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Abstract

This paper proposes a very general time series framework to capture the long-run behaviour of financial series. The suggested model includes linear and non-linear time trends, and stationary and nonstationary processes based on integer and/or fractional degrees of differentiation. Moreover, the spectrum is allowed to contain more than a single pole or singularity, occurring at zero and non-zero (cyclical) frequencies. This model is used to analyse four annual time series with a long span, namely dividends, earnings, interest rates and long-term government bond yields. The results indicate that the four series exhibit fractional integration with one or two poles in the spectrum. A forecasting comparison shows that a model with a non-linear trend along with fractional integration outperforms alternative models over long horizons.

JEL Classification: C22, G1
Keywords: Fractional Integration; Financial Time Series Data; Trends; Cycles

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1. Introduction

The statistical modelling of financial time series data such as asset prices plays an important role in portfolio management. Despite the extensive theoretical and empirical literature of the last thirty years, there is still no consensus on what might be the most adequate model specification for many financial series. For instance, whether asset returns of asset prices are predictable or not is still controversial. While the efficiency market hypothesis suggests that they should follow a random walk (see Fama, 1970; Summers, 1986), other authors have found evidence of mean reversion in their behaviour (see, e.g., Poterba and Summers, 1988 and Fama and French, 1988). The standard econometric approach to settle this issue empirically relies on establishing the (integer) order of integration of the series by carrying out nonstationary unit root tests. More recently, however, the possibility of fractional orders of integration has also been taken into account, with a slow rate of decay. Long memory specifications for realised volatility have been shown frequently to forecast very accurately (see, e.g., Li, 2002 or Martens and Zein, 2004), though in some cases the sum of short-memory (ARMA) specifications appears to forecast as accurately as a long-memory (ARFIMA) model (see, e.g., Pong et al, 2004). Using a fractional model, Caporale and Gil-Alana (2002) find that there is no permanent component in US stock market returns, since the series examined is close to being I(0). Caporale and Gil-Alana (2007) decompose the stochastic process followed by US stock prices into a long-run component described by the fractional differencing parameter (d) and a short-run (ARMA) structure. Empirical support for non-linear asset pricing models (such as the one by Dittmar, 2002) has also been found (see, inter alia, Hossein and Sonnie, 2006).

The present paper takes into account these various strands of the literature on modelling asset prices and proposes a very general time series framework to capture the long-run behaviour of financial data. The suggested model includes linear and non-linear time trends, and stationary and nonstationary processes based on integer and/or fractional degrees of differentiation. Moreover, the spectrum is allowed to contain more than a single pole or singularity, occurring at the zero frequency but also at non-zero (cyclical) frequencies. This model is used to analyse four annual time series with a long span, namely dividends, earnings, interest rates and long-term government bond yields, obtainable from Robert Shiller's homepage (http://www.econ.yale.edu/~shiller/). We are able to show that the selected specifications (with linear and non-linear trends, fractional integration and cyclical fractional integration) have better forecasting properties than alternative models used in the literature to analyse these data.

The structure of the paper is as follows. Section 2 describes the model and the statistical approach employed in the paper. Section 3 presents the empirical analysis, considering first the case with linear trends and then the non-linear one. Section 4 assesses the forecasting performance of the selected models, whilst Section 5 offers some concluding comments.

2. The model

Let us assume that $\{y_t, t = 1, 2, .., T\}$ is the time series we observe. We consider the following model:

$$y_t = f(t) + x_t, \qquad (1)$$

$$(1 - L)^{d_1} (1 - 2\cos w_r L + L^2)^{d_2} x_t = u_t, \qquad (2)$$

$$\phi_p(L)\, u_t = \theta_q(L)\, \varepsilon_t, \qquad (3)$$

where f is a function of time that may be of a linear/non-linear nature; L is the lag operator (i.e., $L^s x_t = x_{t-s}$); d_1 is the order of integration corresponding to the long-run or zero frequency; $w_r = 2\pi/r$, with r representing the number of periods per cycle; d_2 is the order of integration with respect to the non-zero (cyclical) frequency, and u_t is assumed to be an I(0) process, that may follow a general stationary ARMA(p, q) process, where p and q indicate the orders of the autoregressive and moving average components respectively. Note that d_1 and d_2 are allowed be any real values and thus we do not restrict ourselves to integer degrees of differentiation.

The set-up described in (1) – (3) is fairly general, including the standard ARMA model (with or without trends), if $d_1 = d_2 = 0$; the ARIMA case, if d_1 is integer and $d_2 = 0$; the ARFIMA specification, if d_1 is fractional and $d_2 = 0$, along with other more complex representations.

We now focus on equation (2), and first assume that $d_2 = 0$. Then, the spectral density function of x_t is given by:

$$f(\lambda;\ \tau) \;=\; \frac{\sigma^2}{2\pi}\left|\frac{\theta(e^{i\lambda})}{\phi(e^{i\lambda})}\right|^2 \left|1 - e^{i\lambda}\right|^{-2d_1}, \qquad (4)$$

and it contains a pole or singularity at the long-run or zero frequency. Further, note that the polynomial $(1 - L)^{d_1}$ can be expressed in terms of its Binomial expansion, such that, for all real d_1,

$$(1 - L)^{d_1} \;=\; \sum_{j=0}^{\infty}\binom{d_1}{j}(-1)^j L^j \;=\; 1 - \; d_1 L \;+ \frac{d_1(d_1 - 1)}{2}L^2 \;-\; \dots, \qquad (5)$$

implying that the higher is the value of d_1, the higher is the degree of association between observations distant in time. Thus, the parameter d_1 plays a crucial role in determining the degree of persistence of the series. Although the time series literature for very long only considered the cases of integer values of d_1 (stationarity if $d_1 = 0$, and nonstationarity with $d_1 = 1$), more recently fractional values of d_1 have been widely employed when modelling macroeconomic and financial data.[1] Suppose now that $d_1 = 0$ in (2). Then, the process x_t has a spectral density function given by:

$$f(\lambda;\ \tau) \;=\; \frac{\sigma^2}{2\pi}\left|\frac{\theta(e^{i\lambda})}{\phi(e^{i\lambda})}\right|^2 \left|2\left(\cos(\lambda)-\cos(w_r)\right)\right|^{-2d_2} \qquad (6)$$

and is characterised by having a pole at a non-zero frequency. Moreover, the polynomial $(1-2\cos w_r L + L^2)^{d_2}$ can be expressed as a Gegenbauer polynomial, such that, defining $\mu = \cos w_r$, for all $d_2 \neq 0$,

$$(1-2\mu L + L^2)^{-d_2} \;=\; \sum_{j=0}^{\infty} C_{j,d_2}(\mu) L^j, \qquad (7)$$

where $C_{j,d_2}(\mu)$ are orthogonal Gegenbauer polynomial coefficients recursively defined as:

$$C_{0,d_2}(\mu) = 1,$$

$$C_{1,d_2}(\mu) = 2\mu d_2,$$

$$C_{j,d_2}(\mu) = 2\mu\left(\frac{d_2-1}{j}+1\right)C_{j-1,d_2}(\mu) - \left(2\frac{d_2-1}{j}+1\right)C_{j-2,d_2}(\mu), \quad j = 2, 3, ...$$

(see, inter alia, Magnus et al., 1966, or Rainville, 1960, for further details on Gegenbauer polynomials). Gray et al. (1989, 1994) showed that this process is stationary if $d_2 < 0.5$ for $|\mu = \cos w_r| < 1$ and if $d_2 < 0.25$ for $|\mu| = 1$. If $d_2 = 1$, the process is said to contain a unit root cycle (Ahtola and Tiao, 1987; Bierens, 2001); other applications using fractional values of d_2 can be found in Gil-Alana (2001), Ahn, Knopova and Leonenko (2004), Soares and Souza (2006), etc.[2]

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[1] Empirical applications using fractional values of d_1 include Diebold and Rudebusch (1989), Sowell (1992), Gil-Alana and Robinson (1997), etc.

[2] Models with multiple cyclical structures (k-factor Gegenbauer processes) with multiple poles in the spectrum have been examined, among others, by Ferrara and Guegan (2001), Sadek and Khotanzad (2004) and Gil-Alana (2007a).

In the empirical analysis carried out in the following section we use a method developed by Robinson (1994) that enables us to test a model such as (1) - (3). It is a testing procedure based on the Lagrange Multiplier (LM) principle that uses the Whittle function in the frequency domain. It can be used to test the null hypothesis:

$$H_o: \; d \;\equiv\; (d_1, d_2)^T \;=\; (d_{1o}, d_{2o})^T \;\equiv\; d_o, \qquad (8)$$

in (1) – (3) where d_{10} and d_{20} may be any real values, thus encompassing stationary and nonstationary hypotheses. The specific form of the test statistic (denoted by $\hat{R}$) is presented in the appendix. Under very general regularity conditions, Robinson (1994) showed that for this particular version of his tests,

$$\hat{R} \;\to_d\; \chi_2^2, \qquad as \quad T \to \infty. \qquad (9)$$

Thus, unlike in other procedures, we are in a classical large-sample testing situation. A test of (8) will reject H_o against the alternative H_a: $d \neq d_o$ if $\hat{R} > \chi^2_{2,\alpha}$, where Prob (χ_2^2 $> \chi^2_{2,\alpha}$) = α. Furthermore the test is efficient in the Pitman sense against local departures from the null, that is, if the test is implemented against local departures of the form: H_a: $d = d_o + \delta T^{-1/2}$, for $\delta \neq 0$, the limit distribution is a $\chi_2^2(v)$, with a non-centrality parameter v that is optimal under Gaussianity of u_t.

There exist other procedures for estimating and testing the fractionally differenced parameters, some of them also based on the likelihood function. As in other standard large-sample testing situations, Wald and LR test statistics against fractional alternatives will have the same null and local limit theory as the LM tests of Robinson (1994). Ooms (1997) proposed tests based on seasonal fractional models: they are Wald tests, and thus require efficient estimates of the fractional differencing parameters. He used a modified periodogram regression estimation procedure due to Hassler (1994). Also, Hosoya (1997) established the limit theory for long-memory processes with the singularities not restricted at the zero frequency, and proposed a set of quasi log-likelihood statistics to be applied to raw time series.[3] Unlike these previous methods, the tests of Robinson (1994) do not require estimation of the long-memory parameters since the differenced series have short memory under the null. Similarly, with respect to the zero frequency, Sowell (1992) employed a Wald testing procedure, though again this approach requires an efficient estimate of d_1, and while such estimates can be obtained, the LM procedure of Robinson (1994) seems computationally more attractive.[4]

3. Empirical Analysis

The data analysed in this paper have been obtained from Robert Shiller's homepage (http://www.econ.yale.edu/~shiller/). They are described in chapter 26 of Shiller's (1989) book on "Market Volatility", where further details can be found, and are constantly updated and revised. Specifically, they are the following series: dividends (an index), earnings (also an index), one-year interest rate (this series is the result of converting the January and July rates into an annual yield), long-term government bond yield (this is the yield on the 10-year Treasury bonds after 1953). The sample period goes from 1871 to 2006 for the first two series, 2004 for the third one, and 2007 for the fourth one. In all cases, we leave out the last ten observations to use them for the forecasting comparison carried out in Section 4.

[INSERT FIGURE 1 ABOUT HERE]

[3] Models of this form (with a pole at the non-zero frequency) were also considered, among others, by Giraitis, Hidalgo and Robinson (2001), Hidalgo and Soulier (2004) and Hidalgo (2005). These authors assume that the pole in the spectrum is unknown and suggest various parametric and semiparametric methods to estimate the fractional parameter (d_2), along with the frequency of the pole in the spectrum (see also Arteche and Robinson, 2000, and Arteche, 2002).

[4] See also Tanaka (1999) for a time domain representation of Robinson's (1994) tests.

Figure 1 contains plots of the four series. As can be seen, both dividends and earnings appear to be quite stable for about a century, and then increase sharply in the last few decades of the sample. Interest rates and government bond yields fluctuate a lot more throughout the sample, but also seem to increase towards the end of the sample, before a significant fall.

In the following two subsections, we examine first a model with linear trends, and then one with non-linear structures. In both cases we allow for long-range dependence at the zero and non-zero (cyclical) frequencies.

3.1 The case of linear trends

First we consider the case of linear trends, and assume that the model contains two cyclical structures, one for the long-term behaviour of the series and the other for the cyclical component. We allow both components to display long-memory behaviour, and test the null hypothesis in (8),

$$H_o: \; d \;\equiv\; (d_1, d_2)^T \;=\; (d_{1o}, d_{2o})^T \;\equiv\; d_o,$$

in the following model,

$$y_t \;=\; \alpha \;+\; \beta t \;+\; x_t, \qquad (10)$$

$$(1 - L)^{d_1} (1 - 2\cos w_r L \;+\; L^2)^{d_2}\, x_t \;=\; u_t, \qquad (11)$$

under the assumption that the disturbance term u_t is white noise, AR(1) and AR(2) respectively. Higher AR orders were also employed and the results do not substantially alter the conclusions based on these two first orders. In all cases, we test H_o for (d_{1o}, d_{2o})-values from -1 to 3 with 0.01 increments, and r = 2, 3, …, T/2,[5] choosing as

[5] Note that in case of r = 1, the polynomial $(1-2\cos w_r L + L^2)^{d_2}$ becomes $(1-L)^{2d_2}$, implying the existence of a pole at the long run or zero frequency.

estimates of d_1 and d_2 the values of d_{1o} and d_{2o} that produce the lowest statistics. These values should be an approximation to the maximum likelihood estimates, noting that Robinson's (1994) method is based on the Whittle function, which is an approximation to the likelihood function.[6] Given that some of the coefficients in (10) were not significant, we also carried out the computations in a model with only an intercept (i.e. $\beta = 0$ a priori) and with no regressors at all ($\alpha = \beta = 0$ a priori). The results for the four series are displayed in Tables 1A – 4A.

Also, noting that in some cases the order of integration for the cyclical part (d_2) was not statistically significantly different from zero, we also perform the analysis with a single fractional differencing parameter, i.e., employing

$$(1 - L)^{d1} x_t \;=\; u_t, \qquad (12)$$

rather than (11). The results for this case are displayed in Tables 1B – 4B. We describe first of all the results for the trend-cyclical case.

[INSERT TABLES 1A – 4A ABOUT HERE]

The first remark to make is that the parameter r (indicating the number of time periods per cycle) is constrained between 2 and 15 in all cases, being around 8 in the majority of cases. This is consistent with the empirical findings in the business cycle literature (Canova, 1998; Burnside, 1998; King and Rebelo, 1999; etc.) according to which cycles have a periodicity between five and ten years. It is also noteworthy that the order of integration at the long-run or zero frequency (i.e., d_1) is substantially higher than its corresponding value at the cyclical frequency (d_2), especially for earnings and

[6] Several Monte Carlo experiments based on this approach were conducted by Caporale and Gil-Alana (2006), and Gil-Alana (2007). It is shown in these papers that this method correctly determines the orders of integration at the two frequencies for samples of similar size to those employed in this article.

interest rates. For these two series the unit root null cannot be rejected at the long-run frequency (d_1 = 1), while d_2 is found to be strictly below 1 in all cases. In Tables 1A – 4A we report in bold the cases where the null hypothesis of white noise errors cannot be rejected at conventional statistical levels. There are two such cases for dividends, a single one for earnings, six for interest rates and five for government bond yields. Among these selected models we choose the best specification on the basis of LR tests and other likelihood criteria. The selected model for each series is as follows. For dividends,

$$y_t \;=\; -6.891 \;+\; x_t; \quad (1-L)^{1.48}\,(1-2\cos w_{10}L+L^2)^{0.52}\,x_t \;=\; \varepsilon_t. \qquad \text{(1A)}$$

For earnings,

$$\begin{aligned} y_t \;&=\; 10.348 \;+\; x_t; \quad (1-L)^{1.19}\,(1-2\cos w_7L+L^2)^{0.39}\,x_t \;=\; u_t; \\ u_t \;&=\; -0.284\,u_{t-1} \;-\; 0.503u_{t-2} \;+\; \varepsilon_t. \end{aligned} \qquad \text{(2A)}$$

For interest rates,

$$(1-L)^{0.76}\,(1-2\cos w_8L+L^2)^{0.18}\,x_t \;=\; u_t; \quad u_t \;=\; -0.217\,u_{t-1} - 0.211u_{t-2} \;+\; \varepsilon_t. \qquad \text{(3A)}$$

Finally, for the government bond yields,

$$\begin{aligned} y_t \;&=\; -23.090 + \; x_t; \quad (1-L)^{0.96}\,(1-2\cos w_8L+L^2)^{0.28}\,x_t \;=\; u_t. \\ u_t \;&=\; -0.182u_{t-1} \;+\; \varepsilon_t.1 \end{aligned} \qquad \text{(4A)}$$

Considering the confidence bands for the orders of integration of these selected models we see that for dividends and earnings (Tables 1A and 2A), d_1 is strictly above 1 while d_2 is constrained between 0 and 1. On the other hand, for interest rates and bond yields (Tables 3A and 4A) we cannot reject the null hypotheses of d_1 = 1 and d_2 = 0.

[INSERT TABLES 1B – 4B ABOUT HERE]

Next we examine the case of a single pole at the long-run or zero frequency (Tables 1B – 4B). Here we notice that for dividends the order of integration is much higher than 1, being even above 2 in three cases. For earnings and government bond yields, some values are below 1 while others are above 1. Finally, for interest rates, the estimated order of integration is below 1 in all cases and in six out of nine cases the unit root null is rejected in favour of smaller orders of integration. Using this specification, the selected model for dividends is the following:

$$(1-L)^{1.57} x_t = u_t; \qquad u_t = 0.677 u_{t-1} - 0.211 u_{t-2} + \varepsilon_t, \tag{1B}$$

In case of earnings, the chosen specification is:

$$y_t = 0.575 - 0.434 t + x_t; \quad (1-L)^{1.38} x_t = u_t; \quad u_t = 0.023 u_{t-1} - 0.445 u_{t-2} + \varepsilon_t. \tag{2B}$$

For interest rates,

$$(1-L)^{0.64} x_t = u_t; \qquad u_t = 0.244 u_{t-1} - 0.083 u_{t-2} + \varepsilon_t, \tag{3B}$$

and finally, for government bond yields

$$(1-L)^{1.00} x_t = u_t; \qquad u_t = -0.159 u_{t-1} + \varepsilon_t. \tag{4B}$$

According to these models the four series are nonstationary, and the unit root hypothesis is rejected in favour of higher orders of integration in case of the dividend series.

3.2 The case of non-linear trends

Next we allow for possible non-linearities, and assume that the four series exhibit a single break.[7] Figure 1 suggests that there might be a break around 1973, the time of the first oil price crisis. We experimented with a change in the level, in the slope and in both of them, and came to the conclusion that a level change was the most plausible one for government bond yields, while for the remaining three series we allowed for a

[7] Multiple breaks could also be considered. However, we believe that the series examined in this paper can be adequately described including a single structural break. Note that allowing for multiple breaks would result in short subsamples and innaccurate estimates of the coefficients.

change in both level and slope after the break. Specifically, we consider a model of the form:

$$y_t = \alpha_1 I(t < 1973) + \alpha_2 I(t \geq 1973) + x_t, \tag{13}$$

for government bond yields, and

$$y_t = \alpha_1 I(t < 1973) + \alpha_2 I(t \geq 1973) + \beta t I(t \geq 1973) + x_t, \tag{14}$$

for the remaining three series, allowing two fractional structures as in (11) (in Tables 1C – 4C) and with a single fractional differencing polynomial at the zero frequency as in (12) (in Tables 1D – 4D):[8]

[INSERT TABLES 1C – 4D ABOUT HERE]

Starting again with the case of two poles in the spectrum (i.e., using equation 11), the selected models for dividends (with $T^* = 1973$) in Tables 1C – 4C were:

$$\begin{aligned} &y_t = -6.006 I(t<T^*) - 20.204 I(t \geq T^*) + 0.452 t I(t \geq T^*) + x_t; \\ &(1-L)^{1.56} (1 - 2\cos w_{11} L + L^2)^{0.53} x_t = u_t; \quad u_t = 0.308 u_{t-1} + \varepsilon_t. \end{aligned} \tag{1C}$$

For earnings,

$$\begin{aligned} &y_t = -121.003 I(t<T^*) - 104.15 I(t \geq T^*) - 1.039 t I(t \geq T^*) + x_t; \\ &(1-L)^{1.32} (1 - 2\cos w_7 L + L^2)^{0.52} x_t = u_t; \quad u_t = -0.117 u_{t-1} - 0.260 u_{t-2} + \varepsilon_t. \end{aligned} \tag{2C}$$

For interest rates,

$$\begin{aligned} &y_t = -75.817 I(t<T^*) - 63.114 I(t \geq T^*) - 0.636 t I(t \geq T^*) + x_t; \\ &(1-L)^{0.38} (1 - 2\cos w_{15} L + L^2)^{1.23} x_t = u_t; \quad u_t = -0.237 u_{t-1} + \varepsilon_t. \end{aligned} \tag{3C}$$

and finally, for government bond yields,

[8] We also considered other break dates for the four series, and the coefficients in (13) and (14) were insignificant in the majority of the cases.

$$y_t = -25.644 I(t<T^*) - 28.013 I(t \geq T^*) + x_t;$$
$$(1-L)^{0.95} (1 - 2\cos w_3 L + L^2)^{0.14} x_t = u_t; \quad u_t = -0.088 u_{t-1} + \varepsilon_t. \tag{4C}$$

It can be seen that the number of periods per cycle varies substantially depending on the series. Specifically, it is 11 for dividends, 7 for earnings, 15 for interest rates, and 3 for government bond yields. The order of integration at the long-run or zero frequency is higher than the cyclical one for dividends, earnings and government bond yields, while the opposite holds for interest rates. For the first two series, d_1 is significantly higher than 1, while d_2 is in the interval (0, 1) for the three latter series. Surprisingly, for the interest rate d_1 is strictly smaller than 1 while d_2 is significantly above 1.

[INSERT TABLES 1D – 4D ABOUT HERE]

When we assume that there is a single pole occurring at the long-run or zero frequency, (Tables 1D – 4D), the deterministic terms are found to be mostly insignificant in the case of dividends and earnings, while they are all significant in the case of interest rates and government bond yields. The order of integration appears to be highly sensitive to the chosen specification for the disturbance term, especially for dividends and earnings. For instance, for dividends, d_1 is above 2 in case of a white noise u_t; it is 1.24 (and the unit root null cannot be rejected) if u_t is AR(1), and it is strictly smaller than 1 with an AR(2) u_t. The selected models in this case are the following: for dividends,

$$y_t = -1.744 I(t<T^*) - 2.878 I(t \geq T^*) + 0.485 t I(t \geq T^*) + x_t;$$
$$(1-L)^{0.63} x_t = u_t; \quad u_t = 1.372 u_{t-1} - 0.745 u_{t-2} + \varepsilon_t. \tag{1D}$$

For earnings,

$$y_t = 0.231 I(t < T^*) + 0.269 I(t \geq T^*) + 1.641 t I(t \geq T^*) + x_t; \qquad (2D)$$
$$(1-L)^{1.17} x_t = u_t; \quad u_t = 0.162 u_{t-1} - 0.401 u_{t-2} + \varepsilon_t.$$

For interest rates,

$$y_t = 2.751 I(t < T^*) + 7.105 I(t \geq T^*) - 0.258 t I(t \geq T^*) + x_t; \qquad (3D)$$
$$(1-L)^{0.45} x_t = u_t; \quad u_t = 0.333 u_{t-1} + 0.007 u_{t-2} + \varepsilon_t.$$

and for government bond yields,

$$y_t = 5.279 I(t < T^*) + 5.761 I(t \geq T^*) + x_t; \qquad (4D)$$
$$(1-L)^{0.99} x_t = u_t; \quad u_t = -0.183 u_{t-1} - 0.007 u_{t-2} + \varepsilon_t.$$

According to these specifications, dividends, government bond yields and earnings are nonstationary variables, while interest rates is the only one with a fractional differencing parameter in the stationary region (d < 0.5). However, the confidence intervals indicate that dividends is the only series for which the unit root null (d = 1) is rejected.

4. Forecasting performance

This section examines the forecasting performance of the models previously selected. For each of the series we consider the four model specifications given by equations (1A) – (4D). First, we compute the k (=1, 2, …, 10)-ahead prediction errors of each model, obtained by expanding the fractional polynomials in (5) and (7). Tables 5 – 8 report the Root Mean Squared Errors (RMSE) of each specification for each series. It can be seen that for dividends (Table 5) model (1B) appears to be the best based on the 1-period ahead prediction. However, for longer horizons, model (1D) (i.e., a single fractional polynomial at the long-run frequency along with a non-linear trend) seems to perform best in all cases. A similar conclusion is reached for earnings (Table 6) and government bond yields (Table 8). Thus, based on the 1-period ahead predictions, the model with a single fractional differencing polynomial (equations (2B) and (4B)) performs best, while the one with a non-linear trend is preferred in the remaining cases (equations (2D) and (4D)). Finally, for interest rates (Table 7) models (3A) and (3D) are the most adequate ones: based on the 1, 2, 5, 6 and 7-period ahead predictions, the model with a non-linear trend and a single fractional polynomial seems to be the most adequate. However, when the forecasting horizon is 3, 4, 8, 9 and 10 periods ahead, the specification with two fractional polynomials is the preferred one.

[INSERT TABLES 5 – 8 ABOUT HERE]

Overall, for dividends, earnings and government bond yields, the model with a single fractional polynomial at the zero frequency predicts better 1-period ahead; however, for longer horizons, a model with a non-linear trend (and also a fractional process at the zero frequency) outperforms the rival models. For interest rates, the results are slightly more ambiguous: the model with two polynomials (at the zero and the cyclical frequency) seems to be the most adequate one in some cases, but a non-linear model with a single polynomial at the zero frequency appears to be preferable in other cases.

The results presented so far as based on the RMSE. However, this criterion along with other methods such as the Mean Absolute Prediction Error (MAPE), Mean Squared Error (MSE), Mean Absolute Deviation (MAD), etc., is a purely descriptive device.[9] Several statistical tests for comparing different forecasting models are now available. One of them, widely employed in the time series literature, is the asymptotic

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[9] The accuracy of different forecasting methods is a topic of continuing interest and research (see, e.g., Makridakis et al., 1998 and Makridakis and Hibon, 2000, for a review of the forecasting accuracy of competing forecasting models).

test for a zero expected loss differential due to Diebold and Mariano (1995).[10] Harvey, Leybourne and Newbold (1997) note that the Diebold-Mariano test statistic could be seriously over-sized as the prediction horizon increases, and therefore provide a modified Diebold-Mariano test statistic given by:

$$M - DM = DM\sqrt{\frac{n + 1 - 2h + h(h-1)/n}{n}},$$

where DM is the original Diebold-Mariano statistic, h is the prediction horizon and n is the time span for the predictions. Harvey et al. (1997) and Clark and McCracken (2001) show that this modified test statistic performs better than the DM test statistic (though still poorly in finite samples), and also that the power of the test is improved when p-values are computed with a Student t-distribution.

Using the M-DM test statistic, we further evaluate the relative forecast performance of the different models by making pairwise comparisons. We consider 2, 4, 6 and 8-period ahead forecasts on a 10-period horizon. The results are displayed in Tables 9 – 12, and are consistent with the previous ones.

[INSERT TABLES 9 – 12 ABOUT HERE]

In particular, models (1D), (2D), (3D) and (4D) are preferred in most cases, especially based on the 2- and 4-period ahead prediction horizons. Only for interest rates does model (3A) outperform (3D) in some cases.

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[10] An alternative approach is the bootstrap-based test of Ashley (1998), though his method is computationally more intensive.

5. Conclusions

In this paper we have introduced a new time series approach to modelling long-run trends and cycles in financial time series data. The proposed model is general enough to include linear and non-linear trends along with fractional integration at zero and non-zero (cyclical) frequencies. It is based on the testing procedure developed by Robinson (1994) for stationary and nonstationary hypotheses. We have used our framework to investigate the behaviour of four financial time series already examined in many earlier studies. Specifically, we have used the annual dataset including dividends, earnings, interest rates and government bond yields, which was constructed (and is constantly updated) by Robert Shiller.

The results can be summarised as follows. It appears that the four series of interest can be characterised in terms of long-memory processes with two poles in the spectrum, one corresponding to the long-run or zero frequency, and the other one to the cyclical component. The latter exhibits a periodicity ranging between 3 and 15 years depending on the series and the model considered. In general, the order of integration is higher at the zero frequency, implying that the degree of persistence is higher in this component. When non-linear trends are incorporated, the models outperform the linear ones in terms of their forecasting accuracy, especially over longer horizons.

This paper can be extended in several directions. First, multiple cyclical structures of the form advocated by Ferrara and Guegan (2001) and others can be considered. In fact, the interaction between cyclical (fractional) processes may produce autocorrelations decaying in a very complicated way that has not been much investigated yet. Other more complex non-linear structures (like the Threshold AutoRegressive, TAR, Momentum Threshold AutoRegressive, M-TAR or Smooth Transition Autoregressive, STAR-form (see, e.g. Enders and Granger, 1998; Enders and

Siklos, 2001; Skalin and Teräsvirta, 2002) can also be included in the regression model (1). Finally, the date(s) of the structural break(s) can be endogenously determined in the context of the general model described by equations (1) – (3). Future research will address these issues.

Appendix

The test statistic proposed by Robinson (1994) for testing H_o (8) in (1) - (3) is given by:

$$\hat{R} = \frac{T}{\hat{\sigma}^4} \hat{a}' \hat{A}^{-1} \hat{a},$$

where T is the sample size, and

$$\hat{a} = \frac{-2\pi}{T} \sum_j^* \psi(\lambda_j) g(\lambda_j;\hat{\tau})^{-1} I(\lambda_j); \qquad \hat{\sigma}^2 = \sigma^2(\hat{\tau}) = \frac{2\pi}{T} \sum_{j=1}^{T-1} g(\lambda_j;\hat{\tau})^{-1} I(\lambda_j),$$

$$\hat{A} = \frac{2}{T} \left(\sum_j^* \psi(\lambda_j)\psi(\lambda_j)' - \sum_j^* \psi(\lambda_j)\hat{\varepsilon}(\lambda_j)' \left(\sum_j^* \hat{\varepsilon}(\lambda_j)\hat{\varepsilon}(\lambda_j)' \right)^{-1} \sum_j^* \hat{\varepsilon}(\lambda_j)\psi(\lambda_j)' \right)$$

$$\psi(\lambda_j)' = \lfloor \psi_1(\lambda_j), \psi_2(\lambda_j) \rfloor; \qquad \hat{\varepsilon}(\lambda_j) = \frac{\partial}{\partial \tau} \log g(\lambda_j;\hat{\tau}); \qquad \psi_1(\lambda_j) = \log \left| 2 \sin \frac{\lambda_j}{2} \right|;$$

$\psi_2(\lambda_j) = \log \left| 2(\cos\lambda_j - \cos w_r) \right|$, with $\lambda_j = 2\pi j/T$, and the summation in * is over all frequencies which are bounded in the spectrum. $I(\lambda_j)$ is the periodogram of $\hat{u}_t = (1-L)^{d_{1o}} (1 - 2\cos w_r L + L^2)^{d_{2o}} y_t - \hat{\beta}' \bar{z}_t$, with

$$\hat{\beta} = \left(\sum_{t=1}^T \bar{z}_t \bar{z}_t' \right)^{-1} \sum_{t=1}^T \bar{z}_t (1-L)^{d_{1o}} (1 - 2\cos w_r L + L^2)^{d_{2o}} y_t;$$

$\bar{z}_t = (1-L)^{d_{1o}} (1 - 2\cos w_r L + L^2)^{d_{2o}} z_t$, evaluated at $\lambda_j = 2\pi j/T$ and $\hat{\tau} = \arg\min_{\tau \in T^*} \sigma^2(\tau)$, with T^* as a suitable subset of the R^q Euclidean space. Finally, the function g above is a known function coming from the spectral density of u_t:

$$f(\lambda;\tau) = \frac{\sigma^2}{2\pi} g(\lambda;\tau), \qquad -\pi < \lambda \leq \pi.$$

Note that these tests are purely parametric and, therefore, they require specific modelling assumptions about the short-memory specification of u_t. Thus, if u_t is white noise, $g \equiv 1$, and if u_t is an AR process of the form $\phi(L)u_t = \varepsilon_t$, $g = |\phi(e^{i\lambda})|^{-2}$, with $\sigma^2 = V(\varepsilon_t)$, so that the AR coefficients are a function of τ.

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Figure 1: Time series plots

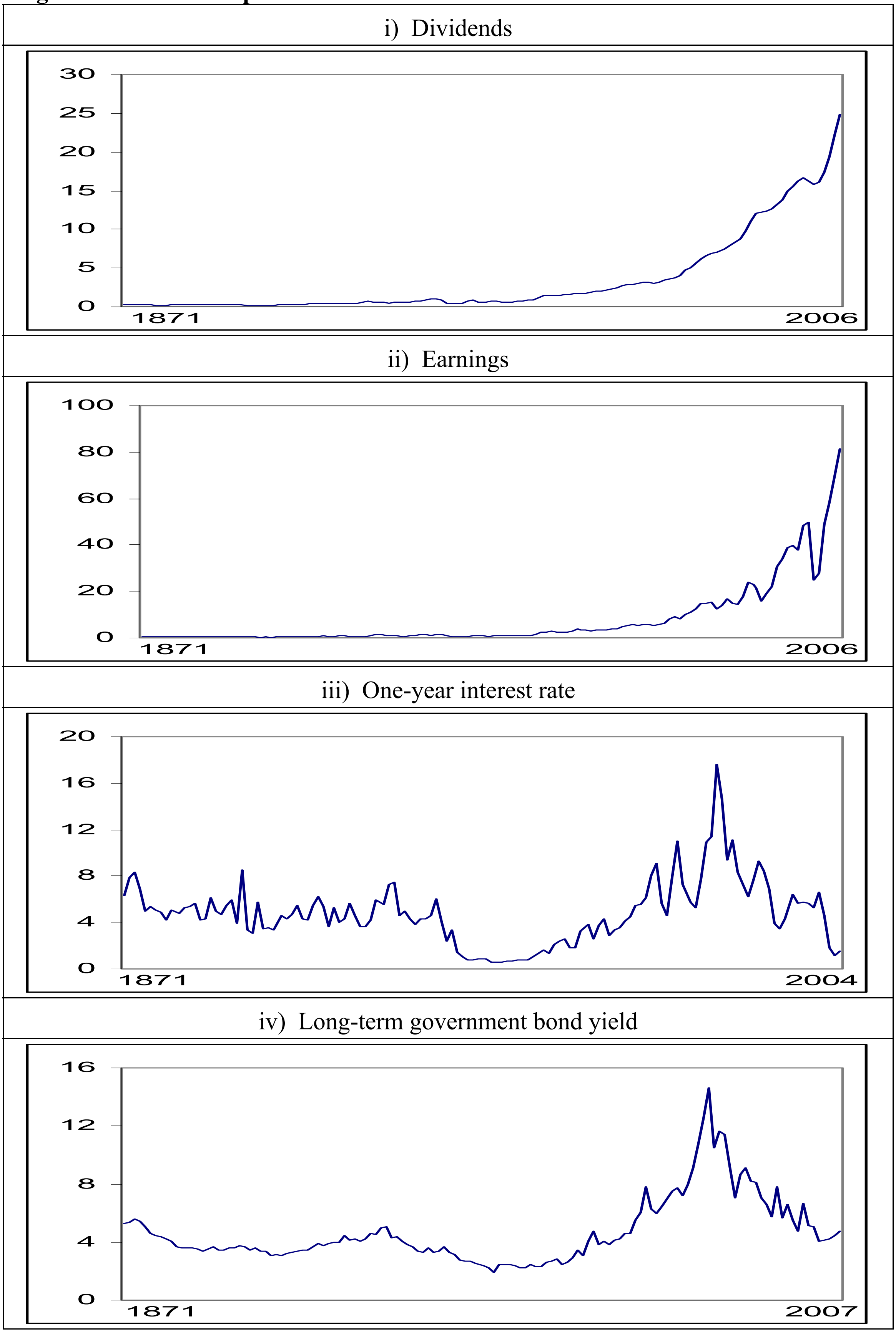

Table 1A: Coefficient estimates with two fractional structures. Series: Dividends

	α	β	d_1	r	d_2	ρ_1	ρ_2
AR(2) with time trend	-462.89 (-61.42)	-0.075 (-0.80)	0.02 [-0.09, 0.14]	8	0.60 [0.47, 1.54]	0.804	-0.420
AR(2) with an intercept	-2.257 (-7.59)	---	1.16 [0.76, 1.54]	15	0.99 [0.33, 1.37]	-0.422	-0.212
AR(2) with no regressors	**---**	**---**	**1.51 [0.84, 1.97]**	**9**	**0.53 [0.03, 0.87]**	**-0.207**	**-0.099**
AR(1) with time trend	-2.023 (-2.23)	-24.501 (-98.28)	0.96 [0.91, 1.04]	13	0.98 [0.93, 1.42]	0.412	---
AR(1) with an intercept	42.283 (11.73)	---	1.47 [1.26, 1.87]	3	0.80 [0.64, 0.96]	-0.091	---
AR(1) with no regressors	---	---	1.36 [1.19, 1.53]	5	0.06 [0.00, 0.14]	0.691	---
White noise with trend	-431.63 (-33.46)	-1.710 (-10.66)	0.13 [0.06, 0.17]	15	1.01 [0.97, 1.07]	---	---
White noise + intercept*	**-6.891 (-9.32)**	**---**	**1.48 [1.31, 1.71]**	**10**	**0.52 [0.30, 0.69]**	**---**	**---**
White noise with no reg.	---	---	1.17 [1.06, 1.35]	12	0.58 [0.47, 0.71]	---	---

In bold, the models for which the null hypothesis of white noise errors cannot be rejected. "*" indicates the best model specification using LR tests and other likelihood criteria.

Table 2A: Coefficient estimates with two fractional structures. Series: Earnings

	α	β	d_1	r	d_2	ρ_1	ρ_2
AR(2) with time trend	-54.618 (-3.45)	-61.003	0.93 [0.86, 1.19]	8	0.73 [0.50, 0.91]	0.174	0.053
AR(2) with an intercept*	**10.348 (0.89)**	**---**	**1.19 [1.04, 1.81]**	**7**	**0.39 [0.09, 0.51]**	**-0.284**	**-0.503**
AR(2) with no regressors	---	---	1.06 [0.60, 1.77]	8	0.10 [-0.37, 0.69]	0.130	-0.371
AR(1) with time trend	-10.996 (-0.98)	-87.225	1.09 [0.89, 1.18]	11	0.74 [0.30, 0.94]	-0.165	---
AR(1) with an intercept	733.08 (11.79)	---	1.20 [0.98, 1.47]	2	0.92 [0.68, 1.07]	-0.069	---
AR(1) with no regressors	---	---	0.79 [0.28, 1.23]	3	0.16 [-0.02, 0.31]	0.560	---
White noise with trend	-51.997 (-3.50)	-63.653	0.95 [0.89, 1.15]	10	0.72 [0.56, 0.97]	---	---
White noise with intercept	-131.86 (-7.26)	---	1.08 [1.01, 1.16]	8	0.42 [0.22, 0.66]	---	---
White noise with no reg.	---	---	0.93 [0.76, 1.21]	6	0.24 [0.01, 0.54]	---	---

In bold, the models for which the null hypothesis of white noise errors cannot be rejected. "*" indicates the best model specification using LR tests and other likelihood criteria.

Table 3A: Coefficient estimates with two fractional structures. Series: Interest rates

	α	β	d_1	r	d_2	ρ_1	ρ_2
AR(2) with time trend	-70.345 (-8.33)	-3.626 (-35.50)	0.68 [0.60, 1.54]	12	0.41 [0.07, 0.62]	0.115	0.137
AR(2) with an intercept	-144.46 (-8.19)	---	1.18 [0.80, 1.64]	4	0.11 [-0.30, 0.29]	0.091	-0.247
AR(2) with no regressors*	**---**	**---**	**0.76 [0.17, 1.11]**	**8**	**0.18 [-0.15, 0.92]**	**-0.217**	**-0.211**
AR(1) with time trend	62.566 (4.27)	14.998 (39.80)	1.17 [0.75, 1.32]	3	0.11 [-0.27, 0.34]	0.075	---
AR(1) with an intercept	**-120.01 (-7.96)**	**---**	**1.08 [0.78, 1.47]**	**2**	**0.09 [-0.05, 0.72]**	**-0.166**	**---**
AR(1) with no regressors	**---**	**---**	**0.53 [0.12, 0.88]**	**8**	**0.05 [-0.13, 0.38]**	**0.204**	**---**
White noise with trend	**-48.096 (-5.45)**	**-2.212 (-26.53)**	**0.74 [0.71, 0.78]**	**15**	**0.19 [-0.04, 0.46]**	**---**	**---**
White noise + intercept	**-83.753 (-6.50)**	**---**	**1.44 [1.32, 1.57]**	**3**	**0.05 [-0.05, 0.29]**	**---**	**---**
White noise with no reg.	**---**	**---**	**0.59 [0.45, 0.84]**	**15**	**0.05 [-0.05, 0.20]**	**---**	**---**

In bold, the models for which the null hypothesis of white noise errors cannot be rejected. "*" indicates the best model specification using LR tests and other likelihood criteria.

Table 4A: Coefficient estimates with two fractional structures. Series: Government bond yields

	α	β	d_1	r	d_2	ρ_1	ρ_2
AR(2) with time trend	**26.951 (2.29)**	**3.889 (17.66)**	**1.20 [0.86, 1.94]**	**8**	**0.09 [-0.29, 0.44]**	**-0.334**	**-0.047**
AR(2) with an intercept	-2.098 (-1.43)	---	0.95 [0.67, 1.09]	11	1.00 [0.40, 1.24]	0.998	-0.453
AR(2) with no regressors	---	---	0.10 [-0.14, 1.11]	4	0.76 [0.16, 1.53]	0.622	0.337
AR(1) with time trend	-84.161 (-10.97)	-4.990 (-54.51)	0.71 [0.61, 1.02]	14	0.42 [0.17, 0.58]	0.489	---
AR(1) with an intercept*	**-23.090 (-4.75)**	**---**	**0.96 [0.88, 1.02]**	**8**	**0.28 [-0.04, 0.87]**	**-0.182**	**---**
AR(1) with no regressors	**---**	**---**	**0.75 [0.39, 1.16]**	**3**	**0.01 [-0.17, 0.36]**	**0.097**	**---**
White noise with trend	-36.744 (-3.53)	-2.563 (-4.123)	0.89 [0.85, 0.94]	15	0.08 [-0.04, 0.39]	---	---
White noise + intercept	**-39.023 (-6.84)**	**---**	**1.30 [1.05, 1.46]**	**3**	**0.09 [-0.04, 0.62]**	**---**	**---**
White noise with no reg.	**---**	**---**	**0.82 [0.70, 1.01]**	**2**	**0.01 [-0.06, 0.12]**	**---**	**---**

In bold, the models for which the null hypothesis of white noise errors cannot be rejected. "*" indicates the best model specification using LR tests and other likelihood criteria.

Table 1B: Coefficient estimates with one fractional structure. Series: Dividends

	α	β	d_1	ρ_1	ρ_2
AR(2) with time trend	0.298 (1.448)	-0.113 (-0.522)	1.68 [1.07, 2.05]	0.577	-0.167
AR(2) with an intercept	0.241 (1.405)	---	1.73 [1.11, 2.05]	0.530	-0.150
AR(2) with no regressors*	**---**	**---**	**1.57 [1.01, 2.00]**	**0.677**	**-0.211**
AR(1) with time trend	0.282 (1.268)	-0.091 (-1.049)	1.34 [1.21, 1.56]	0.699	---
AR(1) with an intercept	0.234 (1.068)	---	1.33 [1.20, 1.52]	0.707	---
AR(1) with no regressors	---	---	1.48 [1.32, 1.62]	0.619	---
White noise with trend	0.218 (0.960))	0.042 (0.127))	2.12 [1.93, 2.33]	---	---
White noise + intercept	0.239 (1.590)	---	2.12 [1.93, 2.35]	---	---
White noise with no reg.	---	---	2.11 [1.93, 2.35]	---	---

In bold, the models for which the null hypothesis of white noise errors cannot be rejected. "*" indicates the best model specification using LR tests and other likelihood criteria.

Table 2B: Coefficient estimates with one fractional structure. Series: Earnings

	α	β	d_1	ρ_1	ρ_2
AR(2) with time trend*	**0.575 (0.185)**	**-0.434 (-0.306)**	**1.38 [0.75, 1.82]**	**0.023**	**-0.445**
AR(2) with an intercept	0.349 (0.115)	---	1.36 [0.83, 1.81]	0.038	-0.438
AR(2) with no regressors	---	---	1.37 [0.93 1.81]	0.031	-0.441
AR(1) with time trend	-3.048 (-0.911)	0.091 (0.981))	0.73 [0.55, 0.98]	0.485	---
AR(1) with an intercept	-2.701 (-0.841)	---	0.70 [0.46, 0.98]	0.519	---
AR(1) with no regressors	---	---	0.81 [0.65, 0.96]	0.413	---
White noise with trend	-0.069 (-0.019)	0.709 (1.293)	1.13 [0.99, 1.34]	---	---
White noise + intercept	0.365 (0.103)	---	1.13 [0.99, 1.34]	---	---
White noise with no reg.	---	---	1.13 [0.99, 1.34]	---	---

In bold, the models for which the null hypothesis of white noise errors cannot be rejected. "*" indicates the best model specification using LR tests and other likelihood criteria.

Table 3B: Coefficient estimates with one fractional structure. Series: Interest rates

	α	β	d_1	ρ_1	ρ_2
AR(2) with time trend	6.332 (4.431)	-0.055 (-1.271)	0.75 [0.53, 1.16]	0.074	-0.149
AR(2) with an intercept	**4.923 (4.071)**	**---**	**0.62 [0.37, 1.16]**	**0.206**	**-0.098**
AR(2) with no regressors*	**---**	**---**	**0.64 [0.19, 1.11]**	**0.244**	**-0.083**
AR(1) with time trend	**5.728 (4.544)**	**-0.053 (-2.261)**	**0.60 [0.40, 0.80]**	**0.207**	**---**
AR(1) with an intercept	**3.751 (3.654)**	**---**	**0.53 [0.39, 0.78]**	**0.289**	**---**
AR(1) with no regressors	---	---	0.53 [0.40, 0.62]	0.340	---
White noise with trend	6.457 (4.487)	-0.019 (-0.471)	0.73 [0.61, 0.91]	---	---
White noise + intercept	**6.301 (4.514)**	**---**	**0.73 [0.61, 0.89]**	**---**	**---**
White noise with no reg.	**---**	**---**	**0.74 [0.64, 0.89]**	**---**	**---**

In bold, the models for which the null hypothesis of white noise errors cannot be rejected. "*" indicates the best model specification using LR tests and other likelihood criteria.

Table 4B: Coefficient estimates with one fractional structure. Series: Government bond yields

	α	β	d_1	ρ_1	ρ_2
AR(2) with time trend	5.333 (7.139)	-0.038 (-0.526)	1.03 [0.80, 1.40]	-0.222	-0.038
AR(2) with an intercept	**5.300 (7.119)**	**---**	**1.03 [0.82, 1.40]**	**-0.222**	**-0.038**
AR(2) with no regressors	**---**	**---**	**1.20 [1.10, 1.81]**	**-0.372**	**-0.134**
AR(1) with time trend	5.324 (7.119)	-0.038 (-0.608)	1.00 [0.86, 1.17]	-0.188	---
AR(1) with an intercept	**5.279 (7.085)**	**---**	**0.99 [0.85, 1.17]**	**-0.180**	**---**
AR(1) with no regressors*	**---**	**---**	**1.00 [0.82, 1.30]**	**-0.159**	**---**
White noise with trend	5.278 (7.091)	0.0001 (0.0034)	0.88 [0.78, 1.01]	---	---
White noise + intercept	**5.278 (7.139)**	**---**	**0.88 [0.80, 0.98]**	**---**	**---**
White noise with no reg.	**---**	**---**	**0.88 [0.81, 0.98]**	**---**	**---**

In bold, the models for which the null hypothesis of white noise errors cannot be rejected. "*" indicates the best model specification using LR tests and other likelihood criteria.

Table 1C: Estimates in the nonlinear case with two fractional structures. Series: Dividends

	α_1	α_2	β_2	d_1	r	d_2	ρ_1	ρ_2
AR(2)	-0.682 (-2.855)	-0.999 (-0.006)	0.0076 (0.0016)	2.15 [1.99, 2.24]	7	1.01 [0.94, 1.09]	-1.059	-0.560
AR(1)*	**-6.066 (-7.180)**	**-20.204 (-0.036)**	**0.452 (-0.028)**	**1.56 [1.41, 1.72]**	**11**	**0.53 [0.42, 0.71]**	**-0.308**	**---**
White Noise	-494.658 (-60.53)	-498.19 (-.325)	-1.580 (-1.890)	0.08 [0.00, 0.11]	13	1.07 [1.01, 1.18]	---	---

In bold, the models for which the null hypothesis of white noise errors cannot be rejected. "*" indicates the best model specification using LR tests and other likelihood criteria.

Table 2C: Estimates in the nonlinear case with two fractional structures. Series: Earnings

	α_1	α_2	β_2	d_1	r	d_2	ρ_1	ρ_2
AR(2)*	**-121.003 (-2.891)**	**-104.15 (-10.30)**	**-1.039 (--3.34)**	**1.32 [1.17, 1.55]**	**7**	**0.52 [0.34, 0.63]**	**-0.117**	**-0.260**
AR(1)	-446.595 (-13.54)	396.206 (11.07)	-20.120 (-0.980)	0.17 [0.11, 0.30]	15	1.33 [1.21, 1.56]	-0.354	---
White Noise	-1090.33 (-62.53)	-1146.4 (-26.58)	-1.276 (-0.611)	0.09 [0.00, 0.13]	7	1.29 [1.17, 1.40]	---	---

In bold, the models for which the null hypothesis of white noise errors cannot be rejected. "*" indicates the best model specification using LR tests and other likelihood criteria.

Table 3C: Estimates in the nonlinear case with two fractional structures. Series: Interest rates

	α_1	α_2	β_2	d_1	r	d_2	ρ_1	ρ_2
AR(2)	68.826 (3.409)	6.006 (0.0013)	1.728 (0.0123)	1.18 [1.04, 1.39]	5	0.63 [0.41, 0.88]	-0.199	-0.284
AR(1)*	**-75.817 (-9.678)**	**-63.114 (-7.089)**	**-0.636 (-4.351)**	**0.38 [0.30, 0.52]**	**15**	**1.23 [1.13, 1.39]**	**-0.237**	**---**
White Noise	-41.014 (-5.193)	-37.947 (-4.720)	-0.851 (-5.028)	0.82 [0.77, 0.89]	3	0.01 [-0.09, 0.16]	---	---

In bold, the models for which the null hypothesis of white noise errors cannot be rejected. "*" indicates the best model specification using LR tests and other likelihood criteria.

Table 4C: Estimates in the nonlinear case with two fractional structures. Series: Government bond yields

	α_1	α_2	d_1	r	d_2	ρ_1	ρ_2
AR(2)	-3.301 (-1.917)	-3.536 (-1.915)	0.95 [0.69, 1.06]	13	0.94 [0.75, 1.13]	-0.955	-0.424
AR(1)*	**-25.644 (-4.446)**	**-28.013 (-4.806)**	**0.95 [0.89, 1.07]**	**3**	**0.14 [0.03, 0.54]**	**-0.088**	**---**
White Noise	**-34.125 (-6.275)**	**-36.569 (-6.608)**	**0.94 [0.86, 1.10]**	**8**	**0.27 [0.08, 0.59]**	**---**	**---**

In bold, the models for which the null hypothesis of white noise errors cannot be rejected. "*" indicates the best model specification using LR tests and other likelihood criteria.

Table 1D: Coefficient estimates with one fractional structure. Series: Dividends

	α_1	α_2	β_2	d_1	ρ_1	ρ_2
AR(2)*	**-1.744** **(-8.287)**	**-2.878** **(-9.185)**	**0.485** **(31.232)**	**0.63** **[0.55, 0.70]**	**1.372**	**-0.745**
AR(1)	0.225 (0.985)	-0.005 (-0.016)	0.459 (5.334)	1.24 [0.86, 1.49]	0.712	---
White noise	0.239 (1.592)	0.270 (1.001)	0.088 (0.263)	2.12 [1.93, 2.35]	---	---

In bold, the models for which the null hypothesis of white noise errors cannot be rejected. "*" indicates the best model specification using LR tests and other likelihood criteria.

Table 2D: Coefficient estimates with one fractional structure. Series: Earnings

	α_1	α_2	β_2	d_1	ρ_1	ρ_2
AR(2)*	**0.231** **(0.073)**	**0.269** **(0.059)**	**1.641** **(1.733)**	**1.17** **[0.51, 1.79]**	**0.162**	**-0.401**
AR(1)	-8.748 (-4.465)	-13.960 (-4.363)	1.572 (11.162)	0.47 [0.30, 0.67]	0.497	---
White noise	-0.080 (-0.023)	-0.440 (-0.088)	1.634 (2.441)	1.03 [0.81, 1.31]	---	---

In bold, the models for which the null hypothesis of white noise errors cannot be rejected. "*" indicates the best model specification using LR tests and other likelihood criteria.

Table 3D: Coefficient estimates with one fractional structure. Series: Interest Rate

	α_1	α_2	β_2	d_1	ρ_1	ρ_2
AR(2)*	**2.751** **(3.330)**	**7.105** **(5.134)**	**-0.258** **(-4.019)**	**0.45** **[0.30, 1.12]**	**0.333**	**0.007**
AR(1)	**2.462** **(3.152)**	**6.890** **(5.159)**	**-0.257** **(-4.142)**	**0.43** **[0.16, 0.74]**	**0.360**	**---**
White noise	**5.668** **(4.328)**	**9.466** **(4.930)**	**-0.259** **(-2.365)**	**0.69** **[0.58, 0.85]**	**---**	**---**

In bold, the models for which the null hypothesis of white noise errors cannot be rejected. "*" indicates the best model specification using LR tests and other likelihood criteria.

Table 4D: Coefficient estimates with one fractional structure. Series: Government bond yields

	α_1	α_2	d_1	ρ_1	ρ_2
AR(2)*	**5.279** **(7.100)**	**5.761** **(5.479)**	**0.99** **[0.80, 1.38]**	**-0.183**	**-0.007**
AR(1)	**5.272** **(6.985)**	**5.760** **(5.398)**	**0.98** **[0.82, 1.17]**	**-0.173**	**---**
White noise	**5.126** **(6.975)**	**5.713** **(5.505)**	**0.87** **[0.78, 0.98]**	**---**	**---**

In bold, the models for which the null hypothesis of white noise errors cannot be rejected. "*" indicates the best model specification using LR tests and other likelihood criteria.

Table 5: RMSE of the k-ahead prediction errors. Series: Dividends

k / Model	(1A)	(1B)	(1C)	(1D)
1	0.006234	**0.000591**	0.008026	0.018123
2	0.031326	0.025251	0.036914	**0.016958**
3	0.030545	0.024227	0.033172	**0.013850**
4	0.031718	0.025917	0.034087	**0.012656**
5	0.034013	0.028231	0.036075	**0.012689**
6	0.036629	0.030663	0.038195	**0.013575**
7	0.039581	0.033446	0.040673	**0.015228**
8	0.042596	0.036330	0.043256	**0.017238**
9	0.045808	0.039465	0.046073	**0.019673**
10	0.049288	0.042921	0.049189	**0.022551**

In bold the lowest value among models for each k-ahead prediction error.

Table 6: RMSE of the k-ahead prediction errors. Series: Earnings

k / Model	(2A)	(2B)	(2C)	(2D)
1	0.009811	**0.002347**	0.128726	0.041816
2	0.043473	0.047612	0.179285	**0.033633**
3	0.045904	0.054964	0.183457	**0.030183**
4	0.063079	0.073397	0.198744	**0.039817**
5	0.067516	0.080890	0.205456	**0.043198**
6	0.073898	0.087810	0.213325	**0.047346**
7	0.084659	0.097785	0.224043	**0.055428**
8	0.096819	0.109013	0.235610	**0.065194**
9	0.110367	0.122034	0.248383	**0.077060**
10	0.126495	0.138105	0.263549	**0.092292**

In bold the lowest value among models for each k-ahead prediction error.

Table 7: RMSE of the k-ahead prediction errors. Series: Interest Rates

k / Model	(3A)	(3B)	(3C)	(3D)
1	0.001353	0.001622	0.077673	**0.000636**
2	0.031361	0.032086	0.107293	**0.031284**
3	**0.057927**	0.058682	0.131547	0.058063
4	**0.085821**	0.086357	0.158054	0.085826
5	0.110345	0.110731	0.182517	**0.110294**
6	0.130518	0.130777	0.203123	**0.130446**
7	0.150488	0.150638	0.223397	**0.150430**
8	**0.167591**	0.167674	0.240997	0.167592
9	**0.180127**	0.180186	0.254282	0.180229
10	**0.188005**	0.188063	0.263042	0.188227

In bold the lowest value among models for each k-ahead prediction error.

Table 8: RMSE of the k-ahead prediction errors. Series: Government bond yields

k / Model	(4A)	(4B)	(4C)	(4D)
1	0.021940	**0.000892**	0.026910	0.006647
2	0.043353	0.022852	0.046478	**0.019229**
3	0.060376	0.040011	0.063574	**0.035646**
4	0.079280	0.059526	0.083255	**0.054984**
5	0.096742	0.077144	0.101070	**0.072477**
6	0.112871	0.093214	0.117255	**0.088455**
7	0.126838	0.106984	0.131291	**0.102144**
8	0.139513	0.119445	0.143957	**0.114540**
9	0.151303	0.131048	0.155693	**0.126092**
10	0.162691	0.142301	0.167085	**0.137308**

In bold the lowest value among models for each k-ahead prediction error.

Table 9: Modified DM statistic: 2-step ahead forecasts

Dividends	1A	1B	1C
1B	**65.654 (1B)**	XXXXXX	XXXXXX
1C	**-3.154 (1A)**	**-14.420 (1B)**	XXXXXX
1D	**4.574 (1D)**	**3.120 (1D)**	**5.217 (1D)**
Earnings	2A	2B	2C
2B	**-3.998 (2A)**	XXXXXX	XXXXXX
2C	**-63.060 (2A)**	**-174.975 (2B)**	XXXXXX
2D	**2.813 (2D)**	**3.130 (2D)**	**17.222 (2D)**
Interest rates	3A	3B	3C
3B	**-3.501 (3A)**	XXXXXX	XXXXXX
3C	**-139.916 (3A)**	**-137.877 (3B)**	XXXXXX
3D	0.565	**2.905 (3D)**	**129.224 (3D)**
Gov bond yield	4A	4B	4C
4B	**125.925 (4B)**	XXXXXX	XXXXXX
4C	**-20.277 (4A)**	**-93.416 (4B)**	XXXXXX
4D	**22.406 (4D)**	**3.069 (4D)**	**28.210 (4D)**

Critical value: 1.833 (95% level, with 9 degrees of freedom). In parentheses, the preferred model in a pairwise comparison.

Table 10: Modified DM statistic: 4-step ahead forecasts

Dividends	1A	1B	1C
1B	**50.144 (1B)**	XXXXXX	XXXXXX
1C	**-2.420 (1A)**	**-11.013 (1B)**	XXXXXX
1D	**3.493 (1D)**	**2.383 (1D)**	**3.985(1D)**
Earnings	2A	2B	2C
2B	**-3.054 (2A)**	XXXXXX	XXXXXX
2C	**-48.163 (2A)**	**-133.64 (2B)**	XXXXXX
2D	**2.148 (2D)**	**2.390 (2D)**	**13.153 (2D)**
Interest rates	3A	3B	3C
3B	**-2.674 (3A)**	XXXXXX	XXXXXX
3C	**-106.863 (3A)**	-105.305	XXXXXX
3D	0.431	**2.219 (3D)**	**98.696 (3D)**
Gov bond yield	4A	4B	4C
4B	**96.177(4B)**	XXXXXX	XXXXXX
4C	**-15.487(4A)**	**-71.347 (4B)**	XXXXXX
4D	**17.113 (4D)**	**2.344 (4D)**	**21.545 (4D)**

Critical value: 1.833 (95% level, with 9 degrees of freedom). In parentheses, the preferred model in a pairwise comparison.

Table 11: Modified DM statistic: 6-step ahead forecasts

Dividends	1A	1B	1C
1B	**34.602 (1B)**	XXXXXX	XXXXXX
1C	-1.662	**-7.600 (1B)**	XXXXXX
1D	**2.410 (1D)**	1.644	**2.750 (1D)**
Earnings	2A	2B	2C
2B	**-2.107 (2A)**	XXXXXX	XXXXXX
2C	**-33.235 (2A)**	**-92.220 (2B)**	XXXXXX
2D	1.482	1.649	**9.076 (2D)**
Interest rates	3A	3B	3C
3B	**-1.845 (3A)**	XXXXXX	XXXXXX
3C	**-73.742 (3A)**	**-72.667 (3B)**	XXXXXX
3D	0.298	1.531	**68.107 (3D)**
Gov bond yield	4A	4B	4C
4B	**66.368 (4B)**	XXXXXX	XXXXXX
4C	**-10.687 (4A)**	**-49.234 (4B)**	XXXXXX
4D	**11.809 (4D)**	1.617	**14.868 (4D)**

Critical value: 1.833 (95% level, with 9 degrees of freedom). In parenthesis, the preferred model in a pairwise comparison.

Table 12: Modified DM statistic: 8-step ahead forecasts

Dividends	1A	1B	1C
1B	**18.952 (1B)**	XXXXXX	XXXXXX
1C	-0.910	**-4.162 (1B)**	XXXXXX
1D	1.320	0.900	1.506
Earnings	2A	2B	2C
2B	-1.154	XXXXXX	XXXXXX
2C	**-18.204 (2A)**	**-50.511 (2B)**	XXXXXX
2D	0.812	0.903	**4.971 (2D)**
Interest rates	3A	3B	3C
3B	-1.010	XXXXXX	XXXXXX
3C	**-40.390 (3A)**	**-39.801 (3B)**	XXXXXX
3D	0.163	0.838	**37.303 (3D)**
Gov bond yield	4A	4B	4C
4B	**36.351 (4B)**	XXXXXX	XXXXXX
4C	**-5.853 (4A)**	**-26.966 (4B)**	XXXXXX
4D	**6.468 (4D)**	0.885	**8.143 (4D)**

Critical value: 1.833 (95% level, with 9 degrees of freedom). In parenthesis, the preferred model in a pairwise comparison.