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LONG-RANGE FORECASTING OF THE S&P 500 STOCK MARKET INDEX USING FRACTIONAL INTEGRATION TECHNIQUES

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ABSTRACT

In this paper we examine the stochastic behaviour of the S&P 5000 stock market index by means of fractional integration techniques. Specifically, we use a parametric method to test I(d) statistical models. Model selection criteria based on out-of-sample forecasting performance suggest that that best model specification is an I(d) process with d higher than 1, implying that the series under examination is nonstationary and non-mean-reverting.

Keywords: Fractional integration; Long memory; Long-range prediction.

JEL Classification: C22, G14

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1. Introduction

Modelling the stochastic behaviour of macroeconomic time series is still controversial. Since it became apparent that deterministic approaches based on linear (or quadratic) functions of time are inappropriate in many cases, stochastic models based on first (or second) differences of the data have been widely used, especially after the seminal paper of Nelson and Plosser (1982), who, following on from Box and Jenkins (1970), showed that many macroeconomic series can be specified in terms of unit roots. They used tests of Fuller (1976) and Dickey and Fuller (1979), and could not reject the hypothesis of a unit root for most of the US series examined. Subsequently, a battery of unit root tests have been developed (e.g., Phillips and Perron, 1988, Kwiatkowski et al., 1992, etc.), providing mixed empirical evidence For example, Perron (1989, 1993) argued that the 1929 stock market crash and the 1973 oil price shock were behind the non-rejections of the unit root null hypothesis, and that once these were taken into account deterministic models could be shown to be preferable. Other authors, such as Christiano, 1992, and Zivot and Andrews, 1992, who estimated models with endogenously determined breaks, reached the opposite conclusion. In the last twenty years, there has been a growing literature that studies the sources of nonstationarity in macroeconomic time series in terms of fractionally differenced processes. Examples are Diebold and Rudebusch (1989), Baillie and Bollerslev (1994), Gil-Alana and Robinson (1997), etc. In this paper we follow this type of approach, using a version of the tests of Robinson (1994) that is suitable to test fractional hypotheses, and using model selection criteria based on out-of-sample forecasting performance show that the S&P 500 stock market index can be specified as an I(d) process with d higher than 1, implying that this series is nonstationary and non-mean-reverting.

The outline of the paper is as follows: Section 2 briefly describes the version of the tests of Robinson (1994) used in this paper. Section 3 applies these tests to the US stock market. In Section 4 we examine different models in order to select the best specification on the basis of various forecasting performance criteria. Section 5 contains some concluding comments.

2. The tests of Robinson and the I(d) hypothesis

For the purpose of the present paper, we define an I(0) process $\{u_t, t = 0, \pm 1, \ldots\}$ as a covariance stationary process, with spectral density function that is positive and finite at the zero frequency. In this context, we say that $\{x_t, t = 0, \pm 1, \ldots\}$ is I(d) if

$$(1 - L)^d \, x_t \;=\; u_t, \qquad t \;=\; 1, 2, \ldots, \tag{1}$$

$$x_t \;=\; 0, \qquad t \;\leq\; 0, \tag{2}$$

where the polynomial in (1) can be expressed in terms of its Binomial expansion such that

$$(1 \;-\; L)^d \;=\; \sum_{j=0}^{\infty} \binom{d}{j} (-1)^j \, L^j \;=\; 1 \;-\; d\,L \;+\; \frac{d\,(d-1)}{2} L^2 \;-\; \ldots.$$

for all real d. If $d > 0$ in (1), x_t is said to be a long-memory process, so called because of the strong association between observations widely separated in time. This type of processes was initially introduced by Granger (1980, 1981), Granger and Joyeux (1980) and Hosking (1981) (though earlier work by Adenstedt, 1974, and Taqqu, 1975, shows an awareness of its representation), and was theoretically justified in terms of aggregation of ARMA series by Robinson (1978), and Granger (1980). Cioczek-Georges and Mandelbrot (1995), Taqqu et al. (1997), Chambers (1998) and Lippi and Zaffaroni (1999) also use aggregation to motivate long-memory processes, while Parke (1999) uses a closely related discrete time error duration model. The fractional differencing parameter d plays a crucial role from both theoretical and empirical viewpoints. If $d < 0.5$, x_t is covariance stationary and mean-reverting, with the effect of the shocks dying away in the long run. If $d \in [0.5, 1)$, x_t is no longer covariance stationary but is still mean reverting, while $d \geq 1$ implies nonstationarity and non-mean-reversion.

Robinson (1994) proposed a Lagrange Multiplier (LM) test of the null hypothesis:

$$H_o : \; d \; = \; d_o . \qquad (3)$$

in a model given by

$$y_t \; = \; \beta' z_t \; + \; x_t , \qquad t \; = \; 1, 2, \ldots, \qquad (4)$$

and (1), for any real value d_o, where y_t is the time series we observe; $\beta = (\beta_1, \ldots, \beta_k)^T$ is a (kx1) vector of unknown parameters; z_t is a (kx1) vector of deterministic regressors that may include, for example, an intercept, (e.g. $z_t \equiv 1$), or an intercept and a linear time trend (when $z_t = (1,t)^T$). Specifically, the test statistic is given by:

$$\hat{r} \; = \; \frac{T^{1/2}}{\hat{\sigma}^2} \hat{A}^{-1/2} \, \hat{a} \qquad (5)$$

where T is the sample size and

$$\hat{a} \; = \; \frac{-2\pi}{T} \sum_{j=1}^{T-1} \psi(\lambda_j) \, g(\lambda_j; \hat{\tau})^{-1} \, I(\lambda_j); \qquad \hat{\sigma}^2 \; = \; \frac{2\pi}{T} \sum_{j=1}^{T-1} g(\lambda_j; \hat{\tau})^{-1} \, I(\lambda_j);$$

$$\hat{A} \; = \; \frac{2}{T} \left(\sum_{j=1}^{T-1} \psi(\lambda_j)^2 \; - \sum_{j=1}^{T-1} \psi(\lambda_j) \, \hat{\varepsilon}(\lambda_j)' \times \left(\sum_{j=1}^{T-1} \hat{\varepsilon}(\lambda_j) \, \hat{\varepsilon}(\lambda_j)' \right)^{-1} \times \sum_{j=1}^{T-1} \hat{\varepsilon}(\lambda_j) \psi(\lambda_j) \right)$$

$$\psi(\lambda_j) = \; \log \left| 2 \sin \frac{\lambda_j}{2} \right|; \qquad \hat{\varepsilon}(\lambda_j) \; = \; \frac{\partial}{\partial \tau} \log g(\lambda_j; \hat{\tau}); \qquad \lambda_j \; = \; \frac{2\pi \; j}{T}; \qquad \hat{\tau} \; = \; \arg\min \sigma^2(\tau).$$

$I(\lambda_j)$ is the periodogram of u_t evaluated under the null, i.e.,

$$\hat{u}_t \; = \; (1 \; - \; L)^{d_o} \, y_t \; - \; \hat{\beta}' w_t ;$$

$$\hat{\beta} \; = \; \left(\sum_{t=1}^{T} w_t \, w_t' \right)^{-1} \sum_{t=1}^{T} w_t \, (1 \; - \; L)^{d_o} \, y_t ; \qquad w_t \; = \; (1 \; - \; L)^{d_o} \, z_t ,$$

and the function g above is a known function coming from the spectral density function of u_t,

$$f(\lambda; \sigma^2; \tau) \; = \; \frac{\sigma^2}{2\pi} g(\lambda; \tau), \qquad -\pi \; < \; \lambda \; \leq \; \pi.$$

Note that these tests are parametric and therefore require specific modelling assumptions about the short-memory specification of u_t. Thus, if u_t is white noise, $g \equiv 1$, and if u_t is an AR process of the form $\phi(L)u_t = \varepsilon_t$, $g = |\phi(e^{i\lambda})|^{-2}$, with $\sigma^2 = V(\varepsilon_t)$, so that the AR coefficients are a function of τ.

Based on the null hypothesis, given by H_o in (3), Robinson (1994) established that under certain regularity conditions:

$$\hat{r} \quad \rightarrow_d \quad N(0,1) \quad as \quad T \rightarrow \infty, \tag{6}$$

and also the Pitman efficiency theory of the tests against local departures from the null. Therefore, we are in a classical large sample-testing situation: an approximate one-sided $100\alpha\%$ level test of H_o (3) against the alternative: H_a: $d > d_o$ ($d < d_o$) will be given by the rule: "Reject H_o if $\hat{r} > z_\alpha$ ($\hat{r} < - z_\alpha$)", where the probability that a standard normal variate exceeds z_α is α. This version of the tests of Robinson (1994) was used in empirical applications in Gil-Alana and Robinson (1997) and Gil-Alana (2000); other versions of these tests, based on seasonal (quarterly and monthly) and cyclical data can be found in Gil-Alana and Robinson (2001) and Gil-Alana (1999, 2001) respectively.

3. Modelling the US stock market

In this section we analyse annual data for a US stock market index, namely the S&P 500 Composite, for the time period 1870 – 2001, discarding the last 10 observations for forecasting purposes.

INSERT FIGURE 1 ABOUT HERE

Figure 1 contains plots of the original series and its first differences, along with the corresponding correlograms and periodograms. Visual inspection suggests that the series is upward trending, increasing very slowly during the first half of the sample, and very rapidly afterwards. The nonstationary nature of this series is also indicated by its correlogram (with values decreasing very slowly), and periodogram, (with a large peak around the smallest frequency). The first-differenced data exhibit a large degree of oscillation in the second part of the sample, and though the series may now be stationary, there are still significant values at the correlogram even at some lags far away from zero, as well as another peak in the periodogram at the zero frequency, which both suggest that some type of long-memory behaviour is still present in the data.

Denoting the time series by y_t, we employ throughout the model given by (1) and (4), with $z_t = (1, t, S_t)^T$, $t \geq 1$, $z_t = (0, 0, 0)^T$ otherwise, and where S_t is a dummy variables, $S_t = t\, I(t > 1929)$, corresponding to the 1929 stock market crash.[1] Thus, under the null hypothesis H_o (3):

$$y_t = \beta_0 + \beta_1 t + \beta_2 S_t + x_t, \qquad t = 1, 2, \ldots \qquad (7)$$

$$(1 - L)^{d_o} x_t = u_t, \qquad t = 1, 2, \ldots . \qquad (8)$$

where we treat separately the cases $\beta_0 = \beta_1 = \beta_2 = 0$ a priori; β_0 unknown and $\beta_1 = \beta_2 = 0$ a priori; β_0, β_1 unknown and $\beta_2 = 0$; and β_0, β_1 and β_2 unknown, i.e., we consider respectively the cases of no regressors in the undifferenced regression (7), an intercept, an intercept and a linear time trend, and an intercept, a linear trend and the dummy variable, and report the test statistic, not merely for the case of $d_o = 1$ (a unit root), but also for $d_o = 0.50, (0.10), 1.50$, thereby including a test for stationarity ($d_o = 0.5$) as well as other fractionally integrated possibilities.

INSERT TABLE 1 ABOUT HERE

The test statistic reported in Table 1 is the one-sided one corresponding to $\hat{r}$ in (5), such that significantly positive values are consistent with orders of integration higher than d_o, whereas significantly negative ones are consistent with alternatives of the form: $d < d_o$. It can be noted in the upper part of Table 1, where u_t is assumed to be white noise, that the value of the test statistic monotonically decreases with d_o. This is to be expected in view of the fact that it is a one-sided statistic. Thus, for example, if H_o (3) is rejected with $d_o = 1$ against alternatives of the form: H_a: $d > 1$, an even more significant result in this direction should be expected when $d_o = 0.75$ or $d_o = 0.50$ are tested. It can be seen that H_o (3) cannot be rejected when $d_o = 1.25$, being rejected for all the remaining values of d_o, including the unit root case. This result is obtained regardless of the deterministic components included in the regression model (7). However, these results might reflect to a large extent the unaccounted for I(0) autocorrelation in u_t; therefore, we also present the results for the case of AR(1) and AR(2) disturbances.[2] In both cases we do not find a monotonic decrease in the value of $\hat{r}$ with respect to d_o if d_o is smaller than 1. This may be due to model misspecification, as argued, for example, in Gil-Alana and Robinson (1997). Note that in the event of misspecification both numerator and denominator of $\hat{r}$ are frequently inflated to varying degrees, $\hat{r}$ being affected in a complicated way. Computing $\hat{r}$ for a range of values of d_o is therefore useful to reveal possible misspecification (although monotonicity does not necessarily represent evidence of correct specification). However, the lack of monotonicity may also be due to the fact that the AR coefficients are Yule-Walker estimates and therefore, although they are smaller than one in absolute value, they can be arbitrarily close to 1. Then they may be capturing the order of integration through, for example, a coefficient of 0.99 in the case of AR(1) disturbances. In fact, we always find monotonicity for values of d_o equal to or higher than 1. Starting with the case of AR(1) disturbances, it can be seen that H_o (3) cannot be rejected when $d_o = 1$ or 1.25,

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[1] Alternative dummy variables for the break were also considered, but the coefficients were insignificantly different from zero in all cases.

while these hypotheses are rejected in favour of higher orders of integration in the case of AR(2) u_t, the null then not being rejected if $d_o = 1.75$ or 2.

INSERT TABLE 2 ABOUT HERE

In order to determine more precisely the order of integration of this series, we perform again Robinson's (1994) tests, but this time for a range of values of $d_o = 0, (0.01), 2$. Table 2 reports, in column 3, the interval of values of d_o where H_o (3) cannot be rejected at the 95% significance level, while column 4 reports the values of d_o (d_o^*) which produce the lowest $|\hat{r}|$ across d_o. The results are shown for each type of I(0) disturbances u_t in (1) and for each type of regressors in z_t in (4). It can be seen that the values are very similar for the different types of regressors used in z_t; however, they are very different for different types of I(0) disturbances. Specifically, if u_t is white noise, the intervals range between 1.13 and 1.43, and d_o^* appears to be 1.25 or 1.26. If u_t is AR(1), the intervals are wider and include the unit root null hypothesis; however, the values of d_o which produce the lowest statistics are higher, ranging now between 1.26 and 1.30. Finally, if the disturbances are AR(2), the orders of integration are much higher, d_o^* lying between 1.92 and 1.95.

INSERT TABLE 3 ABOUT HERE

Table 3 reports the values of the estimated coefficients of each of the twelve selected models according to the results in Table 2. That is, for each type of disturbances u_t (white noise, AR(1) and AR(2)) and each type of regressors, we select the model with the lowest statistic in absolute value corresponding to d_o. The intuition behind this is that, for each u_t and

[2] Other MA and ARMA specifications were also examined, but they are not reported here in view of their poor forecasting performance.

z_t, the model with the lowest $|\hat{r}|$ will be the one with the closest residuals to a white noise process. Although not reported in the table, all the coefficients, except the AR parameters in the case of AR(1) $u_{t,}$ were found to be significantly different from zero. In the following section we compare the forecasting performance of the selected models.

4. Forecasting the S&P 500 stock market index

Long-range forecasts are often a more useful model evaluation criterion than goodness of fit. In this section we use three model selection criteria base on the accuracy of out-of-sample forecasts. The three model selection criteria are the following:

1. *Mean Absolute Percentage Error of Forecasts*:

$$\text{MAPE} = \text{Mean}_{\,t \in T} \frac{|\hat{x}_t - x_t|}{x_t} x100,$$

where T is an index set of time periods t over which the forecasts are made.

2. *Mean Percentage Error of Forecasts*:

$$\text{MPE} = \text{Mean}_{\,t \in T} \frac{(\hat{x}_t - x_t)}{x_t} x100.$$

3. *Root Mean Square Error of Forecasts*:

$$\text{RMSE} = \{\text{Mean}_{\,t \in T} (\hat{x}_t - x_t)^2 \, x100\}^{1/2}.$$

For a discussion of these selection criteria, see Makridakis et al., (1982).

We compare the performance of the selected models from the previous section by their forecasting properties, using the MAPE, MPE and RMSE statistics. The index set T is 1, 3, 5, 7 and 10 forecasts. The parameters of each model are re-estimated at the beginning of each forecast period using all of the observations up to the forecast origin. Table 4 gives the MAPE, MPE and RMSE statistics for each of the twelve models selected in Table 3.

INSERT TABLE 4 ABOUT HERE

Starting with the 1-year out-of-sample forecast, one can see that the best results are produced by specification A1, which is an I(1.26) model with no regressors and white noise u_t. However, when increasing the forecast horizon, other models seem to be preferable. Specifically, when looking at the 3-year forecasts, model B1 (I(1.30) with no regressors and AR(1) u_t) is the most adequate specification according to the MAPE and RMSE statistics, while model B2 (I(1.26) with an intercept and AR(1) u_t) is preferred on the basis of the MPE. The same is true for the other forecast horizons, B1 being the best model according to the MAPE and RMSE statistics, and B2 on the basis of the MPE. Overall, models A1, B1 and B2 appear to be the best specifications in terms of forecasting properties (the order of integration being 1.26 for models A1 and B2, and 1.30 for model B1).

5. Conclusions

In this paper we have examined the stochastic behaviour of the S&P 500 stock market index by means of fractional integration techniques. Specifically, we have used a parametric procedure due to Robinson (1994) which is suitable to test I(d) statistical models. These tests have standard null and local limit distributions and are easy to implement. We find that the unit root hypothesis can be rejected in favour of higher orders of integration. In particular, if the underlying I(0) disturbances are white noise or AR(1), the order of integration is around 1.30, being much higher if u_t is AR(2). We also examined the forecasting properties of the selected models using various measures of forecasting accuracy. We find that a I(1.26) model is the best specification based on for the 1-year forecasts, while a similar process with AR(1) disturbances (with or without intercept) appears to be preferable for longer forecast horizons.

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FIGURE 1

US stock market and first differences, with corresponding correlograms and periodograms

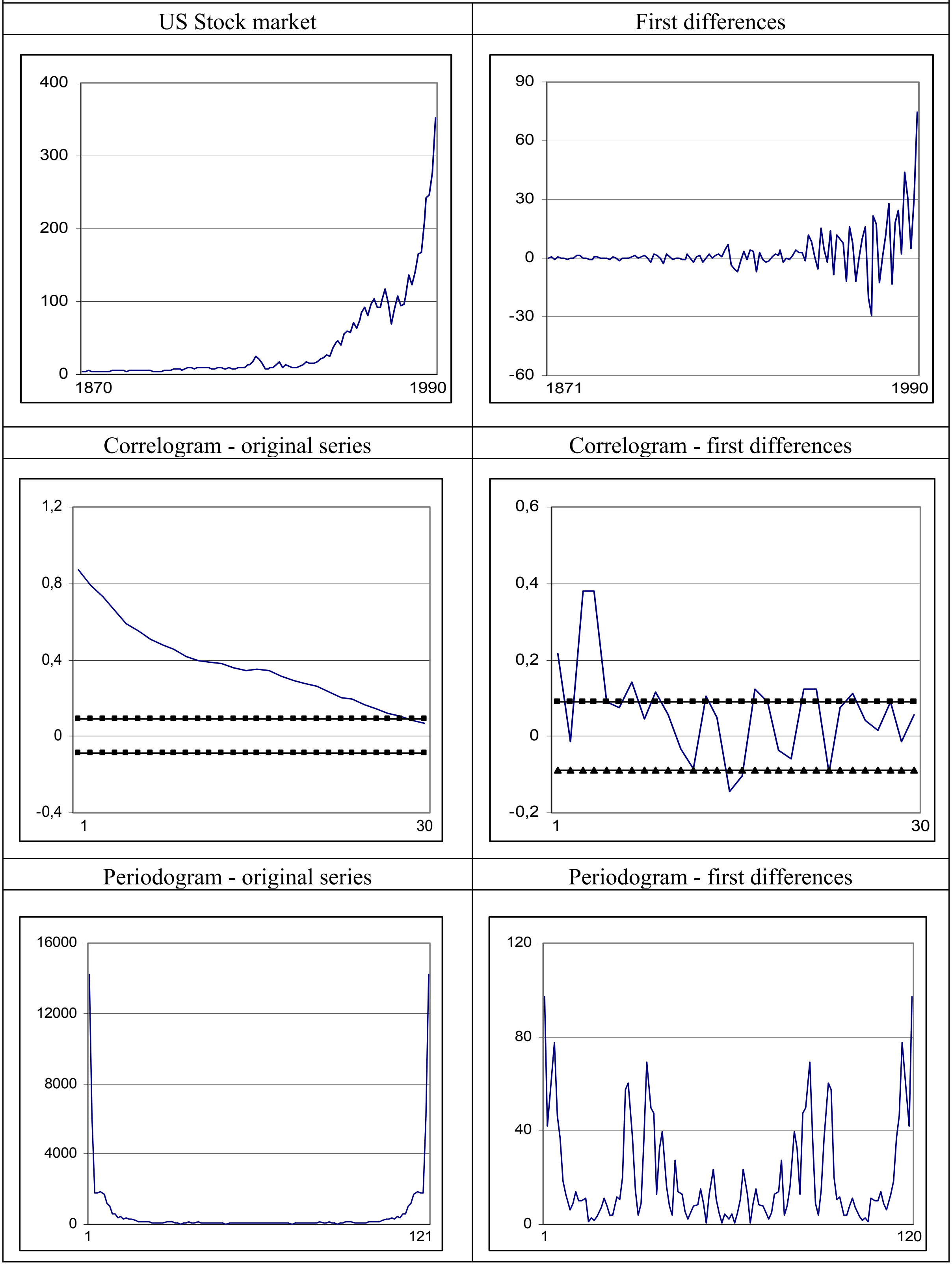

TABLE 1										
Testing the order of integration of the S&P 500 with the tests of Robinson (1994)										
u_t	z_t	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
White noise	---	18.42	16.44	13.36	8.95	3.99	**0.11**	-2.12	-3.25	-3.84
	1	18.42	16.70	14.28	9.42	3.91	**-0.04**	-2.24	-3.33	-3.89
	(1, t)’	20.96	19.46	15.39	9.56	3.91	**-0.03**	-2.14	-3.25	-3.87
	(1, t, S)’	12.95	12.29	10.34	7.15	3.34	**-0.02**	-2.20	-3.32	-3.90
AR (1)	---	-2.67	-3.88	-6.54	-8.34	**0.86**	**0.37**	-1.71	-3.21	-4.01
	1	-2.67	-3.82	-5.89	-6.53	**0.67**	**0.07**	-1.92	-3.34	-4.11
	(1, t)’	-3.26	-3.13	-4.57	-5.53	**0.67**	**0.13**	-1.70	-3.17	-4.06
	(1, t, S)’	-0.02	0.69	-0.56	1.91	**0.93**	**-0.01**	**-1.26**	-2.43	-3.45
AR (2)	---	-3.68	-4.68	-8.81	-12.54	7.63	5.53	4.16	**1.57**	**-0.45**
	1	-3.68	-5.15	-9.73	-15.22	9.93	5.07	3.87	**1.39**	**-0.55**
	(1, t)’	-11.81	-7.09	-10.09	-7.78	6.41	4.61	3.36	**1.28**	**-0.57**
	(1, t, S)’	-6.01	-4.37	-3.67	-1.62	5.01	4.09	3.29	**1.28**	**-0.42**

TABLE 2			
Confidence intervals and values of d_o which produce the lowest statistic in absolute value			
u_t	z_t	Confidence interval	d_o^*
White noise	---	(1.14 - 1.43)	1.26
	1	(1.13 - 1.41)	1.25
	(1, t)’	(1.13 - 1.42)	1.25
	(1, t, S)’	(1.14 - 1.42)	1.26
AR (1)	---	(0.89 - 1.48)	1.30
	1	(0.91 - 1.46)	1.26
	(1, t)’	(0.88 - 1.49)	1.27
	(1, t, S)’	(0.92 - 1.47)	1.26
AR (2)	---	(1.70 - 2.32)	1.93
	1	(1.68 - 2.33)	1.92
	(1, t)’	(1.73 - 2.33)	1.95
	(1, t, S)’	(1.72 - 2.32)	1.94

TABLE 3								
Selected models according to Table 2								
u_t	z_t	Model	d_o^*	β_0	β_1	β_2	α_1	α_2
White noise	---	A1	1.26	---	---	---	---	---
	1	A2	1.25	3.62	---	---	---	---
	(1, t)'	A3	1.25	4.28	-1.11	---	---	---
	(1, t, S)'	A4	1.26	4.96	-2.27	2.42	---	---
AR (1)	---	B1	1.30	---	---	---	-0.04	---
	1	B2	1.26	3.72	---	---	-0.01	---
	(1, t)'	B3	1.27	4.41	-1.25	---	-0.02	---
	(1, t, S)'	B4	1.26	4.96	-0.24	2.42	-0.02	---
AR (2)	---	C1	1.93	---	---	---	-0.45	-0.57
	1	C2	1.92	4.16	---	---	-0.44	-0.57
	(1, t)'	C3	1.95	4.83	-1.33	---	-0.45	-0.58
	(1, t, S)'	C4	1.94	4.84	-1.35	-11.11	-0.46	-0.58

TABLE 4													
MAPE, MPE and RMSE for forecasts of the S&P 500													
F.Horizon	Crit.	A1	A2	A3	A4	B1	B2	B3	B4	C1	C2	C3	C4
1 YEAR	MAPE	**12.76**	13.39	33.95	59.27	12.99	13.47	43.51	26.97	15.06	15.95	42.25	45.60
	MPE	**12.76**	13.39	-33.95	-59.27	12.99	13.47	-43.51	26.97	15.06	15.95	-42.25	-45.60
	RMSE	**483.18**	510.69	836.90	1228.9	493.18	514.27	1001.2	1219.6	585.82	626.81	980.79	1034.2
3 YEAR	MAPE	12.05	12.25	34.74	45.43	**12.03**	12.25	38.92	19.03	18.20	18.27	18.56	18.22
	MPE	-0.80	-3.42	-34.74	-45.43	-0.50	**0.06**	-38.92	18.ñ30	41.2	4.08	4.30	4.17
	RMSE	461.91	468.05	1392.2	1808.2	**459.92**	467.98	155.52	698.93	717.81	720.50	732.31	718.85
5 YEAR	MAPE	8.91	9.03	31.63	41.64	**8.86**	9.02	35.53	18.47	15.70	15.74	16.04	15.78
	MPE	0.09	0.83	-31.63	-41.64	**0.02**	0.04	-35.53	18.03	3.71	3.66	3.81	3.74
	RMSE	354.64	358.26	1335.9	1751.5	**351.80**	357.84	1497.6	745.09	651.49	653.28	666.13	655.27
7 YEAR	MAPE	11.28	11.24	33.13	42.03	**11.20**	11.22	36.59	14.97	15.26	15.28	15.47	15.32
	MPE	-4.84	-4.19	-33.13	-42.03	-4.56	**-4.10**	-36.59	11.10	-5.39	-5.42	-5.29	-5.37
	RMSE	573.43	569.19	1653.5	2068.3	563.64	**567.83**	1815.0	642.80	721.81	722.50	729.58	724.73
10 YEAR	MAPE	11.64	11.59	30.35	37.64	**11.46**	11.54	33.15	12.38	13.31	13.36	13.39	13.34
	MPE	-7.13	-6.65	-30.35	-37.64	-6.82	**-6.56**	-33.15	5.86	-8.60	-8.66	-8.46	-8.57
	RMSE	825.09	818.45	1981.8	2397.0	**607.25**	613.58	2138.3	656.78	778.74	783.52	775.66	778.95