $t_7 \geq t_1$ such that $x(t_7) - p(t_7)x(t_7 - \tau) < 0$, then, for all $t \geq t_7$,

$$x(t) - p(t)x(t-\tau) \leq x(t_7) - p(t_7)x(t_7 - \tau) < 0.$$

This contradicts $x(t) - p(t)x(t-\tau) \in L^1[t_0, \infty)$. Hence $x(t) - p(t)x(t-\tau) > 0$ for all large $t \geq t_1$. From this and the assumption on p, we have $x(t) \geq p(t)x(t-\tau) \geq x(t-\tau)$, which contradicts $x \in L^1[t_0, \infty)$.

Therefore (3.4.14) is bounded oscillatory.

Theorem 3.4.2 Suppose conditions (H_1), (H_3) and (H_4) hold, $q(t) < r(t)$, $\sigma \geq \rho$, $0 \leq p(t) \leq p_1 < 1$ or $1 < p_2 \leq p(t)$, $r(t)$ and $1/a(t)$ are bounded. Then (3.4.14) is bounded almost oscillatory.

Proof Without loss of generality, assume that $x(t)$ is a bounded eventually positive solution. By a proof similar to that of Theorem 3.2.3, we obtain

$\lim_{t\to\infty}(x(t) - p(t)x(t-\tau)) = 0$. Suppose

$$\limsup_{t\to\infty} x(t) = l > 0.$$

So there exists a sequence $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$ and $\lim_{k\to\infty} x(t_k) = l > 0$.

(i) If $0 \le p(t) \le p_1 < 1$, then we have $(1-p_1)l \le 0$ which contradicts $l > 0$ and $1 - p_1 > 0$.

(ii) If $1 < p_2 \le p(t)$, then we have $0 \le (1-p_2)l$ which contradicts $l > 0$ and $p_2 - 1 > 0$.

Therefore, we must have

$$\limsup_{t\to\infty} x(t) = 0.$$

Then $\lim_{t\to\infty} x(t) = 0$ as $x(t)$ is eventually positive. This show that (3.4.14) is bounded almost oscillatory.

Theorem 3.4.3 Suppose conditions (H_1), (H_2) and (H_4) hold, $q(t) > r(t)$, $\sigma < \rho$, $0 \le p(t) \le p_1 < 1$ or $1 < p_2 \le p(t)$, $q(t)$ and $1/a(t)$ are bounded. Then (3.4.14) is bounded almost oscillatory.

Proof Without loss of generality, suppose that $x(t)$ is a bounded eventually positive solution. As in the proof of Theorem 3.2.2, we obtain

$$\lim_{t\to\infty}(x(t) - p(t)x(t-\tau)) = 0.$$

Then the rest follows from the proof of Theorem 3.4.2.

Theorem 3.4.4 Suppose conditions (H_1), (H_2) and (H_4) hold, $q(t) > r(t)$, $\sigma \geq \rho$, $q(t)$ is bounded, $0 \leq p(t) \leq p < 1$, or $1/a(t)$ is bounded and $1 < p_2 \leq p(t)$. Then (3.4.14) is bounded almost oscillatory.

Proof Without loss of generality, suppose $x(t)$ is a bounded eventually positive solution. Let

$$z(t) = a(t)(x(t) - p(t)x(t-\tau))' - \int_{t-\sigma}^{t-\rho} r(s)x(s)ds. \tag{3.4.16}$$

By the reasoning similar to that used in the proof of Theorem 3.2.1, we have $x \in L^1[t_0, \infty)$, $\lim_{t\to\infty} z(t) = c = 0$ and $z'(t) < 0$. So $z(t) > 0$, $(x(t) - p(t)x(t-\tau))' > 0$ and $x(t) - p(t)x(t-\tau)$ is increasing. We claim that $x(t) - p(t)x(t-\tau) < 0$ for $t \geq t_1$. In fact, if there exists a $t_8 \geq t_1$ such that $x(t_8) - p(t_8)x(t_8 - \tau) \geq 0$, then $x(t) - p(t)x(t-\tau) \geq x(t_8 + 1) - p(t_8 + 1)x(t_8 + 1 - \tau) > 0$ for $t \geq t_8 + 1$ which contradicts $x(t) - p(t)x(t-\tau) \in L^1[t_1, \infty)$. Hence $x(t) - p(t)x(t-\tau) < 0$ for all $t \geq t_1$. If $0 \leq p(t) \leq p < 1$ is satisfied, then $x(t) < px(t-\tau)$ for all $t \geq t_1$. This implies that $\lim_{t\to\infty} x(t) = 0$.

If $1/a(t)$ is bounded and $1 < p_2 \leq p(t)$, from the proof of Theorem 3.4.2, we have $\lim_{t\to\infty}(x(t) - p(t)x(t-\tau)) = 0$ and thus $\lim_{t\to\infty} x(t) = 0$.

Therefore, (3.4.14) is bounded almost oscillatory.

3.5 EXAMPLES FOR (3.4.14)

Here, we will give three examples to illustrate the results obtained in last section.

Example 3.5.1 Consider the differential equation

$$\left(\left(1-\frac{1}{t}\right)(x(t)-2x(t-\pi))'\right)'+\frac{3}{t^2}x\left(t-\frac{\pi}{2}\right)-3x(t-\pi)=0. \qquad (3.5.17)$$

Viewing (3.5.17) as (3.4.14), we have $\tau=\pi$, $\sigma=\frac{\pi}{2}<\rho=\pi$,

$$\begin{aligned} a(t) &= 1-\frac{1}{t}, \\ p(t) &= 2>0, \\ q(t) &= \frac{3}{\left(t+\frac{\pi}{2}\right)^2}<r(t)=3 \end{aligned}$$

for $t\geq\pi$. We note that conditions (H_1), (H_3) and (H_4) are satisfied and by Theorem 3.4.1, equation (3.5.17) is bounded oscillatory.

Example 3.5.2 Consider the differential equation

$$\begin{aligned} &\left(\frac{t}{t+1}\left(x(t)-\frac{2(t-2\pi)}{t}x(t-2\pi)\right)'\right)'+\frac{(t^3+t^2-2t-1)(t-\pi)}{2t^2(t+1)^2}x(t-\pi) \\ &\qquad -\frac{(2t+1)\left(t-\frac{3\pi}{2}\right)}{2t(t+1)^2}x\left(t-\frac{3\pi}{2}\right)=0. \end{aligned} \qquad (3.5.18)$$

Viewing (3.5.18) as (3.4.14), we have $\tau=2\pi$, $\sigma=\pi<\rho=3\pi/2$,

$$\begin{aligned} a(t) &= \frac{t}{t+1}, \\ p(t) &= \frac{2(t-2\pi)}{t}\geq 1.2 \quad \text{for } t\geq 5\pi, \\ q(t) &= \frac{t((t+\pi)^3+(t+\pi)^2-2t-2\pi-1)}{2(t+\pi)^2(t+\pi+1)^2}, \\ r(t) &= \frac{t(2t+3\pi+1)}{2(t+3\pi/2)(t+3\pi/2+1)^2}. \end{aligned}$$

Clearly, $q(t) > r(t)$ for large t and $1/a(t)$ and $q(t)$ are bounded. We note that conditions $(H_1), (H_2)$ and (H_4) are satisfied and by Theorem 3.4.3, equation (3.5.18) is bounded almost oscillatory.

Example 3.5.3 Consider the differential equation

$$\left(\frac{t}{t+1}\left(x(t) - \frac{t-2\pi}{2t}x(t-2\pi)\right)'\right)' + \frac{(t^3+t^2-2t-1)(t-2\pi)}{2t^2(t+1)^2}x(t-2\pi)$$
$$-\frac{(2t+1)(t-\pi/2)}{2t(t+1)^2}x(t-\pi/2) = 0. \qquad (3.5.19)$$

Regarding (3.5.19) as (3.4.14), we have $\tau = 2\pi$, $\sigma = 2\pi > \rho = \frac{\pi}{2}$,

$$\begin{aligned}
a(t) &= \frac{t}{t+1}, \\
0 < p(t) &= \frac{t-2\pi}{2t} \leq \frac{1}{2} < 1, \qquad t \geq 3\pi, \\
q(t) &= \frac{t((t+2\pi)^3 + (t+2\pi)^2 - 2t - 4\pi - 1)}{2(t+2\pi)^2(t+2\pi+1)^2}, \\
r(t) &= \frac{t(2t+\pi+1)}{2(t+\pi/2)(t+\pi/2+1)^2}.
\end{aligned}$$

Clearly, $q(t) > r(t)$ for large enough t and $q(t)$ is bounded. Notice that $(H_1), (H_2)$ and (H_4) are satisfied therefore by Theorem 3.4.4, equation (3.5.19) is bounded almost oscillatory.

3.6 CONCLUSION

The objective of this chapter is to study the oscillation of second order non-linear neutral differential equations (3.1.1). We are interested in the nontrivial solutions in this chapter. Since (3.1.1) can be either (3.2.2) or (3.4.14), we investigate (3.2.2) and (3.4.14), separately, rather than (3.1.1) itself. In former case, a function $z(t)$ has been defined as (3.2.3) or (3.2.6). We have managed to establish four sufficient conditions for (3.2.2) to be bounded oscillatory, almost oscillatory, and bounded almost oscillatory. The results have been presented in Theorems 3.2.1-3.2.4. Examples are given to demonstrate the applications of the results in every case. For (3.4.14), we define $z(t)$ as (3.4.15) or (3.4.16). Theorem 3.4.1 is a sufficient condition for (3.4.14) to be bounded oscillatory. Theorems 3.4.2, 3.4.3 and 3.4.4 give sufficient conditions for (3.4.14) to be bounded almost oscillatory. Three examples are presented to illustrate the results.

It is not hard to see that the results in this chapter are more general than the results for linear and nonlinear ordinary differential equations given in the reference papers. For example, the equation

$$(x(t) - px(t-\tau))'' + qx(t-\sigma) = 0$$

with $p(t) \equiv p$ and $q(t) \equiv q > 0$ is a special case of (3.1.1) and was studied in [26]. When $a(t) \equiv 1$ and $0 \leq p(t) < 1$, the results in [19] may cover some of the results in this chapter. However, it does not affect the generality of the results here. Equation (3.1.1) was investigated in [49] when $\delta = 1$ and $g(t,v) = 0$. Since we have assumed $g(t,v) \neq 0$ if $v \neq 0$, the results obtained here are not comparable with those given in [49]. Further improvement are expected in future for the oscillatory conditions and for higher order equations.

Chapter 4

EVEN ORDER DIFFERENCE EQUATIONS

4.1 INTRODUCTION

In this chapter and chapter 5, we are concerned with the nonlinear neutral difference equation

$$\Delta_\tau^m(x(t) - px(t-r)) + f(t, x(g(t))) = 0, \tag{4.1.1}$$

where $m \geq 2$ is a natural number, $p \geq 0$, τ and r are positive constants, $\Delta_\tau x(t) = x(t+\tau) - x(t)$, $0 < g(t) < t$, $g \in C^1([t_0, \infty), R^+)$, $g'(t) > 0$, and $f \in C([t_0, \infty) \times R, R)$. Throughout this chapter and the next we assume that

$$g(t+\tau) \geq g(t) + \tau \qquad \text{for} \quad t \geq t_0 \tag{4.1.2}$$

and

$$f(t,\ u)/u \geq q(t) > 0 \ \text{ for } \ u \neq 0 \ \text{ and some } \ q \in C(R, R^+). \tag{4.1.3}$$

Let $t_0' = \min\{g(t_0),\ t_0 - r\}$ and $I_0 = [t_0',\ t_0]$. A function x is called the *solution* of (4.1.1) with $x(t) = \varphi(t)$ for $t \in I_0$ and $\varphi \in C(I_0, R)$ if it satisfies (4.1.1) for $t \geq t_0$.

The properties of the solutions of (4.1.1) are very different between even order and odd order equations. Therefore, in this chapter we will just investigate equation (4.1.1) with even order. Three cases will be considered here, i.e., $m = 2$, $m = 4$, and m is any even number with $m > 4$. Equation (4.1.1) with the odd order will be discussed in chapter 5.

There has been an increasing interest in study of the oscillation behaviour of solutions of difference equations. See [53] - [39] for examples. Particularly, Stavroulakis [53] considered the delayed difference equation

$$\Delta x_n + p_n x_{n-k} = 0, \qquad n \in \bar{N}$$

and established oscillation criteria for this equation. Zhang and Yan [64] studied difference equations with continuous argument

$$\Delta_\tau y(t) + p(t)y(t-\sigma) = 0$$

and obtained oscillation criteria. Zhang, Yan and Zhao [68] also considered the above equation and established some new results for the oscillation. Zhang, Yan and Choi [63] investigated the equation

$$\Delta y(t) + p(t)H(y(t-\sigma)) = f(t), \qquad t \geq 0,$$

and obtained oscillation criteria for this equation. The system of delayed difference equations

$$\Delta_\tau x_i(t) + \sum_{k=1}^{l}\sum_{j=1}^{n} p_{ijk}x_j(t-\sigma_k) = 0, \qquad i,j \in N(1,n) \text{ and } k \in N(1,l)$$

was studied by Yan and Zhang [59]. Sufficient conditions were obtained for all solutions of this system to be oscillatory. Zhang, Chen and Zhang [66] considered the second order nonlinear difference equation with nonlinear neutral term

$$\Delta(a_n\Delta(x_n + \phi(n, x_{\tau_n}))) + q_n f(x_{g_n}) = 0$$

and obtained oscillation criteria for the above equation. Similarly, Zhang and Zhang [69] discussed

$$\Delta(a(n)\Delta x(n)) + p(n)x(g(n)) = 0$$

and established some sufficient conditions for the above equation to be oscillatory. Zhang, Bi and Chen [65] were concerned with the second order nonlinear difference equation with continuous variable of the form

$$\Delta_\tau^2 x(t) + f(t, x(t-\tau)) = 0$$

and some oscillatory criteria for the above equation were given.

Equation (4.1.1) is a neutral generalization of some of the above equations. Comments made in later sections will confirm this. The qualitative study of solutions of neutral difference equation is developing very fast recently. According to our best knowledge, however, we believe that there is no result about the fourth order, higher even order, third order, and higher odd order neutral difference equation with continuous variable so far.

4.2 PRELIMINARIES

Throughout this chapter, chapter 5 and chapter 6, we use the symbol $\lceil a \rceil$ to denote the smallest integer not less than a. The lemmas in this section will be needed in both section 4.4 and section 4.5. The following lemma can be found in [2] (page 31) and will be needed in the proof of Lemma 4.2.2.

Lemma 4.2.1 *(Discrete Keneser's Theorem)* Let $u(k)$ be defined on $N(a)$, where $a \in N$, and $u(k) > 0$ with $\Delta^m u(k)$ of constant sign on $N(a)$ for any positive integer m and not identically zero. Then, there exists an integer h, $0 \leq$

$h \leq m$, with $m+h$ odd for $\Delta^m u(k) \leq 0$ or $m+h$ even for $\Delta^m u(k) \geq 0$ such that

(i) $h \leq m-1$ implies $(-1)^{h+i}\Delta^i u(k) > 0$ for all $k \in N(a)$, $h \leq i \leq m-1$,

(ii) $h \geq 1$ implies $\Delta^i u(k) > 0$ for all $k \in N(a)$, $1 \leq i \leq h-1$.

To obtain the oscillatory behaviour of solutions for all large enough t, we need to know the features of the difference when t is sufficiently large. By applying the above lemma into the difference with continuous variables, we extend the above result for discrete difference to the following lemma for difference with continuous arguments.

Lemma 4.2.2 Let $y(t)$ be defined on $[t_0, +\infty)$ where $t_0 \in R$, and $y(t) > 0$ with $\Delta_\tau^m y(t)$ of constant sign on $[t_0, +\infty)$ for any positive integer m and not identically zero. Then, there exists an integer h, $0 \leq h \leq m$, with $m+h$ odd for $\Delta_\tau^m y(t) \leq 0$ or $m+h$ even for $\Delta_\tau^m y(t) \geq 0$ such that

(i) $h \leq m-1$ implies $(-1)^{h+i}\Delta_\tau^i y(t) > 0$ for all $t \in [t_0, \infty)$, $h \leq i \leq m-1$,

(ii) $h \geq 1$ implies $\Delta_\tau^i y(t) > 0$ for all $t \in [t_0, +\infty)$, $1 \leq i \leq h-1$.

Proof Let t' be any constant real number in $[t_0, +\infty)$. For this fixed t', by the assumption, we have $y(t' + k\tau)$ defined for any $k \in \{0, 1, \cdots\}$, and $y(t' + k\tau) > 0$ with $\Delta_\tau^m y(t' + k\tau)$ of constant sign for any $k \in \{0, 1, \cdots\}$ and for any positive integer m and not identically zero. Thus, by Lemma 4.2.1, the conclusion holds with the replacement of t by $t' + k\tau$ for all $k \in N$. Since $t' \in [t_0, \infty)$

is arbitrary, we can see that the conclusion holds for $t \in [t_0, +\infty)$.

Lemma 4.2.3 Let $y(t)$ be an m times differentiable function on R_+ of constant sign satisfying $y^{(m)}(t) \not\equiv 0$ and $y^{(m)}(t)y(t) \leq 0$ on $[t_1, \infty)$. Then the following statements hold.

(i) There exists a $t_2 \geq t_1$ such that the functions $y^{(j)}(t)$, $j = 1, 2, \ldots, m-1$, are of constant sign on $[t_2, \infty)$.

(ii) There exists an integer $k < m$ which is odd (even) when m is even (odd), such that

$$y(t)y^{(j)}(t) > 0 \quad \text{for} \quad j = 0, 1, \ldots, k, \;\; t \geq t_2,$$

$$(-1)^{m+j+1}y(t)y^{(j)}(t) > 0 \quad \text{for} \quad j = k+1, \ldots, m, \;\; t \geq t_2.$$

Lemma 4.2.3 can be found in [2] (page 289).

Lemma 4.2.4 Assume that $y(t)$, $y'(t)$, $\ldots$, $y^{(m-1)}(t)$ are absolutely continuous and of constant sign on the interval (t_0, ∞), and assume $y^{(m)}(t)y(t) \geq 0$. Then either $y^{(k)}(t)y(t) \geq 0$, $k = 0, 1, \ldots, m$ or there exists an integer l, $0 \leq l \leq m-2$, which is even (odd) when m is even (odd), such that

$$y^{(k)}(t)y(t) \geq 0, \quad \text{for} \quad k = 0, 1, \ldots, l,$$

$$(-1)^{m+k}y^{(k)}(t)y(t) \geq 0, \quad \text{for} \quad k = l+1, \ldots, m.$$

The above lemma can be found in [15] (page 289).

4.3 SECOND ORDER EQUATION (4.1.1)

In this section, we are mainly concerned with equation (4.1.1) when $m = 2$, i.e.,

$$\Delta_\tau^2(x(t) - px(t-r)) + f(t, x(g(t))) = 0. \tag{4.3.4}$$

We shall give some criteria and remarks in subsection 4.3.1. In subsection 4.3.2, we will present the illustrating examples. To prove the criteria, in subsection 4.3.3 we shall state some lemmas. Following this subsection, the proofs of the theorems will be given in subsection 4.3.4.

Note that this section is a modified version of the published paper [56] under the joint authorship of S. Wu and Z. Hou. This reflects the contribution of the second author in the process of refining the many previous drafts and extending and sharping the original rough results.

4.3.1 OSCILLATION CRITERIA

The assumptions given in section 4.1 guarantee the existence and differentiability of the inverse g^{-1} of g. Let

$$\bar{q}_2(t) = \alpha \min_{t\le s\le t+2\tau}\{q(s)\}\left(\min_{g(t)\le s\le g(t)+2\tau}\{(g^{-1}(s))'\}\right)^2, \tag{4.3.5}$$

where $0 < \alpha < 1$. We shall see below that oscillatory behaviour of the solutions of (4.3.4) can be determined by conditions involving the function $\bar{q}_2$. Let

$$z(t) = \int_t^{t+\tau} ds \int_s^{s+\tau} x(\theta)d\theta,$$

where x denotes any solution of (4.3.4). Then $z''(t) = \Delta_\tau^2 x(t)$.

Theorem 4.3.1 Assume that

$$\sum_{i=0}^{\infty} \bar{q}_2(t' + i\tau) = \infty \tag{4.3.6}$$

for some $t' \geq t_0$. Then every solution of (4.3.4) is either oscillatory or eventually satisfies $|z(t)| < p|z(t-r)|$.

Remark 4.3.1 A special non-neutral case included in (4.3.4) is when $p = 0$. In this case, condition (4.3.6) implies that every solution is oscillatory. In [65], the authors obtained oscillation criteria for a class of equations of the form

$$\Delta_\tau^2 x(t) + f(t, x(t-\sigma)) = 0.$$

We can see that even the special case of our Theorem 4.3.1 can be applied to a larger class of equations than the above.

Theorem 4.3.2 In addition to (4.3.6), we assume that $0 < p < 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_n = \lceil (g(t_1+n\tau)-t_1+k_0 r)/\tau \rceil \leq n$ and

$$\sum_{s=m_n}^{n} (s + 1 - m_n)\bar{q}_2(t_1 + s\tau) \geq \frac{(1-p)p^{k_0}}{1-p^{k_0}} \tag{4.3.7}$$

for large enough n. Then every solution x of (4.3.4) is oscillatory.

Theorem 4.3.3 In addition to (4.3.6), we assume that $p = 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_n = \lceil (g(t_1+n\tau) - t_1 + k_0 r)/\tau \rceil \leq n$

and

$$\sum_{s=m_n}^{n}(s+1-m_n)\bar{q}_2(t_1+s\tau) \geq \frac{1}{k_0} \tag{4.3.8}$$

for large enough n. Then every solution x of (4.3.4) is oscillatory.

Theorem 4.3.4 Under the conditions of Theorem 4.3.2 with the replacement of $0<p<1$ by $p>1$, every bounded solution x of (4.3.4) is oscillatory.

4.3.2 EXAMPLES

In this subsection, two illustrating examples are given to demonstrate the results obtained in last subsection.

Example 4.3.1 Consider the linear difference equation

$$\Delta_\tau^2(x(t)-px(t-r))+\frac{1}{t}x\left(t-\frac{\sigma}{1+\beta t}\right)=0 \tag{4.3.9}$$

for $t \geq 0$, where $p \geq 0$ and $\beta \geq 0$, r, τ and σ are positive constants. Viewing (4.3.9) as (4.3.4), we have $q(t)=1/t$ and $g(t)=t-\sigma/(1+\beta t)$. Then, by (4.3.5), $\bar{q}_2(t)=\alpha/(t+2\tau)$ for $\beta=0$ and

$$\bar{q}_2(t)=\frac{\alpha}{t+2\tau}\left(1-\frac{\sigma\beta}{(1+\beta t)^2+\sigma\beta}\right)^2$$

for $\beta>0$. Since $\bar{q}_2(t) \geq \alpha'/(t+2\tau)$ for some $\alpha'>0$ and all $t \geq 0$, $\bar{q}_2$ satisfies (4.3.6) with $t'=0$. By Theorem 4.3.1, every solution of (4.3.9) is either oscillatory or eventually satisfies $|z(t)|<p|z(t-r)|$. In particular, when $p=0$, every solution of (4.3.9) is oscillatory. It was shown in [65] that every solution of the

equation $\Delta_\tau^2 x(t)+t^{-1}x(t-\sigma)=0$ is oscillatory. Clearly, this equation is a special case of (4.3.9) when $p=\beta=0$.

Example 4.3.2 Consider the difference equation

$$\Delta_\pi^2(x(t)-px(t-\pi))+8x(t-\pi)+\frac{8\sigma}{1+t^2}x^3(t-\pi)=0, \tag{4.3.10}$$

where $\sigma \geq 0$ is a constant. Regarding (4.3.10) as (4.3.4), we have $\tau=\pi$, $r=\pi$, $g(t)=t-\pi$ and $q(t)=8$. Then, for any $\alpha \in (0,1)$, $\bar{q}_2=8\alpha$ by (4.3.5) so (4.3.6) is satisfied. For $p=1$, $k_0=1$ and $t_1=t$, we have $m_n=n$ and

$$\sum_{s=m_n}^{n}(s+1-m_n)\bar{q}_2(t_1+s\tau)=8\alpha>1=\frac{1}{k_0}$$

if $\alpha>1/8$. Moreover, we also have

$$\sum_{s=m_n}^{n}(s+1-m_n)\bar{q}_2(t_1+s\tau)=8\alpha>p=\frac{(1-p)p^{k_0}}{1-p^{k_0}}$$

if $p \in (0,1)\cup(1,8)$ and $\alpha>p/8$. By Theorems 4.3.1-4.3.4 every solution of (4.3.10) is oscillatory if $0 \leq p \leq 1$ and every bounded solution of (4.3.10) is oscillatory if $1<p<8$. If $p>8$, then (4.3.10) with $\sigma=0$ has a bounded positive solution $x(t)=\lambda^t$, where $y=\lambda^\pi$ is a root of $(y-p)(y-1)^2+8$ in $(0,1)$. Also, for $p \geq 1+3\sqrt[3]{2}$ and $\sigma=0$, (4.3.10) has an unbounded positive solution $x(t)=\lambda^t$ for some $\lambda>1$.

4.3.3 SOME LEMMAS

To prove the results given in subsection 4.3.1, we need the following lemmas.

Lemma 4.3.1 Assume that x is an eventually positive (negative) solution of (4.3.4) not satisfying $z(t) - pz(t-r) \to -\infty\,(\infty)$ as $t \to \infty$. Then, with $u(t) = z(t) - pz(t-r)$, $u''(t) < 0\,(> 0)$, $u'(t) > 0\,(< 0)$, u is increasing (decreasing) and satisfies

$$u''(t) + q(t)x(g(t)) \leq 0\,(\geq 0) \tag{4.3.11}$$

for t large enough.

Proof Suppose x is an eventually positive solution. From the assumption and (4.3.4) we have $x(g(t)) > 0$ and (4.3.11) for large enough t. So there is a $T \geq t_0$ such that $u''(t) < 0$ for $t \geq T$. We claim that $u'(t) > 0$ for $t \geq T$. Indeed, if not so, there is a $t_1 \geq T$ such that $u'(t_1) \leq 0$. Since $u''(t) < 0$ for $t \geq T$, we have $u'(t) \leq u'(t_1 + 1) < u'(t_1) \leq 0$ for $t \geq t_1 + 1$. This implies $u(t) \to -\infty$ as $t \to \infty$, which contradicts the assumption. Therefore, $u'(t) > 0$ for $t \geq T$ so that u is increasing. When x is an eventually negative solution, the parallel conclusions within brackets follow obviously.

Lemma 4.3.2 Suppose x is an eventually positive (negative) solution of (4.3.4) not satisfying $z(t) - pz(t-r) \to -\infty\,(\infty)$ as $t \to \infty$. Then $\Delta_\tau^2 u(t) < 0\,(> 0)$, $\Delta_\tau u(t) > 0\,(< 0)$, u is increasing (decreasing) and, for every integer $k \geq 0$, satisfies

$$\Delta_\tau^2 u(t) + \bar{q}_2(t) \sum_{i=0}^{k} p^i u(g(t) - kr) \leq 0\,(\geq 0) \tag{4.3.12}$$

for sufficiently large t.

Proof Suppose x is an eventually positive solution. From Lemma 4.3.1 and the assumption, we have $\Delta_\tau^2 u(t) < 0$, $\Delta_\tau u(t) > 0$, $x(g(t)) > 0$ and (4.3.11) for some $T \geq t_0$ and all $t \geq T$. Then, for $t \geq T$, the assumptions on g and q give

$$\begin{aligned}
& \int_t^{t+\tau} ds \int_s^{s+\tau} x(g(\theta))q(\theta)d\theta \\
\geq & \min_{t \leq l \leq t+2\tau} \{q(l)\} \int_t^{t+\tau} ds \int_s^{s+\tau} x(g(\theta))d\theta \\
\geq & \min_{t \leq l \leq t+2\tau} \{q(l)\} \int_{g(t)}^{g(t+\tau)} (g^{-1}(s))'ds \int_s^{g(g^{-1}(s)+\tau)} x(\theta)(g^{-1}(\theta))'d\theta \\
\geq & \min_{t \leq l \leq t+2\tau} \{q(l)\} \int_{g(t)}^{g(t)+\tau} (g^{-1}(s))'ds \int_s^{s+\tau} x(\theta)(g^{-1}(\theta))'d\theta \\
\geq & \bar{q}_2(t)z(g(t)).
\end{aligned}$$

Hence, integrating (4.3.11), we have

$$\Delta_\tau^2 u(t) + \bar{q}_2(t)z(g(t)) \leq 0, \tag{4.3.13}$$

so

$$\Delta_\tau^2 u(t) + \bar{q}_2(t)\sum_{i=0}^{k-1} p^i u(g(t) - ir) + \bar{q}_2(t)p^k z(g(t) - kr) \leq 0$$

for $t \geq T$ and every integer $k > 0$. Then (4.3.12) follows from this for large enough t as $\bar{q}_2(t)p^k z(g(t) - kr) > 0$ and u is increasing. When x is an eventually negative solution, the conclusions within brackets follow in the same way.

4.3.4 PROOF OF THEOREMS

Here, the proofs of Theorems 4.3.1-4.3.4 will be given.

Proof of Theorem 4.3.1 Suppose the conclusion does not hold. Let x be an eventually positive solution of (4.3.4) not eventually satisfying $z(t) < pz(t-r)$.

By Lemmas 4.3.1 and 4.3.2, there is a $T \geq t_0$ such that $u''(t) < 0$, $u'(t) > 0$ and, for any positive integer k, (4.3.12) holds for $t \geq T$. As $u(t) < 0$ is not eventually satisfied and u is increasing, we may assume $u(t) > 0$ for $t \geq T$. Take $T_1 > T$ such that $g(t) - kr \geq T$ for $t \geq T_1$. Note that $(\Delta_\tau u(t))' = \Delta_\tau u'(t) < 0$ so $\Delta_\tau u(t)$ is decreasing. Hence, by (4.1.2),

$$u(g(t+\tau) - kr) - u(g(t) - kr) \geq \Delta_\tau u(g(t) - kr) > \Delta_\tau u(t) > 0$$

for $t \geq T_1$. Then, by the Riccati transformation

$$v(t) = \frac{\Delta_\tau u(t)}{u(g(t) - kr)}, \tag{4.3.14}$$

we have $v(t) > 0$ and

$$\Delta_\tau v(t) < \frac{\Delta_\tau^2 u(t)}{u(g(t) - kr)} - v(t)v(t+\tau) < 0.$$

Thus, $v(t) > v(t+\tau)$. Further, from (4.3.12),

$$\Delta_\tau v(t) < -\bar{q}_2(t) \sum_{i=0}^{k} p^i - v^2(t+\tau)$$

so

$$\Delta_\tau v(t) + \bar{q}_2(t) \sum_{i=0}^{k} p^i + v^2(t+\tau) < 0 \tag{4.3.15}$$

for $t \geq T_1$. There is an integer $K > 0$ such that $t' + K\tau \geq T_1$. Now replacing t by $t' + j\tau$ and summing up both sides of (4.3.15) for j from K to n, we have

$$v(t' + (n+1)\tau) - v(t' + K\tau) + \sum_{j=K}^{n} \bar{q}_2(t' + j\tau) \sum_{i=0}^{k} p^i + \sum_{j=K}^{n} v^2(t' + (j+1)\tau) < 0.$$

Therefore, for all $n \geq K$,

$$\sum_{j=K}^{n} \bar{q}_2(t' + j\tau) \sum_{i=0}^{k} p^i < v(t' + K\tau) < \infty.$$

This contradicts (4.3.6). Thus, every eventually positive solution x must satisfy $z(t) < pz(t-r)$ eventually. Now let x be an eventually negative solution of (4.3.4) not eventually satisfying $z(t) > pz(t-r)$. Then, from Lemmas 4.3.1 and 4.3.2, the above argument is still valid with necessary changes of "decreasing" to "increasing" and of inequalities to opposite directions before (4.3.14). Then the contradiction proves the conclusion of the theorem.

Proof of Theorem 4.3.2 Suppose the conclusion does not hold. Without loss of generality, let x be an eventually positive solution of (4.3.4). If $\lim_{t\to\infty} u(t) = -\infty$, then $u(t) = z(t) - pz(t-r) \le 0$ for large enough t. Using this repeatedly and by the condition $0 < p < 1$, we obtain $\lim_{t\to\infty} z(t) = 0$ and $\lim_{t\to\infty} u(t) = 0$. This contradiction shows that $u(t) \not\to -\infty$ as $t \to \infty$. Thus, the conclusions of Lemmas 4.3.1 and 4.3.2 hold. From (4.3.13), we have

$$\Delta_\tau^2 u(t) - \frac{\bar{q}_2(t)}{p} u(g(t)+r) + \frac{\bar{q}_2(t)}{p} z(g(t)+r) \le 0.$$

Using the same technique as that used in the proof of Lemma 4.3.2, we obtain

$$\Delta_\tau^2 u(t) - \bar{q}_2(t) \sum_{i=1}^{k} \frac{1}{p^i} u(g(t)+i\tau) \le 0. \tag{4.3.16}$$

As u is increasing and $\sum_{i=1}^k 1/p^i = (1-p^k)/[p^k(1-p)]$, (4.3.16) leads to

$$\Delta_\tau^2 u(t) \le \bar{q}_2(t) \frac{1-p^k}{p^k(1-p)} u(g(t)+kr). \tag{4.3.17}$$

Replacing k by k_0 and t by $t_1 + i\tau$ in (4.3.17) and summing up both sides for i from s to n, we have

$$\Delta_\tau u(t_1 + (n+1)\tau) - \Delta_\tau u(t_1 + s\tau) \le \frac{1-p^{k_0}}{p^{k_0}(1-p)} \sum_{i=s}^{n} \bar{q}_2(t_1+i\tau) u(g(t_1+i\tau) + k_0 r).$$

Then, summing up the above inequality for s from m_n to n, we obtain

$$\Delta_\tau u(t_1 + (n+1)\tau)(n+1-m_n) - u(t_1+(n+1)\tau) + u(t_1+m_n\tau)$$
$$\leq \frac{1-p^{k_0}}{p^{k_0}(1-p)} \sum_{s=m_n}^{n} \sum_{i=s}^{n} \bar{q}_2(t_1+i\tau)u(g(t_1+i\tau)+k_0 r).$$

Combining this with $u(g(t_1+i\tau)+k_0r) \leq u(t_1+m_n\tau)$ gives

$$\Delta_\tau u(t_1+(n+1)\tau)(n+1-m_n) - u(t_1+(n+1)\tau)$$
$$\leq u(t_1+m_n\tau)\left\{\frac{1-p^{k_0}}{p^{k_0}(1-p)} \sum_{s=m_n}^{n} (s+1-m_n)\bar{q}_2(t_1+s\tau) - 1\right\}.$$

This inequality holds for large enough n as (4.3.17) holds for large enough t. By Theorem 4.3.1 and Lemma 4.3.2, $u(t) < 0$ and $\Delta_\tau u(t) > 0$ for large enough t. Hence, from the above inequality, we have

$$\sum_{s=m_n}^{n} (s+1-m_n)\bar{q}_2(t_1+s\tau) < \frac{p^{k_0}(1-p)}{1-p^{k_0}}$$

for sufficiently large n. This contradiction to (4.3.7) shows that every solution of (4.3.4) is oscillatory.

Proof of Theorem 4.3.3 The proof of Theorem 4.3.2 up to (4.3.17) is still valid when $z(t) \to 0$ and $u(t) \to 0$ as $t \to \infty$ is replaced by the boundedness of z and u due to $p = 1$. With $p = 1$, (4.3.16) and (4.3.17) now become

$$\Delta_\tau^2 u(t) - \bar{q}_2(t) \sum_{i=1}^{k_0} u(g(t)+ir) \leq 0 \tag{4.3.18}$$

and

$$\Delta_\tau^2 u(t) \leq k_0 \bar{q}_2(t) u(g(t)+k_0 r). \tag{4.3.19}$$

Replacing t by $t_1 + i\tau$ in (4.3.19) and using the same technique as that in the proof of Theorem 4.3.2, we obtain

$$\Delta_\tau u(t_1 + (n+1)\tau)(n+1-m_n) - u(t_1 + (n+1)\tau)$$

$$\leq u(t_1 + m_n\tau)\left\{k_0 \sum_{s=m_n}^{n} (s - m_n + 1)\bar{q}_2(t_1 + s\tau) - 1\right\}.$$

As $u(t) < 0$ and $\Delta_\tau u(t) > 0$ for large enough t, we must have

$$\sum_{s=m_n}^{n} (s - m_n + 1)\bar{q}_2(t_1 + s\tau) < \frac{1}{k_0}$$

for large enough n, which contradicts (4.3.8). Therefore, every solution of (4.3.4) is oscillatory.

Proof of Theorem 4.3.4 Without loss of generality, suppose (4.3.4) has a bounded eventually positive solution x so z is bounded. Then $\lim_{t\to\infty} u(t) \neq -\infty$. The rest is the same as the proof of Theorem 4.3.2.

4.4 FOURTH ORDER EQUATION (4.1.1)

In this section, we will discuss equation (4.1.1) with $m = 4$, i.e.,

$$\Delta_\tau^4(x(t) - px(t-r)) + f(t, x(g(t))) = 0. \tag{4.4.20}$$

The assumptions given in section 4.1 guarantee the existence and differentiability of the inverse g^{-1} of g. Let

$$\bar{q}_4(t) = \alpha \min_{t\leq s\leq t+4\tau}\{q(s)\}\left(\min_{g(t)\leq s\leq g(t)+4\tau}\{(g^{-1}(s))'\}\right)^4, \tag{4.4.21}$$

where $0 < \alpha < 1$. We will see in subsection (4.4.2) that oscillatory behaviour of the solutions of (4.4.20) can be determined by conditions involving the function $\bar{q}_4$.

To prove the main results, lemmas will be presented in subsection 4.4.1. Following this subsection, we will state the oscillatory criteria in subsection 4.4.2. Examples will be given in subsection 4.4.3 to illustrate the obtained results.

4.4.1 RELATED LEMMAS

The lemmas in this subsection will be needed to establish the oscillatory behaviour of solutions for (4.4.20).

Lemma 4.4.1 Assume that $x(t)$ is an eventually positive (negative) solution of (4.4.20) such that $y(t) = x(t) - px(t-r) > 0\,(< 0)$ eventually. Then $\Delta_\tau y(t) > 0\,(< 0)$, $\Delta_\tau^3 y(t) > 0\,(< 0)$ and $\Delta_\tau^4 y(t) < 0\,(> 0)$ hold eventually.

Proof Suppose $x(t)$ is eventually positive. Note that $x(t) > 0$ and $y(t) > 0$ eventually. By $g(t) < t$, $g'(t) > 0$ and (4.1.2), there exists a $t_1 > t_0$ such that $x(g(t)) > 0$ for all $t > t_1$. Further, from (4.4.20), we have

$$\Delta_\tau^4 y(t) + f(t, x(g(t))) = 0.$$

By (4.1.3), we obtain $f(t, x(g(t))) \geq q(t)x(g(t)) > 0$ for $t > t_1$. Therefore

$$\Delta_\tau^4 y(t) \leq -q(t)x(g(t)) < 0 \qquad (4.4.22)$$

for all large enough t, namely, $\Delta_\tau^4 y(t) < 0$ eventually and $\Delta_\tau^4 y(t)$ is not identically zero. Then, by Lemma 4.2.2, we obtain $\Delta_\tau y(t) > 0$ and $\Delta_\tau^3 y(t) > 0$. Suppose $x(t)$ is eventually negative with $y(t) < 0$ eventually. Then (4.4.22) becomes $\Delta_\tau^4 y(t) \geq -q(t)x(g(t)) > 0$. Replacing $y(t)$ by $-y(t)$ in Lemma 4.2.2, we have $\Delta_\tau y(t) < 0$ and $\Delta_\tau^3 y(t) < 0$.

Lemma 4.4.2 Let the hypothesis of Lemma 4.4.1 be satisfied and let $\bar{q}_4(t)$ be defined by (4.4.21). Set

$$u(t) = \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \int_{t_2}^{t_2+\tau} dt_3 \int_{t_3}^{t_3+\tau} y(\theta)d\theta.$$

Then u satisfies $u^{(4)}(t) = \Delta_\tau^4 y(t) < 0\,(> 0), u(t) > 0\,(< 0), u'(t) > 0\,(< 0)$, $u^{(3)}(t) > 0\,(< 0)$, and

$$\Delta_\tau^4 u(t) + \bar{q}_4(t)u(g(t) - kr)\sum_{i=0}^{k} p^i \leq 0\,(\geq 0)$$

for each fixed natural number k and for all large enough t.

Proof Suppose $x(t)$ is an eventually positive solution. From Lemma 4.4.1, we know $y(t) > 0$ eventually. Then by the definition of $u(t)$, we can see that $u(t) > 0$ and $u^{(4)}(t) = \Delta_\tau^4 y(t) < 0$ eventually.

From (4.4.22), we have

$$\Delta_\tau^4 y(t) + q(t)x(g(t)) \leq 0. \tag{4.4.23}$$

From the definition of $y(t)$ and (4.4.23), we obtain

$$\Delta_\tau^4 y(t) + q(t)\left(y(g(t)) + px(g(t) - r)\right) \leq 0.$$

Repeating the above process, we have

$$\Delta_\tau^4 y(t) + q(t)\sum_{i=0}^{k} p^i\, y(g(t)-ir) + q(t)\, p^{k+1} x(g(t)-(k+1)r) \le 0.$$

Therefore, since $q(t)\, p^{k+1} x(g(t)-(k+1)r) \ge 0$, the above inequality implies

$$\Delta_\tau^4 y(t) + q(t)\sum_{i=0}^{k} p^i y(g(t)-ir) \le 0.$$

Hence,

$$u^{(4)}(t) + q(t)\sum_{i=0}^{k} p^i\, y(g(t)-ir) \le 0. \tag{4.4.24}$$

Then, for large enough t, the assumptions on g and q give

$$\begin{aligned}
&\int_t^{t+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \int_{s_2}^{s_2+\tau} ds_3 \int_{s_3}^{s_3+\tau} y(g(\theta)-ir)q(\theta)d\theta \\
&\ge \min_{t\le l\le t+4\tau}\{q(l)\} \int_t^{t+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \int_{s_2}^{s_2+\tau} ds_3 \int_{s_3}^{s_3+\tau} y(g(\theta)-ir)d\theta \\
&\ge \min_{t\le l\le t+4\tau}\{q(l)\} \int_{g(t)}^{g(t+\tau)} (g^{-1}(s_1))'ds_1 \int_{s_1}^{g(g^{-1}(s_1)+\tau)} (g^{-1}(s_2))'ds_2 \\
&\qquad \int_{s_2}^{g(g^{-1}(s_2)+\tau)} (g^{-1}(s_3))'ds_3 \int_{s_3}^{g(g^{-1}(s_3)+\tau)} y(\theta-ir)(g^{-1}(\theta))'d\theta \\
&\ge \min_{t\le l\le t+4\tau}\{q(l)\} \left(\min_{g(t)\le s\le g(t)+4\tau} (g^{-1}(s))'\right)^4 \int_{g(t)}^{g(t)+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \int_{s_2}^{s_2+\tau} ds_3 \\
&\qquad \int_{s_3}^{s_3+\tau} y(\theta-ir)d\theta \\
&\ge \min_{t\le l\le t+4\tau}\{q(l)\} \left(\min_{g(t)\le s\le g(t)+4\tau} (g^{-1}(s))'\right)^4 u(g(t)-ir) \\
&\ge \bar{q}_4(t)u(g(t)-ir).
\end{aligned}$$

Hence, integrating (4.4.24), we obtain

$$\Delta_\tau^4 u(t) + \bar{q}_4(t)\sum_{i=0}^{k} p^i\, u(g(t)-ir) \le 0.$$

By Lemma 4.2.3, we know that $u'(t) > 0$ and $u^{(3)}(t) > 0$. Thus

$$\Delta_\tau^4 u(t) + \bar{q}_4(t)u(g(t)-kr)\sum_{i=0}^{k} p^i \le 0$$

holds for each fixed natural number k and for all large enough t. Now suppose $x(t)$ is an eventually negative solution. Then from Lemma 4.4.1 and the definition of u we have $y(t) < 0$, $u(t) < 0$ and $u^{(4)}(t) = \Delta_\tau^4 y(t) > 0$ eventually. From the inequalities parallel to (4.4.22) we obtain

$$\Delta_\tau^4 y(t) + q(t)x(g(t)) \geq 0$$

so we have the inequalities

$$u^{(4)}(t) + q(t)\sum_{i=0}^{k} p^i y(g(t) - ir) \geq 0$$

and

$$\Delta_\tau^4 u(t) + \bar{q}_4(t)\sum_{i=0}^{k} p^i u(g(t) - ir) \geq 0.$$

Applying Lemma 4.2.3 to $-u(t)$, we have $u'(t) < 0$, $u^{(3)}(t) < 0$ and

$$\Delta_\tau^4 u(t) + \bar{q}_4(t)u(g(t) - kr)\sum_{i=0}^{k} p^i \geq 0.$$

4.4.2 MAIN RESULTS

By the above lemmas, we are now able to obtain the following theorems.

Theorem 4.4.1 Assume that, for some $t' \geq t_0$,

$$\sum_{i=0}^{n} \bar{q}_4(t' + i\tau) \to \infty \tag{4.4.25}$$

as $n \to \infty$. Then every solution $x(t)$ of (4.4.20) is either oscillatory or, for any $T \geq t_0$ there exists a $t'' > T$ such that $|x(t'')| \leq p|x(t'' - r)|$.

Proof Suppose the conclusion does not hold and let $x(t)$ be an eventually positive solution of (4.4.20). Then $x(t) > 0$ with $x(t) - px(t-r) > 0$ for all large enough t. Let $y(t)$ be as in Lemma 4.4.1 and $u(t)$ be as in Lemma 4.4.2. By Lemma 4.4.2, $u(t)$ satisfies $u^{(4)}(t) = \Delta_\tau^4 y(t) < 0,\ u^{(3)}(t) > 0,\ u(t) > 0, u'(t) > 0$, and, for any positive integer k, there is a $T \geq t_0$ such that

$$\Delta_\tau^4 u(t) + \bar{q}_4(t) u(g(t) - kr) \sum_{i=0}^{k} p^i \leq 0$$

and $u(g(t) - kr) > 0$ hold for $t \geq T$. Note that $\Delta_\tau^4 u(t) < 0$. By Lemma 4.2.2, we know that $\Delta_\tau^3 u(t) > 0$ and $\Delta_\tau u(t) > 0$. Let $v(t) = \Delta_\tau^3 u(t)/u(g(t) - kr)$. Hence, $v(t) > 0$. Thus, by $u'(t) > 0$ and $u^{(3)}(t) > 0$, we have

$$\begin{aligned}
\Delta_\tau v(t) &= v(t+\tau) - v(t) \\
&= \frac{\Delta_\tau^3 u(t+\tau)}{u(g(t+\tau) - kr)} - \frac{\Delta_\tau^3 u(t)}{u(g(t) - kr)} \\
&= \frac{u(g(t) - kr)\Delta_\tau^3 u(t+\tau) - u(g(t+\tau) - kr)\Delta_\tau^3 u(t)}{u(g(t+\tau) - kr) u(g(t) - kr)} \\
&= \frac{u(g(t) - kr)\Delta_\tau^3 u(t+\tau) + u(g(t+\tau) - kr)(\Delta_\tau^4 u(t) - \Delta_\tau^3 u(t+\tau))}{u(g(t+\tau) - kr) u(g(t) - kr)} \\
&= \frac{\Delta_\tau^4 u(t)}{u(g(t) - kr)} - \frac{\Delta_\tau^3 u(t+\tau)\Delta_\tau u(g(t) - kr)}{u(g(t) - kr) u(g(t+\tau) - kr)} \\
&\leq -\bar{q}_4(t) \sum_{i=0}^{k} p^i - v(t+\tau) \frac{\Delta_\tau u(g(t) - kr)}{u(g(t) - kr)} \\
&\leq -\bar{q}_4(t) \sum_{i=0}^{k} p^i.
\end{aligned}$$

Therefore, for large enough $t' > t_0$ satisfying (4.4.25) and $j \in N$, we have

$$\Delta_\tau v(t' + j\tau) + \bar{q}_4(t' + j\tau) \sum_{i=0}^{k} p^i \leq 0. \tag{4.4.26}$$

Summing both sides of (4.4.26) for j from 0 to $n-1$, we have

$$v(t' + n\tau) - v(t') + \sum_{i=0}^{k} p^i \sum_{j=0}^{n-1} \bar{q}_4(t' + j\tau) \leq 0.$$

Thus

$$\sum_{i=0}^{k} p^i \sum_{j=0}^{n-1} \bar{q}_4(t' + j\tau) < v(t') < \infty,$$

which contradicts (4.4.25). Now suppose $x(t)$ is an eventually negative solution of (4.4.20). Then $x(t) < 0$ and $y(t) = x(t) - px(t-r) < 0$ for large enough t. By Lemma 4.4.2, $u(t)$ satisfies $u(t) < 0$, $u'(t) < 0$, $u^{(3)}(t) < 0, u^{(4)}(t) = \Delta_\tau^4 y(t) > 0$, $u(g(t) - kr) < 0$ and

$$\Delta_\tau^4 u(t) + \bar{q}_4(t) u(g(t) - kr) \sum_{i=0}^{k} p^i \geq 0$$

for $t \geq T \geq t_0$. Applying Lemma 4.2.2 to $-u(t)$, we have $\Delta_\tau u(t) < 0$ and $\Delta_\tau^3 u(t) < 0$. These also lead to (4.4.26) and then a contradiction. Therefore, the conclusion of the theorem holds.

The following Theorem 4.4.2 and Corollaries 4.4.1-4.4.3 are for equation (4.4.20) with $0 < p < 1$.

Theorem 4.4.2 In addition to (4.4.25), we assume that $0 < p < 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n) = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \leq n$ for all large enough n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\sum_{i=m_1}^{n} \bar{q}_4(t_1 + i\tau) \geq \frac{p^{k_0}(1-p)}{1-p^{k_0}} \tag{4.4.27}$$

holds for all large k with $n = n_k$, $m_1 = m_1(n_k)$. Then, for every solution $x(t)$ of (4.4.20), either $x(t)$ or $x(t) - px(t-r)$ is oscillatory.

Proof By Theorem 4.4.1, if (4.4.25) holds, we have that every solution $x(t)$ of (4.4.20) is either oscillatory or for any $T \geq t_0$, there exists one $t'' > T$ such that $|x(t'')| \leq p|x(t'' - r)|$.

Assume that (4.4.20) has an eventually positive solution $x(t)$ such that $y(t) = x(t) - px(t-r)$ is not oscillatory. Then, by Theorem 4.4.1 we must have $y(t) \leq 0$ for large enough t.

Since $y(t)$ is not oscillatory, we must have $y(t) < 0$ for all sufficiently large t. Set $z(t) = -y(t)$. Then $z(t) > 0$ and by $\Delta_\tau^4 y(t) = -\Delta_\tau^4(-y(t))$, we find

$$\Delta_\tau^4 z(t) - f(t, x(g(t))) = 0.$$

Further, by the assumption, we have

$$\Delta_\tau^4 z(t) - q(t)x(g(t)) \geq 0, \tag{4.4.28}$$

i.e.,

$$\Delta_\tau^4 z(t) \geq q(t)x(g(t)) > 0.$$

Therefore by Lemma 4.2.2, we have $\Delta_\tau^2 z(t) > 0, |\Delta_\tau z(t)| > 0$ and $|\Delta_\tau^3 z(t)| > 0$.

We claim that $\Delta_\tau z(t) < 0$ and $\Delta_\tau^3 z(t) < 0$. Indeed, if $\Delta_\tau z(t) > 0$, then, since $\Delta_\tau^2 z(t) > 0$, we may assume $\Delta_\tau z(t_1 + k\tau) > l > 0$ for a large enough t_1 and all $k \in N$. Then

$$\sum_{i=0}^{n} \Delta_\tau z(t_1 + i\tau) = z(t_1 + (n+1)\tau) - z(t_1) \geq nl.$$

Let $n \to \infty$, then $z(t_1 + (n+1)\tau) \to +\infty$. We have $\lim_{t\to\infty} x(t) = 0$ by repeating $x(t) < px(t-r)$ for $0 < p < 1$. Thus, by the definition of $z(t)$, we have $\lim_{t\to\infty} z(t) = 0$ which contradicts $z(t_1 + n\tau) \to +\infty$ as $n \to \infty$. Thus, $\Delta_\tau z(t) < 0$. Then, by Lemma 4.2.2, $h = 0$ so $\Delta_\tau^3 z(t) < 0$.

By $z(t) = px(t-r) - x(t)$, we have $x(t) = (x(t+r) + z(t+r))/p$. Substituting this into (4.4.28) we obtain

$$\Delta_\tau^4 z(t) - \frac{q(t)}{p} z(g(t) + r) - \frac{q(t)}{p} x(g(t) + r) \geq 0.$$

By repeating the above process, we have

$$\Delta_\tau^4 z(t) - q(t) \sum_{i=1}^{k} \frac{1}{p^i} z(g(t) + ir) - \frac{q(t)}{p^k} x(g(t) + kr) \geq 0,$$

and hence,

$$\Delta_\tau^4 z(t) - q(t) \sum_{i=1}^{k} \frac{1}{p^i} z(g(t) + ir) > 0 \tag{4.4.29}$$

since $x(g(t) + kr) > 0$. With

$$u(t) = \int_0^\tau ds_1 \int_{s_1}^{s_1+\tau} ds_2 \int_{s_2}^{s_2+\tau} ds_3 \int_{t+s_3}^{t+s_3+\tau} z(\theta) d\theta,$$

we have $u^{(4)}(t) = \Delta_\tau^4 z(t) \geq 0$ and $u(t) > 0$. Moreover,

$$u'(t) = \int_0^\tau ds_1 \int_{s_1}^{s_1+\tau} ds_2 \int_{s_2}^{s_2+\tau} \Delta_\tau z(t + s_3) ds_3.$$

Then $\Delta_\tau z(t) < 0$ implies $u'(t) < 0$.

By the same technique used in the above proof of $u'(t) < 0$, we have $u^{(3)}(t) < 0$ and $u^{(2)}(t) > 0$. Integrating (4.4.29) and using the proof of Lemma 4.4.2 with the replacement of y by z , we have

$$\Delta_\tau^4 u(t) - \bar{q}_4(t) \sum_{i=1}^{k} \frac{1}{p^i} u(g(t) + ir) > 0.$$

As $u(t)$ is decreasing, the above inequality leads to

$$\Delta_\tau^4 u(t) - \bar{q}_4(t)\, u(g(t) + kr) \sum_{i=1}^{k} \frac{1}{p^i} > 0.$$

Since $\sum_{i=1}^{k} 1/p^i = (1-p^k)/(p^k(1-p))$, it follows that

$$\Delta_\tau^4 u(t) \geq \frac{1-p^k}{p^k(1-p)}\,\bar{q}_4(t)\,u(g(t)+kr) > 0.$$

Replacing k by k_0 and t by $t_1 + i\tau$ in the above inequality, we obtain

$$\Delta_\tau^4 u(t_1+i\tau) \geq \frac{1-p^{k_0}}{p^{k_0}(1-p)}\,\bar{q}_4(t_1+i\tau)\,u(g(t_1+i\tau)+k_0 r) \geq 0.$$

Summing up both sides of the above inequality for i from s to n and by $u'(t) < 0$, we have

$$\Delta_\tau^3 u(t_1+(n+1)\tau) - \Delta_\tau^3 u(t_1+s\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)}\,u(g(t_1+n\tau)+k_0 r)\sum_{i=s}^{n}\bar{q}_4(t_1+i\tau),$$

which implies that

$$-\Delta_\tau^3 u(t_1+s\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)}\,u(g(t_1+n\tau)+k_0 r)\sum_{i=s}^{n}\bar{q}_4(t_1+i\tau) \tag{4.4.30}$$

since $\Delta_\tau^3 u(t) < 0$. Further, from the above inequality it follows that

$$-\Delta_\tau^2 u(t_1+(s+1)\tau) + \Delta_\tau^2 u(t_1+s\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)}\,u(g(t_1+n\tau)+k_0 r)\sum_{i=s}^{n}\bar{q}_4(t_1+i\tau),$$

and so,

$$\Delta_\tau^2 u(t_1+s\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)}\,u(g(t_1+n\tau)+k_0 r)\sum_{i=s}^{n}\bar{q}_4(t_1+i\tau).$$

The above inequality implies

$$\Delta_\tau u(t_1+(s+1)\tau) - \Delta_\tau u(t_1+s\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)}\,u(g(t_1+n\tau)+k_0 r)\sum_{i=s}^{n}\bar{q}_4(t_1+i\tau).$$

Further, we obtain

$$-\Delta_\tau u(t_1+s\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)}\,u(g(t_1+n\tau)+k_0 r)\sum_{i=s}^{n}\bar{q}_4(t_1+i\tau),$$

i.e.,

$$-u(t_1+(s+1)\tau)+u(t_1+s\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)}u(g(t_1+n\tau)+k_0r)\sum_{i=s}^{n}\bar{q}_4(t_1+i\tau).$$

Since $u(t) > 0$, from the above inequality it follows that

$$u(t_1+s\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)}u(g(t_1+n\tau)+k_0r)\sum_{i=s}^{n}\bar{q}_4(t_1+i\tau).$$

Note that $g(t_1+n\tau)+k_0r \le t_1+m_1\tau$ and u is decreasing. By taking $s=m_1$, we obtain

$$u(t_1+m_1\tau) > u(t_1+m_1\tau)\frac{1-p^{k_0}}{p^{k_0}(1-p)}\sum_{i=m_1}^{n}\bar{q}_4(t_1+i\tau),$$

i.e.,

$$\sum_{i=m_1}^{n}\bar{q}_4(t_1+i\tau) < \frac{p^{k_0}(1-p)}{1-p^{k_0}}.$$

This inequality contradicts (4.4.27) and, therefore, if $x(t)$ is an eventually positive solution then $x(t)-px(t-r)$ is oscillatory.

Assume that (4.4.20) has an eventually negative solution $x(t)$ such that $y(t) = x(t)-px(t-r)$ is not oscillatory. Then, by Theorem 4.4.1, $x(t) < 0$ and $y(t) > 0$ hold eventually. From (4.4.20) and (4.1.3), $\Delta_\tau^4 y(t) \ge -q(t)x(g(t)) > 0$ for $t \ge T \ge t_0$. Using the same argument as above for $z(t)$, we know that $\Delta_\tau y(t) < 0$, $\Delta_\tau^2 y(t) > 0$ and $\Delta_\tau^3 y(t) < 0$. From the definition of $y(t)$ we have $x(t) = \frac{1}{p}(x(t+r)-y(t+r))$ so

$$\Delta_\tau^4 y(t) - \frac{q(t)}{p}y(g(t)+r) + \frac{q(t)}{p}x(g(t)+r) \ge 0$$

and

$$\Delta_\tau^4 y(t) - q(t)\sum_{i=1}^{k}\frac{1}{p^i}y(g(t)+ir) > 0.$$

Replacing $z(t)$ by $y(t)$ in the definition of $u(t)$ and using the same proof as above, we derive a contradiction to (4.4.27). Therefore, for any nonoscillatory solution $x(t)$, $x(t) - px(t-r)$ must be oscillatory.

Corollary 4.4.1 In addition to (4.4.25), we assume that $0 < p < 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n) = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \leq n$ for all large enough n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\sum_{i=m_1}^{n} (i - m_1 + 1)\bar{q}_4(t_1 + i\tau) \geq \frac{p^{k_0}(1-p)}{1-p^{k_0}} \tag{4.4.31}$$

holds for all large enough k with $n = n_k, m_1 = m_1(n_k)$. Then, for every solution $x(t)$ of (4.4.20), either $x(t)$ or $x(t) - px(t-r)$ is oscillatory.

Proof The proof is the same as that of Theorem 4.4.2 until (4.4.30). Summing up (4.4.30) for s from m_1 to n, we have

$$-\Delta_\tau^2 u(t_1+(n+1)\tau)+\Delta_\tau^2 u(t_1+m_1\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)}\, u(g(t_1+n\tau)+k_0 r) \sum_{s=m_1}^{n} \sum_{i=s}^{n} \bar{q}_4(t_1+i\tau).$$

Since $\Delta_\tau^2 u(t) > 0$, the above inequality implies

$$\Delta_\tau^2 u(t_1 + m_1\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)}\, u(g(t_1 + n\tau) + k_0 r) \sum_{s=m_1}^{n} \sum_{i=s}^{n} \bar{q}_4(t_1 + i\tau). \tag{4.4.32}$$

Further, by the same technique used for the proof of Theorem 4.4.2, we obtain

$$u(t_1 + m_1\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)}\, u(g(t_1 + n\tau) + k_0 r) \sum_{s=m_1}^{n} \sum_{i=s}^{n} \bar{q}_4(t_1 + i\tau).$$

Note $g(t_1 + n\tau) + k_0 r \leq t_1 + m_1\tau$ and u is decreasing. So, from the above inequality it follows that

$$u(t_1 + m_1\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(t_1 + m_1\tau) \sum_{i=m_1}^{n} (i - m_1 + 1)\bar{q}_4(t_1 + i\tau),$$

i.e.,

$$\sum_{i=m_1}^{n}(i-m_1+1)\bar{q}_4(t_1+i\tau) < \frac{p^{k_0}(1-p)}{1-p^{k_0}}.$$

This inequality contradicts (4.4.31). Thus, this contradiction shows that the conclusion holds.

Remark 4.4.1 Compared with (4.4.27), the requirement (4.4.31) for $\bar{q}_4(t)$ is weaker than (4.4.27) since $i - m_1 + 1 \geq 1$ holds in (4.4.31).

Corollary 4.4.2 In addition to (4.4.25), we assume that $0 < p < 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n) = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \leq n$ for all large enough n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\frac{1}{2}\sum_{i=m_1}^{n}(i-m_1+1)(i-m_1+2)\bar{q}_4(t_1+i\tau) \geq \frac{p^{k_0}(1-p)}{1-p^{k_0}} \tag{4.4.33}$$

holds for all large enough k with $n = n_k, m_1 = m_1(n_k)$. Then, for every solution $x(t)$ of (4.4.20), either $x(t)$ or $x(t) - px(t-r)$ is oscillatory.

Proof The proof is the same as that of Corollary 4.4.1 until inequality (4.4.32) with m_1 replaced by $m_2\,(m_1 \leq m_2 \leq n)$. Summing up (4.4.32) for m_2 from m_1 to n, we have

$$\begin{aligned}\Delta_\tau u(t_1+(n+1)\tau) - \Delta_\tau u(t_1+m_1\tau) \;&>\; \frac{1-p^{k_0}}{p^{k_0}(1-p)}u(g(t_1+n\tau)+k_0 r)\times\\ &\sum_{m_2=m_1}^{n}\sum_{s=m_2}^{n}\sum_{i=s}^{n}\bar{q}_4(t_1+i\tau),\end{aligned}$$

and so

$$-\Delta_\tau u(t_1+m_1\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)}u(g(t_1+n\tau)+k_0 r)\sum_{m_2=m_1}^{n}\sum_{s=m_2}^{n}\sum_{i=s}^{n}\bar{q}_4(t_1+i\tau). \tag{4.4.34}$$

Next, by the same technique used for the proof of Theorem 4.4.2, we obtain

$$\begin{aligned}
u(t_1+m_1\tau) &> \frac{1-p^{k_0}}{p^{k_0}(1-p)}u(g(t_1+n\tau)+k_0r)\sum_{m_2=m_1}^{n}\sum_{s=m_2}^{n}\sum_{i=s}^{n}\bar{q}_4(t_1+i\tau) \\
&= \frac{1-p^{k_0}}{p^{k_0}(1-p)}u(g(t_1+n\tau)+k_0r)\times \\
&\quad \left(\sum_{m_2=m_1}^{n}\sum_{i=m_2}^{n}(i-m_2+1)\bar{q}_4(t_1+i\tau)\right) \\
&= \frac{1-p^{k_0}}{p^{k_0}(1-p)}u(g(t_1+n\tau)+k_0r)\times \\
&\quad \left(\sum_{i=m_1}^{n}\bar{q}_4(t_1+i\tau)\sum_{m_2=m_1}^{i}(i-m_2+1)\right).
\end{aligned}$$

Further, since $g(t_1+n\tau)+k_0r \le t_1+m_1\tau$ and u is decreasing, we obtain

$$u(t_1+m_1\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)}u(t_1+m_1\tau)\frac{1}{2}\sum_{i=m_1}^{n}(i-m_1+1)(i-m_1+2)\bar{q}_4(t_1+i\tau),$$

i.e.,

$$\frac{1}{2}\sum_{i=m_1}^{n}(i-m_1+1)(i-m_1+2)\bar{q}_4(t_1+i\tau) < \frac{p^{k_0}(1-p)}{1-p^{k_0}}.$$

This inequality contradicts (4.4.33) and, hence, this contradiction shows that the conclusion holds.

Remark 4.4.2 Note that $(i-m_1+1)(i-m_1+2)/2 \ge i-m_1+1 \ge 1$. Hence, (4.4.33) is weaker than (4.4.27) and (4.4.31) in general.

Corollary 4.4.3 In addition to (4.4.25), we assume that $0<p<1$ and that there are a positive integer k_0 and a $t_1 \ge t_0$ satisfying $m_1(n) = \lceil (g(t_1+n\tau)-t_1+k_0r)/\tau \rceil \le n$ for all large enough n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k\to\infty$ such that

$$\frac{1}{3!}\sum_{i=m_1}^{n}(i-m_1+1)(i-m_1+2)(i-m_1+3)\bar{q}_4(t_1+i\tau) \ge \frac{p^{k_0}(1-p)}{1-p^{k_0}} \qquad (4.4.35)$$

holds for all large enough k with $n = n_k, m_1 = m_1(n_k)$. Then, for every solution $x(t)$ of (4.4.20), either $x(t)$ or $x(t) - px(t-r)$ is oscillatory.

Proof The proof is the same as that of Corollary 4.4.2 until (4.4.34) with $m_1 \le m_2$ replaced by $m_2 \le m_3$. Summing up (4.4.34) for m_2 from m_1 to n, we have

$$\begin{aligned} -u(t_1 + (n+1)\tau) + u(t_1 + m_1\tau) &> \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1 + n\tau) + k_0 r) \times \\ & \sum_{m_2=m_1}^{n} \sum_{m_3=m_2}^{n} \sum_{s=m_3}^{n} \sum_{i=s}^{n} \bar{q}_4(t_1 + i\tau), \end{aligned}$$

which, since $u(t) > 0$, implies

$$u(t_1 + m_1\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1 + n\tau) + k_0 r) \sum_{m_2=m_1}^{n} \sum_{m_3=m_2}^{n} \sum_{s=m_3}^{n} \sum_{i=s}^{n} \bar{q}_4(t_1 + i\tau).$$

Note $g(t_1 + n\tau) + k_0 r \le t_1 + m_1\tau$ and u is decreasing. Then, we obtain

$$\begin{aligned} 1 &> \frac{1-p^{k_0}}{p^{k_0}(1-p)} \sum_{m_2=m_1}^{n} \sum_{m_3=m_2}^{n} \sum_{s=m_3}^{n} \sum_{i=s}^{n} \bar{q}_4(t_1 + i\tau) \\ &> \frac{1-p^{k_0}}{p^{k_0}(1-p)} \left(\sum_{m_2=m_1}^{n} \sum_{i=m_2}^{n} \frac{1}{2}(i - m_2 + 1)(i - m_2 + 2)\bar{q}_4(t_1 + i\tau) \right) \\ &= \frac{1-p^{k_0}}{p^{k_0}(1-p)} \left(\sum_{i=m_1}^{n} \bar{q}_4(t_1 + i\tau) \sum_{m_2=m_1}^{i} \frac{1}{2}(i - m_2 + 1)(i - m_2 + 2) \right) \\ &= \frac{1-p^{k_0}}{p^{k_0}(1-p)} \left(\sum_{i=m_1}^{n} \frac{1}{3!}(i - m_1 + 1)(i - m_1 + 2)(i - m_1 + 3)\bar{q}_4(t_1 + i\tau) \right). \end{aligned}$$

So

$$\frac{1}{3!} \sum_{i=m_1}^{n} (i - m_1 + 1)(i - m_1 + 2)(i - m_1 + 3)\bar{q}_4(t_1 + i\tau) < \frac{p^{k_0}(1-p)}{1-p^{k_0}}.$$

This inequality contradicts (4.4.35) and, thus, this contradiction shows that the conclusion holds.

Remark 4.4.3 Notice that

$$\begin{aligned}\frac{(i-m_1+1)(i-m_1+2)(i-m_1+3)}{3!} &\geq \frac{(i-m_1+2)(i-m_1+1)}{2} \\ &\geq i-m_1+1 \\ &\geq 1\end{aligned}$$

in general. Therefore, (4.4.35) is weaker than (4.4.33), (4.4.31) and (4.4.27).

The following Theorem 4.4.3 and Corollaries 4.4.4-4.4.6 are for equation (4.4.20) with $p = 1$.

Theorem 4.4.3 In addition to (4.4.25), we assume that $p = 1$ and that there exists a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n) = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \leq n$ for all large enough n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\sum_{i=m_1}^{n} \bar{q}_4(t_1 + i\tau) \geq \frac{1}{k_0} \tag{4.4.36}$$

for all large enough k with $n = n_k, m_1 = m_1(n_k)$. Then, for every solution $x(t)$ of (4.4.20), either $x(t)$ or $x(t) - x(t-r)$ is oscillatory.

Proof We refer to the proof of Theorem 4.4.2 line by line with the replacement of p by 1 and

$$\sum_{i=1}^{k} \frac{1}{p^i} = \frac{1-p^k}{p^k(1-p)}$$

by k. Since $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} z(t) = 0$ follow from $x(t) < px(t-\tau)$ for $0 < p < 1$, the boundedness of $x(t)$ and $z(t)$ follows from $x(t) < x(t-\tau)$. Then the proof of Theorem 4.4.2 is still valid here after a minor modification.

Corollary 4.4.4 In addition to (4.4.25), we assume that $p = 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n) = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \leq n$ for all large enough n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\sum_{i=m_1}^{n} (i - m_1 + 1)\bar{q}_4(t_1 + i\tau) \geq \frac{1}{k_0} \tag{4.4.37}$$

holds for all large enough k with $n = n_k, m_1 = m_1(k)$. Then, for every solution $x(t)$ of (4.4.20), either $x(t)$ or $x(t) - x(t-r)$ is oscillatory.

Proof The proof of the Corollary 4.4.1 can be used here almost verbatim after the replacement of $(1 - p^{k_0})/[p^{k_0}(1-p)]$ by k_0.

Corollary 4.4.5 In addition to (4.4.25), we assume that $p = 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n) = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \leq n$ for all large enough n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\frac{1}{2}\sum_{i=m_1}^{n} (i - m_1 + 1)(i - m_1 + 2)\bar{q}_4(t_1 + i\tau) \geq \frac{1}{k_0} \tag{4.4.38}$$

holds for all large enough k with $n = n_k, m_1 = m_1(n_k)$. Then, for every solution $x(t)$ of (4.4.20), either $x(t)$ or $x(t) - x(t-r)$ is oscillatory.

Proof Refer to the proof of the Corollary 4.4.2 with the necessary replacement of $(1 - p^{k_0})/[p^{k_0}(1-p)]$ by k_0.

Corollary 4.4.6 In addition to (4.4.25), we assume that $p = 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n) = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \leq n$ for all large enough n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\frac{1}{3!}\sum_{i=m_1}^{n}(i - m_1 + 1)(i - m_1 + 2)(i - m_1 + 3)\bar{q}_4(t_1 + i\tau) \geq \frac{1}{k_0} \tag{4.4.39}$$

holds for all large enough k with $n = n_k, m_1 = m_1(n_k)$. Then, for every solution $x(t)$ of (4.4.20), either $x(t)$ or $x(t) - x(t-r)$ is oscillatory.

Proof Refer to the proof of Corollary 4.4.3 with the necessary replacement of $(1 - p^{k_0})/[p^{k_0}(1-p)]$ by k_0.

Note that remarks for Corollaries 4.4.4-4.4.6 similar to these for Corollaries 4.4.1-4.4.3 are also true.

The following Theorem 4.4.4 and Corollaries 4.4.7-4.4.9 are for equation 4.4.20 with $p > 1$.

Theorem 4.4.4 In addition to (4.4.25), we assume that $p > 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n) = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \leq n$ for all large enough n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\sum_{i=m_1}^{n} \bar{q}_4(t_1 + i\tau) \geq \frac{p^{k_0}(1-p)}{1 - p^{k_0}} \tag{4.4.40}$$

holds for all large enough k with $n = n_k, m_1 = m_1(n_k)$. Then, for every bounded solution of (4.4.20), either $x(t)$ or $x(t) - px(t-r)$ is oscillatory.

Proof Suppose that $x(t)$ is a bounded eventually positive solution of (4.4.20). The proof of Theorem 4.4.2 is then still valid for Theorem 4.4.4 subject to a few obvious minor changes.

Corollary 4.4.7 In addition to (4.4.25), we assume that $p > 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n) = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \leq n$ for all large enough n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\sum_{i=m_1}^{n} (i - m_1 + 1)\bar{q}_4(t_1 + i\tau) \geq \frac{p^{k_0}(1-p)}{1-p^{k_0}} \tag{4.4.41}$$

holds for all large enough k with $n = n_k, m_1 = m_1(n_k)$. Then, for every bounded solution $x(t)$ of (4.4.20), either $x(t)$ or $x(t) - px(t-r)$ is oscillatory.

Corollary 4.4.8 In addition to (4.4.25), we assume that $p > 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n) = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \leq n$ for all large enough n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\frac{1}{2}\sum_{i=m_1}^{n} (i - m_1 + 1)(i - m_1 + 2)\bar{q}_4(t_1 + i\tau) \geq \frac{p^{k_0}(1-p)}{1-p^{k_0}} \tag{4.4.42}$$

holds for all large enough k with $n = n_k, m_1 = m_1(n_k)$. Then, for every bounded solution $x(t)$ of (4.4.20), either $x(t)$ or $x(t) - px(t-r)$ is oscillatory.

Corollary 4.4.9 In addition to (4.4.25), we assume that $p > 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n) = \lceil (g(t_1 +$

$n\tau) - t_1 + k_0 r)/\tau\rceil \leq n$ for all large enough n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\frac{1}{3!}\sum_{i=m_1}^{n}(i-m_1+1)(i-m_1+2)(i-m_1+3)\bar{q}_4(t_1+i\tau) \geq \frac{p^{k_0}(1-p)}{1-p^{k_0}} \quad (4.4.43)$$

holds for all large enough k with $n = n_k, m_1 = m_1(n_k)$. Then, for every bounded solution $x(t)$ of (4.4.20), either $x(t)$ or $x(t) - px(t-r)$ is oscillatory.

Note that remarks for Corollaries 4.4.7-4.4.9 similar to those for Corollaries 4.4.1-4.4.3 are also true.

4.4.3 EXAMPLES

Two examples will be given in this subsection to demonstrate the results in last subsection.

Example 4.4.1 Consider the linear difference equation

$$\Delta_\tau^4(x(t) - px(t-r)) + \frac{1}{t}x(t - \frac{\sigma}{1+\beta t}) = 0 \quad (4.4.44)$$

for $t > 0$, where p, r, τ and σ are positive constants. Viewing (4.4.44) as (4.4.20), we have $q(t) = 1/t$ and $g(t) = t - \sigma/(1+\beta t)$. Then, according to (4.1.3), $\bar{q}_4(t) = \alpha/(t+4\tau)$ for $\beta = 0$ and

$$\bar{q}_4(t) = \frac{\alpha}{t+4\tau}\left(1 - \frac{\sigma\beta}{(1+\beta t)^2+\sigma\beta}\right)^4$$

for $\beta > 0$. Since $\bar{q}_4(t) \geq \alpha'/(t+4\tau)$ for some $\alpha' > 0$ and all $t \geq 0$, $\bar{q}_4$ satisfies (4.4.25) with $t' = 0$. By Theorem 4.4.1, for every solution $x(t)$ of (4.4.44), either

$x(t)$ is oscillatory or for any $T \geq t_0$ there exists a $t'' > T$ such that $|x(t'')| < p|x(t'' - r)|$. In particular, when $p = 0$, every solution of (4.4.44) is oscillatory.

Example 4.4.2 Consider the difference equation

$$\Delta_\pi^4(x(t) - px(t-\pi)) + 8x(t-\pi) + \frac{8\sigma}{1+t^2}x^3(t-\pi) = 0, \tag{4.4.45}$$

where $\sigma \geq 0$ is a constant. Regarding (4.4.45) as (4.4.20), we have $\tau = \pi$, $r = \pi$, $g(t) = t - \pi$ and $q(t) = 8$. Then, for some $\alpha \in (0,1)$, $\bar{q}_4 = 8\alpha$ by (4.1.3) so (4.4.1) is satisfied. For $p = 1$, $k_0 = 1$ and $t_1 = t$, we have $m_n = n$ and

$$\sum_{s=m_n}^{n}(s+1-m_n)\bar{q}_4(t_1+s\tau) = 8\alpha > 1 = \frac{1}{k_0}$$

if $\alpha > 1/8$. Moreover, we also have

$$\sum_{s=m_n}^{n}(s+1-m_n)\bar{q}_4(t_1+s\tau) = 8\alpha > p = \frac{(1-p)p^{k_0}}{1-p^{k_0}}$$

if $p \in (0,1) \cup (1,8)$ and $\alpha > p/8$. According to Corollaries 4.4.1 and 4.4.4, for every solution $x(t)$ of (4.4.45), either $x(t)$ or $x(t) - px(t-r)$ is oscillatory if $0 < p \leq 1$. Furthermore, by Corollary 4.4.7, for every bounded solution $x(t)$ of (4.4.45), either $x(t)$ or $x(t) - px(t-r)$ is oscillatory if $1 < p < 8$.

4.5 HIGHER EVEN ORDER EQUATION (4.1.1)

In this section, we will deal with equation (4.1.1) of the general form

$$\Delta_\tau^m(x(t) - px(t-r)) + f(t, x(g(t))) = 0, \tag{4.5.46}$$

where $m > 4$ is an even integer.

The assumptions given section 4.1 guarantee the existence and differentiability of the inverse g^{-1} of g. Let

$$\bar{q}_m(t) = \alpha \min_{t \leq s \leq t+m\tau} \{q(s)\} \left(\min_{g(t) \leq s \leq g(t)+m\tau} \{(g^{-1}(s))'\} \right)^m, \tag{4.5.47}$$

where $0 < \alpha < 1$. We shall see that the function $\bar{q}_m$ will play an important role in the oscillatory criteria for the solutions of (4.5.46).

This section is composed of three subsections. In subsection 4.5.1, lemmas will be stated for the proofs of the criteria and the main oscillatory criteria will be given in subsection 4.5.2. Finally, examples will be discussed in subsection 4.5.3.

4.5.1 RELATED LEMMAS

In next subsection, we shall present the following lemmas which will be needed in the proofs in next subsection.

Lemma 4.5.1 Assume that $x(t)$ is an eventually positive (negative) solution of (4.5.46) such that $y(t) = x(t) - px(t-r) > 0\,(< 0)$ eventually. Then $\Delta_\tau y(t) > 0\,(< 0)$ and $\Delta_\tau^{m-1} y(t) > 0\,(< 0)$ hold eventually.

Proof Suppose $x(t) > 0$ and $y(t) > 0$ hold eventually. Due to $g(t) < t$, $g'(t) > 0$ and (4.1.2), there exists a $t_1 > t_0$ such that $x(g(t)) > 0$ for all $t \geq t_1$. Further, (4.5.46) becomes

$$\Delta_\tau^m y(t) + f(t, x(g(t))) = 0.$$

According to (4.1.3), $f(t, x(g(t))) \geq q(t)x(g(t)) > 0$ for $t \geq t_1$ hold. Therefore,

$$\Delta_\tau^m y(t) \leq -q(t)x(g(t)) < 0 \tag{4.5.48}$$

for all large enough t, namely, $\Delta_\tau^m y(t) < 0$ eventually. By Lemma 4.2.2, h could be odd with $1 \leq h \leq m-1$. For all cases, we could obtain $\Delta_\tau y(t) > 0$ and $\Delta_\tau^{m-1} y(t) > 0$ eventually. If $x(t) < 0$ and $y(t) < 0$ hold eventually, then (4.5.48) becomes $\Delta_\tau^m y(t) \geq -q(t)x(g(t)) > 0$. Applying Lemma 4.2.2 to $-y(t)$, we obtain $\Delta_\tau y(t) < 0$ and $\Delta_\tau^{m-1} y(t) < 0$.

Lemma 4.5.2 Let the hypothesis of Lemma 4.5.1 be satisfied. Moreover, let $\bar{q}_m(t)$ be defined by (4.5.47). Set

$$u(t) = \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-2}}^{t_{m-2}+\tau} dt_{m-1} \int_{t_{m-1}}^{t_{m-1}+\tau} y(\theta) d\theta.$$

Then u satisfies $u^{(m)}(t) = \Delta_\tau^m y(t) < 0\,(> 0)$, $u(t) > 0\,(< 0)$, $u'(t) > 0\,(< 0)$, $u^{(m-1)}(t) > 0\,(< 0)$, $\Delta_\tau^{m-1} u(t) > 0\,(< 0)$, and

$$\Delta_\tau^m u(t) + \bar{q}_m(t)\, u(g(t) - kr) \sum_{i=0}^{k} p^i \leq 0\,(\geq 0)$$

for each fixed number k and for all large enough t.

Proof Suppose $x(t) > 0$ and $y(t) > 0$ hold eventually. According to the definition of $u(t)$ and (4.5.48), we can see that $u(t) > 0$, $u^{(m)}(t) = \Delta_\tau^m y(t) < 0$ and

$$\Delta_\tau^m y(t) + q(t)x(g(t)) \leq 0 \tag{4.5.49}$$

for sufficiently large t. Taking into account the definition of $y(t)$, we have

$$\Delta_\tau^m y(t) + q(t)(y(g(t)) + px(g(t) - r)) \leq 0.$$

By repeating the above process k times, we deduce

$$\Delta_\tau^m y(t) + q(t)\sum_{i=0}^{k} p^i y(g(t) - ir) + q(t)p^{k+1}x(g(t) - (k+1)r) \leq 0.$$

Therefore, since $q(t)p^{k+1}x(g(t) - (k+1)r) \geq 0$, it follows that

$$\Delta_\tau^m y(t) + q(t)\sum_{i=0}^{k} p^i\, y(g(t) - ir) \leq 0.$$

Furthermore,

$$u^{(m)}(t) + q(t)\sum_{i=0}^{k} p^i\, y(g(t) - ir) \leq 0. \tag{4.5.50}$$

Then, for large enough t, the assumptions on g and q give

$$\begin{aligned}
&\int_t^{t+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_{m-2} \cdots \int_{s_{m-2}}^{s_{m-2}+\tau} ds_{m-1} \int_{s_{m-1}}^{s_{m-1}+\tau} y(g(\theta) - ir)q(\theta)d\theta \\
&\geq \min_{t\leq l\leq t+m\tau}\{q(l)\} \int_t^{t+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_{m-2} \cdots \int_{s_{m-2}}^{s_{m-2}+\tau} ds_{m-1} \int_{s_{m-1}}^{s_{m-1}+\tau} y(g(\theta) - ir)d\theta \\
&\geq \min_{t\leq l\leq t+m\tau}\{q(l)\} \int_{g(t)}^{g(t+\tau)} (g^{-1}(s_1))'ds_1 \int_{s_1}^{g(g^{-1}(s_1)+\tau)} (g^{-1}(s_2))'ds_2 \cdots \\
&\quad \int_{s_{m-2}}^{g(g^{-1}(s_{m-2})+\tau)} (g^{-1}(s_{m-1}))'ds_{m-1} \int_{s_{m-1}}^{g(g^{-1}(s_{m-1})+\tau)} y(\theta - ir)(g^{-1}(\theta))'d\theta \\
&\geq \min_{t\leq l\leq t+m\tau}\{q(l)\} \left(\min_{g(t)\leq s\leq g(t)+m\tau} (g^{-1}(s))'\right)^m \int_{g(t)}^{g(t)+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \cdots \\
&\quad \int_{s_{m-2}}^{s_{m-2}+\tau} ds_{m-1} \int_{s_{m-1}}^{s_{m-1}+\tau} y(\theta - ir)d\theta \\
&\geq \min_{t\leq l\leq t+m\tau}\{q(l)\} \left(\min_{g(t)\leq s\leq g(t)+m\tau} (g^{-1}(s))'\right)^m u(g(t) - ir) \\
&\geq \bar{q}_m(t)u(g(t) - ir).
\end{aligned}$$

Thus, integration on both sides of (4.5.50) gives

$$\Delta_\tau^m u(t) + \bar{q}_m(t)\sum_{i=0}^{k} p^i\, u(g(t) - ir) \leq 0. \tag{4.5.51}$$

According to the definition of $u(t)$, the equality

$$u'(t) = \int_t^{t+\tau} dt_2 \int_{t_2}^{t_2+\tau} dt_3 \cdots \int_{t_{m-2}}^{t_{m-2}+\tau} dt_{m-1} \int_{t_{m-1}}^{t_{m-1}+\tau} \Delta_\tau y(\theta)d\theta$$

holds. Then it follows from Lemma 4.5.1 that $u'(t) > 0$. Similarly, we have

$$u^{(m-1)}(t) = \int_t^{t+\tau} \Delta_\tau^{m-1} y(\theta) d\theta$$

so $u^{(m-1)}(t) > 0$ from Lemma 4.5.1. Hence,

$$\Delta_\tau^{m-1} u(t) = \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-2}}^{t_{m-2}+\tau} u^{(m-1)}(\theta) d\theta > 0.$$

Further, (4.5.51) implies

$$\Delta_\tau^m u(t) + \bar{q}_m(t)\, u(g(t) - kr) \sum_{i=0}^{k} p^i \leq 0$$

for each fixed natural number k and for all large enough t. If $x(t) < 0$ and $y(t) < 0$ hold eventually, then $u(t) < 0$, $u^{(m)}(t) = \Delta_\tau^m y(t) > 0$ and $\Delta_\tau^m y(t) + q(t) x(g(t)) \geq 0$ for large enough t. Moreover, (4.5.50) becomes

$$u^{(m)}(t) + q(t) \sum_{i=0}^{k} p^i y(g(t) - ir) \geq 0$$

and (4.5.51) becomes

$$\Delta_\tau^m u(t) + \bar{q}_m(t) \sum_{i=0}^{k} p^i u(g(t) - ir) \geq 0.$$

That $u'(t) < 0$ and $u^{(m-1)}(t) < 0$ follow from $\Delta_\tau y(t) < 0$ and $\Delta_\tau^{m-1} y(t) < 0$. Then $\Delta_\tau^{m-1} u(t) < 0$ follows from the integration of $u^{(m-1)}(t)$. Since $u(t)$ is decreasing, each $u(g(t) - ir)$ can be replaced by $u(g(t) - kr)$ in the above inequality.

4.5.2 MAIN RESULTS

Using the above lemmas, we can obtain the following theorems for any even order difference equation in the form (4.5.46). Essentially, the spirit of the theorems is the same as the fourth order. However, it is impossible that a general case could be completely the same as a specific case. We, therefore, shall stress the difference between the general case and the specific case, the fourth order. In addition, we shall not give the proof in details if it is similar to that of the fourth order.

Theorem 4.5.1 Assume that, for some $t' \geq t_0$,

$$\sum_{i=0}^{n} \bar{q}_m(t' + i\tau) \longrightarrow \infty \tag{4.5.52}$$

as $n \longrightarrow \infty$. Then for every solution $x(t)$ of (4.5.46), either $x(t)$ is oscillatory or for any $T \geq t_0$ there exists a $t'' > T$ such that $|x(t'')| \leq p|x(t'' - r)|$.

Proof In essence, the proof is the same as that of Theorem 4.4.1 for the fourth order equations. Thus we shall not give the proof in details but an outline. Let $x(t)$ be a solution of (4.5.46) satisfying $x(t) > 0$ and $x(t) - px(t - r) > 0$ for all large t. Let $y(t)$ be as in Lemma 4.5.1 and $u(t)$ be as in Lemma 4.5.2. Furthermore, for any positive integer k, we have

$$\Delta_\tau^m u(t) + \bar{q}_m(t)u(g(t) - kr)\sum_{i=0}^{k} p^i \leq 0,$$

where $u(g(t) - kr) > 0$. Define the Riccati transformation by

$$v(t) = \frac{\Delta_\tau^{m-1} u(t)}{u(g(t) - kr)}.$$

Notice that $v(t) > 0$. Moreover we deduce

$$\begin{aligned}
\Delta_\tau v(t) &= v(t+\tau) - v(t) \\
&= \frac{\Delta_\tau^{m-1} u(t+\tau)}{u(g(t+\tau)-kr)} - \frac{\Delta_\tau^{m-1} u(t)}{u(g(t)-kr)} \\
&= \frac{u(g(t)-kr)\Delta_\tau^{m-1}u(t+\tau) - u(g(t+\tau)-kr)\Delta_\tau^{m-1}u(t)}{u(g(t+\tau)-kr)u(g(t)-kr)} \\
&= \frac{u(g(t)-kr)\Delta_\tau^{m-1}u(t+\tau) + u(g(t+\tau)-kr)(\Delta_\tau^{m}u(t) - \Delta_\tau^{m-1}u(t+\tau))}{u(g(t+\tau)-kr)u(g(t)-kr)} \\
&\le \frac{\Delta_\tau^m u(t)}{u(g(t)-kr)} - \frac{\Delta_\tau^{m-1}u(t+\tau)\Delta_\tau u(g(t)-kr)}{u(g(t)-kr)u(g(t+\tau)-kr)} \\
&\le -\bar{q}_m(t)\sum_{i=0}^{k} p^i - v(t+\tau)\frac{\Delta_\tau u(g(t)-kr)}{u(g(t)-kr)} \\
&\le -\bar{q}_m(t)\sum_{i=0}^{k} p^i.
\end{aligned}$$

Therefore, there exists a $t' > t_0$ such that

$$\Delta_\tau v(t'+j\tau) + \bar{q}_m(t'+j\tau)\sum_{i=0}^{k} p^i \le 0. \tag{4.5.53}$$

Summing up both sides of (4.5.53) from 0 to n, we have

$$v(t'+(n+1)\tau) - v(t') + \sum_{i=0}^{k} p^i \sum_{j=0}^{n} \bar{q}_m(t'+j\tau) \le 0.$$

Thus

$$\sum_{i=0}^{k} p^i \sum_{j=0}^{n} \bar{q}_m(t'+j\tau) < v(t') < \infty,$$

which leads to a contradiction to (4.5.52). If $x(t)$ is a solution of (4.5.46) satisfying $x(t) < 0$ and $y(t) < 0$ eventually, from Lemmas 4.5.1 and 4.5.2, the above argument about $v(t)$ is still valid and also leads to a contradiction. Therefore, the conclusion of the theorem holds.

The following Theorem 4.5.2 and Corollaries 4.5.1-4.5.2 are for equation (4.5.46) with $0 < p < 1$.

Theorem 4.5.2 In addition to (4.5.52), we further assume that $0 < p < 1$ and there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n) = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \leq n$ for all large enough n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\sum_{i=m_1}^{n} \bar{q}_m(t_1 + i\tau) \geq \frac{p^{k_0}(1-p)}{1-p^{k_0}} \tag{4.5.54}$$

holds for all large enough k with $n = n_k, m_1 = m_1(n_k)$. Then, for every solution $x(t)$ of (4.5.46), either $x(t)$ or $x(t) - px(t-r)$ is oscillatory.

Proof According to Theorem 4.5.1, if (4.5.52) holds, we have that every solution $x(t)$ of (4.5.46) is either oscillatory or for any $T \geq t_0$, there exists one $t'' > T$ such that $|x(t'')| \leq p|x(t''-r)|$.

Assume that (4.5.46) has an eventually positive solution $x(t)$ such that $y(t) = x(t) - px(t-\tau)$ is not oscillatory. Then from Theorem 4.5.1, we deduce that $y(t) < 0$ for all large enough t. Let $z(t) = -y(t)$. Therefore, $z(t) > 0$ and

$$\Delta_\tau^m z(t) - f(t, x(g(t))) = 0.$$

Moreover,

$$\Delta_\tau^m z(t) \geq q(t)x(g(t)) > 0$$

so

$$\Delta_\tau^m z(t) - q(t)x(g(t)) \geq 0. \tag{4.5.55}$$

For $z(t)$, according to Lemma 4.2.2, h is even. So $\Delta_\tau^i z(t) > 0$ for all even number i with $2 \leq i \leq m-2$, and $|\Delta_\tau^j z(t)| > 0$ for all odd number j with $1 \leq j \leq m-1$.

We show that $\Delta_\tau z(t) < 0$. Indeed, if $\Delta_\tau z(t) > 0$, then, since $\Delta_\tau^2 z(t) > 0$, we may assume $\Delta_\tau z(t_1 + k\tau) > l > 0$ for a large enough t_1 and all $k \in N$. Then

$$\sum_{i=0}^{d} \Delta_\tau z(t_1 + i\tau) = z(t_1 + (d+1)\tau) - z(t_1) \geq (d+1)l.$$

Let $d \to \infty$, then $z(t_1 + (d+1)\tau) \to +\infty$. We have $\lim_{t\to\infty} x(t) = 0$ by repeating $x(t) < px(t-r)$ for $0 < p < 1$. Thus, by the definition of $z(t)$, we have $\lim_{t\to\infty} z(t) = 0$ which contradicts $z(t_1 + d\tau) \to +\infty$ as $d \to \infty$. Thus, $\Delta_\tau z(t) < 0$.

So, according to Lemma 4.2.2 again, $h = 0$. Thus, $\Delta_\tau^i z(t) > 0$ for all even number i with $2 \leq i \leq m-2$, and $\Delta_\tau^j z(t) < 0$ for all odd number j with $1 \leq j \leq m-1$.

Notice $x(t) = (x(t+r) + z(t+r))/p$. Hence, from (4.5.55), it follows that

$$\Delta_\tau^m z(t) - \frac{q(t)}{p} z(g(t)+r) - \frac{q(t)}{p} x(g(t)+r) \geq 0,$$

and further

$$\Delta_\tau^m z(t) - q(t) \sum_{i=1}^{k} \frac{1}{p^i} z(g(t)+ir) - \frac{q(t)}{p^k} x(g(t)+kr) \geq 0.$$

So,

$$\Delta_\tau^m z(t) - q(t) \sum_{i=1}^{k} \frac{1}{p^i} z(g(t)+ir) > 0 \tag{4.5.56}$$

since $x(g(t)+kr) > 0$. Let

$$u(t) = \int_0^\tau ds_1 \int_{s_1}^{s_1+\tau} ds_2 \cdots \int_{s_{m-2}}^{s_{m-2}+\tau} ds_{m-1} \int_{t+s_{m-1}}^{t+s_{m-1}+\tau} z(\theta)d\theta.$$

Then we have $u^{(m)}(t) > 0$ and $u(t) > 0$. Since

$$u'(t) = \int_0^\tau ds_1 \int_{s_1}^{s_1+\tau} ds_2 \cdots \int_{s_{m-2}}^{s_{m-2}+\tau} \Delta_\tau z(t+s_{m-1}) ds_{m-1},$$

then $\Delta_\tau z(t) < 0$ implies $u'(t) < 0$. Moreover, $u^{(i)}(t) > 0$ for all even number i with $2 \le i \le m-2$, and $u^{(j)}(t) < 0$ for all odd number j with $1 \le j \le m-1$.

Integrating (4.5.56) and from the proof of Lemma 4.5.2 with the replacement of $y(t)$ by $z(t)$, we have

$$\Delta_\tau^m u(t) - \bar{q}_m(t) \sum_{i=1}^{k} \frac{1}{p^i} u(g(t) + ir) > 0,$$

which leads to

$$\Delta_\tau^m u(t) - \bar{q}_m(t)\, u(g(t) + kr) \sum_{i=1}^{k} \frac{1}{p^i} > 0.$$

Due to $\sum_{i=1}^k 1/p^i = (1-p^k)/(p^k(1-p))$, we deduce that

$$\Delta_\tau^m u(t) \ge \frac{1-p^k}{p^k(1-p)} \bar{q}_m(t)\, u(g(t) + kr) > 0.$$

Replacements of k by k_0 and t by $t_1 + i\tau$ in the above inequalities yield

$$\Delta_\tau^m u(t_1 + i\tau) \ge \frac{1-p^{k_0}}{p^{k_0}(1-p)} \bar{q}_m(t_1 + i\tau)\, u(g(t_1 + i\tau) + k_0 r).$$

Summing up both sides of the above inequality for i from s to n and since $u'(t) < 0$, we have

$$\Delta_\tau^{m-1} u(t_1 + (n+1)\tau) - \Delta_\tau^{m-1} u(t_1 + s\tau) \ge \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1 + n\tau) + k_0 r) \sum_{i=s}^{n} \bar{q}_m(t_1 + i\tau),$$

which implies

$$-\Delta_\tau^{m-1} u(t_1 + s\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1 + n\tau) + k_0 r) \sum_{i=s}^{n} \bar{q}_m(t_1 + i\tau) \qquad (4.5.57)$$

due to $\Delta_\tau^{m-1} u(t) < 0$. For the above inequality, we will reduce the order of $\Delta_\tau^j u(t_1 + s\tau)$ by rewriting it as $\Delta_\tau^{j-1} u(t + (s+1)\tau) - \Delta_\tau^{j-1} u(t + s\tau)$ for any $j = 1, 2, \cdots, n-1$. Taking into account the fact that all even terms are positive and all odd terms are negative, we will write off all the negative terms from the left hand side of this inequality. It yields then

$$u(t_1 + s\tau) > \frac{1 - p^{k_0}}{p^{k_0}(1-p)} u(g(t_1 + n\tau) + k_0 r) \sum_{i=s}^{n} \bar{q}_m(t_1 + i\tau).$$

Since $g(t_1 + n\tau) + k_0 r \leq t_1 + m_1\tau$ and u is decreasing, by taking $s = m_1$, we obtain

$$u(t_1 + m_1\tau) > u(t_1 + m_1\tau) \frac{1 - p^{k_0}}{p^{k_0}(1-p)} \sum_{i=m_1}^{n} \bar{q}_m(t_1 + i\tau),$$

i.e.,

$$\sum_{i=m_1}^{n} \bar{q}_m(t_1 + i\tau) < \frac{p^{k_0}(1-p)}{1 - p^{k_0}}.$$

This inequality contradicts (4.5.54). If $x(t)$ is an eventually negative solution such that $y(t)$ is not oscillatory, then $y(t) > 0$ holds eventually. The above reasoning with an obvious minor modification also leads to a contradiction. Therefore, for every solution $x(t)$, either $x(t)$ or $y(t)$ is oscillatory.

Corollary 4.5.1 In addition to (4.5.52), we assume that $0 < p < 1$ and there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n) = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \leq n$ for all sufficiently large n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\sum_{i=m_1}^{n} (i - m_1 + 1)\bar{q}_m(t_1 + i\tau) \geq \frac{p^{k_0}(1-p)}{1 - p^{k_0}} \tag{4.5.58}$$

holds for all large enough k with $n = n_k$, $m_1 = m_1(n_k)$. Then, for every solution $x(t)$ of (4.5.46), either $x(t)$ or $x(t) - px(t-r)$ is oscillatory.

Proof Without loss of generality, we suppose (4.5.46) has an eventually positive solution $x(t)$ such that $y(t) = x(t) - px(t-r)$ is not oscillatory. The proof is the same as that of Theorem 4.5.2 until (4.5.57). By the same technique we reduce the order of the difference on the left hand side of this inequality down to the second order and it yields

$$\Delta_\tau^2 u(t_1 + s\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1 + n\tau) + k_0 r) \sum_{i=s}^{n} \bar{q}_m(t_1 + i\tau).$$

Summing up the above inequality for s from m_1 to n, we have

$$\Delta_\tau u(t_1+(n+1)\tau)-\Delta_\tau u(t_1+m_1\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1+n\tau)+k_0 r) \sum_{s=m_1}^{n} \sum_{i=s}^{n} \bar{q}_m(t_1+i\tau).$$

Due to $\Delta_\tau u(t) < 0$, it follows from the above inequality that

$$-\Delta_\tau u(t_1 + m_1\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1 + n\tau) + k_0 r) \sum_{s=m_1}^{n} \sum_{i=s}^{n} \bar{q}_m(t_1 + i\tau),$$

so

$$u(t_1 + m_1\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1 + n\tau) + k_0 r) \sum_{s=m_1}^{n} \sum_{i=s}^{n} \bar{q}_m(t_1 + i\tau).$$

According to $g(t_1 + n\tau) + k_0 r \le t_1 + m_1\tau$ and u is decreasing, it follows that

$$u(t_1 + m_1\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(t_1 + m_1\tau) \sum_{i=m_1}^{n} (i - m_1 + 1)\bar{q}_m(t_1 + i\tau),$$

i.e.,

$$\sum_{i=m_1}^{n} (i - m_1 + 1)\bar{q}_m(t_1 + i\tau) < \frac{p^{k_0}(1-p)}{1-p^{k_0}}.$$

This inequality contradicts (4.5.58). Thus, this contradiction shows that the conclusion holds.

Remark 4.5.1 Compared with (4.5.54), the requirement (4.5.58) for $\bar{q}_m(t)$ is weaker than (4.5.54) since $(i - m_1 + 1) \ge 1$ holds in (4.5.58).

In the next corollary, we consider more general case for any l, $1 \le l < m$.

Corollary 4.5.2 In addition to (4.5.52), we assume that $0 < p < 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n) = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \leq n$ for all sufficiently large n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ and an integer $l\,(1 \leq l < m)$ such that

$$\frac{1}{l!}\sum_{i=m_1}^{n}(i - m_1 + 1)(i - m_1 + 2)\cdots(i - m_1 + l)\bar{q}_m(t_1 + i\tau) \geq \frac{p^{k_0}(1-p)}{1 - p^{k_0}} \quad (4.5.59)$$

holds for all large enough k with $n = n_k$, $m_1 = m_1(n_k)$. Then, for every solution $x(t)$ of (4.5.46), either $x(t)$ or $x(t) - px(t-r)$ is oscillatory.

Proof The proof is the same as that of Theorem 4.5.2 until (4.5.57). We reduce the order of the difference at the left hand of this inequality down to the lth order as we did in the proof of Theorem 4.5.2. Since $1 \leq l < m$, if l is odd, we obtain

$$-\Delta_\tau^l u(t_1 + s\tau) > \frac{1 - p^{k_0}}{p^{k_0}(1-p)} u(g(t_1 + n\tau) + k_0 r)\sum_{i=s}^{n}\bar{q}_m(t_1 + i\tau),$$

and if l is even,

$$\Delta_\tau^l u(t_1 + s\tau) > \frac{1 - p^{k_0}}{p^{k_0}(1-p)} u(g(t_1 + n\tau) + k_0 r)\sum_{i=s}^{n}\bar{q}_m(t_1 + i\tau).$$

We can reach the same conclusion for the above two cases. Thus, we only give the details of the proof when l is odd. Summing up the above inequality for s from m_l to n, we have

$$\begin{aligned}-\Delta_\tau^{l-1}u(t_1 + (n+1)\tau) &+ \Delta_\tau^{l-1}u(t_1 + m_l\tau) \\ &> \frac{1 - p^{k_0}}{p^{k_0}(1-p)} u(g(t_1 + n\tau) + k_0 r)\sum_{s=m_l}^{n}\sum_{i=s}^{n}\bar{q}_m(t_1 + i\tau).\end{aligned}$$

Since $\Delta_\tau^{l-1}u(t) > 0$, the above inequality implies

$$\Delta_\tau^{l-1}u(t_1 + m_l\tau) > \frac{1 - p^{k_0}}{p^{k_0}(1-p)} u(g(t_1 + n\tau) + k_0 r)\sum_{s=m_l}^{n}\sum_{i=s}^{n}\bar{q}_m(t_1 + i\tau).$$

Further, by repeating the above procedure, we obtain

$$u(t_1+m_1\tau) > \frac{1-p^{k_0}}{p^{k_0}(1-p)} u(g(t_1+n\tau)+k_0 r) \sum_{m_2=m_1}^{n} \sum_{m_3=m_2}^{n} \cdots \sum_{s=m_l}^{n} \sum_{i=s}^{n} \bar{q}_m(t_1+i\tau).$$

Due to $g(t_1+n\tau)+k_0 r \leq t_1+m_1\tau$ and u is decreasing, it yields

$$\begin{aligned}
1 \quad > \quad & \frac{1-p^{k_0}}{p^{k_0}(1-p)} \sum_{m_2=m_1}^{n} \sum_{m_3=m_2}^{n} \cdots \sum_{s=m_l}^{n} \sum_{i=s}^{n} \bar{q}_m(t_1+i\tau) \\
= \quad & \frac{1-p^{k_0}}{p^{k_0}(1-p)} \Bigg(\sum_{m_2=m_1}^{n} \sum_{m_3=m_2}^{n} \cdots \\
& \sum_{i=m_{l-1}}^{n} \frac{1}{2!}(i-m_{l-1}+1)(i-m_{l-1}+2)\bar{q}_m(t_1+i\tau) \Bigg) \\
\cdots \quad & \cdots \\
= \quad & \frac{1-p^{k_0}}{p^{k_0}(1-p)} \Bigg(\sum_{i=m_1}^{n} \bar{q}_m(t_1+i\tau) \sum_{m_2=m_1}^{i} \frac{1}{(l-1)!}(i-m_2+1)(i-m_2+2) \times \\
& \cdots (i-m_2+(l-1)) \Bigg) \\
= \quad & \frac{1-p^{k_0}}{p^{k_0}(1-p)} \Bigg(\sum_{i=m_1}^{n} \frac{1}{l!}(i-m_1+1)(i-m_1+2) \times \cdots (i-m_1+l)\bar{q}_m(t_1+i\tau) \Bigg),
\end{aligned}$$

i.e.,

$$\frac{1}{l!} \sum_{i=m_1}^{n} (i-m_1+1)\cdots(i-m_1+l)\bar{q}_m(t_1+i\tau) < \frac{p^{k_0}(1-p)}{1-p^{k_0}}.$$

This inequality contradicts (4.5.59). Thus, this contradiction shows that the conclusion holds.

Remark 4.5.2 Note that (4.5.59) coincides with (4.5.58) for $l=1$. For $l>1$,

$$\frac{(i-m_1+1)(i-m_1+2)\cdots(i-m_1+l)}{l!} \geq i-m_1+1 \geq 1$$

hold. Thus, (4.5.59) is weaker than (4.5.58) and (4.5.54) in general.

The following Theorem 4.5.3 and Corollaries 4.5.3-4.5.4 are for equation (4.5.46) with $p=1$.

Theorem 4.5.3 In addition to (4.5.52), we assume that $p = 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n) = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \leq n$ for all sufficiently large n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\sum_{i=m_1}^{n} \bar{q}_m(t_1 + i\tau) \geq \frac{1}{k_0} \tag{4.5.60}$$

for large enough k with $n = n_k$, $m_1 = m_1(n)$. Then, for every solution $x(t)$ of (4.5.46), either $x(t)$ or $x(t) - x(t-r)$ is oscillatory.

Proof The proof is similar to that of Theorem 4.5.2. However, the proof of the feature of $z(t)$ is different from that of Theorem 4.5.2 due to $p = 1$. We, hence, just outline the proof about the feature of $z(t)$. For $z(t)$, by Lemma 4.2.2, we notice h could be even with $2 \leq h \leq m-2$. So $\Delta_\tau^i z(t) > 0$ for all even number i with $2 \leq i \leq m-2$, and $|\Delta_\tau^j z(t)| > 0$ for all odd number j with $1 \leq j \leq m-1$.

If $\Delta_\tau z(t) > 0$, from the proof of Theorem 4.5.2 we have $z(t_1 + d\tau) \to +\infty$ as $d \to \infty$ for some $t_1 \geq t_0$. Since $p = 1$, from $0 < x(t) < x(t-r)$, we know that $x(t)$ is bounded on $[t_0, \infty)$. Thus, $z(t)$ is bounded on $[t_0, \infty)$. This contradicts $z(t_1 + d\tau) \to +\infty$ as $d \to \infty$. Thus, $\Delta_\tau z(t) < 0$. So, according to Lemma 4.2.2 again, $h = 0$. Thus, $\Delta_\tau^i z(t) > 0$ for all even number i with $2 \leq i \leq m-2$, and $\Delta_\tau^j z(t) < 0$ for all odd number j with $1 \leq j \leq m-1$.

The rest is the same as the proof of Theorem 4.5.2 with the necessary replacement of $p^{k_0}(1-p)/(1-p^{k_0})$ by $1/k_0$.

The proofs of the following corollaries are very similar to those of Corollaries 4.5.1-4.5.2 except minor changes. Thus, we will omit the proofs.

Corollary 4.5.3 In addition to (4.5.52), we assume that $p = 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n_k) = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \leq n$ for all sufficiently large n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\sum_{i=m_1}^{n} (i - m_1 + 1)\bar{q}_m(t_1 + i\tau) \geq \frac{1}{k_0} \tag{4.5.61}$$

for large enough k with $n = n_k$, $m_1 = m_1(n_k)$. Then, for every solution $x(t)$ of (4.5.46), either $x(t)$ or $x(t) - x(t - r)$ is oscillatory.

Corollary 4.5.4 In addition to (4.5.52), we assume that $p = 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n_k) = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \leq n$ for all sufficiently large n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ and an integer $l\,(1 \leq l \leq m - 1)$ such that

$$\frac{1}{l!} \sum_{i=m_1}^{n} (i - m_1 + 1)(i - m_1 + 2) \cdots (i - m_1 + l)\bar{q}_m(t_1 + i\tau) \geq \frac{1}{k_0} \tag{4.5.62}$$

for large enough k with $n = n_k$, $m_1 = m_1(n_k)$. Then, for every solution $x(t)$ of (4.5.46), either $x(t)$ or $x(t) - x(t - r)$ is oscillatory.

Suppose that $x(t)$ is a bounded eventually positive solution of (4.5.52). The proof of Theorem 4.5.2 is then still valid for Theorem 4.5.4 subject to a few obvious minor changes. Therefore, we will omit the proof of the following results for equation (4.5.46) with $p > 1$.

Theorem 4.5.4 In addition to (4.5.52), we assume that $p > 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n_k) = \lceil (g(t_1+n\tau)-t_1+k_0 r)/\tau \rceil \leq n$ for all large enough n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\sum_{i=m_1}^{n} \bar{q}_m(t_1 + i\tau) \geq \frac{p^{k_0}(1-p)}{1-p^{k_0}} \tag{4.5.63}$$

holds for all large enough k with $n = n_k, m_1 = m_1(n_k)$. Then, for every bounded solution $x(t)$ of (4.5.46), either $x(t)$ or $x(t) - px(t-r)$ is oscillatory.

Corollary 4.5.5 In addition to (4.5.52), we assume that $p > 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n) = \lceil (g(t_1 + n\tau) - t_1 + k_0 r)/\tau \rceil \leq n$ for all large enough n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\sum_{i=m_1}^{n} (i - m_1 + 1)\bar{q}_m(t_1 + i\tau) \geq \frac{p^{k_0}(1-p)}{1-p^{k_0}} \tag{4.5.64}$$

holds for all large enough k with $n = n_k, m_1 = m_1(n_k)$. Then, for every bounded solution $x(t)$ of (4.5.46), either $x(t)$ or $x(t) - px(t-r)$ is oscillatory.

Corollary 4.5.6 In addition to (4.5.52), we assume that $p > 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n) = \lceil (g(t_1+n\tau)-t_1+k_0 r)/\tau \rceil \leq n$ for all sufficiently large n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ and an integer $l\,(1 \leq l < m)$ such that

$$\frac{1}{l!}\sum_{i=m_1}^{n} (i-m_1+1)(i-m_1+2)\cdots(i-m_1+l)\,\bar{q}_m(t_1+i\tau) \geq \frac{p^{k_0}(1-p)}{1-p^{k_0}} \tag{4.5.65}$$

holds for all large enough k with $n = n_k$, $m_1 = m_1(n_k)$. Then, for every bounded solution $x(t)$ of equation (4.5.46), either $x(t)$ or $x(t) - px(t-r)$ is oscillatory.

Remark 4.5.3 Note that

$$\frac{(i-m_1+1)(i-m_1+2)\cdots(i-m_1+l-1)}{(l-1)!} \geq i-m_1+1 \geq 1$$

holds. Thus, (4.5.65) is weaker that (4.5.64) and (4.5.63) in general.

Corollary 4.5.7 In addition to (4.5.52), we assume that $p > 1$ and that there is a positive integer k_0 and a $t_1 \geq t_0$ satisfying $m_1(n) = \lceil (g(t_1+n\tau)-t_1+k_0 r)/\tau \rceil \leq n$ for all sufficiently large n. Moreover, there is a sequence $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\frac{1}{(n-1)!}\sum_{i=m_1}^{n}\frac{(i-m_1+n-1)!}{(i-m_1)!}\bar{q}_m(t_1+i\tau) \geq \frac{p^{k_0}(1-p)}{1-p^{k_0}} \tag{4.5.66}$$

holds for all large enough k with $n = n_k$, $m_1 = m_1(n_k)$. Then, for every bounded solution $x(t)$ of (4.5.46), either $x(t)$ or $x(t) - px(t-r)$ is oscillatory.

4.5.3 EXAMPLES

Three illustrating examples will be given in this subsection to demonstrate the applications of the results given in last subsection.

Example 4.5.1 Consider the linear difference equation

$$\Delta_\tau^{2n}(x(t) - px(t-r)) + \frac{1}{t}x(t - \frac{\sigma}{1+\beta t}) = 0 \tag{4.5.67}$$

for $t > 0$, where n is a positive integer, $p \geq 0$, $\beta \geq 0$, r, τ and σ are positive constants. Viewing (4.5.67) as (4.5.46), we have $q(t) = 1/t$ and $g(t) = t - \sigma/(1+\beta t)$. Then, according to (4.5.47), $\bar{q}_{2n}(t) = \alpha/(t+2n\tau)$ for $\beta = 0$ and

$$\bar{q}_{2n}(t) = \frac{\alpha}{t+2n\tau}\left(1 - \frac{\sigma\beta}{(1+\beta t)^2+\sigma\beta}\right)^{2n}$$

for $\beta > 0$. Since $\bar{q}_{2n}(t) \geq \alpha'/(t + 2n\tau)$ for some $\alpha' > 0$ and all $t \geq 0$, $\bar{q}_{2n}$ satisfies (4.5.52) with $t' = 0$. By Theorem 4.5.1, for every solution $x(t)$ of (4.5.67), either $x(t)$ is oscillatory or for any $T \geq t_0$ there exists a $t'' > T$ such that $|x(t'')| < p|x(t'' - r)|$. In particular, when $p = 0$, every solution of (4.5.67) is oscillatory.

Example 4.5.2 Consider the difference equation

$$\Delta_\pi^{2n}(x(t) - px(t - \pi)) + 8x(t - \pi) + \frac{8\sigma}{1 + t^2} x^3(t - \pi) = 0, \tag{4.5.68}$$

where $\sigma \geq 0$ is a constant. Regarding (4.5.68) as (4.5.46), we have $\tau = \pi$, $r = \pi$, $g(t) = t - \pi$ and $q(t) = 8$. Then, for some $\alpha \in (0, 1)$, $\bar{q}_{2n} = 8\alpha$ by (4.5.47) so (4.5.52) is satisfied. For $p = 1$, $k_0 = 1$ and $t_1 = t$, we have $m_l = l$ and

$$\sum_{s=m_l}^{l} (s + 1 - m_l)\bar{q}_{2n}(t_1 + s\tau) = 8\alpha > 1 = \frac{1}{k_0}$$

if $\alpha > 1/8$. Moreover, we also have

$$\sum_{s=m_l}^{l} (s + 1 - m_l)\bar{q}_{2n}(t_1 + s\tau) = 8\alpha > p = \frac{(1 - p)p^{k_0}}{1 - p^{k_0}}$$

if $p \in (0, 1) \cup (1, 8)$ and $\alpha > p/8$. According to Theorems 4.5.2-4.5.3, for every solution $x(t)$ of (4.5.68), either $x(t)$ or $x(t) - px(t - r)$ is oscillatory if $0 < p \leq 1$. Furthermore, by Theorem 4.5.4, for every bounded solution $x(t)$ of (4.5.68), either $x(t)$ or $x(t) - px(t - r)$ is oscillatory if $1 < p < 8$.

Example 4.5.3 Consider the difference equation

$$\Delta_\tau^{2n}(x(t) - x(t - r)) + 2^{2n+1} x(t - 3) = 0, \tag{4.5.69}$$

where τ and r are positive odd integers. Viewing (4.5.69) as (4.5.46), we have $g(t) = t-3$ and $q(t) = 2^{2n+1}$. Then, for some $\alpha \in (0,1)$, $\bar{q}_{2n} = \alpha 2^{2n+1}$ by (4.5.47) so (4.5.52) is satisfied. For $p = 1$, $k_0 = 1$ and $t_1 = t$, we have $m_l = l$ and

$$\sum_{s=m_l}^{l} (s+1-m_l)\bar{q}_{2n}(t_1 + s\tau) = \alpha 2^{2n+1} > 1 = \frac{1}{k_0}$$

if $\alpha > 2^{-(2n+1)}$. According to Theorems 4.5.3, for every solution $x(t)$ of (4.5.69), either $x(t)$ or $x(t) - x(t-r)$ is oscillatory.

4.6 CONCLUSION

In this chapter we concentrate on neutral difference equation (4.1.1) of even order. Here we just give a brief summary and a detailed summary will be presented at the end of chapter 5 in order to compare the difference of the results between the even order and the odd order of (4.1.1).

From the known results about the difference of discrete argument we have obtained the similar results on the difference of continuous variable. By defining the new functions, we have transformed (4.1.1) to difference equations/inequalites without neutral term. Furthermore, applying Riccati transformation to (4.1.1), we have obtained sufficient conditions for x to be oscillatory or $x(t) - px(t-r)$ to have constant sign. The results are given in three separate cases according to the value of p, i.e., $0 < p < 1$, $p = 1$, and $p > 1$. We managed to establish weaker oscillatory criteria in each case.

Chapter 5

ODD ORDER DIFFERENCE EQUATIONS

5.1 INTRODUCTION

Our main interest in this chapter is to investigate the bounded solutions of equations (4.1.1) with $m \geq 3$ being an odd integer. In chapter 4, we discussed (4.1.1) with $m \geq 2$ an even integer. We shall adopt an approach here different from that for even order equations due to the difference between the format of odd order and even order equations.

A well-known result on differential inequalities will be stated in section 5.2 since it will be needed in later sections. Section 5.3 will consist of three subsections for the third order equations with subsection 5.3.1 covering some lemmas, subsection 5.3.2 the main results and subsection 5.3.3 some illustrating examples. Section 5.4 will be presented for higher order equations in a way similar to section 5.3. Finally, we will close this chapter with a conclusion section.

5.2 PRELIMINARIES

The following lemma is about the inequality of the form

$$x'(t) + q(t)x(\tau(t)) \leq 0, \tag{5.2.1}$$

where $q, \tau \in C([t_0, \infty), R^+)$, $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$. Let

$$\eta = \liminf_{t\to\infty} \int_{\tau(t)}^{t} q(s)ds.$$

Lemma 5.2.1 Assume that τ is nondecreasing, $0 \leq \eta \leq e^{-1}$, and $x(t)$ is an eventually positive function satisfying (5.2.1). Set

$$r = \liminf_{t\to\infty} \frac{x(t)}{x(\tau(t))}.$$

Then r satisfies

$$\frac{1-\eta-\sqrt{1-2\eta-\eta^2}}{2} \le r \le 1.$$

The above lemma can be found in [15] (page 18).

5.3 THIRD ORDER EQUATION (4.1.1)

In this section, we deal with equation (4.1.1) with $m = 3$, i.e.,

$$\Delta_\tau^3(x(t) - px(t-r)) + f(t, x(g(t))) = 0. \tag{5.3.2}$$

For the convenience of later use, let

$$\bar{q}_3(t) = \alpha \min_{t\le s\le t+3\tau}\{q(s)\}\left(\min_{g(t)\le s\le g(t)+3\tau}\{(g^{-1}(s))'\}\right)^3, \tag{5.3.3}$$

where $0 < \alpha < 1$. We shall see from the following subsections that $\bar{q}_3$ will play an important role in the oscillatory criteria for (5.3.2).

The oscillation of (5.3.2) will be considered in this section when $0 \le p < 1$ or $p > 1$ and sufficient conditions will be obtained for the bounded solutions of (5.3.2) to be oscillatory.

5.3.1 RELATED LEMMAS

The properties of the bounded solutions of (5.3.2) are in some way determined by the values of p. Thus we investigate the solutions when $0 \le p < 1$ and $p > 1$ respectively.

Lemma 5.3.1 Let $0 \le p < 1$. Assume that $x(t)$ is a bounded and eventually positive (negative) solution of (5.3.2) with $z(t) = x(t) - px(t-r)$ and $\liminf_{t\to\infty} z(t) \ge 0$ ($\limsup_{t\to\infty} z(t) \le 0$). Let

$$y(t) = \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \int_{t_2}^{t_2+\tau} z(\theta)d\theta.$$

Then $y(t) > 0\,(< 0)$, $y^{(3)}(t) < 0\,(> 0)$, $y''(t) > 0\,(< 0)$, and $y'(t) < 0\,(> 0)$ eventually. Moreover,

$$\Delta_\tau^3 y(t) + \bar{q}_3(t)\sum_{i=0}^{n} p^i y(g(t) - ir) < 0\,(> 0) \tag{5.3.4}$$

holds for each fixed natural number $n \ge 0$ and for all large enough t.

Proof Suppose $x(t)$ is a bounded and eventually positive solution. Note that $g(t) < t$, $g'(t) > 0$ and (4.1.2) hold, so there exists a $t_1 > t_0$ such that $x(g(t)) > 0$ for all $t \ge t_1$. From (5.3.2) it follows that

$$\Delta_\tau^3 z(t) + f(t, x(g(t))) = 0.$$

By (4.1.3), we obtain $f(t, x(g(t))) \ge q(t)x(g(t)) > 0$ for $t \ge t_1$. Therefore,

$$y^{(3)}(t) + q(t)x(g(t)) \le 0 \tag{5.3.5}$$

for $t \ge t_1$. According to $q(t)x(g(t)) > 0$ and (5.3.5), $y^{(3)}(t) < 0$ for all $t \ge t_1$. Thus, $y''(t)$ is decreasing so either $y''(t) > 0$ for $t \ge t_1$ or there is a $t_2 > t_1$ such that $y''(t) < 0$ for $t \geq t_2$. Suppose $y''(t) < 0$ for $t \geq t_2$. Then $y'(t)$ is decreasing and

$$y'(t) = y'(t_2) + \int_{t_2}^{t} y''(s)ds \leq y'(t_2) + y''(t_2)(t - t_2) \to -\infty$$

as $t \to \infty$. Thus, there is a $t_3 > t_2$ such that $y'(t) \leq y'(t_3) < 0$ for $t \geq t_3$. This implies that

$$y(t) = y(t_3) + \int_{t_3}^{t} y'(s)ds \to -\infty$$

as $t \to \infty$, a contradiction to the boundedness of y since both x and z are bounded. Therefore we have $y''(t) > 0$ for $t \geq t_1$. From this we know that $y'(t)$ is increasing so either $y'(t) < 0$ for all $t \geq t_1$ or there is a $t_4 > t_1$ such that $y'(t) \geq y'(t_4) > 0$ for $t \geq t_4$. If the latter holds then

$$y(t) = y(t_4) + \int_{t_4}^{t} y'(s)ds \geq y(t_4) + y'(t_4)(t - t_4) \to \infty$$

as $t \to \infty$, a contradiction again to the boundedness of y. Therefore we must have $y'(t) < 0$ for all $t \geq t_1$. This shows that $y(t)$ is decreasing so either $y(t) > 0$ for $t \geq t_1$ or there is a $t_5 > t_1$ such that $y(t) \leq y(t_5) < 0$ for $t \geq t_5$. In the latter case, then

$$\begin{aligned}
\int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \int_{t_2}^{t_2+\tau} x(\theta)d\theta &= y(t) + p\int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \int_{t_2}^{t_2+\tau} x(\theta - r)d\theta \\
&\leq \quad y(t_5) + p\int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \int_{t_2}^{t_2+\tau} x(\theta - r)d\theta \\
&\dots \\
&\leq \quad y(t_5)\sum_{i=0}^{h-1} p^i + p^h \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \int_{t_2}^{t_2+\tau} x(\theta - hr)d\theta \\
&\leq \quad \frac{y(t_5)(1-p^h)}{1-p} + p^h M\tau^3
\end{aligned}$$

for $t \geq t_5 + hr$, where $M = \sup_{t \geq t_0}\{x(t)\}$ and h is any integer with $h \geq 1$. Let $t \to \infty$ so $h \to \infty$ as well, $p^h M\tau^3$ then is arbitrarily small since $0 \leq p < 1$. Thus,

$$\int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \int_{t_2}^{t_2+\tau} x(\theta)d\theta < 0,$$

which contradicts the assumption that $x(t)$ is eventually positive. Therefore, we must have $y(t) > 0$ for $t \geq t_1$.

From (5.3.5), it follows that

$$\Delta_\tau^3 z(t) + q(t)z(g(t)) + pq(t)x(g(t) - r) \leq 0.$$

By the definition of $z(t)$, the above inequality gives

$$\Delta_\tau^3 z(t) + q(t)z(g(t)) + pq(t)z(g(t) - r) + p^2 q(t)x(g(t) - 2r) \leq 0.$$

Proceeding in the same way as the above, we obtain

$$\Delta_\tau^3 z(t) + q(t)\sum_{i=0}^{n} p^i z(g(t) - ir) + p^{n+1} q(t)x(g(t) - (n+1)r) \leq 0.$$

Since $q(t)p^{n+1}x(g(t) - (n+1)r) > 0$ when t is large enough, the above inequality implies

$$\Delta_\tau^3 z(t) + q(t)\sum_{i=0}^{n} p^i z(g(t) - ir) < 0.$$

In order to integrate the above inequality, we show that $z(t)$ is eventually positive. This is true if $p = 0$ since $z(t) = x(t)$ in this case. Now suppose $0 < p < 1$. Since $y^{(3)}(t) = \Delta_\tau^3 z(t) < 0$ for $t \geq t_1$,

$$\Delta_\tau^2 z(t + (h+1)\tau) - \Delta_\tau^2 z(t + h\tau) = \Delta_\tau^3 z(t + h\tau) < 0$$

so $\Delta_\tau^2 z(t + h\tau)$ is decreasing as h increases. By the boundedness of $x(t)$ we know that $\lim_{h\to\infty} \Delta_\tau^2 z(t + h\tau)$ exists. If $\lim_{h\to\infty} \Delta_\tau^2 z(t + h\tau) = S(t) \neq 0$, then

$$\Delta_\tau z(t + (h+1)\tau) = \Delta_\tau z(t) + \sum_{k=0}^{h} \Delta_\tau^2 z(t + k\tau) \to -\infty \text{ or } \infty$$

as $h \to \infty$, a contradiction to the boundedness of $\Delta_\tau z(t)$. Thus, for each $t \geq t_1$, $\Delta_\tau^2 z(t+h\tau)$ is decreasing and tends to 0 as $h \to \infty$. Similarly, $\Delta_\tau z(t+h\tau)$ is increasing as h increases and $\Delta_\tau z(t+h\tau) \to 0$ as $h \to \infty$; $z(t+h\tau)$ is decreasing as h increases so $\lim_{h\to\infty} z(t+h\tau)$ exists for each $t \geq t_1$. Then $z(t+h\tau)$ is decreasing and, by assumption, $\lim_{h\to\infty} z(t+h\tau) \geq 0$ so $z(t+h\tau) > 0$ for all $t \geq t_1$ and $h \geq 1$.

Integrating $q(t)z(g(t)-ir)$, by the assumptions on g and q, we obtain

$$\begin{aligned}
&\int_t^{t+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \int_{s_2}^{s_2+\tau} z(g(\theta)-ir)q(\theta)d\theta \\
&\geq \min_{t\leq s\leq t+3\tau}\{q(s)\} \int_t^{t+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \int_{s_2}^{s_2+\tau} z(g(\theta)-ir)d\theta \\
&\geq \min_{t\leq s\leq t+3\tau}\{q(s)\} \int_{g(t)}^{g(t+\tau)} (g^{-1}(s_1))' ds_1 \int_{s_1}^{g(g^{-1}(s_1)+\tau)} (g^{-1}(s_2))' ds_2 \\
&\qquad \int_{s_2}^{g(g^{-1}(s_2)+\tau)} z(\theta-ir)(g^{-1}(\theta))' d\theta \\
&\geq \min_{t\leq s\leq t+3\tau}\{q(s)\} \left(\min_{g(t)\leq s\leq g(t)+3\tau}\left\{(g^{-1}(s))'\right\}\right)^3 \int_{g(t)}^{g(t)+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \\
&\qquad \int_{s_2}^{s_2+\tau} z(\theta-ir)d\theta \\
&\geq \min_{t\leq s\leq t+3\tau}\{q(s)\} \left(\min_{g(t)\leq s\leq g(t)+3\tau}\left\{(g^{-1}(s))'\right\}\right)^3 y(g(t)-ir) \\
&\geq \bar{q}_3(t) y(g(t)-ir).
\end{aligned}$$

Therefore, it follows that

$$\Delta_\tau^3 y(t) + \bar{q}_3(t) \sum_{i=0}^{n} p^i y(g(t)-ir) < 0$$

holds for each fixed natural number n and for all large enough t. If $x(t)$ is a bounded and eventually negative solution, then the corresponding inequalities in the above proof reverse to the opposite so the conclusion within brackets follows.

Lemma 5.3.2 Let $0 \le p < 1$ and $r = k\tau$ for a positive integer k. Assume that $x(t)$ is a bounded and eventually positive (negative) solution of (5.3.2). Let

$$\begin{aligned} z(t) &= x(t) - px(t-r), \\ y(t) &= \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \int_{t_2}^{t_2+\tau} z(\theta)d\theta. \end{aligned}$$

Then the conclusion of Lemma 5.3.1 holds.

Proof The proof is the same as that of Lemma 5.3.1 until $\lim_{h\to\infty} z(t+h\tau)$ exists for each $t \ge t_1$. Suppose there is a $t' > t_1$ such that $\lim_{h\to\infty} z(t'+h\tau) = \delta < 0$. Then $z(t'+h\tau) \le \delta/2 < 0$ for $h \ge h_1 > 0$. So

$$\begin{aligned} x(t'+k(h+h_1)\tau) &= z(t'+k(h+h_1)\tau) + px(t'+k(h+h_1)\tau - k\tau) \\ &\le \frac{1}{2}\delta + px(t'+k(h-1+h_1)\tau) \\ &\le \frac{1}{2}\delta(1+p+\cdots+p^{h-1}) + p^h x(t'+kh_1\tau) < 0 \end{aligned}$$

for large h, a contradiction to the assumption that x is eventually positive. Therefore $\lim_{h\to\infty} z(t+h\tau) \ge 0$ for any $t \ge t_1$. Since $z(t+h\tau)$ is decreasing as h increases, $z(t) > 0$ for all $t \ge t_1$. The rest of the proof is the same as that of Lemma 5.3.1.

Lemma 5.3.3 Under the assumptions of Lemma 5.3.1 or Lemma 5.3.2, let

$$v(t) = \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \int_{t_2}^{t_2+\tau} y(\theta)d\theta.$$

Then $v(t) > 0\,(< 0)$, $v^{(3)}(t) < 0\,(> 0)$, $v''(t) > 0\,(< 0)$, and $v'(t) < 0\,(> 0)$ eventually. Moreover,

$$v^{(3)}(t) + \frac{1}{\tau^3}\bar{q}_3(t)\sum_{i=0}^{n} p^i v(g(t) - ir) < 0\,(> 0) \tag{5.3.6}$$

holds for each fixed natural number n and for all large enough t.

Proof Suppose $x(t)$ is a bounded and eventually positive solution. By the definition of $v(t)$, due to $y(t) > 0$, we have $v(t) > 0$ for all $t \geq t_1$. Further, we have

$$v'(t) = \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \int_{t_2}^{t_2+\tau} y'(\theta)d\theta.$$

Since $y'(t) < 0$, then $v'(t) < 0$. Similarly, we have $v''(t) > 0$ and $v^{(3)}(t) < 0$. Note that $v^{(3)}(t) = \Delta_\tau^3 y(t)$. Since $y'(t) < 0$,

$$\begin{aligned}
v(g(t) - ir) &= \int_{g(t)}^{g(t)+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \int_{t_2}^{t_2+\tau} y(\theta - ir)d\theta \\
&\leq \int_{g(t)}^{g(t)+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \int_{t_2}^{t_2+\tau} y(t_2 - ir)d\theta \\
&\leq \tau \int_{g(t)}^{g(t)+\tau} dt_1 \int_{t_1}^{t_1+\tau} y(t_2 - ir)dt_2 \\
&\leq \tau^2 \int_{g(t)}^{g(t)+\tau} y(t_1 - ir)dt_1 \\
&\leq \tau^3 y(g(t) - ir).
\end{aligned}$$

Hence, from (5.3.4), we obtain

$$v^{(3)}(t) + \frac{1}{\tau^3}\bar{q}_3(t)\sum_{i=0}^{n} p^i v(g(t) - ir) < 0$$

for each fixed natural number n and for all large enough t. If $x(t)$ is a bounded and eventually negative solution, then the conclusion within the brackets follows from the same argument with an obvious modification.

Lemma 5.3.4 Under the assumptions of Lemma 5.3.3, for each $t \geq t_1$, there is a $\theta \in (g(t), t)$ such that

$$|v'(g(t))| > \frac{(t - g(t))^2}{2}|v^{(3)}(\theta)|. \tag{5.3.7}$$

Proof Under the assumptions of Lemma 5.3.3, we know that $v'(t)$, $-v''(t)$ and $v^{(3)}(t)$ have the same sign. By Taylor's formula, we have

$$v'(g(t)) = v'(t) + v''(t)(g(t) - t) + \frac{1}{2}v^{(3)}(\theta)(g(t) - t)^2$$

for some $\theta \in (g(t), t)$ and (5.3.7) follows immediately.

The following three lemmas are for the solutions of (5.3.2) with $p > 1$.

Lemma 5.3.5 Let $r = k\tau$, $k \in N$. Assume that $x(t)$ is a bounded and eventually positive (negative) solution of (5.3.2) with $p > 1$. Let

$$\begin{aligned} z(t) &= x(t) - px(t-r), \\ y(t) &= \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \int_{t_2}^{t_2+\tau} z(\theta)d\theta. \end{aligned}$$

Then $y(t) < 0\,(> 0)$, $y^{(3)}(t) < 0\,(> 0)$, $y''(t) > 0\,(< 0)$, and $y'(t) < 0\,(> 0)$ eventually. Moreover,

$$\Delta_\tau^3 y(t) - \bar{q}_3(t) \sum_{i=1}^{n} \frac{1}{p^i} y(g(t) + ir) < 0\,(> 0) \tag{5.3.8}$$

for each fixed integer $n \geq 1$ and all large enough t.

Proof Suppose $x(t)$ is a bounded and eventually positive solution. By $g(t) < t$, $g'(t) > 0$ and (4.1.2), from the assumptions, there exists a $t_1 > t_0$ such that $x(g(t)) > 0$ for all $t \geq t_1$. Note that

$$\Delta_\tau^3 z(t) + f(t, x(g(t))) = 0.$$

According to (4.1.3), we may assume $f(t, x(g(t))) \geq q(t)x(g(t)) > 0$ for all $t \geq t_1$. Therefore

$$\Delta_\tau^3 z(t) + q(t)x(g(t)) \leq 0 \tag{5.3.9}$$

for $t \geq t_1$. By the definition of $y(t)$, we notice $y^{(3)}(t) = \Delta_\tau^3 z(t)$. Thus, from (5.3.9) it follows that

$$y^{(3)}(t) + q(t)x(g(t)) \leq 0 \tag{5.3.10}$$

for $t \geq t_1$. By $q(t)x(g(t)) > 0$, $y^{(3)}(t) < 0$ holds for $t \geq t_1$. Thus, $y''(t)$ is decreasing so either $y''(t) > 0$ for $t \geq t_1$ or there is a $t_2 > t_1$ such that $y''(t) \leq y''(t_2) < 0$ for $t \geq t_2$. By the same procedure as that used in the proof of Lemma 5.3.1, we have $y''(t) > 0$ and $y'(t) < 0$ for $t \geq t_1$. So $y(t)$ is decreasing and either $y(t) > 0$ for $t \geq t_1$ or there is a $t_3 > t_1$ such that $y(t) \leq y(t_3) < 0$ for $t \geq t_3$. Now we claim that $y(t) \leq y(t_3) < 0$ for all $t \geq t_3$. To prove this, we consider the feature of $z(t)$ at first. Since $y^{(3)}(t) = \Delta_\tau^3 z(t) < 0$ for $t \geq t_1$, by the same reasoning as that used in the proof of Lemma 5.3.1, we know that $z(t + h\tau)$ is decreasing for each fixed $t \geq t_1$ as h increases. Suppose there is a $t' > t_1$ such that $z(t' + h\tau) > 0$ for all $h \geq 1$. Under $r = k\tau$, we then have $z(t' + hr) > 0$ for all $h \geq 1$ so

$$x(t' + hr) > px(t' + (h-1)r) > p^2 x(t' + (h-2)r) > \cdots > p^h x(t')$$

for all $h \geq 1$. So $x(t'+hr) \to \infty$ as $h \to \infty$, a contradiction to the boundedness of x. Therefore, for each $t \in [t_1, t_1+\tau]$, $z(t+h\tau)$ is decreasing as h increases and there is an integer $H(t) > 0$ such that $z(t + h\tau) < z(t + H(t)\tau) < 0$ for all $h > H(t)$. Since $z(t)$ is continuous for each $t' \in [t_1, t_1 + \tau]$, there is an open interval $I(t')$ such that $z(t+h\tau) < z(t+H(t')\tau) < 0$ hold for all $t \in I(t')$ and $h > H(t')$. Since $[t_1, t_1 + \tau]$ is compact and $\{I(t') : t' \in [t_1, t_1 + \tau]\}$ is an open cover of $[t_1, t_1 + \tau]$, there is a finite subset of $\{I(t') : t' \in [t_1, t_1 + \tau]\}$ covering $[t_1, t_1 + \tau]$. Therefore, there is a $K > 0$ such that

$$z(t + h\tau) \le z(t + K\tau) < 0$$

for all $t \in [t_1, t_1 + \tau]$ and all $h \ge K$. Hence, there is a $t_3 > t_1$ such that $z(t) < 0$ so that $y(t) < 0$ for all $t \ge t_3$.

From (5.3.9), we have

$$\Delta_\tau^3 z(t) - \frac{q(t)}{p} z(g(t) + r) + \frac{q(t)}{p} x(g(t) + r) \le 0.$$

According to the definition of $z(t)$, it follows from the above inequality that

$$\Delta_\tau^3 z(t) - \frac{q(t)}{p} z(g(t) + r) + \frac{q(t)}{p}(-\frac{1}{p} z(g(t) + 2r) + \frac{1}{p} x(g(t) + 2r)) \le 0.$$

Proceeding in the same procedure as the above, we obtain

$$\Delta_\tau^3 z(t) - q(t) \sum_{i=1}^{n} \frac{1}{p^i} z(g(t) + ir) + q(t) \frac{1}{p^n} x(g(t) + nr) \le 0.$$

Since $q(t)x(g(t) + nr) > 0$ for all large enough t, the above inequality yields

$$\Delta_\tau^3 z(t) - q(t) \sum_{i=1}^{n} \frac{1}{p^i} z(g(t) + ir) < 0.$$

Integrating $q(t)z(g(t) + ir)$, by the assumptions on p and g, we obtain

$$\begin{aligned}
&\int_t^{t+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \int_{s_2}^{s_2+\tau} z(g(\theta) + ir) q(\theta) d\theta \\
&\le \min_{t \le l \le t+3\tau} \{q(l)\} \int_t^{t+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \int_{s_2}^{s_2+\tau} z(g(\theta) + ir) d\theta \\
&\le \min_{t \le l \le t+3\tau} \{q(l)\} \int_{g(t)}^{g(t+\tau)} (g^{-1}(s_1))' ds_1 \int_{s_1}^{g(g^{-1}(s_1)+\tau)} (g^{-1}(s_2))' ds_2 \\
&\qquad \int_{s_2}^{g(g^{-1}(s_2)+\tau)} z(\theta + ir)(g^{-1}(\theta))' d\theta \\
&\le \min_{t \le l \le t+3\tau} \{q(l)\} \left(\min_{g(t) \le s \le g(t)+3\tau} (g^{-1}(s))' \right)^3 \\
&\qquad \times \int_{g(t)}^{g(t)+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \int_{s_2}^{s_2+\tau} z(\theta + ir) d\theta \\
&\le \bar{q}_3(t) y(g(t) + ir).
\end{aligned}$$

Therefore, we have

$$\Delta_\tau^3 y(t) - \bar{q}_3(t)\sum_{i=1}^{n}\frac{1}{p^i}y(g(t)+ir) < 0$$

for each fixed integer $n \geq 1$ and for all large enough t. If $x(t)$ is a bounded and eventually negative solution, then the conclusion within brackets follows from the above proof with minor modifications.

Lemma 5.3.6 Under the assumptions of Lemma 5.3.5, let

$$v(t) = \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \int_{t_2}^{t_2+\tau} y(\theta)d\theta.$$

Then $v(t) < 0\,(> 0)$, $v^{(3)}(t) < 0\,(> 0)$, $v''(t) > 0\,(< 0)$, and $v'(t) < 0\,(> 0)$ eventually. Moreover,

$$v^{(3)}(t) - \frac{1}{\tau^3}\bar{q}_3(t)\sum_{i=1}^{n}\frac{1}{p^i}v(g(t) - 3\tau + ir) < 0\,(> 0) \tag{5.3.11}$$

for each fixed integer $n \geq 1$ and all large enough t.

Proof By the definition of $v(t)$ we know that $y(t)$ and $v(t)$ have the same sign for all large enough t. Further, we have

$$v'(t) = \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \int_{t_2}^{t_2+\tau} y'(\theta)d\theta.$$

Thus, $y'(t)$ and $v'(t)$ have the same sign. Similarly, $v''(t)$ and $y''(t)$ have the same sign and $v^{(3)}(t)$ and $y^{(3)}(t)$ have the same sign. Note that $v^{(3)}(t) = \Delta_\tau^3 y(t)$. If $y'(t) < 0$, then

$$\begin{aligned}
v(g(t)+ir) &= \int_{g(t)}^{g(t)+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \int_{t_2}^{t_2+\tau} y(\theta+ir)d\theta \\
&\geq \int_{g(t)}^{g(t)+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \int_{t_2}^{t_2+\tau} y(t_2+\tau+ir)d\theta \\
&\geq \tau \int_{g(t)}^{g(t)+\tau} dt_1 \int_{t_1}^{t_1+\tau} y(t_2+\tau+ir)dt_2 \\
&\geq \tau^2 \int_{g(t)}^{g(t)+\tau} y(t_1+2\tau+ir)dt_1 \\
&\geq \tau^3 y(g(t)+3\tau+ir).
\end{aligned}$$

Hence, from (5.3.8), we have

$$v^{(3)}(t) - \frac{1}{\tau^3}\bar{q}_3(t)\sum_{i=1}^{n}\frac{1}{p^i}v(g(t)-3\tau+ir) < 0$$

for each fixed integer $n \geq 1$ and for all large enough t. If $y'(t) > 0$, then "$\geq$" and "$<$" are replaced by "$\leq$" and "$>$" respectively in the above inequalities.

Lemma 5.3.7 Assume that $x(t)$ is an eventually positive (negative) and bounded solution of (5.3.2). Let $z(t)$ and $v(t)$ be defined as in Lemmas 5.3.5 and 5.3.6. Then, under the assumptions of Lemma 5.3.5 for any $t \geq t_1$, there is a $\theta \in (g(t), t)$ such that

$$|v'(g(t))| > \frac{(t-g(t))^2}{2}|v^{(3)}(\theta)|. \tag{5.3.12}$$

Proof Under the assumptions, from Lemma 5.3.6, we know that $v'(t)$, $-v''(t)$ and $v^{(3)}(t)$ have the same sign. By Taylor's formula, we have

$$v'(g(t)) = v'(t) + v''(t)(g(t)-t) + \frac{1}{2}v^{(3)}(\theta)(g(t)-t)^2,$$

where $g(t) \leq \theta \leq t$. Then (5.3.12) follows.

5.3.2 MAIN RESULTS

Using the above lemmas, we shall obtain the following sufficient conditions for the bounded solutions of (5.3.2) to be oscillatory. Let

$$\beta_{31} = \inf_{t \geq T_3} \left\{ \frac{(g^{-1}(t) - t)^2 \bar{q}_3(g^{-1}(t))}{2\tau^3} \right\} \tag{5.3.13}$$

and

$$\beta_{32} = \inf_{t \geq T_3} \left\{ \frac{(g^{-1}(t) - t)^2 \bar{q}_3(t)}{2\tau^3} \right\}, \tag{5.3.14}$$

where $T_3 \geq t_0$ is sufficiently large. Note that both β_{31} and β_{32} are nondecreasing as T_3 increases.

Theorem 5.3.1 Assume that (5.3.2) with $0 < p < 1$ satisfies

$$r\beta_{31} \sum_{i=1}^{n} ip^i \geq 1 \tag{5.3.15}$$

and

$$0 \leq \liminf_{t \to \infty} \int_{t-r}^{t} (g^{-1}(s) - s)^2 \bar{q}_3(g^{-1}(s)) ds \leq \frac{2\tau^3(1-p)e^{-1}}{p - p^{n+1}} \tag{5.3.16}$$

for some integer $n \geq 1$. Also assume that $\bar{q}_3(t)$ given by (5.3.3) is nonincreasing. Then, for every bounded solution $x(t)$ of (5.3.2), either $x(t)$ is oscillatory or $\liminf_{t\to\infty}(|x(t)| - p|x(t-r)|) < 0$.

Proof Suppose the conclusion does not hold. Let $x(t)$ be an eventually positive and bounded solution of (5.3.2) with $\liminf_{t\to\infty}(x(t) - px(t-r)) \geq 0$. Let $y(t)$ be defined as in Lemma 5.3.1 and $v(t)$ be defined as in Lemma 5.3.3. From

Lemma 5.3.3, we know that $v(t) > 0$, $v^{(3)}(t) = \Delta_\tau^3 y(t) < 0$, $v''(t) > 0$, $v'(t) < 0$ and

$$v^{(3)}(t) + \frac{1}{\tau^3}\bar{q}_3(t)\sum_{i=0}^{n} p^i v(g(t) - ir) < 0$$

holds for any natural number $n \geq 0$ and for all large enough t. From Lemma 5.3.4 we know that

$$v'(g(t))\frac{2}{(t-g(t))^2} < v^{(3)}(\theta)$$

for some $\theta \in (g(t), t)$. Since $\bar{q}_3(\theta) \geq \bar{q}_3(t)$ by assumption and

$$v(g(\theta) - ir) \geq v(g(t) - ir),$$

from (5.3.6) with t replaced by θ, it follows that

$$v'(g(t))\frac{2}{(t-g(t))^2} + \frac{1}{\tau^3}\bar{q}_3(t)\sum_{i=0}^{n} p^i v(g(t) - ir) < 0,$$

i.e.,

$$v'(g(t)) + \frac{(t-g(t))^2}{2\tau^3}\bar{q}_3(t)\sum_{i=0}^{n} p^i v(g(t) - ir) < 0. \tag{5.3.17}$$

With the replacement of t by $g^{-1}(t)$, (5.3.17) becomes

$$v'(t) + \frac{(g^{-1}(t)-t)^2}{2\tau^3}\bar{q}_3(g^{-1}(t))\sum_{i=0}^{n} p^i v(t - ir) < 0. \tag{5.3.18}$$

We assume that (5.3.18) holds for $t \geq t_1 \geq t_0$. Since β_{31} defined by (5.3.13) is nondecreasing as T_3 increasing, if β_{31} for a fixed T_3 satisfies (5.3.15) then β_{31} for any larger T_3 also satisfies (5.3.15). Thus, without loss of generality, we may assume that $T_3 \geq t_1 + nr$. Let

$$w(t) = \frac{-v'(t)}{v(t)}.$$

Note that $w(t) > 0$ and $v(t) = v(T_3)\exp\int_{T_3}^{t} -w(\theta)d\theta$ for all $t \geq T_3$. From (5.3.18) we have

$$w(t) > \frac{(g^{-1}(t)-t)^2\bar{q}_3(g^{-1}(t))}{2\tau^3}\sum_{i=0}^{n} p^i \exp\int_{t-ir}^{t} w(s)ds, \tag{5.3.19}$$

i.e.,

$$w(t) > \frac{Q_{31}(t)}{2\tau^3}\sum_{i=0}^{n} p^i \exp\int_{t-ir}^{t} w(s)ds \tag{5.3.20}$$

for all $t \geq T_3$, where $Q_{31}(t) = (g^{-1}(t)-t)^2\bar{q}_3(g^{-1}(t)) > 0$.

Let $w_0(t) \equiv 0$ for $t \geq T_3 - nr$. And for each $k \in \bar{N}$ and $t \geq T_3 + nkr$, let

$$w_{k+1}(t) = \frac{Q_{31}(t)}{2\tau^3}\sum_{i=0}^{n} p^i \exp\int_{t-ir}^{t} w_k(s)ds.$$

Let

$$\alpha_{31k} = \inf_{t \geq T_3+(k-1)nr}\{w_k(t)\} \qquad k \in \bar{N}.$$

Then

$$\alpha_{31k+1} \geq \inf_{t\geq T_3}\left\{Q_{31}(t)\cdot\frac{1}{2\tau^3}\sum_{i=0}^{n} p^i e^{ir\alpha_{31k}}\right\} = \beta_{31}\sum_{i=0}^{n} p^i e^{ir\alpha_{31k}}.$$

Since $e^{ir\alpha_{31k}} \geq 1 + ir\alpha_{31k} > ir\alpha_{31}$, by (5.3.15), $\{\alpha_{31k}\}$ is an increasing sequence.

Suppose

$$\lim_{k\to\infty}\alpha_{31k} = \rho_{31} < \infty.$$

So $\rho_{31} \geq \beta_{31}\sum_{i=0}^{n} p^i e^{ir\rho_{31}}$. Let

$$F_{31}(x) = \beta_{31}\sum_{i=0}^{n} p^i e^{irx} - x.$$

Then $F_{31}'(x) = \beta_{31}\sum_{i=1}^{n} irp^i e^{irx} - 1$ and $F_{31}''(x) > 0$, so $F_{31}'(x)$ is increasing. Since $F_{31}'(0) = \beta_{31}\sum_{i=0}^{n} irp^i - 1 \geq 0$ by (5.3.15), then $F_{31}'(x) > 0$ for $x > 0$. Hence $F_{31}(x)$ is increasing. Thus, from $F_{31}(0) = \beta_{31}\sum_{i=0}^{n} p^i > 0$ we have

$F_{31}(x) > 0$ for all $x \geq 0$. This shows that no positive number ρ_{31} satisfies $\rho_{31} \geq \beta_{31} \sum_{i=0}^{n} p^i e^{ir\rho_{31}}$. Therefore, we must have $\alpha_{31k} \to \infty$ as $k \to \infty$. Note that $w(t) \geq w_{k+1}(t) \geq \alpha_{31k+1}$ for $t \geq T_3 + nkr$. Thus $w(t) \to \infty$ as $t \to \infty$. From this it follows that

$$\lim_{t\to\infty} \int_t^{t+r} w(s)ds = \infty.$$

By the definition of $w(t)$ we have

$$\frac{v(t)}{v(t+r)} = \exp \int_t^{t+r} w(s)ds \to \infty \quad \text{as} \quad t \to \infty,$$

i.e.,

$$\lim_{t\to\infty} \frac{v(t)}{v(t+r)} = \infty. \tag{5.3.21}$$

On the other hand, since $v'(t) < 0$ and $v(t) > 0$, it follows from (5.3.18)(by dropping the $i = 0$ term) that

$$\begin{aligned} v'(t) &< -\frac{(g^{-1}(t)-t)^2}{2\tau^3}\bar{q}_3(g^{-1}(t)) \sum_{i=1}^{n} p^i v(t-ir) \\ &\leq -\frac{(g^{-1}(t)-t)^2(p-p^{n+1})}{2\tau^3(1-p)}\bar{q}_3(g^{-1}(t))v(t-r). \end{aligned} \tag{5.3.22}$$

By (5.3.16) and Lemma 5.2.1,

$$\liminf_{t\to\infty} \frac{v(t)}{v(t-r)} \in (0,1].$$

Thus, $v(t+r)/v(t)$ has a positive lower bound so $v(t)/v(t+r)$ has a positive upper bound, a contradiction to (5.3.21). If $x(t)$ is a bounded and eventually negative solution with $\limsup_{t\to\infty}(x(t) - px(t-r)) \leq 0$, the above proof with obvious minor changes also leads to a contradiction. Therefore, the conclusion of the theorem holds.

Corollary 5.3.1 The conclusion of Theorem 5.3.1 still holds if (5.3.16) is replaced by

$$0 \leq \liminf_{t\to\infty} \int_{t-r}^{t} (g^{-1}(s) - s)^2 \bar{q}_3(g^{-1}(s))ds \leq \frac{2\tau^3}{ep}. \tag{5.3.23}$$

Proof The proof is the same as that of Theorem 5.3.1 except (5.3.22). The conclusion still holds if (5.3.22) is replaced by

$$v'(t) < -\frac{(g^{-1}(t) - t)^2}{2\tau^3} \bar{q}_3(g^{-1}(t))pv(t-r).$$

Corollary 5.3.2 Assume that (5.3.2) with $0 < p < 1$ satisfies

$$r\beta_{32} \sum_{i=1}^{n} ip^i \geq 1 \tag{5.3.24}$$

and

$$0 \leq \liminf_{t\to\infty} \int_{t-r}^{t} (g^{-1}(s) - s)^2 \bar{q}_3(s)ds \leq \frac{2\tau^3(1-p)e^{-1}}{p - p^{n+1}} \tag{5.3.25}$$

for some integer $n \geq 1$. Also assume that $\bar{q}_3(t)$ given by (5.3.3) is nondecreasing. Then the conclusion of Theorem 5.3.1 holds.

Proof The proof of Theorem 5.3.1 is still valid after the replacement of $\bar{q}_3(\theta) \geq \bar{q}_3(t)$ by $\bar{q}_3(\theta) \geq \bar{q}_3(g(t))$.

Corollary 5.3.3 The conclusion of Corollary 5.3.2 still holds if (5.3.25) is replaced by

$$0 \leq \liminf_{t\to\infty} \int_{t-r}^{t} (g^{-1}(s) - s)^2 \bar{q}_3(s)ds \leq \frac{2\tau^3}{ep}. \tag{5.3.26}$$

The proof of Corollary 5.3.3 is similar to that of Corollary 5.3.1.

Corollary 5.3.4 Assume $0 < p < 1$ and $r = k\tau$, $k \in N$. Under the assumptions of either Theorem 5.3.1 or Corollary 5.3.1 or Corollary 5.3.2 or Corollary 5.3.3, every bounded solution $x(t)$ of (5.3.2) is oscillatory.

Proof The proof of Theorem 5.3.1 is still valid after the replacement of Lemma 5.3.1 by Lemma 5.3.2.

The next results are for the bounded solutions of (5.3.2) with $p > 1$.

Theorem 5.3.2 Assume that $p > 1$, $r = k\tau$, $k \in N$, $r \geq t + 3\tau - g(t)$ and

$$r\beta_{32} \sum_{i=1}^{n} \frac{(i-1)}{p^i} \geq 1 \tag{5.3.27}$$

for some integer $n \geq 2$. Also assume that $\bar{q}_3(t)$ given by (5.3.3) is nondecreasing. Then every bounded solution $x(t)$ of (5.3.2) is oscillatory.

Proof Suppose the conclusion does not hold. Let $x(t)$ be an eventually positive and bounded solution of (5.3.2). Let $y(t)$ be defined as in Lemma 5.3.5 and $v(t)$ be defined as in Lemma 5.3.6. By Lemma 5.3.6, we know that $v(t) < 0$, $v^{(3)}(t) = \Delta_\tau^3 y(t) < 0$, $v''(t) > 0$, $v'(t) < 0$, and

$$v^{(3)}(t) - \frac{1}{\tau^3}\bar{q}_3(t) \sum_{i=1}^{n} \frac{1}{p^i} v(g(t) - 3\tau + ir) < 0$$

holds for any fixed integer $n \geq 1$ and for all large enough t. By Lemma 5.3.7, we know that

$$v'(g(t)) \frac{2}{(t - g(t))^2} < v^{(3)}(\theta)$$

for some $\theta \in (g(t), t)$. Since $\bar{q}_3(\theta) \geq \bar{q}_3(g(t))$ and $v(g(\theta) - 3\tau + ir) \leq v(g(g(t)) - 3\tau + ir)$, from (5.3.11) with t replaced by θ it follows that

$$v'(g(t))\frac{2}{(t-g(t))^2} - \frac{1}{\tau^3}\bar{q}_3(g(t))\sum_{i=1}^{n}\frac{1}{p^i}v(g(g(t)) - 3\tau + ir) < 0,$$

i.e.,

$$v'(g(t)) - \frac{(t-g(t))^2}{2\tau^3}\bar{q}_3(g(t))\sum_{i=1}^{n}\frac{1}{p^i}v(g(g(t)) - 3\tau + ir) < 0. \tag{5.3.28}$$

With the replacement of t by $g^{-1}(t)$, (5.3.28) yields

$$v'(t) - \frac{(g^{-1}(t)-t)^2}{2\tau^3}\bar{q}_3(t)\sum_{i=1}^{n}\frac{1}{p^i}v(g(t) - 3\tau + ir) < 0. \tag{5.3.29}$$

We assume that (5.3.29) holds for $t \geq t_1 \geq t_0$ and T_3 in β_{32} defined by (5.3.14) satisfied $T_3 \geq t_1 + nr$. Let

$$w(t) = \frac{v'(t)}{v(t)}.$$

Note that $w(t) > 0$ and $v(t) = v(t')\exp\int_{t'}^{t} w(\theta)d\theta$ for all t, $t' \geq T_3$. From (5.3.29) we have

$$w(t) > \frac{(g^{-1}(t)-t)^2\bar{q}_3(t)}{2\tau^3}\sum_{i=1}^{n}\frac{1}{p^i}\exp\int_{t}^{g(t)-3\tau+ir} w(s)ds, \tag{5.3.30}$$

i.e.,

$$w(t) > \frac{Q_{32}(t)}{2\tau^3}\sum_{i=1}^{n}\frac{1}{p^i}\exp\int_{t}^{g(t)-3\tau+ir} w(s)ds \tag{5.3.31}$$

for all $t \geq T_3$, where $Q_{32}(t) = (g^{-1}(t)-t)^2\bar{q}_3(t) > 0$.

Let $w_0(t) \equiv 0$ for $t \geq T_3$. And for each $k \in \bar{N}$ and $t \geq T_3$, let

$$w_{k+1}(t) = \frac{Q_{32}(t)}{2\tau^3}\sum_{i=1}^{n}\frac{1}{p^i}\exp\int_{t}^{g(t)-3\tau+ir} w_k(s)ds$$

and

$$\alpha_{32k} = \inf_{t \geq T_3}\{w_k(t)\}, \qquad k \in \bar{N}.$$

Then $w(t) \geq w_{k+1}(t) \geq w_k(t) \geq \alpha_{32k}$ for all $k \in \bar{N}$ and $t \geq T_3$. Then, by assumption,

$$\begin{aligned} \alpha_{32k+1} &\geq \inf_{t\geq T_3}\left\{Q_{32}(t)\cdot\frac{1}{2\tau^3}\sum_{i=1}^{n}\frac{1}{p^i}e^{(g(t)-t-3\tau+i r)\alpha_{32k}}\right\} \\ &= \beta_{32}\sum_{i=1}^{n}\frac{1}{p^i}e^{(i-1)r\alpha_{32k}}. \end{aligned}$$

Since $\{\alpha_{32k}\}$ is a bounded increasing sequence, we suppose

$$\lim_{k\to\infty}\alpha_{32k} = \rho_{32} < \infty.$$

So $\rho_{32} \geq \beta_{32}\sum_{i=1}^{n}(e^{(i-1)r\rho_{32}}/p^i)$. Let

$$F_{32}(x) = \beta_{32}\sum_{i=1}^{n}\frac{e^{(i-1)rx}}{p^i} - x.$$

Then $F_{32}'(x) = \beta_{32}\sum_{i=2}^{n}((i-1)re^{(i-1)rx}/p^i) - 1$ and $F_{32}''(x) > 0$, so $F_{32}'(x)$ is increasing. Since $F_{32}'(0) = \beta_{32}\sum_{i=1}^{n}(i-1)r/p^i - 1 \geq 0$ by (5.3.27), then $F_{32}'(x) > 0$ for $x > 0$. Hence $F_{32}(x)$ is increasing. Thus, from $F_{32}(0) = \beta_{32}\sum_{i=1}^{n}1/p^i > 0$ we have $F_{32}(x) > 0$ for all $x \geq 0$. This shows that no positive number ρ_{32} satisfies $\rho_{32} \geq \beta_{32}\sum_{i=1}^{n}(e^{(i-1)r\rho_{32}}/p^i)$. This contradiction shows that if $x(t)$ is a bounded solution of (5.3.2) then $x(t)$ cannot be eventually positive. If $x(t)$ is assumed to be a bounded and eventually negative solution, then the above reasoning with obvious changes also leads to a contradiction. Therefore, every bounded solution of (5.3.2) must be oscillatory.

Corollary 5.3.5 Assume that $p > 1$, $r = k\tau$, $k \in N$, $r \geq t + 3\tau - g(t)$ and

$$r\beta_{31}\sum_{i=1}^{n}\frac{(i-1)}{p^i} \geq 1 \tag{5.3.32}$$

for some integer $n \geq 2$. Also assume that $\bar{q}_3(t)$ given by (5.3.3) is nonincreasing. Then every bounded solution $x(t)$ of (5.3.2) is oscillatory.

Proof The proof of the Theorem 5.3.2 is still valid after the replacement of $\bar{q}_3(\theta) \geq \bar{q}_3(g(t))$ by $\bar{q}_3(\theta) \geq \bar{q}_3(t)$.

5.3.3 EXAMPLES

Two examples will be given in this section to illustrate the results in the above section. The first one illustrates Corollary 5.3.1 for a difference equation with $0 < p < 1$. The second example demonstrates Theorem 5.3.2.

Example 5.3.1 Consider the difference equation

$$\Delta_2^3\left(x(t) - \frac{2}{3}x(t-1)\right) + \left(1 + \frac{1}{t}\right)x(t-4) = 0 \tag{5.3.33}$$

for $t > 0$. Viewing (5.3.33) as (5.3.2), we have $\tau = 2$, $0 < p = 2/3 < 1$, $r = 1$, $q(t) = 1+\frac{1}{t}$ and $g(t) = t-4$. Then, according to (5.3.3), $\bar{q}_3(t) = \alpha\cdot(1+1/(t+6))$ and is nonincreasing. And $\beta_{31} = \alpha$. So when $n = 3$, by (5.3.15) we have

$$\alpha\sum_{i=1}^{3} i\left(\frac{2}{3}\right)^i = \alpha \times \left(\frac{2}{3} + \frac{8}{9} + \frac{8}{9}\right) = \frac{22}{9}\alpha \geq 1$$

for some $\alpha \in [9/22, 1)$. Also (5.3.23) is satisfied since

$$\begin{aligned} 0 &\leq \liminf_{t\to\infty} \int_{t-1}^{t} 4^2 \times \alpha\left(1 + \frac{1}{s+10}\right) ds \\ &= \int_{-1}^{0} \lim_{t\to\infty} 4^2 \times \alpha\left(1 + \frac{1}{t+s+10}\right) ds = 16\alpha \end{aligned}$$

and

$$\frac{24}{e} = \frac{2 \times 2^3}{\frac{2e}{3}} \geq 16\alpha$$

holds for any $\alpha \in [9/22, 3/(2e)]$. By Corollary 5.3.1, for every bounded solution $x(t)$ of (5.3.33), either $x(t)$ is oscillatory or

$$\liminf_{t\to\infty} \left(|x(t)| - \frac{2}{3}|x(t-1)| \right) < 0.$$

Example 5.3.2 Consider the difference equation

$$\Delta_{\pi}^{3} \left(x(t) - \frac{4}{3}x(t-4\pi) \right) + 8x(t-\pi) + \frac{8\sigma}{1+t^2}x^3(t-\pi) = 0, \qquad (5.3.34)$$

for $t \geq 0$, where σ is a positive constant. Regarding (5.3.34) as (5.3.2), we have $\tau = \pi, p = 4/3 > 1$, $r = 4\pi$, $g(t) = t - \pi$ and $q(t) = 8$. Note that $r = 4\tau$, $r = t + 3\tau - g(t)$. Then, for some $\alpha \in (0,1)$, $\bar{q}_3 = 8\alpha$ by (5.3.3). Then

$$\beta_{32} = \inf_{t \geq T_3} \left\{ \frac{\pi^2 \times 8\alpha}{2 \times \pi^3} \right\} = \frac{4\alpha}{\pi}$$

and

$$\beta_{32} \sum_{i=1}^{3} \frac{(i-1)r}{p^i} = \frac{4\alpha}{\pi} \times 4\pi \times \left(\left(\frac{3}{4}\right)^2 + 2 \times \left(\frac{3}{4}\right)^3 \right) = \frac{45}{2}\alpha \geq 1$$

holds if $\alpha \in [2/45, 1)$. Thus (5.3.27) is satisfied for $n = 3$. Therefore, by Theorem 5.3.2, every bounded solution $x(t)$ of (5.3.34) is oscillatory.

5.4 HIGHER ODD ORDER EQUATION (4.1.1)

In this section, we consider equation (4.1.1) with $m > 3$ being an odd integer, i.e.,

$$\Delta_\tau^m(x(t) - px(t-r)) + f(t, x(g(t))) = 0. \tag{5.4.35}$$

For the convenience of later use, let

$$\bar{q}_m(t) = \alpha \min_{t \le s \le t+m\tau}\{q(s)\}\left(\min_{g(t)\le s\le g(t)+m\tau}\{(g^{-1}(s))'\}\right)^m, \tag{5.4.36}$$

where $0 < \alpha < 1$. We shall see from the following parts that the function $\bar{q}_m$ will play an important role in the oscillatory criteria for (5.4.35).

The oscillation of (5.4.35) will be considered in two separate cases when $0 \le p < 1$ and $p > 1$. Some lemmas will be given to make the proof of the main results ready at first. Secondly, based on some results of differential equation or inequality, sufficient conditions will be obtained for the bounded solutions of (5.4.35) to be oscillatory. To illustrate the main results, some examples will be given in the last subsection.

5.4.1 RELATED LEMMAS

To obtain the main results, we need to prove the following lemmas first.

Lemma 5.4.1 Let $0 \le p < 1$. Assume that $x(t)$ is a bounded and eventually positive (negative) solution of (5.4.35) with $z(t) = x(t) - px(t-r)$ and

$\liminf_{t\to\infty} z(t) \ge 0\,(\limsup_{t\to\infty} z(t) \le 0)$. Let

$$y(t) = \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} z(\theta)d\theta.$$

Then $y(t) > 0\,(< 0)$, $(-1)^k y^{(k)}(t) > 0\,(< 0)$ for $1 \le k \le m$ eventually. Moreover,

$$\Delta_\tau^m y(t) + \bar{q}_m(t) \sum_{i=0}^{n} p^i y(g(t) - ir) < 0\,(> 0) \tag{5.4.37}$$

holds for any fixed natural number n and for all large enough t.

Proof Suppose $x(t)$ is a bounded and eventually positive solution. Notice that $g(t) < t$ and $g'(t) > 0$ for all $t \ge t_0$. So there exists a $t_1 > t_0$ such that $x(g(t)) > 0$ for all $t \ge t_1$. From (5.4.35) it follows that

$$\Delta_\tau^m z(t) + f(t, x(g(t))) = 0.$$

By (4.1.3), we have $f(t, x(g(t))) \ge q(t)x(g(t)) > 0$ for $t \ge t_1$. Therefore

$$y^{(m)}(t) + q(t)x(g(t)) \le 0 \tag{5.4.38}$$

for $t \ge t_1$. According to $q(t)x(g(t)) > 0$, $y^{(m)}(t) < 0$ for all $t \ge t_1$. Thus, $y^{(m-1)}(t)$ is decreasing so either $y^{(m-1)}(t) > 0$ for all $t \ge t_1$ or $y^{(m-1)}(t) \le y^{(m-1)}(t_2) < 0$ for some $t_2 > t_1$ and for all $t \ge t_2$. If the latter holds then

$$y^{(m-k)}(t) \to -\infty, \qquad k = 2, 3, \cdots, m,$$

as $t \to \infty$, a contradiction to the boundedness of x and z. Therefore we have $y^{(m-1)}(t) > 0$ for all $t \ge t_1$. Thus, $y^{(m-2)}(t)$ is increasing so either $y^{(m-2)}(t) < 0$ for all $t \ge t_1$ or $y^{(m-2)}(t) \ge y^{(m-2)}(t_3) > 0$ for some $t_3 \ge t_1$ and all $t \ge t_3$. If the latter holds then

$$y^{(m-k)}(t) \to \infty, \qquad k = 3, 4, \cdots, m,$$

as $t \to \infty$, a contradiction again to the boundedness of x and z. Hence, we must have $y^{(m-2)}(t) < 0$ for all $t \geq t_1$. Repeating the above process, we obtain $(-1)^k y^{(k)}(t) > 0$ for $1 \leq k \leq m$ and all $t \geq t_1$. Therefore, $y(t)$ is decreasing so either $y(t) > 0$ for all $t \geq t_1$ or there is a $t_4 \geq t_1$ such that $y(t) \leq y(t_4) < 0$ for $t \geq t_4$. Suppose the latter case holds. Then

$$\begin{aligned}
& \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} x(\theta) d\theta \\
= \; & y(t) + p\int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} x(\theta - r) d\theta \\
\leq \; & y(t_4) + p\int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} x(\theta - r) d\theta \\
& \cdots \\
\leq \; & y(t_4)\sum_{i=0}^{s-1} p^i + p^s \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} x(\theta - sr) d\theta \\
\leq \; & \frac{y(t_4)(1-p^s)}{1-p} + p^s M \tau^m
\end{aligned}$$

for $t \geq t_4 + sr$, where $M = \sup_{t \geq t_0} x(t)$ and s is any positive integer. Let $s \to \infty$ so $t \to \infty$ as well, $p^s M \tau^m$ then is arbitrarily small due to $0 \leq p < 1$. Thus,

$$\int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} x(\theta) d\theta < 0,$$

which contradicts the assumption that $x(t)$ is eventually positive. Therefore, we must have $y(t) > 0$ for all $t \geq t_1$.

From (5.4.38) it follows that

$$\Delta_\tau^m z(t) + q(t)z(g(t)) + pq(t)x(g(t) - r) \leq 0.$$

According to the definition of $z(t)$, the above inequality becomes

$$\Delta_\tau^m z(t) + q(t)z(g(t)) + pq(t)z(g(t) - r) + p^2 q(t)x(g(t) - 2r) \leq 0.$$

Proceeding in the same way as the above, we have

$$\Delta_\tau^m z(t) + q(t)\sum_{i=0}^{n} p^i z(g(t) - ir) + p^{n+1} q(t)x(g(t) - (n+1)r) \le 0.$$

Since $q(t)p^{n+1}x(g(t) - (n+1)r) > 0$ when t is large enough, the above inequality implies that

$$\Delta_\tau^m z(t) + q(t)\sum_{i=0}^{n} p^i z(g(t) - ir) < 0.$$

In order to integrate the above inequality, we need to show that $z(t)$ is positive. If $p = 0$ then $z(t) = x(t) > 0$ holds eventually. Now suppose $0 < p < 1$. By the same reasoning as that used in the proof of Lemma 5.3.1, we have $z(t + h\tau) > 0$ for all $t \ge t_1$ and $h \ge 1$.

Integrating $q(t)z(g(t) - ir)$, by the assumptions on g and q, we obtain

$$\begin{aligned}
&\int_t^{t+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \cdots \int_{s_{m-1}}^{s_{m-1}+\tau} z(g(\theta) - ir)q(\theta)d\theta \\
&\ge \min_{t\le s\le t+m\tau}\{q(s)\} \int_t^{t+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \cdots \int_{s_{m-1}}^{s_{m-1}+\tau} z(g(\theta) - ir)d\theta \\
&\ge \min_{t\le s\le t+m\tau}\{q(s)\} \int_{g(t)}^{g(t+\tau)} (g^{-1}(s_1))'ds_1 \int_{s_1}^{g(g^{-1}(s_1)+\tau)} (g^{-1}(s_2))'ds_2 \cdots \\
&\qquad \int_{s_{m-1}}^{g(g^{-1}(s_{m-1})+\tau)} z(\theta - ir)(g^{-1}(\theta))'d\theta \\
&\ge \min_{t\le s\le t+m\tau}\{q(s)\} \left(\min_{g(t)\le s\le g(t)+m\tau} (g^{-1}(s))'\right)^m \int_{g(t)}^{g(t)+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \cdots \\
&\qquad \int_{s_{m-1}}^{s_{m-1}+\tau} z(\theta - ir)d\theta \\
&\ge \min_{t\le s\le t+m\tau}\{q(s)\} \left(\min_{g(t)\le s\le g(t)+m\tau} (g^{-1}(s))'\right)^m y(g(t) - ir) \\
&\ge \bar{q}_m(t)y(g(t) - ir).
\end{aligned}$$

Therefore, it follows that

$$\Delta_\tau^m y(t) + \bar{q}_m(t)\sum_{i=0}^{n} p^i y(g(t) - ir) < 0$$

holds for any fixed natural number n and for all large enough t. If $x(t)$ is a bounded and eventually negative solution, then the above proof with obvious changes shows the conclusion within brackets.

Lemma 5.4.2 Let $0 \le p < 1$ and $r = k\tau$. Assume that $x(t)$ is a bounded and eventually positive (negative) solution of (5.4.35). Let

$$\begin{aligned} z(t) &= x(t) - px(t-r), \\ y(t) &= \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} z(\theta) d\theta. \end{aligned}$$

Then the conclusion of Lemma 5.4.1 holds.

Proof The proof of Lemma 5.3.2 for the third order is still valid after the replacement of the third order by any higher odd order $m \ge 3$.

Lemma 5.4.3 Under the assumptions of Lemma 5.4.1 or Lemma 5.4.2, let

$$v(t) = \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} y(\theta) d\theta.$$

Then $v(t) > 0 (< 0)$, $(-1)^k v^{(k)}(t) > 0 (< 0)$ for $1 \le k \le m$ eventually. Moreover,

$$v^{(m)}(t) + \frac{1}{\tau^m} \bar{q}_m(t) \sum_{i=0}^{n} p^i v(g(t) - ir) < 0 (> 0) \tag{5.4.39}$$

holds for any fixed natural number n and for all large enough t.

Proof By the definition of $v(t)$, $v(t)$ has the same sign as $y(t)$ for all $t \ge t_1$. Furthermore, we have

$$v'(t) = \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} y'(\theta) d\theta.$$

Then $v'(t)$ has the same sign as $y'(t)$. Similarly, $v^{(j)}(t)$ has the same sign as $y^{(j)}(t)$ for all $j = 1, 2, \cdots, m$. Notice also that $v^{(m)}(t) = \Delta_\tau^m y(t)$. If $y'(t) < 0$, then

$$\begin{aligned}
v(g(t) - ir) &= \int_{g(t)}^{g(t)+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} y(\theta - ir) d\theta \\
&\leq \int_{g(t)}^{g(t)+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} y(t_{m-1} - ir) d\theta \\
&\leq \tau \int_{g(t)}^{g(t)+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-2}}^{t_{m-2}+\tau} y(t_{m-1} - ir) dt_{m-1} \\
\cdots \ \cdots \\
&\leq \tau^{m-1} \int_{g(t)}^{g(t)+\tau} y(t_1 - ir) dt_1 \\
&\leq \tau^m y(g(t) - ir).
\end{aligned}$$

Hence, from (5.4.37) it follows that

$$v^{(m)}(t) + \frac{1}{\tau^m} \bar{q}_m(t) \sum_{i=0}^{n} p^i v(g(t) - ir) < 0$$

holds for any fixed natural number n and for all large enough t. If $y'(t) > 0$ then $v(g(t) - ir) \geq \tau^m y(g(t) - ir)$ so

$$v^{(m)}(t) + \frac{1}{\tau^m} \bar{q}_m(t) \sum_{i=0}^{n} p^i v(g(t) - ir) > 0.$$

Lemma 5.4.4 Under the assumptions of Lemma 5.4.3, for each $t \geq t_1$ there is a $\theta \in (g(t), t)$ such that

$$|v'(g(t))| > \frac{(t - g(t))^{m-1}}{(m-1)!} |v^{(m)}(\theta)|. \tag{5.4.40}$$

Proof Under the assumptions of Lemma 5.4.3, we know that $(-1)^j v^{(j)}(t)$ for $j = 1, 2, \cdots, m$ have the same sign. According to Taylor's Formula, we have

$$\begin{aligned} v'(g(t)) &= v'(t) + v''(t)(g(t) - t) + \frac{1}{2}v^{(3)}(t)(g(t) - t)^2 + \cdots \\ &\quad + \frac{1}{(m-1)!}v^{(m)}(\theta)(g(t) - t)^{m-1} \end{aligned}$$

for some $\theta \in (g(t), t)$ and (5.4.40) follows immediately.

The next lemmas are for the bounded solutions of (5.4.35) with $p > 1$.

Lemma 5.4.5 Let $p > 1$ and $r = k\tau$, $k \in N$. Assume that $x(t)$ is a bounded and eventually positive (negative) solution of (5.4.35). Let

$$\begin{aligned} z(t) &= x(t) - px(t - r), \\ y(t) &= \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} z(\theta) d\theta. \end{aligned}$$

Then $y(t) < 0\,(> 0)$, $(-1)^k y^{(k)}(t) > 0\,(< 0)$ for $1 \le k \le m$ eventually. Moreover,

$$\Delta_\tau^m y(t) - \bar{q}_m(t) \sum_{i=1}^{n} \frac{1}{p^i} y(g(t) + ir) < 0\,(> 0) \tag{5.4.41}$$

holds for any fixed integer $n \ge 1$ and for all large enough t.

Proof Suppose $x(t)$ is a bounded and eventually positive solution. Since $g(t) < t$ and $g'(t) > 0$, from the assumptions, there exists a $t_1 > t_0$ such that $x(g(t)) > 0$ for all $t \ge t_1$. Notice also that

$$\Delta_\tau^m z(t) + f(t, x(g(t))) = 0.$$

According to (4.1.3), we have $f(t, x(g(t))) \ge q(t)x(g(t)) > 0$ for $t \ge t_1$. Therefore

$$\Delta_\tau^m z(t) + q(t)x(g(t)) \le 0 \tag{5.4.42}$$

for $t \geq t_1$. By the definition of $y(t)$, $y^{(m)}(t) = \Delta_\tau^m z(t)$. Thus, from (5.4.42) it follows that

$$y^{(m)}(t) + q(t)x(g(t)) \leq 0 \tag{5.4.43}$$

for $t \geq t_1$. Due to $q(t)x(g(t)) > 0$, $y^{(m)}(t) < 0$ for all $t \geq t_1$. From the proof of Lemma 5.4.1 we know that $(-1)^k y^{(k)}(t) > 0$ holds for $1 \leq k \leq m$ and all $t \geq t_1$. Thus, $y(t)$ is decreasing. We now prove that $y(t) < 0$ for all $t \geq t_1$. Since $y^{(m)}(t) = \Delta_\tau^m z(t)$ for all $t \geq t_1$, from the proof of Lemma 5.3.1 we know that $z(t + h\tau)$ is decreasing for each fixed $t \geq t_1$ as h increases. From the proof of Lemma 5.3.5, we know that $z(t) < 0$ so that $y(t) < 0$ for some $t_2 \geq t_1$ and all $t \geq t_2$.

From (5.4.42), we have

$$\Delta_\tau^m z(t) - \frac{q(t)}{p} z(g(t) + r) + \frac{q(t)}{p} x(g(t) + r) \leq 0.$$

According to the definition of $z(t)$, it follows from the above inequality that

$$\Delta_\tau^m z(t) - \frac{q(t)}{p} z(g(t) + r) + \frac{q(t)}{p} \left(-\frac{1}{p} z(g(t) + 2r) + \frac{1}{p} x(g(t) + 2r) \right) \leq 0.$$

Repeating the above procedure, we obtain

$$\Delta_\tau^m z(t) - q(t) \sum_{i=1}^{n} \frac{1}{p^i} z(g(t) + ir) + q(t) \frac{1}{p^n} x(g(t) + nr) \leq 0.$$

Since $q(t)x(g(t) + nr) > 0$ for sufficiently large t, we have

$$\Delta_\tau^m z(t) - q(t) \sum_{i=1}^{n} \frac{1}{p^i} z(g(t) + ir) < 0.$$

Integrating $q(t)z(g(t)+ir)$, by the assumptions on p and g, we obtain

$$\begin{aligned}
&\int_t^{t+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \cdots \int_{s_{m-1}}^{s_{m-1}+\tau} z(g(\theta)+ir)q(\theta)d\theta \\
\leq &\min_{t\leq l\leq t+m\tau}\{q(l)\} \int_t^{t+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \cdots \int_{s_{m-1}}^{s_{m-1}+\tau} z(g(\theta)+ir)d\theta \\
\leq &\min_{t\leq l\leq t+m\tau}\{q(l)\} \int_{g(t)}^{g(t+\tau)} (g^{-1}(s_1))' ds_1 \int_{s_1}^{g(g^{-1}(s_1)+\tau)} (g^{-1}(s_2))' ds_2 \cdots \\
&\int_{s_{m-1}}^{g(g^{-1}(s_{m-1})+\tau)} z(\theta+ir)(g^{-1}(\theta))' d\theta \\
\leq &\min_{t\leq l\leq t+m\tau}\{q(l)\} \left(\min_{g(t)\leq s\leq g(t)+m\tau}(g^{-1}(s))'\right)^m \int_{g(t)}^{g(t)+\tau} ds_1 \int_{s_1}^{s_1+\tau} ds_2 \cdots \\
&\int_{s_{m-1}}^{s_{m-1}+\tau} z(\theta+ir)d\theta \\
\leq &\min_{t\leq l\leq t+m\tau}\{q(l)\} \left(\min_{g(t)\leq s\leq g(t)+m\tau}(g^{-1}(s))'\right)^m y(g(t)+ir) \\
\leq &\ \bar{q}_m(t)y(g(t)+ir).
\end{aligned}$$

Therefore,

$$\Delta_\tau^m y(t) - \bar{q}_m(t)\sum_{i=1}^n \frac{1}{p^i} y(g(t)+ir) < 0$$

holds for any fixed integer $n \geq 1$ and for all large enough t. If $x(t)$ is a bounded and eventually negative solution, then the conclusion within brackets follows from the above proof with minor modification.

Lemma 5.4.6 Under the assumptions of Lemma 5.4.5, let

$$v(t) = \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} y(\theta)d\theta.$$

Then $v(t) < 0\,(> 0)$, $(-1)^k v^{(k)}(t) > 0\,(< 0)$ for $1 \leq k \leq m$ eventually. Moreover,

$$v^{(m)}(t) - \frac{1}{\tau^m}\bar{q}_m(t)\sum_{i=1}^n \frac{1}{p^i} v(g(t)-m\tau+ir) < 0\,(> 0) \tag{5.4.44}$$

holds for any fixed integer $n \geq 1$ and for all large enough t.

Proof By the definition of $v(t)$, $v(t)$ has the same sign as $y(t)$. Further, we have

$$v'(t) = \int_t^{t+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} y'(\theta)d\theta.$$

Then $v'(t)$ has the same sign as $y'(t)$. Similarly, $(-1)^k v^{(k)}(t)$ for $1 \leq k \leq m$ and $(-1)^j y^{(j)}(t)$ for $1 \leq j \leq m$ all have the same sign. Note also that $v^{(m)}(t) = \Delta_\tau^m y(t)$. If $y'(t) < 0$, then

$$\begin{aligned}
v(g(t)+ir) &= \int_{g(t)}^{g(t)+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} y(\theta+ir)d\theta \\
&\geq \int_{g(t)}^{g(t)+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+\tau} y(t_{m-1}+\tau+ir)d\theta \\
&\geq \tau \int_{g(t)}^{g(t)+\tau} dt_1 \int_{t_1}^{t_1+\tau} dt_2 \cdots \int_{t_{m-2}}^{t_{m-2}+\tau} y(t_{m-1}+\tau+ir)dt_{m-1} \\
&\cdots\cdots \\
&\geq \tau^{m-1} \int_{g(t)}^{g(t)+\tau} y(t_1+(m-1)\tau+ir)dt_1 \\
&\geq \tau^m y(g(t)+m\tau+ir).
\end{aligned}$$

Hence, from (5.4.41) we have

$$v^{(m)}(t) - \frac{1}{\tau^m}\bar{q}_m(t)\sum_{i=1}^{n} \frac{1}{p^i} v(g(t)-m\tau+ir) < 0$$

for any fixed integer $n \geq 1$ and for all large enough t. If $y'(t) > 0$ then $0 < v(g(t)+ir) \leq \tau^m y(g(t)+m\tau+ir)$ so

$$v^{(m)}(t) - \frac{1}{\tau^m}\bar{q}_m(t)\sum_{i=1}^{n} \frac{1}{p^i} v(g(t)-m\tau+ir) > 0.$$

Lemma 5.4.7 Assume that $x(t)$ is an eventually positive (negative) and bounded solution of (5.4.35). Let $z(t)$ and $v(t)$ be defined as in Lemma 5.4.5 and Lemma 5.4.6.

Then, under the assumptions of Lemma 5.4.5, for any given $t \geq t_1$, there is a $\theta \in (g(t), t)$ such that

$$|v'(g(t))| > \frac{(t-g(t))^{m-1}}{(m-1)!}|v^{(m)}(\theta)|. \tag{5.4.45}$$

Proof The proof of Lemma 5.4.4 is still valid for Lemma 5.4.7.

5.4.2 MAIN RESULTS

Using the related lemmas above, we shall obtain the following sufficient conditions for the bounded solutions of (5.4.35) to be oscillatory. Let

$$\beta_{m1} = \inf_{t \geq T_m} \left\{ \frac{(g^{-1}(t)-t)^{m-1}\bar{q}_m(g^{-1}(t))}{(m-1)!\tau^m} \right\} \tag{5.4.46}$$

and

$$\beta_{m2} = \inf_{t \geq T_m} \left\{ \frac{(g^{-1}(t)-t)^{m-1}\bar{q}_m(t)}{(m-1)!\tau^m} \right\}, \tag{5.4.47}$$

where $T_m \geq t_0$ is sufficiently large.

Theorem 5.4.1 Assume that (5.4.35) with $0 < p < 1$ satisfies

$$r\beta_{m1}\sum_{i=1}^{n} ip^i \geq 1 \tag{5.4.48}$$

and

$$0 \leq \liminf_{t\to\infty} \int_{t-r}^{t} (g^{-1}(s)-s)^{m-1}\bar{q}_m(g^{-1}(s))ds \leq \frac{(m-1)!\tau^m(1-p)e^{-1}}{p-p^{n+1}} \tag{5.4.49}$$

for some integer $n \geq 1$. Also assume that $\bar{q}_m(t)$ given by (5.4.36) is nonincreasing. Then, for every bounded solution $x(t)$ of (5.4.35), either $x(t)$ is oscillatory or $\liminf_{t\to\infty}(|x(t)| - p|x(t-r)|) < 0$.

Proof Suppose the conclusion is not true. Let $x(t)$ be an eventually positive and bounded solution of (5.4.35) with $\liminf_{t\to\infty}(x(t) - px(t-r)) \geq 0$. Let $y(t)$ be defined as in Lemma 5.4.1 and $v(t)$ be defined as in Lemma 5.4.3. By Lemma 5.4.3, we know that $v(t) > 0$, $(-1)^k v^{(k)}(t) > 0$ for $1 \leq k \leq m$ and (5.4.39), i.e.,

$$v^{(m)}(t) + \frac{1}{\tau^m}\bar{q}_m(t)\sum_{i=0}^{n} p^i v(g(t) - ir) < 0$$

holds for any fixed natural number n and for all large enough t. By Lemma 5.4.4, we know that

$$v'(g(t))\frac{(m-1)!}{(t-g(t))^{m-1}} < v^{(m)}(\theta)$$

for some $\theta \in (g(t), t)$. Since $\bar{q}_m(\theta) \geq \bar{q}_m(t)$ by assumption and

$$v(g(\theta) - ir) \geq v(g(t) - ir),$$

from (5.4.39) with t replaced by θ, it follows that

$$v'(g(t))\frac{(m-1)!}{(t-g(t))^{m-1}} + \frac{1}{\tau^m}\bar{q}_m(t)\sum_{i=0}^{n} p^i v(g(t) - ir) < 0$$

i.e.,

$$v'(g(t)) + \frac{(t-g(t))^{m-1}}{(m-1)!\tau^m}\bar{q}_m(t)\sum_{i=0}^{n} p^i v(g(t) - ir) < 0. \tag{5.4.50}$$

With the replacement of t by $g^{-1}(t)$, (5.4.50) yields

$$v'(t) + \frac{(g^{-1}(t)-t)^{m-1}}{(m-1)!\tau^m}\bar{q}_m(g^{-1}(t))\sum_{i=0}^{n} p^i v(t - ir) < 0. \tag{5.4.51}$$

Assume that $(-1)^k v^{(k)}(t) > 0$ and (5.4.51) hold for $0 \leq k \leq m$ and $t \geq t_1 \geq t_0$. Without loss of generality, we may assume $T_m \geq t_1 + nr$. Let

$$w(t) = \frac{-v'(t)}{v(t)}.$$

Note that $w(t) > 0$ and $v(t) = v(T_m) \exp \int_{T_m}^{t} -w(\theta)d\theta$ for all $t \geq T_m \geq t_1 + nr$. From (5.4.51) it follows that

$$w(t) > \frac{(g^{-1}(t) - t)^{m-1}\bar{q}_m(g^{-1}(t))}{(m-1)!\tau^m} \sum_{i=0}^{n} p^i \exp \int_{t-ir}^{t} w(s)ds, \tag{5.4.52}$$

i.e.,

$$w(t) > \frac{Q_{m1}(t)}{(m-1)!\tau^m} \sum_{i=0}^{n} p^i \exp \int_{t-ir}^{t} w(s)ds \tag{5.4.53}$$

for all $t \geq T_m$, where $Q_{m1}(t) = (g^{-1}(t) - t)^{m-1}\bar{q}_m(g^{-1}(t)) > 0$.

Let $w_0(t) \equiv 0$ for $t \geq T_m - nr$ and let

$$w_{k+1}(t) = \frac{Q_{m1}(t)}{(m-1)!\tau^m} \sum_{i=0}^{n} p^i \exp \int_{t-ir}^{t} w_k(s)ds$$

for each $k \in N$ and $t \geq T_m + nkr$. Let

$$\alpha_{m1k} = \inf_{t \geq T_m + (k-1)nr} \{w_k(t)\}, \quad k \in \bar{N}.$$

Then

$$\alpha_{m1k+1} \geq \inf_{t \geq T_m} \left\{ \frac{Q_{m1}(t)}{(m-1)!\tau^m} \sum_{i=0}^{n} p^i e^{ir\alpha_{m1k}} \right\} = \beta_{m1} \sum_{i=0}^{n} p^i e^{ir\alpha_{m1k}}.$$

Since (5.4.53), (5.4.48) and the definition of $\{\alpha_{m1k}\}$ imply that $\{\alpha_{m1k}\}$ is an increasing sequence, by the same procedure as that used in the proof of Theorem 5.3.1 we have $\alpha_{m1k} \to \infty$ as $k \to \infty$. Notice also that

$$w(t) \geq w_{k+1}(t) \geq \alpha_{m1k+1} \quad \text{for} \quad t \geq T_m + nkr.$$

Thus $w(t) \to \infty$ as $t \to \infty$, which implies

$$\frac{v(t)}{v(t+r)} = \exp \int_{t}^{t+r} w(s)ds \to \infty \quad \text{as} \quad t \to \infty. \tag{5.4.54}$$

On the other hand, since $v'(t) < 0$ and $v(t) > 0$, (5.4.51) yields (by dropping the $i = 0$ term) that

$$\begin{aligned} v'(t) &< -\frac{(g^{-1}(t)-t)^{m-1}}{(m-1)!\tau^m}\bar{q}_m(g^{-1}(t))\sum_{i=1}^{n} p^i v(t-ir) \\ &< -\frac{(g^{-1}(t)-t)^{m-1}}{(m-1)!\tau^m}\cdot\frac{p-p^{n+1}}{1-p}\cdot\bar{q}_m(g^{-1}(t))v(t-r). \end{aligned} \tag{5.4.55}$$

By (5.4.49) and Lemma 5.2.1,

$$\liminf_{t\to\infty}\frac{v(t)}{v(t-r)} \in (0,1].$$

Thus $v(t+r)/v(t)$ has a positive lower bound so $v(t)/v(t+r)$ has a positive upper bound. This contradicts (5.4.54). Assume that $x(t)$ is an eventually negative and bounded solution of (5.4.35) with $\limsup_{t\to\infty}(x(t) - px(t-r)) \leq 0$. Then the above proof with a minor modification also leads to a contradiction. Therefore, the conclusion of the theorem holds.

Corollary 5.4.1 The conclusion of Theorem 5.4.1 still holds if (5.4.49) is replaced by

$$0 \leq \liminf_{t\to\infty}\int_{t-r}^{t}(g^{-1}(s)-s)^{m-1}\bar{q}_m(g^{-1}(s))ds \leq \frac{(m-1)!\tau^m}{ep}. \tag{5.4.56}$$

Proof The proof is the same as that of Theorem 5.4.1 except (5.4.55). The conclusion still holds if (5.4.55) is replaced by

$$v'(t) < -\frac{(g^{-1}(t)-t)^{m-1}}{\tau^m(m-1)!}\bar{q}_m(g^{-1}(t))pv(t-r).$$

Corollary 5.4.2 Assume that (5.4.35) with $0 < p < 1$ satisfies

$$r\beta_{m2}\sum_{i=1}^{n} ip^i \geq 1 \tag{5.4.57}$$

and

$$0 \leq \liminf_{t\to\infty}\int_{t-r}^{t}(g^{-1}(s)-s)^{m-1}\bar{q}_m(s)ds \leq \frac{(m-1)!\tau^m(1-p)e^{-1}}{p-p^{n+1}} \tag{5.4.58}$$

for some integer $n \geq 1$. Also assume that $\bar{q}_m(t)$ given by (5.4.36) is nondecreasing. Then the conclusion of Theorem 5.4.1 holds.

Proof The proof of Theorem 5.4.1 is still valid after the replacement of $\bar{q}_m(\theta) \geq \bar{q}_m(t)$ by $\bar{q}_m(\theta) \geq \bar{q}_m(g(t))$.

Corollary 5.4.3 The conclusion of Corollary 5.4.2 still holds if (5.4.58) is replaced by

$$0 \leq \liminf_{t\to\infty}\int_{t-r}^{t}(g^{-1}(s)-s)^{m-1}\bar{q}_m(s)ds \leq \frac{(m-1)!\tau^m}{ep}. \tag{5.4.59}$$

The proof of Corollary 5.4.3 is similar to that of Corollary 5.4.1.

Corollary 5.4.4 Assume $0 < p < 1$ and $r = k\tau$. Under the assumptions of either Theorem 5.4.1 or Corollary 5.4.1 or Corollary 5.4.2 or Corollary 5.4.3, every bounded solution $x(t)$ of (5.4.35) is oscillatory.

Proof The proof of Theorem 5.4.1 is still valid after the replacement of Lemma 5.4.1 by Lemma 5.4.2.

The following results are for the bounded solutions of (5.4.35) with $p > 1$.

Theorem 5.4.2 Assume that $p > 1$, $r = k\tau$, $k \in N$, $r \geq t + m\tau - g(t)$ and

$$r\beta_{m2}\sum_{i=1}^{n}\frac{(i-1)}{p^i} \geq 1 \tag{5.4.60}$$

for some integer $n \geq 2$. Also assume that $\bar{q}_m(t)$ given by (5.4.36) is nondecreasing. Then every bounded solution $x(t)$ of (5.4.35) is oscillatory.

Proof Suppose the conclusion is not true. Without loss of generality, assume that (5.4.35) has an eventually positive and bounded solution $x(t)$. And let $y(t)$ be defined as in Lemma 5.4.5 and $v(t)$ be defined as in Lemma 5.4.6. By Lemma 5.4.6, we know that $v(t) < 0$, $(-1)^k v^{(k)}(t) > 0$ for $1 \leq k \leq m$, and (5.4.44), i.e.,

$$v^{(m)}(t) - \frac{1}{\tau^m}\bar{q}_m(t)\sum_{i=1}^{n}\frac{1}{p^i}v(g(t) - m\tau + ir) < 0$$

holds for any fixed integer $n \geq 1$ and for all large enough t. By Lemma 5.4.7, we know that

$$v'(g(t))\frac{(m-1)!}{(t-g(t))^{m-1}} < v^{(m)}(\theta)$$

for some $\theta \in (g(t), t)$. Since $\bar{q}_m(\theta) \geq \bar{q}_m(g(t))$ and $v(g(\theta) - ir) \leq v(g(g(t)) - ir)$, with the replacement of t by θ, (5.4.44) yields

$$v'(g(t))\frac{(m-1)!}{(t-g(t))^{m-1}} - \frac{1}{\tau^m}\bar{q}_m(g(t))\sum_{i=1}^{n}\frac{1}{p^i}v(g(g(t)) - m\tau + ir) \leq 0$$

i.e.,

$$v'(g(t)) - \frac{(t-g(t))^{m-1}}{(m-1)!\tau^m}\bar{q}_m(g(t))\sum_{i=1}^{n}\frac{1}{p^i}v(g(g(t)) - m\tau + ir) \leq 0. \tag{5.4.61}$$

With the replacement of t by $g^{-1}(t)$, (5.4.61) becomes

$$v'(t) - \frac{(g^{-1}(t)-t)^{m-1}}{(m-1)!\tau^m}\bar{q}_m(t)\sum_{i=1}^{n}\frac{1}{p^i}v(g(t)-m\tau+ir) \leq 0. \qquad (5.4.62)$$

Assume that $v(t)$, $(-1)^k v^{(k)}(t) > 0\ (1 \leq k \leq m)$ and (5.4.62) hold for $t \geq t_1 \geq t_0$ and, without loss of generality, that $T_m \geq t_1 + nr$. Let

$$w(t) = \frac{v'(t)}{v(t)}.$$

Note that $w(t) > 0$ and $v(t) = v(t')\exp\int_{t'}^{t} w(\theta)d\theta$ for all $t,\ t' \geq T_m$. From (5.4.62), we have

$$w(t) \geq \frac{(g^{-1}(t)-t)^{m-1}\bar{q}_m(t)}{(m-1)!\tau^m}\sum_{i=1}^{n}\frac{1}{p^i}\exp\int_{t}^{g(t)-m\tau+ir} w(s)ds, \qquad (5.4.63)$$

i.e.,

$$w(t) \geq \frac{Q_{m2}(t)}{(m-1)!\tau^m}\sum_{i=1}^{n}\frac{1}{p^i}\exp\int_{t}^{g(t)-m\tau+ir} w(s)ds \qquad (5.4.64)$$

for all $t \geq T_m$, where $Q_{m2}(t) = (g^{-1}(t)-t)^{m-1}\bar{q}_m(t) > 0$.

Let $w_0(t) \equiv 0$ for $t \geq T_m$. And for each $k \in \bar{N}$ and $t \geq T_m$, let

$$w_{k+1}(t) = \frac{Q_{m2}(t)}{(m-1)!\tau^m}\sum_{i=1}^{n}\frac{1}{p^i}\exp\int_{t}^{g(t)-m\tau+ir} w_k(s)ds$$

and

$$\alpha_{m2k} = \inf_{t\geq T_m}\{w_k(t)\}, \qquad k \in \bar{N}.$$

So $w(t) \geq w_{k+1}(t) \geq w_k(t) \geq \alpha_{m2k}$ for all $k \in N$ and $t \geq T_m$. By assumption, we therefore have

$$\begin{aligned}\alpha_{m2k+1} &\geq \inf_{t\geq T_m}\left\{Q_{m2}(t)\cdot\frac{1}{(m-1)!\tau^m}\sum_{i=1}^{n}\frac{1}{p^i}e^{[g(t)-t-m\tau+ir]\alpha_{m2k}}\right\}\\ &\geq \beta_{m2}\sum_{i=1}^{n}\frac{1}{p^i}e^{(i-1)r\alpha_{m2k}}.\end{aligned}$$

Note that $\{\alpha_{m2k}\}$ is a bounded nondecreasing sequence and suppose that

$$\lim_{k\to\infty} \alpha_{m2k} = \rho_{m2} < \infty. \tag{5.4.65}$$

So $\rho_{m2} \geq \beta_{m2} \sum_{i=1}^{n} (e^{(i-1)r\rho_{m2}}/p^i)$. By (5.4.60) and the same reasoning as that used in the proof of Theorem 5.3.2, we reach that no positive number ρ_{m2} satisfies $\rho_{m2} \geq \beta_{m2} \sum_{i=1}^{n} (e^{(i-1)r\rho_{m2}}/p^i)$. This contradiction shows that the conclusion holds.

Corollary 5.4.5 Assume that $p > 1$, $r = k\tau$, $k \in N$, $r \geq t + m\tau - g(t)$ and

$$r\beta_{m1} \sum_{i=1}^{n} \frac{(i-1)}{p^i} \geq 1 \tag{5.4.66}$$

for some integer $n \geq 2$. Also assume that $\bar{q}_m(t)$ given by (5.4.36) is nonincreasing. Then every bounded solution $x(t)$ of (5.4.35) is oscillatory.

Proof The proof of Theorem 5.4.2 is still valid after the replacement of $\bar{q}_m(\theta) \geq \bar{q}_m(g(t))$ by $\bar{q}_m(\theta) \geq \bar{q}_m(t)$.

5.4.3 EXAMPLES

Three examples will be given in this section to demonstrate the applications of the results obtained. From (5.4.46) and (5.4.47) it is clear that both β_{m1} and β_{m2} are nondecreasing functions of T_m. The following examples show that β_{m1} and β_{m2} may be independent of T_m or increasing functions of T_m.

Example 5.4.1 Consider the difference equation

$$\Delta_1^m \left(x(t) - \frac{1}{2}x(t-1)\right) + \left((m-1)! + \frac{1}{t}\right) x(t-1) = 0 \qquad (5.4.67)$$

for $t > 0$, where m is an odd positive integer $m \geq 3$. Viewing (5.4.67) as (5.4.35), we have $\tau = 1$, $0 < p = 1/2 < 1$, $r = 1$, $q(t) = (m-1)! + 1/t$ and $g(t) = t - 1$. Then, according to (5.4.36),

$$\bar{q}_m(t) = \alpha\left((m-1)! + \frac{1}{t+m}\right).$$

So

$$\beta_{m1} = \inf_{t \geq T_m} \left\{ \frac{(t+1-t)^{m-1} \cdot \alpha\left((m-1)! + \frac{1}{t+m+1}\right)}{(m-1)! \cdot 1^m} \right\} = \alpha$$

with $T_m \geq 3$. Since

$$\beta_{m1} \sum_{i=1}^{3} irp^i = \alpha \cdot \left(\frac{1}{2} + 2 \times \frac{1}{4} + 3 \times \frac{1}{8}\right) = \frac{11\alpha}{8} \geq 1$$

holds for $\alpha \in [8/11, 1)$ and

$$\begin{aligned} 0 &\leq \liminf_{t\to\infty} \int_{t-1}^{t} (s+1-s)^{m-1} \cdot \alpha\left((m-1)! + \frac{1}{s+m+1}\right) ds = \alpha \cdot (m-1)! \\ &\leq \frac{2 \cdot (m-1)!}{e} \end{aligned}$$

holds for any $\alpha \in (0, 2/e]$, (5.4.48) and (5.4.56) are satisfied for $n = 3$ and $\alpha \in [8/11, 2/e]$. Since $r = 1 = \tau$, by Corollaries 5.4.1 and 5.4.4, every bounded solution $x(t)$ of (5.4.67) is oscillatory.

Example 5.4.2 Consider the difference equation

$$\Delta_{\frac{\pi}{m}}^m \left(x(t) - 2x(t-4\pi)\right) + 8x(t-\pi) + \frac{8\sigma}{1+t^2} x^3(t-\pi) = 0, \qquad (5.4.68)$$

for $t > 0$, where m is an odd positive integer with $m \geq 3$ and σ is a positive real number. Regarding (5.4.68) as (5.4.35), we have $\tau = \pi/m$, $p = 2$, $r = 4\pi$, $g(t) = t - \pi$ and $q(t) = 8$. Then, for some $\alpha \in (0,1)$, $\bar{q}_m = 8\alpha$ by (5.4.36). Moreover, $r \geq t + m\tau - g(t)$ and $r = k\tau$ are satisfied. In addition,

$$\beta_{m2} = \inf_{t \geq T_m} \left\{ \frac{8\alpha \cdot (t + \pi - t)^{m-1}}{(m-1)!(\frac{\pi}{m})^m} \right\} = \frac{8m^m\alpha}{\pi(m-1)!},$$

where $T_m \geq 12\pi$. So (5.4.60) is satisfied since

$$\beta_{m2} \sum_{i=1}^{3} \frac{4\pi(i-1)}{p^i} = \frac{8m^m\alpha}{\pi(m-1)!} \times 4\pi \times \left(\frac{1}{2^2} + \frac{2}{2^3}\right) = \alpha\frac{16m^m}{(m-1)!} \geq 1$$

holds for $\alpha \in [(m-1)!/(16m^m), 1)$. By Theorem 5.4.2, every bounded solution $x(t)$ of (5.4.68) is oscillatory.

Example 5.4.3 Consider the difference equation

$$\Delta^m_{\frac{\pi}{m}} (x(t) - 2x(t - 2\pi)) + e^{-\frac{\sigma}{t}} x(t - \pi) = 0, \tag{5.4.69}$$

for $t > 0$, where m is an odd positive integer with $m \geq 3$ and σ is a positive constant. Regarding (5.4.69) as (5.4.35), we have $\tau = \pi/m$, $p = 2$, $r = 2\pi$, $g(t) = t - \pi$ and $q(t) = e^{-\frac{\sigma}{t}}$. Then, for some $\alpha \in (0,1)$, $\bar{q}_m = \alpha e^{-\frac{\sigma}{t}}$ by (5.4.36). Moreover, $r \geq t + m\tau - g(t)$ and $r = k\tau$ are satisfied. In addition,

$$\beta_{m2} = \inf_{t \geq T_m} \left\{ \frac{\alpha e^{-\frac{\sigma}{t}} \cdot (t + \pi - t)^{m-1}}{(m-1)!(\frac{\pi}{m})^m} \right\} = \frac{m^m\alpha}{e^{\sigma/T_m}\pi(m-1)!} \to \frac{m^m\alpha}{\pi(m-1)!},$$

as $T_m \to \infty$. So (5.4.60) is satisfied when T_m is large enough since

$$\beta_{m2} \sum_{i=1}^{3} \frac{2\pi(i-1)}{p^i} \to \frac{\alpha m^m}{\pi(m-1)!} \times 2\pi \times \left(\frac{1}{2^2} + \frac{2}{2^3}\right) = \alpha\frac{m^m}{(m-1)!} > 1$$

as $T_m \to \infty$ for $\alpha \in ((m-1)!/(m^m), 1)$. By Theorem 5.4.2, every bounded solution $x(t)$ of (5.4.69) is oscillatory.

5.5 CONCLUSION AND SUMMARY

Our objective of this chapter and chapter 4 is to study a particular class of neutral difference equation of the form

$$\Delta_\tau^m(x(t) - px(t-r)) + f(t, x(g(t))) = 0,$$

where $m \geq 2$ is a natural number, $p \geq 0$, τ and r are positive constants, $\Delta_\tau x(t) = x(t+\tau) - x(t)$, $0 < g(t) < t$, $g \in C^1([t_0, \infty), R^+)$, $g'(t) > 0$, and $f \in C([t_0, \infty) \times R, R)$. Under the assumptions, the existence and uniqueness of solutions are guaranteed. We have concentrated on the oscillatory behaviour of the nontrivial solutions as t tends to ∞.

According to our best knowledge, some known results are available only for the first order of this type of neutral difference equations and special cases of second order equations. Even though the results we have obtained incorporate those known results as special instances, our study has been inspired by them and some of our results could be regarded as the generalization of some previous results. Due to that techniques used in dealing with the oscillatory behaviour of solutions of the even order equations are very much different from those for the odd order equations, we have dealt with them separately.

In chapter 4, we have focused on the even order equations which are composed of the second, fourth and higher even order equations. Section 4.3 devotes to the second order equation (4.3.4). To establish oscillatory criteria, $\bar{q}_2(t)$ has been defined as in (4.3.5) and $z(t)$ has been defined by

$$z(t) = \int_t^{t+\tau} ds \int_s^{s+\tau} x(\theta) d\theta.$$

By constructing a Ricatti transformation, we have obtained a sufficient condition (4.3.6) in Theorem 4.3.1 for every solution $x(t)$ of (4.3.4), either $x(t)$ to be oscillatory or eventually satisfy $|z(t)| < p|z(t-r)|$. Furthermore, in Theorem 4.3.2, we have gained a sufficient condition (4.3.7) for (4.3.4) with $0 < p < 1$ to be oscillatory. Condition (4.3.7) still holds when $p > 1$. Similarly, condition (4.3.8) in Theorem 4.3.3 has been obtained for (4.3.4) to be oscillatory when $p = 1$. At the end of this section, two examples have been given to demonstrate the applications of the obtained results and to show the generality of the obtained results.

In section 4.4 we have concentrated on equation (4.4.20). The organization of this section is the same as that of section 4.3 but the arguments are rather more complicated. Let $\bar{q}_4(t)$ be as in (4.4.21). We have defined $y(t)$ and $u(t)$ as in Lemmas 4.4.1 and 4.4.2, respectively, and obtained their qualitative features as t tends to ∞. Through a Ricatti transformation, equation (4.4.20) has been converted to a first order inequality and then condition (4.4.25) has been established in Theorem 4.4.1 for every solution $x(t)$, either $x(t)$ to be oscillatory or for any $T \geq t_0$, there exists a $t'' > T$ such that $|x(t)| \leq p|x(t''-r)|$. Moreover, we have obtained four more conditions in each case of $0 < p < 1$, $p = 1$ and $p > 1$. When $0 < p < 1$, condition (4.4.27) has been established in Theorem 4.4.2 for every solution $x(t)$, either $x(t)$ or $x(t) - px(t-r)$ is oscillatory. Based on Theorem 4.4.2, three more weaker conditions (4.4.31), (4.4.33), and (4.4.35) have been gained in Corollaries 4.4.1, 4.4.2, and 4.4.3, respectively. By the same procedure as before, we have condition (4.4.36) in Theorem 4.4.3 when $p = 1$, which has been extended to conditions (4.4.37), (4.4.38), (4.4.39) in Corollaries 4.4.4, 4.4.5, 4.4.6, respectively, and condition (4.4.40) in Theorem 4.4.4 when $p > 1$, which has been extended to conditions (4.4.41), (4.4.42), (4.4.43) in Corollaries 4.4.7, 4.4.8, 4.4.9. Two illustrating examples have been given at the end of this section.

In section 4.5, we have focused on equation (4.5.46). The idea in this section was inspired by the previous two sections and could be viewed as a generalization of them. However, as a general case, the reasonings in this section are much more complicated. Let $\bar{q}_m(t)$ be defined as in (4.5.47). In addition, let $y(t)$ be as in Lemma 4.5.1 and $u(t)$ be as in Lemma 4.5.2. Based on the known results, the properties of $y(t)$ and $u(t)$ have been gained. By a Riccati transform, we have converted equation (4.5.46) to the first order inequality (4.5.53) then obtained sufficient condition (4.5.52) in Theorem 4.5.1 for every solution $x(t)$, either $x(t)$ to be oscillatory or for any $T \geq t_0$, there exists a $t'' > T$ such that $|x(t)| \leq p|x(t''-r)|$. When $0 < p < 1$, condition (4.5.54) has been obtained in Theorem 4.5.2 for $x(t)$ or $x(t) - px(t-r)$ to be oscillatory. Basis on Theorem 4.5.2, we have managed to have two weaker conditions (4.5.58) and (4.5.59) in Corollaries 4.5.1 and 4.5.2, respectively. In a same way, when $p = 1$ we have condition (4.5.60) in Theorem 4.5.3, condition (4.5.61) in Corollary 4.5.3, and condition (4.5.62) in Corollary 4.5.4 for $x(t)$ or $x(t) - px(t-r)$ be oscillatory, and when $p > 1$, we have condition (4.5.63) in Theorem 4.5.4, condition (4.5.64) in Corollary 4.5.5, and condition (4.5.65) in Corollary 4.5.6 for $x(t)$ or $x(t) - px(t-r)$ to be oscillatory. Three illustrating examples have been given at the end of this section to demonstrate the applications of the obtained results.

In chapter 5, we have focused on equation (4.1.1) with odd $m \geq 3$. We investigated the third order equations at first then the higher odd order equations. The obtained results for the odd order equations are weaker than those of the even order equations. In this chapter, oscillatory criteria are just for the bounded solutions except $p = 1$.

In section 5.3, we have concentrated on the third order equation (5.3.2). Let $\bar{q}_3(t)$ be as in (5.3.3). When $0 < p < 1$, we have defined $z(t)$ and $y(t)$ as in Lemma 5.3.1 and obtained their qualitative properties as t tends to ∞. By defining $v(t)$ as in Lemma 5.3.3 and Taylor's formula, we have converted equation (5.3.2) to the first order differential inequality (5.3.18). Using the known results about differential equations/inequalites, we have obtained sufficient conditions (5.3.15) and (5.3.16) in Theorem 5.3.1 for every bounded solution $x(t)$, either $x(t)$ to be oscillatory or $\liminf_{t\to\infty}(|x(t)| - p|x(t-r)|) < 0$. From Theorem 5.3.1, four corollaries have been gained. The conclusion of this theorem are still valid if (5.3.16) is replaced by (5.3.23) in Corollary 5.3.1. Furthermore, if conditions (5.3.24) and (5.3.25) in Corollary 5.3.2 or conditions (5.3.24) and (5.3.26) in Corollary 5.3.3 hold, then the conclusion of Theorem 5.3.1 holds as well. Corollary 5.3.4 is about the special case $r = k\tau$. When $p > 1$, by the same procedure, we have defined $z(t)$ and $y(t)$ as in Lemma 5.3.5 and $v(t)$ as in Lemma 5.3.6 and obtained their qualitative features as t tends to ∞. By (5.3.12) in Corollary 5.3.7, equation (5.3.2) has been transformed to the first order differential inequality (5.3.29). Basis on the known results about differential equations/inequalites and constructing the sequence $\{\alpha_{32k}\}$, we have obtained (5.3.27) in Theorem 5.3.2 for every bounded solution x to be oscillatory. From

Theorem 5.3.2, we have Corollary 5.3.5. Two illustrating examples have been given at the end of this section.

Section 5.4 devotes to the higher odd order equation (5.4.35). The structure of this section is the same as that of the last section. The ideas were inspired by the third order equations so the results could be regarded as a generalization. But, the arguments are rather more complicated. Let $\bar{q}_m(t)$ be as in (5.4.36). When $p > 1$, let $z(t)$ and $y(t)$ be as in Lemma 5.4.1 and $v(t)$ be as in Lemma 5.4.3. From the features of $v(t)$, we have gained (5.4.40) in Lemma 5.4.4, by which equation (5.4.35) has been converted to the differential inequality (5.4.51). From the known results on differential equations/inequalites, we have conditions (5.4.48) and (5.4.49) in Theorem 5.4.1 for every bounded solution $x(t)$, either $x(t)$ be oscillatory or $\lim_{t\to\infty}(|x(t)| - p|x(t-r)|) < 0$. Basis on Theorem 5.4.1, Corollaries 5.4.1-5.4.4 have been gained. When $p > 1$, we have defined $z(t)$ and $y(t)$ as in Lemma 5.4.5 and $v(t)$ as in Lemma 5.4.6. From these two lemmas, we have obtained (5.4.45) in Lemma 5.4.7, by which equation (5.4.35) has been converted to differential inequality (5.4.62). By the known results, we have obtained sufficient condition (5.4.60) in Theorem 5.4.2 for every bounded solution x to be oscillatory. Basis on Theorem 5.4.2, Corollary 5.4.5 has been obtained. Three illustrating examples have been given at the end of this section.

Chapter 6

HIGHER ORDER NONLINEAR DIFFERENCE EQUATIONS

6.1 INTRODUCTION

In this chapter, even order nonlinear neutral difference equations of the form

$$\Delta^{m-1}(a_n\Delta(x_n+\varphi(n,x_{\tau_n})))+q_nf(x_{g_n})=0 \tag{6.1.1}$$

are considered, where m is an even positive integer, $n \geq n_0$, $\{\tau_n\}$ and $\{g_n\}$ are nondecreasing sequences of nonnegative integers with $\tau_n \leq n$, $g_n \leq n$, $\lim_{n\to\infty}\tau_n = \infty$, $\lim_{n\to\infty} g_n = \infty$, $\{a_n\}$ and $\{q_n\}$ are sequences of real numbers with $a_n > 0$, $q_n \geq 0$ and $q_n \not\equiv 0$, and $f : R \to R$ and $\varphi : R^2 \to R$ are functions. Assume that the following conditions always hold throughout this chapter:

$$\sum_{n=n_0}^{\infty}\frac{1}{a_n}=\infty, \tag{6.1.2}$$

$$0<\frac{\varphi(n,u)}{u}\leq p_n<1 \qquad \text{for} \quad u\neq 0, \tag{6.1.3}$$

$$\frac{f(u)}{u}\geq \varepsilon_0>0 \qquad \text{for} \quad u\neq 0. \tag{6.1.4}$$

There has been an increasing interest in the study of oscillation for the solutions of higher order difference equations recently. For instance, Zaffer and Dahiya [61] studied the equation

$$\Delta(a_n\Delta^{m-1}(x_n+p_nx_{n-k}))+\delta q_nf(x_{\sigma_n})=0,$$

where $\delta=\pm 1$, Thandapani, Sundaram, and Lalli [54] investigated

$$\Delta^m x(n)+q(n)f(x(\sigma(n)))h(\Delta^{m-1}x(\delta(n)))=0, \quad n\in \bar{N}, \ m \text{ is even},$$

as well as the forced difference equation

$$\Delta^m x(n)+q(n)f(x(\sigma(n)))h(\Delta^{m-1}x(\delta(n)))=e(n), \quad n\in N, \ m \text{ is even},$$

and Graef, etc. [24] discussed the higher order neutral delay difference equation

$$\Delta^m(x_{n-m+1} + p_{n-m+1}x_{n-m+1-k}) + \delta f(n, x_{n-l}) = 0,$$

where $\delta = \pm 1$. In addition, Yan and Liu [58] studied the fourth order difference equations of the form

$$\Delta^2(r_n \Delta^2 x_n) + f(n, x_n) = 0.$$

Note that the highest order difference term in each of the above equations is linear whereas φ in (6.1.1) may be nonlinear. Oscillation criteria will be established, which completely cover the results of Zhang and Li [67] and Zhang and Zhang [69] as special cases. Notice also that this chapter is a modified version of the published paper [40] under the joint authorship of Z. Liu, S. Wu and Z. Zhang, in an alphabet order. This reflects the contribution from the first and third authors in the process of refining the previous drafts and developing and sharping the original results, but the main idea and results belong to the second author.

In this chapter, the wording is slightly different from the published paper in order to be consistent with the previous chapters. At first, we will present some related lemmas in section 6.2. In section 6.3 we shall state the main results and give their proofs as well. In section 6.4 examples will be given to illustrate the obtained oscillatory criteria. At last, we will finish this chapter with a conclusion.

6.2 RELATED LEMMAS

To prove the main results given in section 6.3, we need the following lemmas.

Lemma 6.2.1 Assume that the positive sequence $\{y_n\}$ satisfies $\Delta^m y_n \leq 0$ but $\Delta^m y_n$ is not identically zero, where m is an even positive integer with $m \geq 2$. Then there exist an integer k and an $n_1 \geq n_0$ such that $1 \leq k \leq m-1$, $m+k$ is odd, and

$$\begin{aligned} \Delta^j y_n &> 0, \qquad j = 1, 2, \ldots, k, \\ (-1)^{j+1}\Delta^j y_n &> 0, \qquad j = k+1, \ldots, m-1, \end{aligned}$$

for $n \geq n_1$.

Lemma 6.2.2 Under the assumptions of Lemma 6.2.1, there exists an $n_2 \geq n_1$ such that

$$y_n \geq \frac{(n-n_2)^{(m-1)}}{(m-1)!}\Delta^{m-1} y_{2^{m-k-1}n}$$

holds for $n \geq n_2$.

The above two lemmas can be found in [2] (P31 and P33).

Lemma 6.2.3 Assume that $\{x_n\}$ is an eventually positive solution of (6.1.1). Let $z_n = x_n + \varphi(n, x_{\tau_n})$. Then there exists an $n_3 \geq n_2$ such that

$$\Delta z_{r_n} \geq \frac{(r_n - n_3)^{m-2}}{a_{r_n}(m-2)!}\Delta^{(m-2)}(a_{g_n}\Delta z_{g_n}) \tag{6.2.5}$$

for $r_n \geq n_3$, where $r_n = \left\lceil \frac{g_n}{2^{m-2}} \right\rceil$.

Proof From (6.1.1) and the assumptions, we have $\Delta^{m-1}(a_n\Delta z_n) \le 0$. So $\Delta^{m-2}(a_n\Delta z_n)$ is nonincreasing. If $m = 2$ then $\Delta^{m-2}(a_n\Delta z_n) = a_n\Delta z_n$ is nonincreasing. If there is an $\hat{n}$ such that $\Delta z_n < 0$ for all $n \ge \hat{n}$ then $a_n\Delta z_n \le a_{\hat{n}}\Delta z_{\hat{n}} < 0$. According to the definition of $z(n)$, we have $x_{n+1} + \varphi(n+1, x_{\tau_{n+1}}) - (x_n + \varphi(n, x_{\tau_n})) = \Delta z_n$. Then

$$\begin{aligned} x_{n+1} + \varphi(n+1, x_{\tau_{n+1}}) &= x_{\hat{n}} + \varphi(\hat{n}, x_{\tau_{\hat{n}}}) + \sum_{s=\hat{n}}^{n} \Delta z_s \\ &\le x_{\hat{n}} + \varphi(\hat{n}, x_{\tau_{\hat{n}}}) + a_{\hat{n}}\Delta z_{\hat{n}} \sum_{s=\hat{n}}^{n} \frac{1}{a_s}. \end{aligned}$$

By (6.1.2), $x_{n+1} < x_{\hat{n}} + \varphi(\hat{n}, x_{\tau_{\hat{n}}}) + a_{\hat{n}}\Delta z_{\hat{n}} \sum_{s=\hat{n}}^{n} \frac{1}{a_s} \to -\infty$ as $n \to \infty$, a contradiction to the positiveness of x_n. Therefore, $\Delta z_n > 0$ for $n \ge n_0$. We now show that $a_n\Delta z_n$ is eventually positive for even $m > 2$. Since $\Delta^{m-2}(a_n\Delta z_n)$ is nonincreasing, we have either $\Delta^{m-2}(a_n\Delta z_n) > 0$ for all $n \ge n_0$ or $\Delta^{m-2}(a_n\Delta z_n) \le \Delta^{m-2}(a_{n_1}\Delta z_{n_1}) < 0$ for some $n_1 \ge n_0$ and all $n \ge n_1$. In the latter case, we see that

$$\Delta^{m-3}(a_n\Delta z_n) = \Delta^{m-3}(a_{n_1}\Delta z_{n_1}) + \sum_{s=n_1}^{n-1} \Delta^{m-2}(a_s\Delta z_s) \to -\infty$$

and $\Delta^i(a_n\Delta z_n) \to -\infty\,(0 \le i \le m-3)$ as $n \to \infty$. This again will lead to a contradiction to the positiveness of x_n. Therefore, $\Delta^{m-2}(a_n\Delta z_n)$ is positive and nonincreasing. Thus, $\Delta^{m-3}(a_n\Delta z_n)$ is increasing with either $\Delta^{m-3}(a_n\Delta z_n) < 0$ for all $n \ge n_0$ or $\Delta^{m-3}(a_n\Delta z_n) \ge \Delta^{m-3}(a_{n_2}\Delta z_{n_2}) > 0$ for some $n_2 \ge n_0$ and all $n \ge n_2$. In the latter case, we have $\Delta^i(a_n\Delta z_n) \to \infty\,(0 \le i \le m-4)$ as $n \to \infty$ so $a_n\Delta z_n > 0$ holds eventually. In the former case, viewing $m-2$ as m, we obtain the required conclusion from the same reasoning as above.

From Lemma 6.2.2, there is an $n_3 \geq n_2$ such that for $r_n \geq n_3$

$$\begin{aligned}
a_{r_n}\Delta z_{r_n} &\geq \frac{(r_n-n_3)^{(m-2)}}{(m-2)!}\Delta^{m-2}(a_{2^{m-k-2}r_n}\Delta z_{2^{m-k-2}r_n}) \\
&\geq \frac{(r_n-n_3)^{(m-2)}}{(m-2)!}\Delta^{m-2}(a_{2^{m-2}r_n}\Delta z_{2^{m-2}r_n}) \\
&\geq \frac{(r_n-n_3)^{(m-2)}}{(m-2)!}\Delta^{m-2}(a_{g_n}\Delta z_{g_n}).
\end{aligned}$$

Therefore, we have

$$\Delta z_{r_n} \geq \frac{(r_n-n_3)^{(m-2)}}{a_{r_n}(m-2)!}\Delta^{m-2}(a_{g_n}\Delta z_{g_n})$$

for $r_n \geq n_3$.

Lemma 6.2.4 Assume that $\{x_n\}$ is an eventually positive solution of (6.1.1). Let $\{A_n\}$ be an arbitrary positive sequence. Then there exists an $n_4 \geq n_3$ such that Riccati difference inequality

$$\Delta u_n + Q_n + \frac{A_n B_n u_{n+1}^2}{A_{n+1}^2} \leq 0 \quad \text{for} \quad r_n \geq n_4 \tag{6.2.6}$$

has a solution $\{u_n\}$, where

$$\begin{aligned}
Q_n &= A_n\left\{\varepsilon_0 q_n(1-p_{g_n}) + \frac{(\Delta A_n)^2}{4A_n^2 B_n} + \Delta\left(\frac{\Delta A_{n-1}}{2A_{n-1}B_{n-1}}\right)\right\}, \\
B_n &= \frac{(r_n-n_4)^{(m-2)}}{a_{r_n}(m-2)!}.
\end{aligned}$$

Proof From (6.1.1), (6.1.3) and (6.1.4), it follows that

$$\Delta^{m-1}(a_n\Delta z_n) + q_n f(x_{g_n}) = 0,$$

$$\Delta^{m-1}(a_n\Delta z_n) + \varepsilon_0 q_n x_{g_n} \leq 0,$$

$$\Delta^{m-1}(a_n\Delta z_n) + \varepsilon_0 q_n(z_{g_n} - \varphi(g_n, x_{\tau_{g_n}})) \leq 0,$$

$$\Delta^{m-1}(a_n\Delta z_n) + \varepsilon_0 q_n(z_{g_n} - p_{g_n} z_{\tau_{g_n}}) \leq 0.$$

Since $\Delta z_n > 0$, $\tau_{g_n} \le g_n$ and $g_n \ge r_n$, we have $z_{\tau_{g_n}} \le z_{g_n}$ and $z_{g_n} \ge z_{r_n}$. So

$$\Delta^{m-1}(a_n \Delta z_n) + \varepsilon_0 q_n (1 - p_{g_n}) z_{g_n} \le 0$$

and

$$\Delta^{m-1}(a_n \Delta z_n) + \varepsilon_0 q_n (1 - p_{g_n}) z_{r_n} \le 0. \tag{6.2.7}$$

Let

$$u_n = A_n \left\{ \frac{\Delta^{m-2}(a_n \Delta z_n)}{z_{r_n}} - \frac{\Delta A_{n-1}}{2A_{n-1}B_{n-1}} \right\}.$$

Note that $\{z_n\}$ is increasing and $\{\Delta^{m-2}(a_n\Delta z_n)\}$ is decreasing. In addition, by the definition of r_n we can always choose a subsequence of r_n to replace r_n if necessary, thus we assume that

$$r_n = r_{n+1} - 1 = \cdots = r_{n+h} - h = \cdots .$$

Therefore, by (6.2.7) and Lemma 6.2.3, there is an $n_4 \ge n_3$ such that for $r_n \ge n_4$ we have

$$\begin{aligned}
\Delta u_n &= \frac{\Delta A_n}{A_{n+1}} u_{n+1} + A_n \frac{z_{r_{n+1}} \Delta^{m-1}(a_n \Delta z_n) - \Delta z_{r_n} \Delta^{m-2}(a_{n+1}\Delta z_{n+1})}{z_{r_{n+1}} z_{r_n}} \\
&\quad - A_n \Delta \left(\frac{\Delta A_{n-1}}{2A_{n-1}B_{n-1}} \right) \\
&\le \frac{\Delta A_n}{A_{n+1}} u_{n+1} - A_n \varepsilon_0 q_n (1 - p_{g_n}) - \frac{A_n \Delta^{m-2}(a_{n+1}\Delta z_{n+1}) \Delta^{m-2}(a_{g_n} \Delta z_{g_n}) B_n}{z_{r_n} z_{r_{n+1}}} \\
&\quad - A_n \Delta \left(\frac{\Delta A_{n-1}}{2A_{n-1}B_{n-1}} \right) \\
&\le -A_n \left(\varepsilon_0 q_n (1 - p_{g_n}) + \Delta \left(\frac{\Delta A_{n-1}}{2A_{n-1}B_{n-1}} \right) \right) + \frac{\Delta A_n}{A_{n+1}} u_{n+1} \\
&\quad - A_n B_n \left(\frac{\Delta^{m-2}(a_{n+1}\Delta z_{n+1})}{z_{r_{n+1}}} \right)^2 \\
&= -A_n \left(\varepsilon_0 q_n (1 - p_{g_n}) + \Delta \left(\frac{\Delta A_{n-1}}{2A_{n-1}B_{n-1}} \right) \right) + \frac{\Delta A_n}{A_{n+1}} u_{n+1} \\
&\quad - A_n B_n \left(\frac{u_{n+1}}{A_{n+1}} + \frac{\Delta A_n}{2A_n B_n} \right)^2 \\
&= -A_n \left\{ \varepsilon_0 q_n (1 - p_{g_n}) + \frac{(\Delta A_n)^2}{4A_n^2 B_n} + \Delta \left(\frac{\Delta A_{n-1}}{2A_{n-1}B_{n-1}} \right) \right\} - \frac{A_n B_n}{A_{n+1}^2} u_{n+1}^2.
\end{aligned}$$

Therefore,

$$\Delta u_n + Q_n + \frac{A_n B_n u_{n+1}^2}{A_{n+1}^2} \le 0 \quad \text{for} \quad r_n \ge n_4.$$

So the conclusion of the Lemma holds.

Lemma 6.2.5 Under the assumptions of Lemma 6.2.4, further assume that

$$\sum_{s=n}^{\infty} \frac{A_s B_s}{A_{s+1}^2} = \infty \tag{6.2.8}$$

and $P_n^{(0)} = \sum_{s=n}^{\infty} Q_s$ is a constant number or tends to ∞. Then there exists an $n_5 \ge n_4$ such that for $r_n \ge n_5$,

$$u_n \ge P_n^{(0)} + \sum_{s=n}^{\infty} \frac{A_s B_s u_{s+1}^2}{A_{s+1}^2}, \tag{6.2.9}$$

$$P_n^{(0)} = \sum_{s=n}^{\infty} Q_s < \infty, \tag{6.2.10}$$

and

$$\sum_{s=n}^{\infty} \frac{A_s B_s u_{s+1}^2}{A_{s+1}^2} < \infty. \tag{6.2.11}$$

Proof Note that from Lemma 6.2.4 (6.2.6) is valid for $r_n \ge n_4$. We will show that (6.2.11) holds at first. In fact, if

$$\sum_{s=n}^{\infty} \frac{A_s B_s u_{s+1}^2}{A_{s+1}^2} = \infty, \tag{6.2.12}$$

then in view of the definition of $P_n^{(0)}$ and (6.2.12), there exists an $N_1 > n$ for any fixed n such that for $\xi \ge N_1$,

$$\begin{aligned} u_{\xi+1} &\le u_n - \sum_{s=n}^{\xi} Q_s - \sum_{s=n}^{N_1-1} \frac{A_s B_s u_{s+1}^2}{A_{s+1}^2} - \sum_{s=N_1}^{\xi} \frac{A_s B_s u_{s+1}^2}{A_{s+1}^2} \\ &\le -1 - \sum_{s=N_1}^{\xi} \frac{A_s B_s u_{s+1}^2}{A_{s+1}^2}. \end{aligned}$$

Therefore we have

$$\sum_{s=N_1}^{\xi} \frac{A_s B_s u_{s+1}}{A_{s+1}^2} \le -\sum_{s=N_1}^{\xi} \frac{A_s B_s}{A_{s+1}^2} - \sum_{s=N_1}^{\xi} \frac{A_s B_s}{A_{s+1}^2} \sum_{i=N_1}^{s} \frac{A_i B_i u_{i+1}^2}{A_{i+1}^2}.$$

Let

$$v_\xi = \sum_{s=N_1}^{\xi} \frac{A_s B_s u_{s+1}}{A_{s+1}^2}.$$

According to discrete Cauchy-Schmartz inequality, we have

$$\sum_{i=N_1}^{s} \frac{A_i B_i u_{i+1}^2}{A_{i+1}^2} \ge v_s^2 \left(\sum_{i=N_1}^{s} \frac{A_i B_i}{A_{i+1}^2} \right)^{-1}.$$

So

$$v_\xi \le -\sum_{i=N_1}^{\xi} \frac{A_i B_i}{A_{i+1}^2} - \sum_{s=N_1}^{\xi} \frac{A_s B_s v_s^2}{A_{s+1}^2} \left(\sum_{i=N_1}^{s} \frac{A_i B_i}{A_{i+1}^2} \right)^{-1} \equiv H_\xi.$$

Note that $H_\xi < 0$, $H_\xi \to -\infty$ as $\xi \to \infty$ by (6.2.8), and

$$\sum_{s=N_1}^{\xi} \frac{A_s B_s}{A_{s+1}^2} \le |H_\xi| \le |v_\xi|.$$

Clearly,

$$\Delta H_\xi = -\frac{A_{\xi+1} B_{\xi+1}}{A_{\xi+2}^2} - \frac{A_{\xi+1} B_{\xi+1} v_{\xi+1}^2}{A_{\xi+2}^2} \left(\sum_{s=N_1}^{\xi+1} \frac{A_s B_s}{A_{s+1}^2} \right)^{-1} < 0.$$

Thus $\{H_\xi\}$ is decreasing and

$$\frac{\Delta H_\xi}{H_\xi H_{\xi+1}} \le \frac{\Delta H_\xi}{H_{\xi+1}^2} \le \frac{\Delta H_\xi}{v_{\xi+1}^2} \le -\frac{A_{\xi+1} B_{\xi+1}}{A_{\xi+2}^2} \left(\sum_{i=N_1}^{\xi+1} \frac{A_i B_i}{A_{i+1}^2} \right)^{-1}. \tag{6.2.13}$$

Summation of (6.2.13) for ξ from $N-1$ to ξ, we obtain

$$\begin{aligned} \frac{1}{H_{N-1}} - \frac{1}{H_{\xi+1}} &\le -\sum_{s=N}^{\xi+1} \frac{A_s B_s}{A_{s+1}^2} \left(\sum_{i=N_1}^{s} \frac{A_i B_i}{A_{i+1}^2} \right)^{-1} \\ &\le -\sum_{s=N}^{\xi+1} \frac{A_s B_s}{A_{s+1}^2} \left(\sum_{i=N_1}^{\xi+1} \frac{A_i B_i}{A_{i+1}^2} \right)^{-1} \equiv G_\xi. \end{aligned}$$

In view of (6.2.8), $G_\xi \to -1$ and $(1/H_{\xi+1}) \to 0$ as $\xi \to \infty$. Hence

$$\frac{1}{H_{N-1}} \le -1$$

for large enough N, which contradicts $H_n \to -\infty$ as $n \to \infty$. Hence (6.2.11) holds. Therefore, we have

$$u_n \geq \limsup_{\xi\to\infty} u_\xi + P_n^{(0)} + \sum_{s=n}^{\infty} \frac{A_s B_s u_{s+1}^2}{A_{s+1}^2}.$$

We will prove that $\limsup_{\xi\to\infty} u_\xi \geq 0$. Indeed, if $\limsup_{\xi\to\infty} u_\xi < 0$, then there must be an $l > 0$ and an $N_2 \geq N_1$ such that $u_\xi \leq -l$ for $\xi \geq N_2$, so

$$\sum_{s=n}^{\infty} \frac{A_s B_s u_{s+1}^2}{A_{s+1}^2} \geq l^2 \sum_{s=n}^{\infty} \frac{A_s B_s}{A_{s+1}^2} \to \infty,$$

a contradiction to (6.2.11). Hence we have $\limsup_{\xi\to\infty} u_\xi \geq 0$. Furthermore, there is an $n_5 \geq n_4$ such that for $r_n \geq n_5$, (6.2.9) holds and (6.2.10) follows from (6.2.9). Therefore we have showed that the conclusion of the Lemma holds.

Lemma 6.2.6 Under the assumptions of Lemma 6.2.5, further assume $P_n^{(0)} \geq 0$ eventually. Then there exist an $n_6 \geq n_5$ and a sequence $\{h_n\}$ such that for every positive integer k,

$$h_n \geq P_n^{(k)} + \sum_{s=n}^{\infty} \frac{A_s B_s h_{s+1}^2}{A_{s+1}^2} R_{s,n}^{(k)} \quad \text{for} \quad r_n \geq n_6, \tag{6.2.14}$$

where

$$\begin{aligned} P_n^{(k)} &= \sum_{s=n}^{\infty} \frac{A_s B_s \left(P_{s+1}^{(k-1)}\right)^2}{A_{s+1}^2} R_{s,n}^{(k)}, \\ R_{s,n}^{(k)} &= \prod_{i=n}^{s-1} \left(1 + \frac{2A_i B_i P_{i+1}^{(k-1)}}{A_{i+1}^2}\right), \quad \prod_{n}^{n-1} = 1. \end{aligned}$$

Proof By Lemma 6.2.5, there exists a sequence $\{u_n\}$ such that (6.2.9) and (6.2.11) hold. Let

$$h_n = \sum_{s=n}^{\infty} \frac{A_s B_s u_{s+1}^2}{A_{s+1}^2}$$

then we have $\Delta h_n = -A_n B_n u_{n+1}^2 / A_{n+1}^2$. Let

$$R_{s,n}^{(1)} = \prod_{i=n}^{s-1} \left(1 + \frac{2A_i B_i P_{i+1}^{(0)}}{A_{i+1}^2} \right)$$

and

$$\Delta\left(h_s R_{s,n}^{(1)}\right) - h_{s+1} \frac{2A_s B_s P_{s+1}^{(0)}}{A_{s+1}^2} R_{s,n}^{(1)} = -\frac{A_s B_s u_{s+1}^2}{A_{s+1}^2} R_{s,n}^{(1)}. \tag{6.2.15}$$

Summation (6.2.15) for s from n to $N-1$, we have

$$\sum_{s=n}^{N-1} \Delta\left(h_s R_{s,n}^{(1)}\right) - \sum_{s=n}^{N-1} h_{s+1} \frac{2A_s B_s P_{s+1}^{(0)}}{A_{s+1}^2} R_{s,n}^{(1)} = -\sum_{s=n}^{N-1} \frac{A_s B_s u_{s+1}^2}{A_{s+1}^2} R_{s,n}^{(1)},$$

i.e.,

$$h_N R_{N,n}^{(1)} - h_n - \sum_{s=n}^{N-1} h_{s+1} \frac{2A_s B_s P_{s+1}^{(0)}}{A_{s+1}^2} R_{s,n}^{(1)} = -\sum_{s=n}^{N-1} \frac{A_s B_s u_{s+1}^2}{A_{s+1}^2} R_{s,n}^{(1)}.$$

So

$$h_n \geq \sum_{s=n}^{N-1} \frac{A_s B_s}{A_{s+1}^2} R_{s,n}^{(1)} \left(u_{s+1}^2 - 2h_{s+1} P_{s+1}^{(0)} \right).$$

By Lemma 6.2.5, we have $u_{s+1} \geq P_{s+1}^{(0)} + h_{s+1} \geq 0$. Hence $u_{s+1}^2 \geq \left(P_{s+1}^{(0)}\right)^2 + h_{s+1}^2 + 2P_{s+1}^{(0)} h_{s+1}$. Then we obtain

$$h_n \geq \sum_{s=n}^{N-1} \frac{A_s B_s}{A_{s+1}^2} R_{s,n}^{(1)} \left(\left(P_{s+1}^{(0)}\right)^2 + h_{s+1}^2 \right).$$

Therefore

$$\begin{aligned} h_n &\geq \sum_{s=n}^{\infty} \frac{A_s B_s}{A_{s+1}^2} R_{s,n}^{(1)} \left(\left(P_{s+1}^{(0)}\right)^2 + h_{s+1}^2 \right) \\ &= P_n^{(1)} + \sum_{s=n}^{\infty} \frac{A_s B_s h_{s+1}^2}{A_{s+1}^2} R_{s,n}^{(1)}. \end{aligned}$$

Next we will prove that (6.2.14) holds for $k = 2$. Replacing $R_{s,n}^{(1)}$ by $R_{s,n}^{(2)}$ in (6.2.15) and following the same reasoning as the above, we have

$$h_n \geq \sum_{s=n}^{N-1} \frac{A_s B_s}{A_{s+1}^2} R_{s,n}^{(2)} \left(u_{s+1}^2 - 2h_{s+1} P_{s+1}^{(1)} \right).$$

Note that $h_n \geq P_n^{(1)}$. According to (6.2.9) and (6.2.11), there is an $n_6 \geq n_5$ such that $u_n \geq 2h_n$ for $r_n \geq n_6$. So $u_n \geq h_n + P_n^{(1)}$. Furthermore,

$$\begin{aligned} h_n &\geq \sum_{s=n}^{\infty} \frac{A_s B_s}{A_{s+1}^2} R_{s,n}^{(2)} \left(\left(P_{s+1}^{(1)}\right)^2 + h_{s+1}^2 \right) \\ &= P_n^{(2)} + \sum_{s=n}^{\infty} \frac{A_s B_s h_{s+1}^2}{A_{s+1}^2} R_{s,n}^{(2)}. \end{aligned}$$

By the same argument as the above, we have

$$h_n \geq P_n^{(k)} + \sum_{s=n}^{\infty} \frac{A_s B_s h_{s+1}^2}{A_{s+1}^2} R_{s,n}^{(k)} \tag{6.2.16}$$

for all positive integer k and $r_n \geq n_6$.

Lemma 6.2.7 Under the assumptions of Lemma 6.2.6, for every positive integer k, the following inequality holds

$$\limsup_{n \to \infty} P_n^{(k)} \prod_{s=n_6}^{n-1} \left(1 + \frac{4A_s B_s P_{s+1}^{(k-1)}}{A_{s+1}^2} \right) < \infty. \tag{6.2.17}$$

Proof Notice that from Lemma 6.2.6, (6.2.14) holds for $k \in N$. Let

$$w_n = \sum_{s=n}^{\infty} \frac{A_s B_s h_{s+1}^2}{A_{s+1}^2} R_{s,n}^{(k)}.$$

From (6.2.14) it follows that

$$h_n \geq P_n^{(k)} + w_n.$$

So

$$-\Delta w_n = \frac{A_n B_n h_{n+1}^2}{A_{n+1}^2} \geq \frac{4A_n B_n P_{n+1}^{(k)} w_{n+1}}{A_{n+1}^2},$$

i.e.,

$$w_n - w_{n+1} \geq \frac{4A_n B_n P_{n+1}^{(k)} w_{n+1}}{A_{n+1}^2}.$$

The above inequality implies that

$$w_{n+1} \leq w_n \left(1 + \frac{4A_nB_nP_{n+1}^{(k)}}{A_{n+1}^2}\right)^{-1} \leq w_{n_6} \prod_{s=n_6}^{n} \left(1 + \frac{4A_sB_sP_{s+1}^{(k)}}{A_{s+1}^2}\right)^{-1}.$$

From the proof of Lemma 6.2.6 and the definition of w_n, we have

$$w_n = \sum_{s=n}^{\infty} \frac{A_sB_sh_{s+1}^2}{A_{s+1}^2} R_{s,n}^{(k)} \geq \sum_{s=n}^{\infty} \frac{A_sB_s(P_n^{(k)})^2}{A_{s+1}^2} R_{s,n}^{(k)} \geq P_n^{(k+1)}.$$

Thus

$$P_n^{(k+1)} \leq w_{n_6} \prod_{s=n_6}^{n} \left(1 + \frac{4A_sB_sP_{s+1}^{(k)}}{A_{s+1}^2}\right)^{-1}$$

i.e.,

$$P_n^{(k+1)} \prod_{s=n_6}^{n-1} \left(1 + \frac{4A_sB_sP_{s+1}^{(k)}}{A_{s+1}^2}\right) \leq w_{n_6}.$$

Therefore,

$$\limsup_{k\to\infty} P_n^{(k)} \prod_{s=n_6}^{n-1} \left(1 + \frac{4A_sB_sP_{s+1}^{(k-1)}}{A_{s+1}^2}\right) < \infty.$$

6.3 MAIN RESULTS

Using the above lammas, we will be able to obtain the following sufficient conditions for all the solutions of (6.1.1) to be oscillatory.

Theorem 6.3.1 Assume that there exists a positive sequence $\{A_n\}$ such that

$$\sum_{s=n}^{\infty} \frac{A_sB_s}{A_{s+1}^2} = \infty \tag{6.3.18}$$

and

$$\sum_{s=n}^{\infty} Q_s = \infty, \tag{6.3.19}$$

where B_s and Q_s are defined in Lemma 6.2.4. Then (6.1.1) is oscillatory.

Proof Without loss of generality, suppose that $\{x_n\}$ is an eventually positive solution of (6.1.1). Then all conditions of Lemma 6.2.5 are satisfied. From Lemma 6.2.5 we have

$$\sum_{s=n}^{\infty} Q_s < \infty,$$

a contradiction to (6.3.19). Thus, this contradiction shows that (6.1.1) is oscillatory.

Theorem 6.3.2 Assume that there exists a positive sequence $\{A_n\}$ such that (6.3.18) holds, for Q_s defined in Lemma 6.2.4, $\{\sum_{i=1}^{n} Q_i\}$ is convergent to a positive number, and

$$\sum_{s=n}^{\infty} \frac{A_s B_s \left(P_{s+1}^{(0)}\right)^2}{A_{s+1}^2} \prod_{i=n}^{s-1} \left(1 + \frac{2A_i B_i P_{i+1}^{(0)}}{A_{i+1}^2}\right) = \infty. \tag{6.3.20}$$

Then (6.1.1) is oscillatory.

Proof Suppose that the conclusion does not hold and (6.1.1) is nonoscillatory. Without loss of generality, suppose that $\{x_n\}$ is an eventually positive solution of (6.1.1). The conditions of Lemma 6.2.6 are met. So (6.2.14) holds for $r_n \geq n_6$, a contradiction to (6.3.20). Therefore, the contradiction proves that the conclusion of the theorem holds.

Theorem 6.3.3 Assume that there exists a positive sequence $\{A_n\}$ such that $P_n^{(0)} \geq 0$ eventually, (6.3.18) holds, and for some positive integer k_0

$$P_n^{(k)} < \infty, \quad k = 0, 1, 2, \cdots, k_0 - 1, \tag{6.3.21}$$

and $P_n^{(k_0)}$ does not exists. Then (6.1.1) is oscillatory.

Proof The proof of Theorem 6.3.2 is still valid after the replacement of a contraction to (6.3.20) by a contradiction to (6.3.21).

Theorem 6.3.4 Assume that there exists a positive sequence $\{A_n\}$ such that $P_n^{(0)} \geq 0$ eventually and (6.3.18) holds. In addition, there exists a positive integer k_0 such that

$$\limsup_{n\to\infty} P_n^{(k_0)} \prod_{s=n_0}^{n-1} \left(1 + \frac{4A_sB_sP_{s+1}^{(k_0-1)}}{A_{s+1}^2}\right) = \infty. \tag{6.3.22}$$

Then (6.1.1) is oscillatory.

Proof The proof is the same as that of Theorem 6.3.2. The conclusion holds if the contradiction to (6.3.20) is replaced by a contradiction to (6.3.22).

Theorem 6.3.5 Assume that there exists a positive sequence $\{A_n\}$ such that $P_n^{(0)} \geq 0$ eventually and (6.3.18) holds. In addition, there is a positive integer k_0 such that

$$\lim_{n\to\infty} \sum_{s=n_0}^{n} \prod_{i=n_0}^{s-1} \left(1 + \frac{4A_iB_iP_{i+1}^{(k_0-1)}}{A_{i+1}^2}\right)^{-1} < \infty \tag{6.3.23}$$

and

$$\lim_{n\to\infty} \sum_{s=n_0}^{n} P_s^{(k_0)} = \infty. \tag{6.3.24}$$

Then (6.1.1) is oscillatory.

Proof Without loss of generality, suppose that $\{x_n\}$ is an eventually positive solution of (6.1.1). Then all conditions of Lemma 6.2.7 are satisfied. By the proof of Lemma 6.2.7, we have

$$\begin{aligned} P_n^{(k_0)} &\leq w_{n_0} \prod_{n_0}^{n-1} \left(1 + \frac{4A_s B_s P_{s+1}^{(k_0-1)}}{A_{s+1}^2}\right)^{-1}, \\ \sum_{s=n_0}^{n} P_s^{(k_0)} &\leq w_{n_0} \sum_{s=n_0}^{n} \prod_{i=n_0}^{s-1} \left(1 + \frac{4A_i B_i P_{i+1}^{(k_0-1)}}{A_{i+1}^2}\right)^{-1}, \end{aligned}$$

which contradict (6.3.23) and (6.3.24). The contradiction shows that the conclusion holds.

To further study the oscillation of (6.1.1), we construct the following sequence $\{\alpha_n^{(l)}\}$ for any sequence $\{P_n^{(k)}\}$. Set

$$\begin{aligned} \alpha_n^{(0)} &= P_n^{(k)}, \\ \alpha_n^{(1)} &= \sum_{s=n}^{\infty} \frac{A_s B_s (\alpha_{s+1}^{(0)})^2}{A_{s+1}^2} R_{s,n}^{(k)}, \\ \alpha_n^{(l+1)} &= \sum_{s=n}^{\infty} \frac{A_s B_s (\alpha_{s+1}^{(0)} + \alpha_{s+1}^{(l)})^2}{A_{s+1}^2} R_{s,n}^{(k)}. \end{aligned} \tag{6.3.25}$$

If every term in (6.3.25) is defined, then we have $\alpha_n^{(l+1)} \geq \alpha_n^{(l)}$ and

$$\lim_{n\to\infty} \alpha_n^{(l)} = 0.$$

Theorem 6.3.6 Assume that the assumptions of Lemma 6.2.6 are satisfied. If (6.1.1) has a non-oscillatory solution and all $\alpha_n^{(l)}$ in (6.3.25) are defined, then

$$\lim_{l\to\infty} \alpha_n^{(l)} = \alpha_n. \tag{6.3.26}$$

Proof Without loss of generality, suppose $\{x_n\}$ is an eventually positive solution of (6.1.1). By Lemma 6.2.6, there are an $n_2 \geq n_1$ and a sequence $\{w_n\}$ such that

$$w_n \geq P_n^{(k)} + \sum_{s=n}^{\infty} \frac{A_s B_s v_{s+1}^2}{A_{s+1}^2} R_{s,n}^{(k)} \quad \text{for} \quad n \geq n_2$$

and

$$P_n^{(k)} = \sum_{s=n}^{\infty} \frac{A_s B_s (P_{s+1}^{(k-1)})^2}{A_{s+1}^2} R_{s,n}^{(k)} < \infty.$$

Hence we have

$$\begin{aligned} w_n \geq P_n^{(k)} &= \alpha_n^{(0)}, \\ w_{n+1}^2 &\geq (\alpha_{n+1}^{(0)})^2, \\ \alpha_n^{(1)} &\leq \sum_{s=n}^{\infty} \frac{A_s B_s w_{s+1}^2}{A_{s+1}^2} R_{s,n}^{(k)}. \end{aligned}$$

Hence $w_n \geq \alpha_n^{(0)} + \alpha_n^{(1)}$. By mathematical induction, we have

$$\begin{aligned} w_n &\geq \alpha_n^{(0)} + \alpha_n^{(l)}, \\ \alpha_n^{(l)} &\leq \alpha_n^{(l+1)} \leq w_n. \end{aligned}$$

Therefore, all $\alpha_n^{(l)}$ in (6.3.25) are defined and (6.3.26) holds. From theorem 6.3.6, we can easily obtain the following theorems.

Theorem 6.3.7 Assume that there exists a positive sequence $\{A_n\}$ such that $P_n^{(0)} \geq 0$ eventually and (6.3.18) holds. In addition, there exists a nonnegative integer k_0 such that one of the following conditions is satisfied.

(i) There exists a nonnegative integer l_0 such that $\alpha_n^{(l)}$, $l = 0, 1, 2, \cdots, l_0 - 1$ are defined but $\alpha_n^{(l_0)}$ does not exist.

(ii) All $\alpha_n^{(l)}$ in (6.3.25) are defined. In addition, for every sufficiently large n there exists $n^* \geq n$ such that $\lim_{l\to\infty} \alpha_{n^*}^{(l)} = \infty$.

Then (6.1.1) is oscillatory.

Proof Suppose (6.1.1) is nonoscillatory and, without loss of generality, x_n is an eventually positive solution of (6.1.1). So all condition of Theorem 6.3.7 are satisfied then (6.3.26) holds, which contradicts (i) or (ii). The contradictions show that the conclusion hold.

Theorem 6.3.8 Under the assumptions of Theorem 6.3.7, and one of the following conditions is satisfied. For some k_0, there exists a nonnegative integer l_0 such that

$$\limsup_{n\to\infty} \alpha_n^{(l_0)} \prod_{s=n_0}^{n-1} \left(1 + \frac{4A_sB_sP_{s+1}^{(k_0)}}{A_{s+1}^2}\right) = \infty \tag{6.3.27}$$

or

$$\limsup_{n\to\infty} \alpha_n \prod_{s=n_0}^{n-1} \left(1 + \frac{4A_sB_sP_{s+1}^{(k_0)}}{A_{s+1}^2}\right) = \infty. \tag{6.3.28}$$

Then (6.1.1) is oscillatory.

Proof Similar to Lemma 6.2.7, we obtain

$$\alpha_n^{(l_0)} \prod_{s=n_0}^{n-1} \left(1 + \frac{4A_sB_sP_{s+1}^{(k_0)}}{A_{s+1}^2}\right) \leq w_{n_0} < \infty$$

and

$$\alpha_n \prod_{s=n_0}^{n-1} \left(1 + \frac{4A_sB_sP_{s+1}^{(k_0)}}{A_{s+1}^2}\right) \leq w_{n_0} < \infty,$$

which contradict (6.3.27) and (6.3.28). Therefore (6.1.1) is oscillatory.

Theorem 6.3.9 Assume that the assumptions of Theorem 6.3.7 are met. In addition,

$$\lim_{n\to\infty}\sum_{s=n_0}^{n}\prod_{i=n_0}^{s-1}\left(1+\frac{4A_iB_iP_{i+1}^{(k_0)}}{A_{i+1}^2}\right)^{-1}<\infty. \tag{6.3.29}$$

And there exists a l_0 such that

$$\lim_{n\to\infty}\sum_{s=n_0}^{n}\alpha_s^{(l_0)}=\infty. \tag{6.3.30}$$

Then (6.1.1) is oscillatory.

Proof Suppose that $\{x_n\}$ is an eventually positive solution of equation (6.1.1). Then all conditions of Lemma 6.2.7 are satisfied. Similar to the proof of Lemma 6.2.7, we have

$$\alpha_n^{(l_0)}\le w_{n_0}\prod_{i=n_0}^{n-1}\left(1+\frac{4A_iB_iP_{i+1}^{(k_0)}}{A_{i+1}^2}\right)^{-1}. \tag{6.3.31}$$

Summation both sides of (6.3.30) for s from n_0 to n, we have

$$\sum_{s=n_0}^{n}\alpha_s^{(l_0)}\le\sum_{s=n_0}^{n}w_{n_0}\prod_{i=n_0}^{s-1}\left(1+\frac{4A_iB_iP_{i+1}^{(k_0)}}{A_{i+1}^2}\right)^{-1},$$

which contradicts (6.3.29) and (6.3.30). Therefore, (6.1.1) is oscillatory.

Remark 6.3.1 Our conclusions also hold for the mixed difference equations with $\tau_n\le n$ and $n\le g_n\le 2^{m-2}n$, all results in [67] and [69] are included and extended.

Remark 6.3.2 By equivalence behavior of between in convergence of the series $\sum_{s=n}^{\infty}f(s-n_1)$, $\sum_{s=n}^{\infty}f(s)$ and $B_n=(r_n-n_1)^{(m-2)}/(a_{r_n}(m-2)!)$ in our theorems, they can be replaced by

$$\bar{B}_n = \frac{(g_n)^{(m-2)}}{a_{\tau_n}(m-2)!(2^{m-2})^{(m-2)}}.$$

6.4 EXAMPLES

Here, two examples will be given in this section to demonstrate the results obtained in last section.

Example 6.4.1 Consider the fourth order difference equation

$$\Delta^4(x_n + \frac{1}{2}x_{n-1}) + \frac{\lambda}{2(n-1)^3}x_{n-1} = 0. \tag{6.4.32}$$

Regarding (6.4.32) as (6.1.1), we have $m-1 = 4$, $a_n \equiv 1$, $q_n = \lambda/(2(n-1)^3)$, $\tau_n = g_n = n-1$ and $\lambda > 0$. Choose $A_n \equiv 1$, $Q_n = \lambda/(2(n-1)^3)$, $\bar{B}_n = (n-1)^2/24$. Then we have

$$\begin{aligned}
P_n^{(0)} &= \frac{\lambda}{2}\sum_{s=n}^{\infty}\frac{1}{(s-1)^3} = \frac{\lambda}{4(n-2)^2},\\
P_n^{(1)} &= \frac{\lambda}{4}\sum_{s=n}^{\infty}\frac{(s-1)^2}{24((s-1)^2)^2}\prod_{i=n}^{s-1}\left(1+\frac{\lambda(i-1)^2}{48(i-1)^2}\right)\\
&= \frac{\lambda}{4}\sum_{s=n}^{\infty}\frac{1}{24(s-1)^2}\left(1+\frac{\lambda}{48}\right)^{s-n}.
\end{aligned}$$

So $P_n^{(1)} \to \infty$ as $n \to \infty$. By Theorem 6.3.4, (6.4.32) is oscillatory.

Example 6.4.2 Consider the fourth order difference equation

$$\Delta^4\left(x_n + \frac{1}{2}x_{n-1}\right) + \frac{\lambda}{2(n-1)^4}x_{n-1} = 0. \tag{6.4.33}$$

Regarding (6.4.33) as (6.1.1), we have $m-1=4$, $a_n \equiv 1$, $q_n = \lambda/(2(n-1)^4)$, $\tau_n = g_n = n-1$. Choose $A_n \equiv 1$, $Q_n = \lambda/(2(n-1)^4)$, $\bar{B}_n = (n-1)^2/24$. Then we have

$$\begin{aligned}
P_n^{(0)} &= \frac{\lambda}{2}\sum_{s=n}^{\infty}\frac{1}{(s-1)^4} = \frac{\lambda}{6(n-2)^3}, \\
P_n^{(1)} &= \frac{\lambda}{36}\sum_{s=n}^{\infty}\frac{(s-1)^2}{24(s-1)^3(s-3)}\prod_{i=n}^{s-1}\left(1+\frac{\lambda}{72(i-3)}\right) \\
&\geq \frac{a\lceil 10\rceil}{(s-1)^{(3-\frac{\lambda}{72})}}, \\
P_n^{(1)}\prod_{s=n_1}^{n-1}\left(1+\frac{\lambda}{36(s-3)}\right) &\geq \frac{b}{(n-1)^{3-\frac{\lambda}{18}}},
\end{aligned}$$

where a and b are positive constants. When $\lambda > 54$,

$$P_n^{(1)}\prod_{s=n_1}^{n-1}\left(1+\frac{\lambda}{36(s-3)}\right) \to \infty \text{ as } n \to \infty.$$

By Theorem 6.3.3, when $\lambda > 54$ (6.4.33) is oscillatory.

6.5 CONCLUSION

The objective of this chapter is to investigate (6.1.1) with a nonlinear neutral term. To establish the oscillatory criteria, we have managed to construct a Riccati type inequality (6.2.6) in Lemma 6.2.4 based on the results about the features of solutions of (6.1.1). Even better, we have obtained two more Riccati type difference inequalities (6.2.9) and (6.2.14) in Lemma 6.2.5 and Lemma 6.2.6, respectively. Based on these inequalities, we have managed to develop five criteria for (6.1.1) to be oscillatory. The criteria have been presented in Theorem 6.3.1 -6.3.5. In addition, to deep the study of (6.1.1) we have constructed another sequence $\alpha_n^{(l)}$. From this sequence, we have managed to establish three more oscillatory criteria, referring to Theorem 6.3.8 and 6.3.9. Examples are given to demonstrate the results obtained in section 6.4.

The results in this chapter have completely covered the results in [67] and [69] as special cases. Since a_n is a sequence, (6.1.1) is more general than the equations in [61] and [54] except the forced equation in [54]. On the other hand, the results in this chapter are not ideal. There are new effort should be made in future to improve the results obtained here.

Chapter 7

CONCLUSION

The objective of this thesis was to investigate the oscillatory and asymptotic behaviour of the solutions of certain particular classes of equations when t tends to infinity. In this chapter, we briefly summarise what we have achieved so far and then point out directions for further investigation.

The nonautonomous first order differential system of the form

$$x_i'(t) = b_i(t)x_i(t)\left(1 - \sum_{j=1}^{n} a_{ij}(t)x_j(t)\right), \quad i \in N(1,n)$$

has been studied in chapter 2. Here the functions $a_{ij}(t)$ and $b_i(t)$ are continuous on R and bounded above and below by strictly positive numbers. By adapting the previous results about the canonical equations to the above system, a less restrictive improved condition just involving coefficients has been obtained for a particular type of solutions to be globally stable when t is sufficiently large.

Second order nonlinear neutral differential equations having the form

$$(a(t)(x(t) + \delta p(t)x(t-\tau))')' + f(t, x(t-\sigma)) - g(t, x(t-\rho)) = 0$$

were investigated in chapter 3. Here $\delta = +1$ or -1, $t \geq t_0$, $a(t)$ is a continuously differentiable function, $p(t)$ is a continuous bounded function with $a(t) > 0$, $p(t) \geq 0$, $f(t,u)$ and $g(t,v)$ are continuous functions, the constants τ, σ, $\rho \in [0,\infty)$. We are interested in nontrivial solutions on $[t_0, \infty)$. Under the assumption of existence and uniqueness, we have concentrated on the oscillation of solutions. Sufficient conditions in terms of $a(t)$, $q(t)$, $r(t)$, $p(t)$, σ and ρ have been achieved for the solutions to be bounded oscillatory, almost oscillatory, and bounded almost oscillatory. Our results are more general than and coincident with some of the previous studies though we did not find out the ideal conditions for the equations to be oscillatory.

Nonlinear neutral difference equations of the form

$$\Delta_\tau^m(x(t) - px(t-r)) + f(t, x(g(t))) = 0$$

were studied in chapter 4 and chapter 5. Here $m \geq 2$ is a natural number, $p \geq 0$, τ and r are positive constants, $g \in C^1([t_0, \infty), R^+)$, $g'(t) > 0$, and $f \in C([t_0, \infty) \times R, R)$. Under the assumption of existence and uniqueness, we have focused on the oscillatory behaviour of solutions when t tends to infinity. Chapter 4 focuses on the even order equations while chapter 5 devotes to the odd order equations. Different techniques are needed to obtain oscillatory criteria for the two kinds of equations. In chapter 4, by applying the available theory for discrete differences to the differences with continuous argument, oscillatory criteria involving the function $\bar{q}_i$ ($i = 2, 4, 2n$) defined by (4.3.5), (4.4.21) or (4.5.47) for the second order, fourth order, and higher even order equations have been obtained, respectively. For the odd order equations in chapter 5, sufficient conditions for the third order and higher odd order equations have been established for the bounded solutions to be oscillatory.

Even order difference equations with a nonlinear neutral term having the form

$$\Delta^{m-1}(a_n\Delta(x_n + \varphi(n, x_{\tau_n}))) + q_n f(x_{g_n}) = 0$$

were discussed in chapter 6. Here m is an even positive integer, $n \geq n_0$, $\{\tau_n\}$ and $\{g_n\}$ are nondecreasing sequences of nonnegative integers with $\tau_n \leq n$, $g_n \leq n$, $\lim_{n\to\infty} \tau_n = \infty$, $\lim_{n\to\infty} g_n = \infty$, $\{a_n\}$ and $\{q_n\}$ are sequences of real numbers with $a_n > 0$, $q_n \geq 0$ and $q_n \not\equiv 0$, and $f : R \to R$ and $\varphi : R^2 \to R$ are functions. We are interested in the oscillatory behaviour when n tends to infinity under the assumption of existence and uniqueness. By applying the previous results, Riccati transformation, and Riccati inequalities to the above equations, various criteria have been obtained for the solutions to be oscillatory.

Our work has been inspired by the previous studies as we mentioned in each part so that our results can be regarded as the generalizations of the available results.

There are some outstanding problems for the differential equations we have discussed in chapter 3 and new effort should be made to improve the existing theory. We list some of them below, which may require completely different techniques.

(i) **Higher order equations.** For the higher order equation

$$(a(t)(x(t) + \delta p(t)x(t-\tau))')^{(n)} + f(t, x(t-\sigma)) - g(t, x(t-\rho)) = 0,$$

where n is a positive integer, whether oscillatory criteria analogous to those given in chapter 3 could be obtained by the same method needs further investigation.

(ii) **Corresponding difference equations.** For the corresponding second order difference equations

$$\Delta_\theta(a(t)\Delta_\theta(x(t) + \delta p(t)x(t-\tau))) + f(t, x(t-\sigma)) - g(t, x(t-\rho)) = 0,$$

under the same assumptions as in chapter 3, oscillatory criteria similar to those given in chapter 3 might be achieved by the same method after the replacement of the derivatives by the differences. However, to carry out this work, new techniques might be needed.

(iii) **Weaker conditions for the oscillatory solutions.** The boundedness of the solution x plays an important role in the proofs of oscillation criteria and the proofs would fail without this condition. Sufficient conditions for all solutions to be oscillatory demand new methods.

In chapter 4 and 5, nonlinear neutral difference equation (4.1.1) was studied. There are some outstanding problems for further investigation. For instance, it is natural to explore the corresponding nonautonomous equations in the future. For the corresponding nonautonomous difference equation

$$\Delta_\tau^m(x(t) - p(t)x(t-r)) + f(t, x(g(t))) = 0,$$

oscillatory criteria for the even order equations will be expected by the same methods in chapter 4 under the assumptions $0 < p(t) < p_1 < 1$ or $p(t) > p_2 > 1$. Without such assumptions on $p(t)$, oscillatory criteria might be gained by using different methods. For the odd order nonautonomous equations, new techniques might be demanded to establish oscillatory criteria even under the assumptions $0 < p(t) < p_1 < 1$ or $p(t) > p_2 > 1$.

Some conditions in chapter 4 are not very general and new efforts are needed to make to generalize the obtained results. Some examples are given as follows.

(i) In Theorem 4.5.1 (P84), we did not have the ideal sufficient conditions for (4.1.1) to be oscillatory. In its proof, we suppose that $x(t) - px(t-r) > 0$

when t is sufficiently large and the proof fails without it. New techniques are needed to generalize the criteria for (4.1.1).

(ii) Oscillatory theorems except Theorem 4.5.1 have been gained in three separate cases according to $p > 1$, $p = 1$ and $0 < p < 1$. The value of p plays an important part in the proofs. Therefore, new methods and further investigation are needed to establish the more general criteria only with $p > 0$.

(iii) In theorems except Theorem 4.5.1, the sufficient conditions are involving a number k_0 for solutions to be oscillatory. In practice, it is not easy to find such a number k_0. New methods are needed to improve the gained criteria.

For the odd order equations in chapter 5, our results are not as good as those in chapter 4. Further investigation is needed to improve the existing results by new methods. First of all, various criteria in chapter 5 are for the bounded solutions to be oscillatory. New methods are demanded to establish oscillatory criteria for all solutions. Secondly, in the definitions of β_{m1} and β_{m2} (P133), a $T_m \geq t_0$ being large enough is required. In application, it is not easy to find such a T_m to fulfill the requirements. Thirdly, in theorem 5.4.1 (P133), conditions (5.4.48) and (5.4.49) are demanded to hold at the same time. If we rewrite condition (5.4.49), then we can see it involving β_{m1} as well. The requirements for β_{m1} in both inequalities are in the opposite directions. Thus the applications of this oscillatory criteria are very limited. Finally, when $p > 1$, we have obtained the oscillatory criteria just for certain particular classes of equations with $r =$

$k\tau$ and $r \geq t+3\tau-g(t)$. Less restrictive conditions are required for improvement.

For the oscillatory criteria given in chapter 6, in practice, it is difficult to use them since a sequence $\{A_n\}$ needs to be built at the first place. By adopting the methods in chapter 4 and 5, weaker conditions might be obtained.

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