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AERODYNAMIC DESIGN OF ANNULAR DUCTS

A THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

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ABSTRACT

This thesis presents mathematical and numerical methods for designing axisymmetric annular ducts having geometries capable of supporting fluid flows with prescribed performance characteristics. Three basic numerical methods of solution are given and are used to obtain results to the known exact solutions for a class of axisymmetric, irrotational incompressible flow regimes.
Examination is made into the type of boundary conditions appropriate to control boundary layer behaviour and a new mixed boundary condition is derived to accomplish this. The technique is extended to cater for a class of swirling flows by investigating the derivation of a boundary layer approximation and further development allows the application of numerical iterative techniques to compressible flows with vorticity.
These methods are especially suited to generating duct geometries with predetermined flow characteristics.

INTRODUCTION

This thesis presents mathematical and numerical methods for designing axisymmetric annular ducts capable of supporting fluid flow regimes with predetermined flow features. The primary approach has been to develop techniques whereby duct geometries may be generated having prescribed performance characteristics (an example being the avoidance of boundary layer separation).

Initially the equations for axisymmetric, inviscid flow are mapped into the (Φ,Υ) 'DESIGN PLANE' (DP) in which the 'stream' and 'velocity potential' functions replace the spacial coordinates as the independent variables.

These design plane equations ('DPE') form the basis of all subsequent solution schemes. This set is reducible to a second order, non-linear, partial differential equation (PDE) in the radial coordinate y (a similar equation being available for the axial coordinate 'x').

Exact solutions to the DPE exist in the simplest case of irrotational incompressible flow. These solutions are given with some detail and are used as a test to validate the routines subsequently used to obtain numerical solutions for other flow configurations and boundary conditions. The PDE is approximated by finite difference forms and three iterative solution techniques are presented to obtain numerical results for the exact solutions namely

(1) Point Iterative Methods;

(2) Matrix Formulations;

(3) Numerical Technique based on an Integral Equation of a function of a complex variable.

In obtaining the numerical results for comparison with the 'exact' solutions, the boundary values of the space coordinates (x,y) are known and could be used as a B.C to determine the flow over the complete Φ,Ψ domain. However, alternative to this usual procedure of prescribing the known values of y (or x) on the boundaries to solve the PDE in 'y', the equivalent and corresponding invariant distribution in the speed is applied.
Then, for some initial and arbitrary boundary distribution of 'y', the invariant speed distribution of 'q' (known from the exact solution) is used to calculate sucessive and varying boundary distributions of 'y' until both the boundary and field distributions in 'y' converge.
Computer programs were developed to obtain numerical results to all forms of the exact solutions and the results were found to be accurate to 10^{-4} relative error in the distributions of x,y,q. However the five basic duct geometries and associated flows produced by these solutions are not considered suitable for applications to annular duct flow and due to the non-linearity of the equations the technique of superposition of solutions is not available to us. Given the current unavailability of other 'exact' solutions it follows that further progress in determining duct geometries and corresponding flow patterns can most likiely be made via a numerical approach.
An important consideration in the design of annular ducts is the behaviour of the boundary layers (B.L) and in particular the avoidance of their separation would be an advantageous design feature. Randomly prescribed velocity distributions (PVD) do not

necessarily yield boundary layers having this characteristic and it is not apparent what form an invariant boundary condition might have in order to 'control' the B.L behaviour or any other flow feature in this way. However some feasable distribution is required on the duct walls in order to produce acceptable duct contours.
The methods derived by Stratford (Refs.11,12) for predicting the point of separation of the two dimensional plane B.L are here extended to the axisymmetric case by use of a transform due to Mangler (Ref.15). This yields a new 'mixed' boundary condition which may be imposed at the duct walls to give velocity distributions which are on (or below) the point of separating for both the laminar and turbulent B.L.
The inclusion of this condition into the general numerical iterative scheme allows the calculation of duct shapes with this flow feature. A further transform is derived which enables this condition to be applied to a class of swirling flows having non-skewed B.Ls. The computer program is extended to cater for alternative B.Cs including (a) accelerating flows, (b) flows with sections of constant velocity and/or radius on either or both walls. The methods developed for incompressible, irrotational flows are then widened to include the case of flows with with arbitrary distributions of vorticity and speed across the duct at inlet.
The laws governing the transport of vorticity through the duct are included in the general numerical scheme and the results compared with the irrotational case. This vorticity is generated by prescribing a non-uniform axial velocity across the duct at inlet based on a parabolic profile together with a swirl compenent of

velocity of the form (a.y + b/y) both of which may independently contribute to non-zero vorticity distribution in the flow.
These profiles may be varied at will to show the variation of duct shape with change in velocity and vorticity distribution.
Finally the technique is extended to cater for compressible isentropic flow of a gas with constant specific heats and the numerical methods allow the effects of compressibility to be included in the solution schemes.
Again it is possible to prescribe an arbitrary distribution in the parameters of state (p, ρ, T) at some station of the flow (stagnation conditions say) to give some variation of the state variables throughout the transition region. In view of the substantial degree of flexibility afforded by this approach and the wide choice of B.Cs available, these methods could form the basis for a substantial amount of numerical experimentation to determine the interaction and effect of the numerous parameters that may affect the flow and hence the geometry of annular ducts.

CHAPTER 1

In steady, axisymmetric, inviscid, irrotational flow the condition of continuity and the absence of vorticity are sufficient to define the familiar stream and potential functions. Constant values of the stream function, Υ, in steady flow coincide with the stream lines and these together with lines of constant Φ form an orthogonal coordinate system over the flow field.
Solutions to the equations of flow are traditionally derived in either the (x,y) or hodograph plane, however by utilizing the definitions of the Υ, Φ functions, the equations of plane and axisymmetric flow in the (x,y) plane may be mapped to an equivalent set in the (Φ,Υ) domain.
Laidler and Walkden (Ref. 10), Cousins (Ref. 6) et al. have used this approach to derive numerical solutions for inviscid, irrotational flow fields through axisymmetric ducts subject to a variety of boundary conditions on the duct walls.
The most obvious feature of this method is that the potential and stream functions now become the independent variables rather than the space cordinates (x,y,z or (Θ)).
Laidler & Walkden obtained numerical solutions to the design problem of generating shapes for (non-annular) ducts subject to a fixed and prescribed velocity distribution varying as a cubic in arc length on the casing. Their inference was that it should be possible to design 'quite short' ducts having 'almost' uniform inlet and outlet conditions satisfying this fixed distribution on the casing.

In Ref. 6, Cousins has obtained solutions for distributions of y (or x) prescribed at equal delta Y, Φ intervals on annular duct boundaries to determine the geometry for flow past a point source. Having found the values of the radial coordinate 'y' throughout the flow field, the corresponding distributions of 'x' and the speed 'q' are derived.

The ability to deal with rotational and compressible flows in a comparable (Φ^*, Y^*) domain would extend to such flows this advantage of prescribing arbitrary boundary distributions of x,y,q or F(x,y,q) (an arbitrary function). It has proved possible to widen the definition of the design plane to cater for compressible flows with vorticity. The overall approach is to define an orthogonal coordinate system based on the differential relationships between vorticity, density and speed.

This allows us the freedom to prescribe arbitrary functions of x,y,q on the flow boundaries. Suitable numerical formulation of the flow equations then provide the solution to the flow problem in the design plane.

The Generalized Design Plane Equations

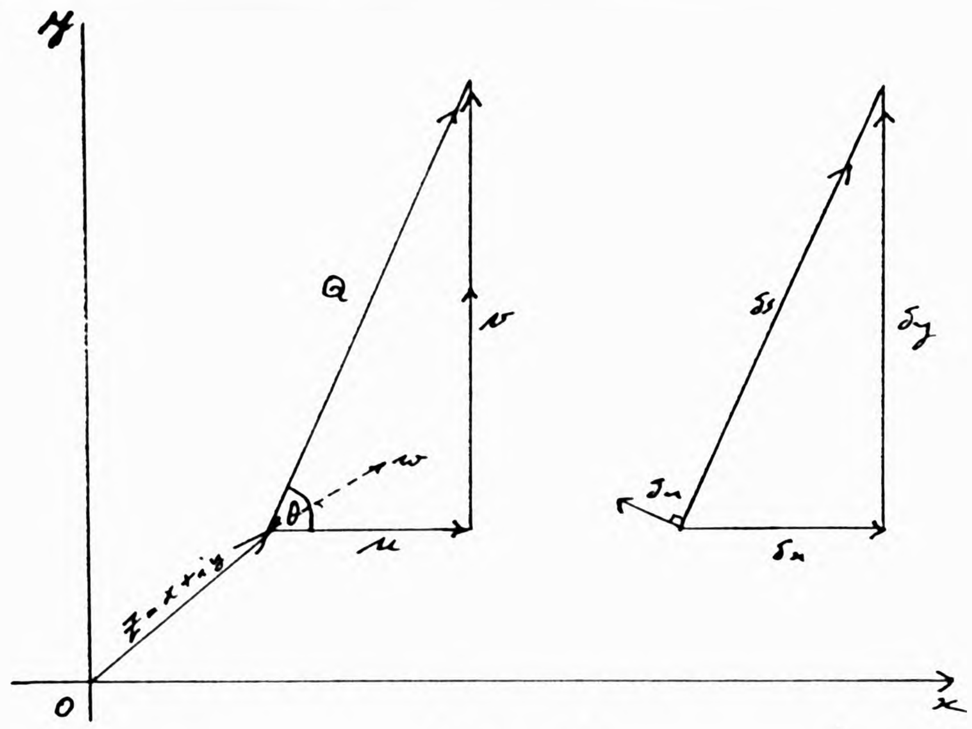


FIG 1.1

Consider the flow given by the complex velocity

$$Q = u - i.v = q.e^{-i\theta}$$

$$q^2 = u^2 + v^2$$

where

$$z = x + i.y \;;\; z^* = x - i.y \;;\; i = (-1)^{.5}$$

and θ is the angle that the flow with speed q makes with the x-axis.

Defining $\quad \epsilon^* = u_x + v_y \qquad [1.2.1]$

and $\quad \Omega^* = v_x - u_y \qquad [1.2.2]$

Then

$$Q_{z^*} = \tfrac{1}{2}.[\,\partial/\partial x + i.\partial/\partial y\,].(\,u - i.v\,)$$

$$= \tfrac{1}{2}.[\,(\,u_x + v_y\,) - i.(\,v_x - u_y\,)\,]$$

$$= \tfrac{1}{2}.[\,\epsilon^* - i.\Omega^*\,] \qquad [1.2.6]$$

where ϵ^* is the rate of expansion of the fluid and Ω^* the component of vorticity perpendicular to the (x,y) plane, and 's' and 'n' are metrics parallel and perpendicular to the flow lines of Q.

By using the idea of two comparison flows an orthogonal coordinate set is defined over the flow field of Q.

(1) Consider the first comparison flow defined by the complex velocity $Q_1 = \Upsilon_n . e^{-i\theta} = q_1 . e^{-i\theta} = u_1 - i.v_1$; $q_1 = \Upsilon_n$. [1.3.1]

Since flows Q and Q_1 have a common direction at all points of the (x,y) plane then the metrics 's' and 'n' are also parallel and perpendicular to the flow lines of Q_1.

Suppose that $\Upsilon = \Upsilon(x,y)$ is a real function of x and y and that the rate of change of Υ in the 's' direction is zero.

Now $x_s = \text{Cos}\theta$; $y_s = \text{Sin}\theta$; $x_n = -\text{Sin}\theta$; $y_n = \text{Cos}\theta$

and since $\Upsilon_s = 0$ and $\Upsilon_n = q_1$ [1.4.1]

Then
$$\Upsilon_s = \Upsilon_x . x_s + \Upsilon_y . y_s$$
$$= \Upsilon_x . \text{Cos}\theta + \Upsilon_y . \text{Sin}\theta = 0$$

Therefore $\Upsilon_x = -(\text{Sin}\theta/\text{Cos}\theta).\Upsilon_y$ or $\Upsilon_y = -(\text{Cos}\theta/\text{Sin}\theta).\Upsilon_x$

Therefore
$$\Upsilon_n = \Upsilon_x . x_n + \Upsilon_y . y_n$$
$$= -\text{Sin}\theta.\Upsilon_x + \text{Cos}\theta.\Upsilon_y$$
$$= -\text{Sin}\theta.\Upsilon_x + \text{Cos}\theta.(-\text{Cos}\theta/\text{Sin}\theta).\Upsilon_x$$
$$= -(1/\text{Sin}\theta).\Upsilon_x = (1/\text{Cos}\theta).\Upsilon_y$$

which gives the rate of change of Υ normal to 's'.

Hence
$$Q_1 = \Upsilon_n . e^{-i\theta} = (1/\text{Cos}\theta).\Upsilon_y . (\text{Cos}\theta - i.\text{Sin}\theta)$$
$$= \Upsilon_y - i.(\text{Sin}\theta/\text{Cos}\theta).\Upsilon_y$$
$$= \Upsilon_y + i.\Upsilon_x = 2.i.\Upsilon_z$$
$$= u_1 - i.v_1$$

Hence $u_1 = \Upsilon_y$; $v_1 = -\Upsilon_x$ with $q_1^2 = u_1^2 + v_1^2$

Also $\epsilon_1{}^* = (u_1)_x + (v_1)_y = \Upsilon_{xy} - \Upsilon_{yx} = 0$ [1.4.2a]

and
$$\Omega_1{}^* = (v_1)_x - (u_1)_y = -\Upsilon_{xx} - \Upsilon_{yy} = -\nabla^2(\Upsilon)$$
$$= -4.\Upsilon_{zz^*} = 2.i.Q_{z^*} \qquad [1.4.2b]$$

Thus, this flow has zero rate of expansion ϵ^*_1 but non-zero vorticity component Ω^*_1.

(2) Consider now a second flow defined by the complex velocity

$$Q_2 = \Phi_s . e^{-i\theta} = q_2 . e^{-i\theta} = u_2 - i.v_2 \ ; \ q_2 = \Phi_s \qquad [1.4.3]$$

where $\Phi = \Phi(x,y)$ is real and let the rate of change of Φ normal to 's' be set equal to zero.

Thus in this flow

$$\Phi_n = 0 \quad \text{and} \quad \Phi_s = q_2 \qquad [1.4.4]$$

$\Rightarrow \quad \Phi_n = -\text{Sin}\theta.\Phi_x + \text{Cos}\,\theta.\Phi_y = 0$

$\Rightarrow \quad \Phi_y = (\text{sin}\theta/\text{Cos}\theta).\Phi_x \quad \text{or} \quad \Phi_x = (\text{Cos}\theta/\text{Sin}\theta).\Phi_y$

Hence $\Phi_s = \text{Cos}\theta.\Phi_x + \text{Sin}\theta.\Phi_y$

$\qquad = (1/\text{Cos}\theta).\Phi_x = (1/\text{Sin}\theta).\Phi_y$

$\Rightarrow \quad Q_2 = (1/\text{Cos}\theta).\Phi_x .(\text{cos}\theta - i.\text{Sin}\theta)$

$\qquad = \Phi_x - (\text{Sin}\theta/\text{Cos}\theta).\Phi_y = \Phi_x - i.\Phi_y = 2.\Phi_z$

$\Rightarrow \quad u_2 = \Phi_x \quad ; \quad v_2 = \Phi_y$

and $\quad \epsilon_2^* = (u_2)_x + (v_2)_y = \Phi_{xx} + \Phi_{yy} = \nabla^2(\Phi) \qquad [1.4.3a]$

$\quad \Omega_2^* = (v_2)_x - (u_2)_y = \Phi_{xy} - \Phi_{yx} = 0 \qquad [1.4.3b]$

Lines of constant Φ and Υ defined by these two subsidiary flows form an orthogonal family of curves over the domain of the flow field of 'Q' since if

$$\Phi^*(x,y) = 0 \quad \text{and} \quad \Upsilon^*(x,y) = 0$$

are a pair of the set of orthogonal curves then

$$\Upsilon^*_x + \Upsilon^*_y .(dy/dx)_\Psi = 0 \quad \text{and} \quad \Phi^*_x + \Phi^*_y .(dy/dx)_\Phi = 0$$

Hence $(dy/dx)_\Psi = - \Upsilon^*_x/\Upsilon^*_y$ and $(dy/dx)_\Phi = - \Phi^*_x/\Phi^*_y$

But the derivatives of $\Upsilon(x,y)$ and $\Phi(x,y)$ along and normal to the streamwise direction are respectively zero.

thus $\Psi^*_s = \mathrm{Cos}\Theta.\Psi^*_x + \mathrm{Sin}\Theta.\Psi^*_y = 0$

and $\Phi^*_n = -\mathrm{Sin}\Theta.\Phi^*_x + \mathrm{Cos}\Theta.\Phi^*_y = 0$

Hence $\Psi^*_x / \Psi^*_y = -\mathrm{Sin}\Theta/\mathrm{Cos}\Theta$ and $\Phi^*_x / \Phi^*_y = \mathrm{Cos}\Theta/\mathrm{Sin}\Theta$

Thus
$$(dy/dx)_\psi . (dy/dx)_\Phi = (- \Psi^*_x . \Psi^*_y).(-\Phi^*_x/\Phi^*_y)$$
$$= (\sin\Theta/\mathrm{Cos}\Theta).(-\mathrm{Cos}\Theta/\mathrm{Sin}\Theta)$$
$$= -1 \ .$$

Hence the lines of constant Ψ and Φ form an orthogonal set. The rate of change of Ψ along 's' is known and equal to zero but the distribution of Ψ (i.e the speed q_1 in the direction of the normal, 'n') is unspecified. Similarly in the case of the flow with speed q_2 the distribution of Φ along 'n' (normal to 's' direction) is zero but its distribution along 's' is not yet determined. Once these distributions are specified the corresponding ones in $\in^*$ and Ω^* are defined by equations [1.4.2b] and [1.4.3a].

The Intrinsic Flow Equations.

By considering the differential of the flow speed 'q' in the directions defined by the coordinate system (Φ, Υ), relationships between speed (q), direction (Θ), vorticity (Ω^*) and the rate of expansion (ϵ^*) can be established.

For any function F

$$F_s + i.F_n = (\text{Cos}\Theta.F_x + \text{Sin}\Theta.F_y) + i.(-\text{Sin}\Theta.F_x + \text{Cos}\Theta.F_y)$$
$$= (\text{Cos}\Theta - i.\text{Sin}\Theta).F_x + i.(\text{Cos}\Theta - i.\text{Sin}\Theta).F_y$$
$$= e^{-i\theta}(F_x + i.F_y) = 2.e^{-i\theta}.F_{z*}$$

Applying this differential operator to the function ln(Q) we have

$$(\ln(Q))_s + i.(\ln(Q))_n = 2.e^{-i\theta}.(\ln(Q))_{z*}$$
$$= 2.e^{-i\theta}.Q^{-1}.Q_{z*}$$
$$= 2.e^{-i\theta}.q^{-1}.e^{i\theta}.Q_{z*}$$
$$= (2/q).[\epsilon^* - i.\Omega^*]/2 \quad \text{(from 1.2.6)}$$
$$= (1/q).[\epsilon^* - i.\Omega^*] \qquad [1.5.1]$$

Now, the alternative expansion of the left hand side of [1.5.1] gives

$$(\ln(Q))_s + i.(\ln(Q))_n = (\ln(q.e^{-i\theta}))_s + i.(\ln(q.e^{-i\theta}))_n$$
$$= (\ln(q) - i.\Theta)_s + i.(\ln(q) - i.\Theta)_n$$
$$= [\ (\ln(q))_s + \Theta_n\ ] + i.[\ (\ln(q))_n - \Theta_s\ ]$$

On equating real and imaginary parts

$$(\ln(q))_s + \Theta_n = \epsilon^*/q \qquad [1.5.2]$$
$$(\ln(q))_n - \Theta_s = -\Omega^*/q \qquad [1.5.3]$$

Application of this differential identity to the two subsidiary flows Q_1 and Q_2 defined above yield the following relationships;

$$(\ln(q_1))_s + \Theta_n = \epsilon^*_1/q_1 = \emptyset \qquad [1.5.4] \text{ (from 1.4.2a)}$$
$$(\ln(q_1))_n - \Theta_s = -\Omega^*_1/q_1 \qquad [1.5.5]$$

$$(\ln(q_2))_s + \Theta_n = \epsilon^*_2/q_2 \qquad [1.5.6]$$

$$(\ln(q_2))_n - \Theta_s = -\Omega^*_2/q_2 = 0 \qquad [1.5.7] \text{ (from 1.4.3b)}$$

Now the derivatives with respect to 's' and 'n' may be replaced by those with respect to Φ and Ψ as follows. For any function F

$$F_s = F_\Phi.\Phi_s + F_\Psi.\Psi_s = F_\Phi.\Phi_s + 0 = q_2.F_\Phi \quad (\text{Since } \Psi_s = 0 \text{ By 1.4.1}) \qquad [1.5.8]$$

$$F_n = F_\Phi.\Phi_n + F_\Psi.\Psi_n = 0 + F_\Psi.\Psi_n = q_1.F_\Psi \quad (\text{Since } \Phi_n = 0 \text{ by 1.4.4}) \qquad [1.5.9]$$

Replacing derivatives with respect to s and n by those with respect Φ and Ψ gives

$$q_2.(\ln(q))_\Phi + q_1.\Theta_\Psi = \epsilon^*/q \qquad [1.6.1]$$

$$q_1.(\ln(q))_\Psi - q_2.\Theta_\Phi = -\Omega^*/q \qquad [1.6.2]$$

$$q_2.(\ln(q_1))_\Phi + q_1.\Theta_\Psi = 0 \qquad [1.6.3]$$

$$q_1.(\ln(q_1))_\Psi - q_2.\Theta_\Phi = -\Omega^*_1/q_1 \qquad [1.6.4]$$

$$q_2.(\ln(q_2))_\Phi + q_1.\Theta_\Psi = \epsilon^*_2/q_2 \qquad [1.6.5]$$

$$q_1.(\ln(q_2))_\Psi - q_2.\Theta_\Phi = 0 \qquad [1.6.6]$$

Eliminating 'Θ' between the equation pairs [1.6.1] & [1.6.3]; [1.6.2] & [1.6.6] ; [1.6.3] & [1.6.6] gives after some rearrangement

$$q_2.(\ln(q/q_1))_\Phi = \epsilon^*/q \qquad [1.6.7]$$

$$q_1.(\ln(q/q_2))_\Psi = -\Omega^*/q \qquad [1.6.8]$$

$$[q_1/q_2.(\ln(q_2))_\Psi]_\Psi + [q_2/q_1.(\ln(q_1))_\Phi]_\Phi = 0 \qquad [1.6.9]$$

Defining the ratio of the speeds of the flows as

$$A = q/q_1 \ ; \ B = q/q_2 \ \text{ and } A/B = q_2/q_1 \qquad [1.6.10]$$

Then [1.5.8] and [1.5.9] may be written as

$$F_s = (q/B).F_\Phi \quad [1.6.11] \ ; \quad F_n = (q/A).F_\Psi \qquad [1.6.12]$$

then the set [1.6.7] to [1.6.9] may be writen as

$$[\ \ln(A)\ ]_{\Phi} = \epsilon^{*}.B/q^{2} \qquad [1.7.1]$$

$$[\ \ln(B)\ ]_{\Upsilon} = -\Omega^{*}.A/q^{2} \qquad [1.7.2]$$

$$[(B/A).(\ln(q/B))_{\Upsilon}\ ]_{\Upsilon} + [(A/B).(\ln(q/A))_{\Phi}\ ]_{\Phi} = 0 \qquad [1.7.3]$$

If q, ϵ^{*}, Ω^{*} are considered as known functions of Φ, Υ and A and B are known along one Φ and Υ characteristic respectively, then equations [1.7.1] and [1.7.2] enable the distributions of A and B to be calculated over the whole (Φ,Υ) plane. If in turn, A and B are now considered known throughout the domain then equation [1.7.3] allows q to be determined together with the corresponding distributions of ϵ^{*} and Ω^{*} via equations [1.2.1] and [1.2.2].
The interdependency of this set allied to a suitable iterative numerical scheme will allow the evaluation of the A, B, q, x and y distributions over the complete flow field in the (Φ,Υ) plane.

Alternative Derivation of an Equivalent Set of Equations

With x (or y) as The Dependent Variable.

Equation [1.7.3] may be expressed in an alternative form in terms of either of the space coordinates x or y instead of the speed q.

Since $z = x + i.y \; ; \; dz = dx + i.dy$ 1.

and from geometrical considerations

$$ds = dx.\cos\theta + dy.\sin\theta \qquad 2.$$

$$dn = -dx.\sin\theta + dy.\cos\theta \qquad 3.$$

Hence from 2. and 3.

$$dx = ds.\cos\theta - dn.\sin\theta \qquad 4.$$

$$dy = ds.\sin\theta + dn.\cos\theta \qquad 5.$$

$$\begin{aligned} \Rightarrow \quad dz &= ds.\cos\theta - dn.\sin\theta + i.ds.\sin\theta + i.dn.\cos\theta \\ &= ds.(\cos\theta + i.\sin\theta) + i.dn.(\cos\theta + i.\sin\theta) \\ &= (ds + i.dn).(\cos\theta + i.\sin\theta) \\ &= e^{i\theta}.(ds + i.dn) \qquad 6. \end{aligned}$$

From equations [1.4.3] and [1.6.10]

$$d\Phi = q_2.ds = (q/B).ds \qquad 7.$$

and $d\Psi = q_1.dn = (q/A).dn$ 8.

$$\Rightarrow \quad dz = e^{i\theta}.(B/q.d\Phi + A/q.d\Psi)$$

$$dz = e^{i\theta}.(B.d\Phi + A.d\Psi)/q \qquad 9.$$

$$\begin{aligned} \Rightarrow \quad & z_\Phi = B.e^{i\theta}/q \quad \text{(i)} \qquad z_\Psi = i.A.e^{i\theta}/q \quad \text{(ii)} \\ \Rightarrow \quad & x_\Phi = B.\cos\theta/q \quad \text{(iii)} \qquad x_\Psi = -A.\sin\theta/q \quad \text{(iv)} \\ & y_\Phi = B.\sin\theta/q \quad \text{(v)} \qquad y_\Psi = A.\cos\theta/q \quad \text{(vi)} \end{aligned} \qquad [1.7.4]$$

Eliminating θ gives

$$\begin{aligned} & x_\Phi = (B/A).y_\Psi \quad ; \quad x_\Psi = -(A/B).y_\Phi \\ \text{or} \quad & y_\Psi = (A/B).x_\Phi \quad ; \quad y_\Phi = -(B/A).x_\Psi \end{aligned} \qquad [1.7.5]$$

Eliminating x or alternatively y from [1.7.5] leads to the fundamental equations

$$[\,(B/A).y_{\Psi}\,]_{\Psi} \;+\; [\,(A/B).y_{\Phi}\,]_{\Phi} \;=\; 0 \qquad [1.11.1]$$

$$[\,(B/A).x_{\Psi}\,]_{\Psi} \;+\; [\,(A/B).x_{\Phi}\,]_{\Phi} \;=\; 0 \qquad [1.11.2]$$

Either of the above may be used in place of [1.7.3].

For the purpose of determining the distribution of q in equations [1.7.1] and [1.7.2] which are to be used in conjunction with [1.11.1], q may be found by eliminating Θ from [1.7.4].

Thus

$$\left.\begin{aligned}
&(x_{\Phi})^2/B^2 + (y_{\Phi})^2/B^2 = && 1.\\
&(x_{\Phi})^2/B^2 + (x_{\Phi})^2/A^2 = && 2.\\
&(y_{\Psi})^2/A^2 + (y_{\Psi})^2/B^2 = && 3.\\
&(y_{\Psi})^2/A^2 + (x_{\Psi})^2/A^2 = 1/q^2 && 4.
\end{aligned}\right\} \qquad [1.11.3]$$

It can be shown (from [1.7.5] and 2 & 3 above)

$$dx = (B/A).y_{\Psi}.d\Phi \;-\; (A/B).y_{\Phi}.d\Psi \qquad [1.11.4]$$

$$dy = -(B/A).x_{\Psi}.d\Phi \;+\; (A/B).x_{\Phi}.d\Psi \qquad [1.11.5]$$

Velocity components are given by

$$\left.\begin{aligned}
u &= q^2.x_{\Phi}/B = q^2.y_{\Psi}/A \quad \text{(i)}\\
v &= q^2.y_{\Phi}/B = -q^2.x_{\Psi}/A \quad \text{(ii)}
\end{aligned}\right\} \qquad [1.11.6]$$

And

$$\left.\begin{aligned}
\Psi_x &= -v/A \quad ; \quad \Psi_y = u/A\\
\Phi_x &= u/B \quad ; \quad \Phi_y = v/B
\end{aligned}\right\} \qquad [1.11.7]$$

Since $\quad d\Psi = (u/A).dy \;-\; (v/a).dx \quad ; \quad d\Phi = (v/B).dy \;+\; (u/B).dx$

Further algebraic and differential relationships are given in the appendices and will be refered to as necessary.

Equations [1.11.1] (or [1.11.2]) together with [1.7.2] and [1.7.3] will yield the flow solution in terms of the distribution of 'y' or 'x' with the distribution in 'q' being derived via [1.11.3].

In this chapter the generalized design plane equations are applied in conjunction with the standard flow equations to an incompressible, irrotational, invicid, axisymmetric flow with zero body forces.

The equations of motion for such a flow are

$$u.u_x + v.u_y = -(1/\rho_0).p_x \qquad [2.1]$$

$$u.v_x + v.v_y - w^2/y = -(1/\rho_0).p_y \qquad [2.2]$$

$$u.w_x + v.w_y + v.w/y = 0 \qquad [2.3]$$

$$(y.u)_x + (y.v)_y = 0 \qquad [2.4]$$

with the vorticity vector given by

$$\Omega^* = [\ (y.w)_y\ ].\underline{x} + [\ -(w)_x\ ].\underline{y} + [\ v_x - u_y\ ].\tilde{\underline{\theta}} \qquad [2.5]$$

The generalized design plane equations (see Chapter 1; [1.7.1] & [1.7.2] & [1.11.2]) are

$$[\ \ln(\ A\ )\ ]_\Phi = \epsilon^*.B/q^2 \qquad [1.7.1]$$

$$[\ \ln(\ B\ )\ ]_\Psi = -\Omega^*.A/q^2 \qquad [1.7.2]$$

$$[\ (B/A).y_\Psi\ ]_\Psi + [\ (A/B).y_\Phi\ ]_\Phi = 0 \qquad [1.11.1]$$

where $\epsilon^* = u_x + v_y$ and $\Omega^* = v_y - u_x$

Expressions are derived from the flow equations [2.1-2.5] above for the quantities ϵ^* and Ω^* and substituted into [1.7.1] and [1.7.2] whence the functions A and B are evaluated. Substitution into [1.11.1] gives an equation for solution in 'y'.

Swirl Velocity.

Since the vorticity vector is zero then the individual components are zero and from [2.5]

$(y.w)_y = 0$; $-w_x = 0$; $w_\theta = 0$ by virtue of axial symmetry.

Hence $(y.w)_y = 0$; $(y.w)_x = 0$; $(y.w)_\theta = 0$

Therefore $(y.w)$ is constant throughout the flow field.

Hence $y.w = k_0$ (say) where k_0 is constant [2.6]

Thus, for irotational flow, the swirl velocity is of the form

$$w = k_0/y \quad [2.7]$$

The Functions A and ϵ^*

From the equation of continuity [2.4]

$$(y.u)_x + (y.v)_y = 0$$

Hence $y.(u_x + v_y) = -(u.y_x + v.y_y)$

Hence
$$\begin{aligned}
\epsilon^* = u_x + v_y &= -(u.y_x + v.y_y)/y \\
&= -(u.(\ln y)_x + v.(\ln y)_y) \\
&= -[q.\cos\theta.(\ln y)_x + q.\sin\theta.(\ln y)_y] \\
&= -q.[x_s.(\ln y)_x + y_s.(\ln y)_y] \\
&= -q.(\ln y)_s \\
&= -q.q/B.(\ln y)_\Phi \quad \text{(from [1.6.11])}
\end{aligned}$$

Therefore $\epsilon^* = -q^2/B.(\ln y)_\Phi$ or $(\ln y)_\Phi = -B.\epsilon^*/q^2$

Substituting for ϵ^* into equation [1.7.1] gives

$$[\ln(A)]_\Phi = \epsilon^*.B/q^2 = -[\ln(y)]_\Phi$$

Therefore $[\ln(A.y)]_\Phi = 0$

$A.y = g_1(\Psi)$ where $g_1(\Psi)$ is an arbitrary function of Ψ. It follows that the function A is given by

$$A = g_1(\Psi)/y \quad [2.8]$$

The functions B and Ω^*

From the θ component of the expression for vorticity we have $\Omega^* = v_x - u_y = 0$

but from [1.7.2] $[\ln(B)]_\Psi = -\Omega^*.A/q^2$

=> $[\ln(B)]_\Psi = 0$

=> $B = g_2(\Phi)$ where $g_2(\Phi)$ is an arbitrary function of Φ.

Since both $g_1(\Upsilon)$ and $g_2(\Phi)$ are arbitrary we may set

$$-g_1(\Upsilon) = g_2(\Phi) = 1$$

Hence $$A = 1/y \quad ; \quad B = 1 \qquad [2.9]$$

Substituting these expressions for A and B into equations [1.11.1 to 7] gives the following set;

$$[\, y.y_{\Psi} \,]_{\Psi} + [\, (1/y).y_{\Phi} \,]_{\Phi} = 0 \qquad [2.10.1]$$

$$(x_{\Phi})^2 + (y_{\Phi})^2 = (x_{\Phi})^2 + (y.x_{\Psi})^2$$

$$(y.y_{\Psi})^2 + (y_{\Phi})^2 = (y.y_{\Psi})^2 + (y.x_{\Psi})^2$$

$$= 1/q^2 \qquad [2.10.2]$$

Further substitution into [1.11.4] and [1.11.5] gives the physical coordinates as

$$dx = y.y_{\Psi}.d\Phi - (1/y).y_{\Phi}.d\Upsilon$$

$$dy = -y.x_{\Psi}.d\Phi + (1/y).x_{\Phi}.d\Upsilon$$

Therefore $x_{\Psi} = -(1/y).y_{\Phi} \quad ; \quad x_{\Phi} = y.y_{\Psi}$

and $y_{\Psi} = (1/y).x_{\Phi} \quad ; \quad y_{\Phi} = -y.x_{\Psi} \qquad [2.10.3]$

and [1.11.6] yields the velocity components

$$u = q^2.x_{\Phi} = q^2.y.y_{\Psi}$$

$$v = q^2.y_{\Phi} = -q^2.y.x_{\Psi} \qquad [2.10.4]$$

with $\Upsilon_x = -y.v \quad ; \quad \Upsilon_y = y.u$

$$\Phi_x = u \quad ; \quad \Phi_y = v \qquad [2.10.5]$$

$$d\Upsilon = u.y.dy - v.y.dx$$

$$d\Phi = v.dy + u.dx \qquad [2.10.6]$$

There is no loss of generality in choosing $g_1(\Upsilon) = g_2(\Phi) = 1$. For suppose that $A = g_1(\Upsilon)/y$ and $B = g_2(\Phi)$ then the basic equation [1.11.1] becomes

$$[\, (g_2/g_1).y.y_{\Psi} \,]_{\Psi} + [\, (g_1/g_2).y_{\Phi}/y \,]_{\Phi} = 0$$

Since g_2 and g_1 are, respectively, functions of Φ and Υ alone, then we may write, $g_2.[(1/g_1).y_\Psi.y]_\Psi + g_1.[(1/g_2).y_\Phi/y]_\Phi = \emptyset$

Hence

$$(1/g_1).[(1/g_1).y.y_\Psi]_\Psi + (1/g_2).[(1/g_2).y/y_\Phi]_\Phi = \emptyset \qquad [2.11]$$

provided g_1 and g_2 are non zero.

Define $d\Upsilon^* = g_1(\Upsilon).d\Upsilon$ and $d\Phi^* = g_2(\Phi).d\Phi$

Hence $\Upsilon^*_\Psi = g_1(\Upsilon)$ and $\Phi^*_\Phi = g_2(\Phi)$

Hence for any function F

$$F_\Upsilon = F_{\Psi^*}.\Upsilon^*_\Psi = g_1(\Upsilon).F_{\Psi^*}$$

$$F_\Phi = F_{\Phi^*}.\Phi^*_\Phi = g_2(\Upsilon).F_{\Phi^*}$$

$$\Rightarrow \quad F_{\Psi^*} = (1/g_1(\Upsilon)).F_\Psi \quad ; \quad F_{\Phi^*} = (1/g_1(\Phi)).F_\Phi$$

Substituting into [2.11] gives

$$[y.y_{\Psi^*}]_{\Psi^*} + [(1/y).y_{\Phi^*}]_{\Phi^*} = \emptyset$$

which is identical in form to [2.1Ø.1]

Similarly the equations set [2.1Ø.#] will transform into matching forms. The choice of different functions $g_1(\Upsilon)$ and $g_2(\Phi)$ merely implies a mapping from some plane $(\Phi^{(1)},\Upsilon^{(1)})$ to another plane $(\Phi^{(2)},\Upsilon^{(2)})$ (say).

Change of Dependent Variable.

The equations may be written in a more convenient form by making the substitution

$$r = y^2 \qquad [2.12.\emptyset]$$

Thus the equation set [2.1Ø.#] becomes

$$r_{\Psi\Psi} + (\ln r)_{\Phi\Phi} = 0 \qquad [2.12.1]$$

$$(r_\Psi)^2 + (1/r).(r_\Phi)^2 = \qquad \text{(i)}$$

$$4r.(x_\Psi)^2 + 4(x_\Phi)^2 = \qquad \text{(ii)} \qquad [2.12.2]$$

$$(r_\Psi)^2 + 4r.(x_\Psi)^2 = \qquad \text{(iii)}$$

$$(1/r).(r_\Phi)^2 + 4(x_\Phi)^2 = 4/q^2 \qquad \text{(iv)}$$

$$dx = [r_\Psi.d\Phi - (\ln r)_\Phi.d\Psi]/2 \qquad \text{(i)} \qquad [2.12.3]$$

$$dy = -\,r^{.5}.x_\Psi.d\Phi + r^{-.5}x_\Phi.d\Psi \qquad \text{(ii)}$$

$$x_\Phi = r_\Psi/2 \quad ; \quad x_\Psi = -(\ln r_\Phi)/2 \qquad [2.12.4]$$

$$u = q^2.r_\Psi/2 = q^2.x_\Phi \qquad \text{(i)} \qquad [2.12.5]$$

$$v = -r^{.5}.q^2.x_\Psi = r^{-.5}.q^2.r_\Phi/2 \qquad \text{(ii)}$$

$$ds = d\Phi/q \quad ; \quad dn = d\Psi/(q.r^{.5}) \qquad [2.12.6]$$

Once the distribution of 'r' has been obtained by solving equation [2.12.1] the distribution in x may be derived via [2.12.4]. Substitution for x,r (which are now known) into any of [2.12.2] will yield the speed 'q' with its components in the 'x' and 'y' directions given by [2.12.5]. Exact solutions of the separation of variable type do exist for equation [2.12.1] and their general form has been given by Cousins (Ref. 6) and reference made to the unsuitability in applying them to flows through annular ducts.

However in the interests of having a set of exact solutions in closed form which may be used as a basis to test numerical techniques, the specific form of the solutions to [2.12.1] are now derived and a description of the corresponding flow patterns given.

The 'Exact' Solutions By Separation of Variable.

In discussing the exact solutions it is convenient to make the following substitution

Let $z = 2x$; $Q = 2/q$

This transform is a particular case of a more general one used in deriving further numerical solutions to the basic equations and with this mapping the governing equations become

$$r_{\Psi\Psi} + (\ln r)_{\Phi\Phi} = 0 \qquad [2.12.1]$$

$$r_\Psi^2 + r_\Phi^2/r = r.z_\Psi^2 + z_\Phi^2 = r_\Psi^2 + r.z_\Psi^2 = r_\Phi^2/r + z_\Phi^2 = Q^2 \qquad [2.12.2.a]$$

$$z_\Phi = r_\Psi \quad ; \quad z_\Psi = -(\ln r)_\Phi \qquad [2.12.4.a]$$

Separating the variables let

$$r(\Phi,\Upsilon) = P(\Upsilon).F(\Phi)$$

Substitution into [2.12.1] gives

$$(F.P)_{\Psi\Psi} + (\ln(F.P))_{\Phi\Phi} = 0$$

Hence $P_{\Psi\Psi} = -(1/F).(\ln(F))_{\Phi\Phi} = k_1$ (k_1 = constant)

=> $P_{\Psi\Psi} = k_1$ [2.13]

and $(1/F).(\ln F)_{\Phi\Phi} = -k_1$ [2.14]

Equation [2.13] may be integrated directly to give

$$P(\Upsilon) = k_1.\Upsilon^2/2 + p_2\Upsilon + p_3$$

where p_2 and p_3 are arbitrary constants.

Equation [2.14] yields five independent solutions for the function $F(\Phi)$ and the corresponding 'z' coordinate is derived via [2.12.4.a].

Case(1) $k_1 = 0$;

If $k_1 = 0$ then $(\ln F)_{\Phi\Phi} = 0$

$$(\ln F) = k_2.\Phi + k_3$$

Hence $F_1 = F(\Phi) = \exp(k_2.\Phi + k_3) =$

$$= k_4.\exp(k_2.\Phi) \quad (\text{where } k_4 = \exp(k_3) > 0)$$

Case(2) $k_1 \neq 0$

Let $U = \ln(F) \Rightarrow U_\Phi = F_\Phi/F$ and $F = e^U$

Therefore equation [2.14] becomes

$$U_{\Phi\Phi} = -k_1.e^U$$

$$\Rightarrow \quad U_{\Phi\Phi}.U_\Phi = -k_1.e^U.U_\Phi$$

$$\Rightarrow \quad (U_\Phi^{\ 2}/2)_\Phi = -k_1.(e^U)_\Phi$$

$$\Rightarrow \quad U_\Phi^{\ 2} = 2.(k_5 - k_1.e^U) \quad (\text{where } k_5 = \text{arbitrary constant})$$

$$\Rightarrow \quad [(1/F).F_\Phi]^2 = 2.(k_5 - k_1.e^U) = 2.(k_5 - k_1.F)$$

$$\Rightarrow \quad F_\Phi = a.F.(k_5 - k_1.F)^{.5} \quad \text{where } a = \pm 2^{.5}$$

$$\Rightarrow \quad d\Phi = dF/[a.F.(k_5 - k_1.F)^{.5}]$$

Let $F = V^2 \Rightarrow dF = 2.V.dV$.

$$d\Phi = 2.V.dV/[a.V^2.(k_5 - k_1.V^2)^{.5}] = a.dV/[V.(k_5 - k_1.V^2)^{.5}]$$

Integrating with respect to Φ gives

$$\Phi + k_6 = a.I(k_5,k_1) \text{ where } I(k_5,k_1) = \int dV/[v.(k_5 - k_1.v^2)^{.5}]$$

Different combinations of the constants k_5 and k_1 lead to the following evaluations of the integral $I(k_5,k_1)$

where $M_1 = m_1/m_2$ and $\Phi_1 = \Phi + k_6$ (k_6 = arbitrary constant)

n	k_5	k_1	$I(k_5,k_1)$	$F_n = V_n^{\ 2}$
1	0	$m_2^{\ 2}$	$i/(m_2.V)$	$-2/[m_2.\Phi_1]^2$
2	0	$-m_2^{\ 2}$	$1/(m_2.V)$	$2/[m_2.\Phi_1]^2$
3	$m_1^{\ 2}$	$m_2^{\ 2}$	$\mathrm{Sech}^{-1}[V/M_1]/m_1$	$M_1^{\ 2}\mathrm{Sech}^2[m_1.\Phi_1/a]$
4	$m_1^{\ 2}$	$-m_2^{\ 2}$	$\mathrm{Cosech}^{-1}[V/M_1]2m_1$	$M_1^{\ 2}\mathrm{Cosech}^2[m_1.\Phi_1/a]$
5	$-m_1^{\ 2}$	$m_2^{\ 2}$	$\mathrm{Cosech}^{-1}[V/M_1]/i.m_1$	$M_1^{\ 2}\mathrm{Cosech}^2[i.m_1.\Phi_1/a]$
6	$-m_1^{\ 2}$	$-m_1^{\ 2}$	$\mathrm{Sech}^{-1}[V/M_1]/i.m_1$	$M_1^{\ 2}\mathrm{Sech}^2[i.m_1.\Phi_1/a]$

Table 2.1

Applying the standard relationships between the hyperbolic functions and their trigonometric counterparts and absorbing the alternative sign in the constant 'a' into 'm_1' the complete solution for 'r' in [2.12.1] is obtained by combining the $P_n[\Upsilon]$ functions from [2.15] with the $F_n[\Phi]$ in the table above. Thus since (i) $\mathrm{Cosech}^2(-X) = \mathrm{Cosech}^2(X)$, (ii) $\mathrm{Cosech}^2(i.X) = -\mathrm{Cosec}^2(X)$ (iii) $\mathrm{Sech}^2(i.X) = \mathrm{Sec}^2(X)$.

Then linear substitutions for Υ and Φ (Φ_1) of the form

$$\Upsilon^* = a_1 + a_2.\Upsilon$$

and

$$\Phi^* = b_1 + b_2.\Phi$$

(where a_1, a_2, b_1, b_2 are constants depending on the arbitrary quantities m_1, m_2 etc.) allow the solutions for 'r' to be written as listed below (the sub. and superscripts having been dropped).

Exact Solutions for $r_{\Upsilon\Upsilon} + (\ln r)_{\Phi\Phi} = \emptyset$

n	$P_n[\Upsilon]$	$F_n[\Phi]$
Ø	$(a - \Upsilon^2)$	e^{Φ}
1	$(a - \Upsilon^2)$	Φ^{-2}
2	$(a - \Upsilon^2)$	Φ^{-2}
3	$(a + \Upsilon^2)$	$\mathrm{Sech}^2(\Phi)$
4	$(a - \Upsilon^2)$	$\mathrm{Cosech}^2(\Phi)$
5	$(a - \Upsilon^2)$	$\mathrm{Cosec}^2(\Phi)$
6	$(a - \Upsilon^2)$	$\mathrm{Sec}^2(\Phi)$

Table 2.2

n	r_n	x_n
Ø	$r_Ø = \Upsilon . e^{\Phi}$	$z_0 = (e^{\Phi} - \Upsilon)/2$
1	$r_1 = (a - \Upsilon^2).\Phi^{-2}$	$z_1 = \Upsilon/\Phi + b$
2	$r_2 = (a - \Upsilon^2).\Phi^{-2}$	$z_2 = -\Upsilon/\Phi + b$
3	$r_3 = (a + \Upsilon^2).\mathrm{Sech}^2(\Phi)$	$z_3 = \Upsilon.\mathrm{Than}(\Phi) + b$
4	$r_4 = (a - \Upsilon^2).\mathrm{Cosech}^2(\Phi)$	$z_4 = \Upsilon.\mathrm{Coth}(\Phi) + b$
5	$r_5 = (a - \Upsilon^2).\mathrm{Cosec}^2(\Phi)$	$z_5 = \Upsilon.\mathrm{Cot}(\Phi) + b$
6	$r_6 = (a - \Upsilon^2).\mathrm{Sec}^2(\Phi)$	$z_6 = -\Upsilon.\mathrm{Tan}(\Phi) + b$

Table 2.2 (contd.)

Derivation of the 'x' coordinate solutions are made via equation [2.12.4.a].

Thus from the solution for r_4 we have (droping subscripts)

$r = (a - \Upsilon^2).\mathrm{Cosech}^2(\Phi)$; $\ln r = \ln(a - \Upsilon^2) + 2.\ln(\mathrm{Cosech}(\Phi))$

From [2.12.4a] $\quad z_{\Phi} = r_{\Psi} = 2.x_{\Phi} \quad ; \quad z_{\Psi} = -(\ln r)_{\Phi} = 2.x_{\Psi}$

But $\quad r_{\Psi} = -2.\Upsilon.\mathrm{Cosech}^2(\Phi) \quad ; \quad (\ln r)_{\Phi} = -2.\mathrm{Coth}(\Phi)$

=> $\quad x_{\Phi} = -\Upsilon.\mathrm{Cosech}^2(\Phi) \quad ; \quad x_{\Psi} = \mathrm{Coth}(\Phi)$

Integrating w.r.t Φ gives ; Integrating w.r.t Υ gives

$x = \Upsilon.\mathrm{Coth}(\Phi) + G^*(\Upsilon)$ (say) ; $x = \Upsilon.\mathrm{Coth}(\Phi) + H^*(\Phi)$ (say)

Comparing the two forms for 'x' we have $G^*(\Upsilon) = H^*(\Phi) =$ constant.

Hence $\quad x = \Upsilon.\mathrm{Coth}(\Phi) + b$

Expressions for the other 'x' solutions are derived in a similar manner. The constants in the x_i solutions can be eliminated without loss of generality by the substitution

$$x_i^* = x_i - b$$

Solutions 'r_5' and 'r_6' are in fact identical as can be seen by

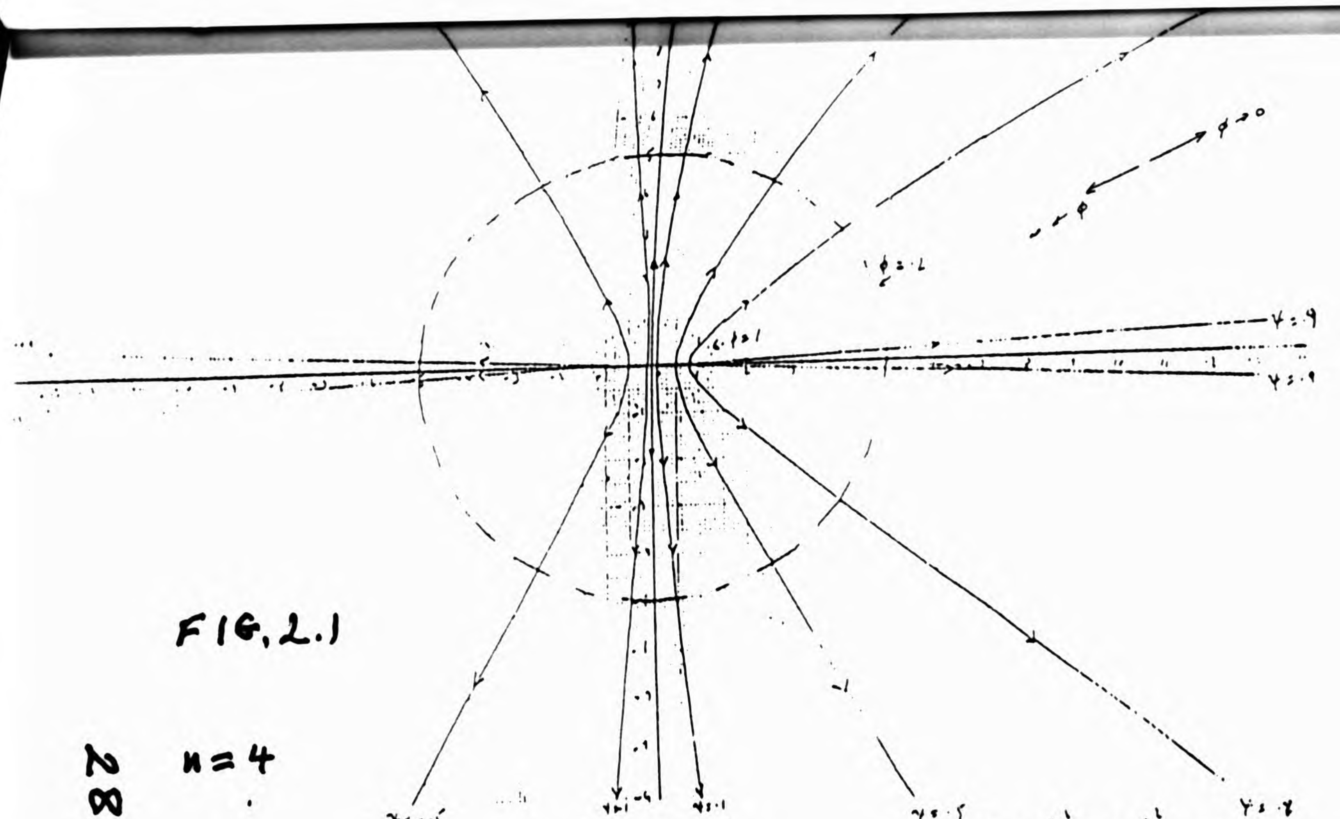


FIG. 2.1

n = 4

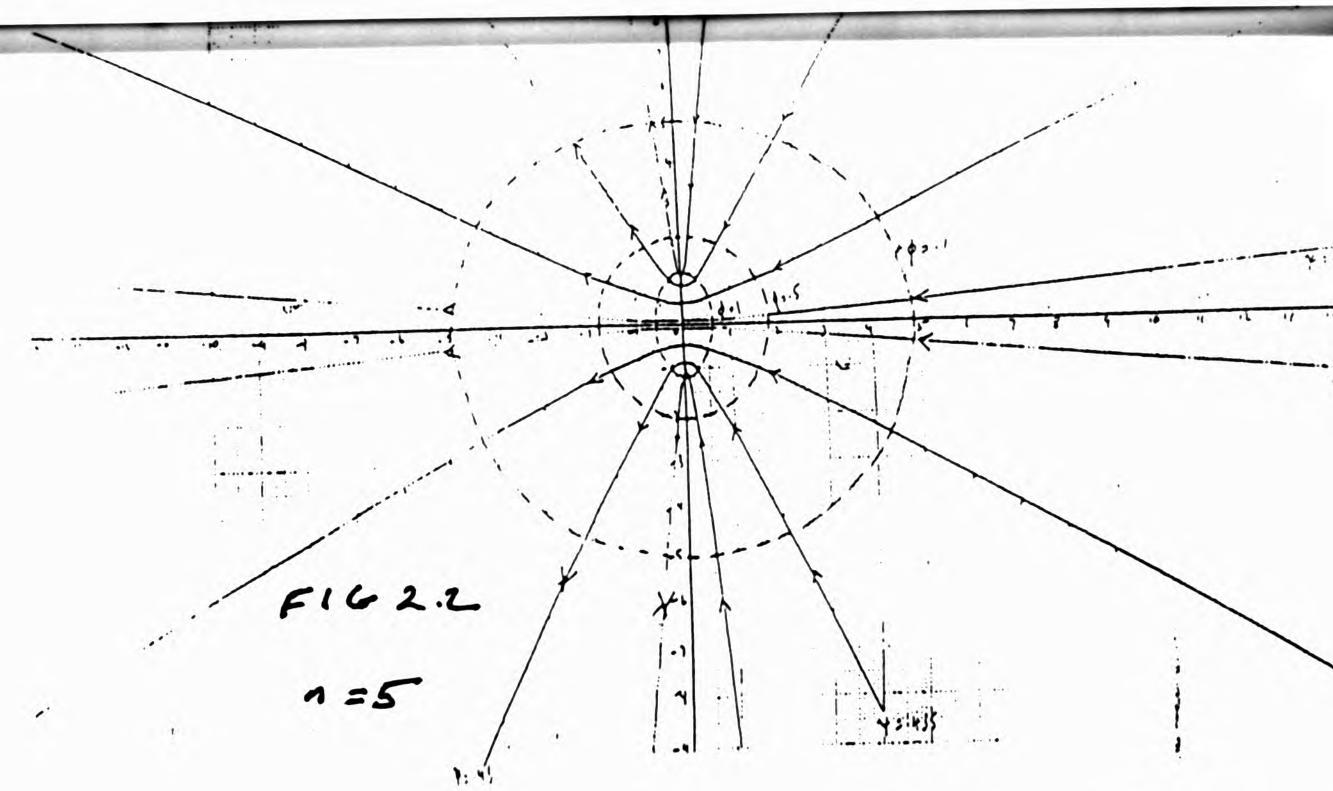


FIG 2.2

n = 5

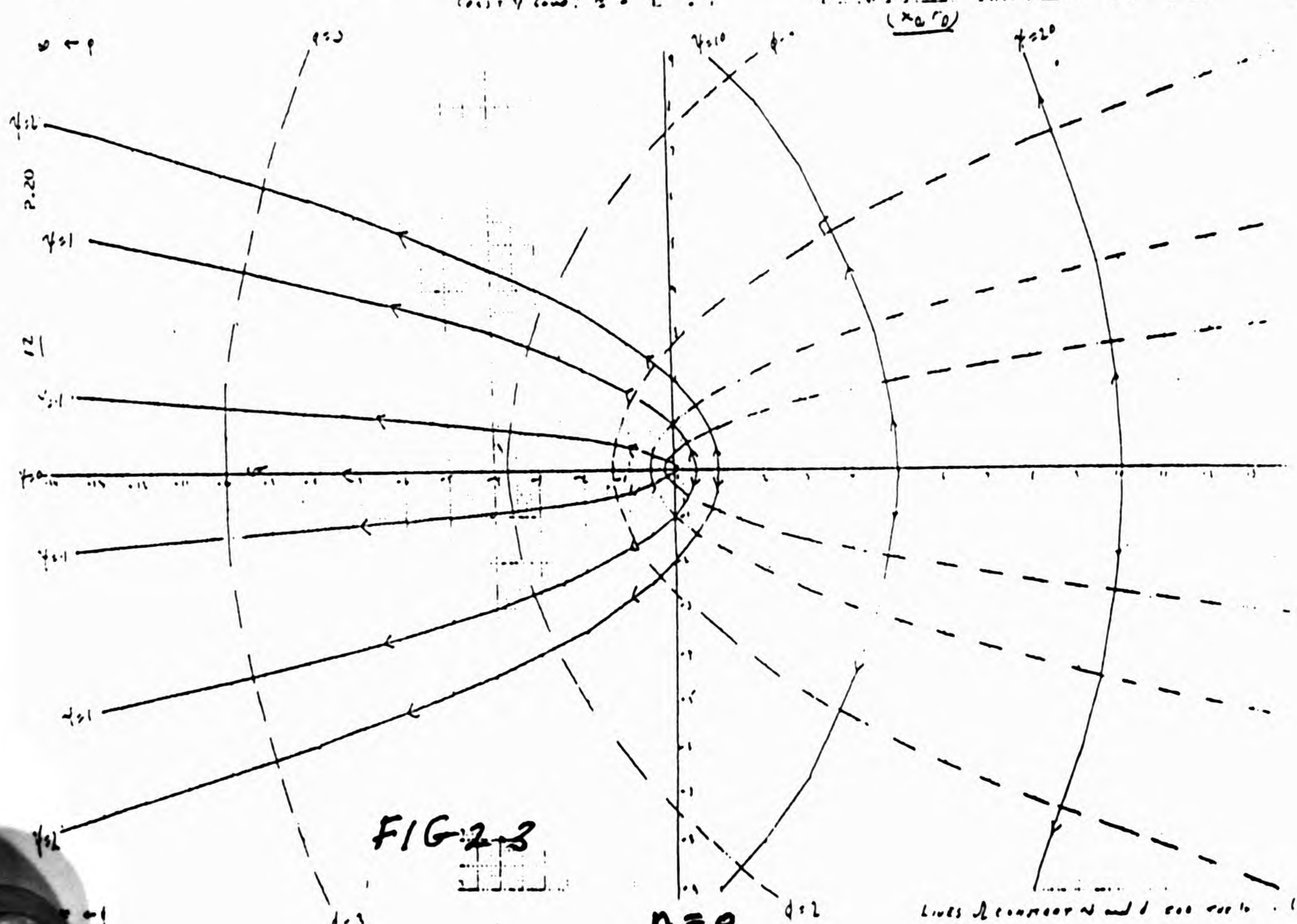


FIG 2.3

n = 0

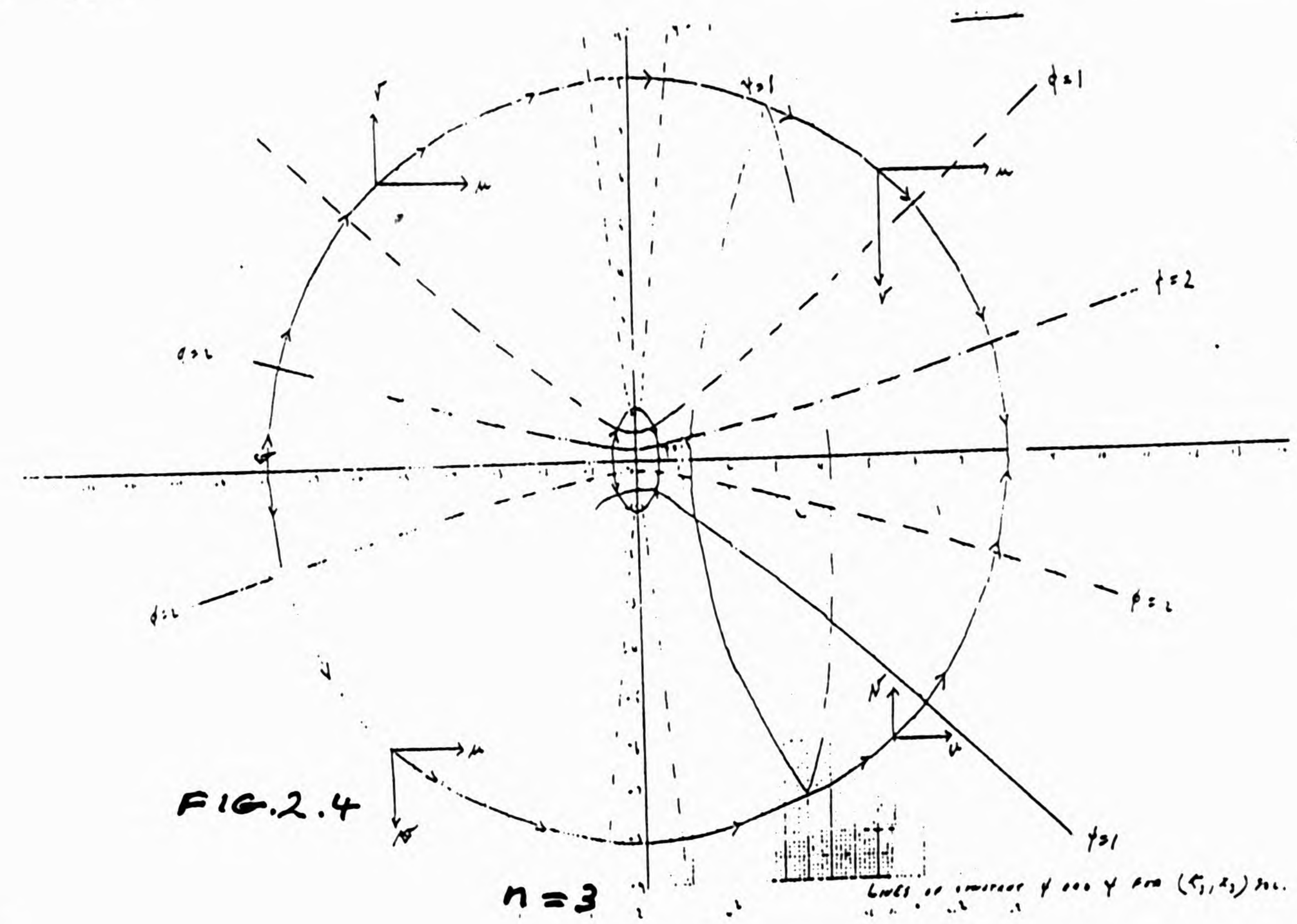


FIG. 2.4

n = 3

making the substitution $\Phi = \pi/2 - \Phi^*$

$\Rightarrow \quad Tan(\Phi) = Tan(\pi/2 - \Phi^*) = Cot(\Phi^*)$

and $Sec^2(\Phi) = Sec^2(\pi/2 - \Phi^*) = Cosec^2(\Phi^*)$

$\Rightarrow \quad x_6 = -\Upsilon.Tan(\Phi) = -Cot(\Phi^*)$

and $r_6 = (a - \Upsilon^2).Sec^2(\Phi) = (a - \Upsilon^2).Cosec^2(\Phi^*)$

which is the same form as solution 'r_5' and need not be considered separately. Similarly for 'r_1' and 'r_2'.

Surfaces of constant Φ and Υ are found by eliminating Φ and Υ from the coordinate forms in table 2.2. These surfaces form, in general, pairs of families of orthogonal confocal conics symmetric about both the x and y axes. All flows are source/sink flows having point or line singularities where one or more of the velocity components becomes infinite. By considering the change of sign of the velocity components u,v with respect to the x and y axes along lines of constant Υ and Φ, the flow patterns may be determined as shown in Figs 2.1, 2.2, 2.3, 2.4, the solid and dotted lines denoting lines of constant Υ and Φ respectively.

Orthogonal Φ-Υ Lines.

n	Lines of Constant Υ	Lines of Constant Φ
0	$y^2 = 4.(\Upsilon/2).(x - (-\Upsilon/2))$	: $y^2 = 4.(e^{\Phi}/2).(e^{\Phi}/2 - x)$
1	$y = (\pm).((a-\Upsilon^2)^{.5}/\Upsilon).x$	: $x^2 + y^2 = a/\Phi^2$
2	"	"
3	$x^2/\Upsilon^2 + y^2/(a + \Upsilon^2) = 1$	: $y^2/(aSech^2\Phi) - x^2/(aThan^2\Phi) = 1$
4	$x^2/\Upsilon^2 - y^2/(a - \Upsilon^2) = 1$	: $y^2/(aCosech^2\Phi) + x^2/(aCoth^2\Phi) = 1$
5	$y^2/(a - \Upsilon^2) - x^2/\Upsilon^2 = 1$	: $y^2/(aCosec^2\Phi) + x^2/(a.Cot^2\Phi) = 1$
6	"	"

n	Range of Φ	Range of Υ	Range of 'a'
0	$\Upsilon >= 0$:	$-\infty < \Phi < \infty$	
1,2	$0 =< \Upsilon =< a^{.5}$:	$0 =< \Phi =< \infty$:	$a >= 0$
3	$-a =< \Upsilon^2$:	$-\infty =< \Phi =< \infty$:	
4	$0 =< \Upsilon =< a^{.5}$:	$-\infty =< \Phi =< \infty$:	$a >= 0$
5,6	$0 =< \Upsilon =< a^{.5}$:	$k.\pi =< \Phi =< (k+1).\pi$:	$a >= 0$

The speed and velocity components are calculated from [2.12.2 and 5].

Thus from $q^2 = 4.(r_\Psi^{\ 2} + r_\Phi^{\ 2}/r)^{-1}$: $u = q^2.r_\Psi/2$: $v = q^2.r^{.5}.r_\Phi/2$

The speed and velocity components are given by

n	q^2	u	v
0	$4.(e^{2\Phi} + \Upsilon.e^{\Phi})^{-1}$	: $q^2.(e^{\Phi}/2)$	: $q^2(\Upsilon.e^{\Phi})^{.5}/2$
1	Φ^4/a	: $-q^2.(\Upsilon.\Phi^{-2}).a^{.5}$	: $-q^2(\Upsilon.\Phi^{-2}).a^{.5}$
2	"	: "	: "
3	$(\mathrm{sech}^2\Phi.(\Upsilon^2 + a.\mathrm{Than}^2\Phi))^{-1}$	: $q^2.\Upsilon.\mathrm{Sech}^2\Phi$	: $-q^2(a+\Upsilon^2)^{.5}.\mathrm{Sech}\Phi.\mathrm{Than}\Phi$
4	$(\mathrm{Cosech}^2\Phi.(a.\mathrm{Coth}^2\Phi-\Upsilon^2))^{-1}$	: $-q^2\Upsilon\mathrm{Cosech}^2\Phi$	: $-q^2(a-\Upsilon^2)^{.5}\mathrm{Cosech}\Phi\mathrm{Coth}\Phi$
5	$(\mathrm{Cosec}^2\Phi.(\Upsilon^2 + a.\mathrm{Cot}^2\Phi))^{-1}$	: $-q^2.\Upsilon.\mathrm{Cosec}^2\Phi$	: $-q^2(a-\Upsilon^2)^{.5}.\mathrm{Cosec}\Phi.\mathrm{Cot}\Phi$
6	"	"	"

The solutions of the form '$y^2 = P(\Upsilon).F(\Phi)$', although exhibiting interesting properties in themselves, do not have flow geometries of the type usually associated with annular ducts .

Further since the non-linearity of of the equation precludes the use of the process of super-position of solutions, it is necessary to develop numerical methods to obtain further solutions to the flow equations in the design plane. The three methods used are (i) Point Iteration (ii) Matrix Formulation (iii) Integral Equation of a complex variable and are presented in the next chapters.

CHAPTER 3

In this chapter two numerical iterative techniques for solving the equations of incompressible, irrotational flow are described together with their theoretical justification where necessary. The two methods discussed are (1) Point iteration, (2) Matrix Formulation. The discrete forms of the equations are given and used to obtain numerical solutions to the fundamental equations which are compared for accuracy with the exact ones derived in Chapter 2. The nature of the boundary conditions is examined and an acceleration procedure is given which will improve the rate of convergence of the iteration.

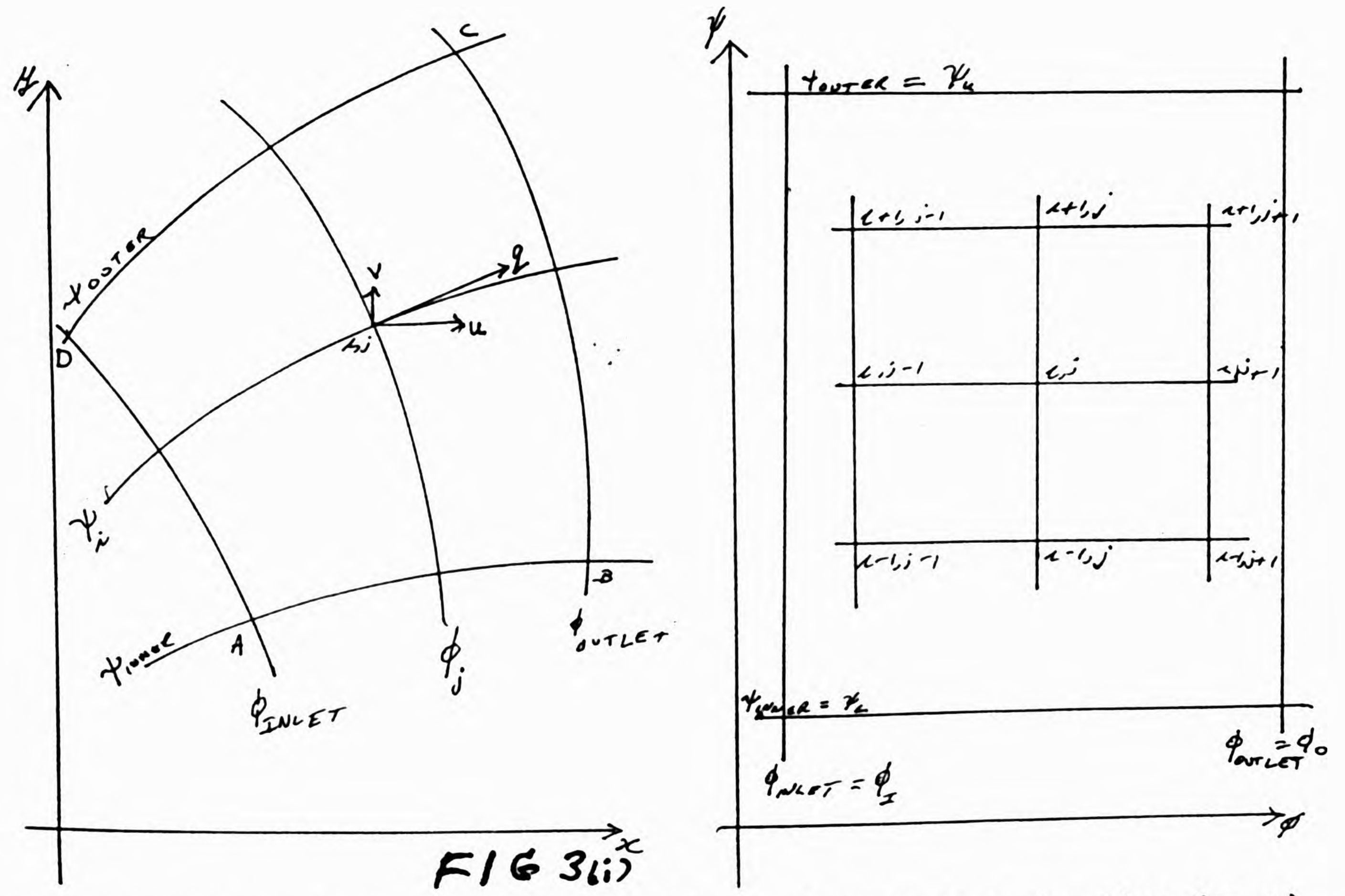


FIG 3(i)

Consider a typical section of a flow of speed q in the (x,y) plane represented by the strip ABCD and its counterpart A'B'C'D' in the (Φ,Ψ) plane. The strip ABCD is bounded by curves along lines of constant Φ and Ψ with Φ_I and Φ_O representing the inlet and outlet stations of the flow respectively and Ψ_L and Ψ_U forming the inner hub and outer casing boundaries. The transform of Chapter 1 (Equations [1.11.1] et seq.) maps the strip ABCD into a rectangular domain A'B'C'D' in which the lines of constant Φ and Ψ form an orthogonal coordinate system with the speed $q = (u^2 + v^2)^{.5}$ having, by definition, no component in the Ψ direction.

The rectangle A'B'C'D is sectioned by an n by m mesh at equal $d\Phi$ and $d\Psi$ intervals. Since Φ_I, Φ_O, Ψ_L and Ψ_U are arbitrary the following transform is employed to map A'B'C'D' onto the unit square;

$$r = c_1 . r_1 \quad ; \quad x = c_2 . x_1 \quad ; \quad q = c_3 / q_1$$

$$u = c_3 / u_1 \quad ; \quad v = c_3 / v_1 \quad ; \quad w = c_3 / w_1 \quad ; \quad Q = (u^2 + v^2 + w^2)^{.5} = c_3 / Q_1$$

$$\Psi_1 = (\Psi - c_4)/c_5 \quad ; \quad \Phi_1 = (\Phi - c_6)/c_7$$

where $c_4 = \Psi_L$; $c_5 = \Psi_U - \Psi_L$; $c_6 = \Phi_I$; $c_7 = \Phi_O - \Phi_I$

and $c_1 = (c_5/c_7)^2$; $c_2 = (c_5/(2.c_7))$; $c_3 = 2.c_7^2/c_5$

The transform is linear in the variables x, r, Φ, Ψ the velocities, however, being replaced by their reciprocals.

The differential coefficients of the transform are given by

$$d/d\Psi = (1/c_5).d/d\Psi_1 \quad ; \quad d/d\Phi = (1/c_7).d/d\Phi_1$$

$$d^2/d\Psi^2 = (1/c_5^2).d^2/d\Psi_1^2 \quad ; \quad d^2/d\Phi^2 = (1/c_7^2).d^2/d\Phi^2$$

Thus the set of equations [2.12.1] to [2.12.5] becomes (dropping the sub-scripts)

$$r_{\Psi\Psi} + (\ln r)_{\Phi\Phi} = 0 \qquad [3.2]$$

$$(r_\Psi)^2 + (r_\Phi)^2/r = r.(x_\Psi)^2 + (x_\Phi)^2 =$$

$$(r_\Psi)^2 + r.(x_\Psi)^2 = (r_\Phi)^2/r + (x_\Phi)^2 = q^2 \qquad [3.3]$$

$$dx = (r_{\Psi}).d\Phi - (\ln(r)_{\Phi}).d\Psi \qquad [3.4]$$

$$r_{\Psi} = x_{\Phi} \quad (i) \qquad ; \quad x_{\Psi} = -(\ln r)_{\Phi} \quad (ii) \qquad [3.5]$$

$$u^{-1} = q^{-2}.x_{\Phi} = q^{-2}.r_{\Psi}$$

$$v^{-1} = -q^{-2}.x_{\Psi} = r^{.5}.q^{-2}.r_{\Phi} \qquad [3.6]$$

$$0 =< \Psi =< 1 \qquad ; \qquad 0 =< \Phi =< 1 \qquad [3.7]$$

A suitable finite difference representation of equations [3.2] to [3.7] will yield numerical solutions for comparison with their exact counterparts.

Boundary Conditions.

Inlet

At inlet, on $\Phi = \Phi_I$, the values of the coordinates r, x are calculated from the exact solution at equal delta Ψ intervals across the duct and remain fixed throughout the iteration.

Outlet

As for inlet but at $\Phi = \Phi_O$.

Inner and Outer Wall Conditions.

On the duct walls, represented by Ψ_U and Ψ_L in the (Φ,Ψ) plane, the speed 'q' is known from the exact solutions and may be calculated at equal delta Φ intervals along the duct walls from inlet, Φ_I, to outlet, Φ_O. This speed distribution on the walls remains invariant throughout the iteration but, given some initial distribution in the radial coordinate, $r(0)$ say, on the duct walls, this invariant speed distribution will imply a distribution in 'r' on the duct walls varying continuously as the iteration proceeds.

Finite Difference Forms

Denoting the value of any function $F(\Upsilon,\Phi)$ at the point (Υ_i, Φ_j) in the (Υ,Φ) plane by $F_{i,j}$, then we may approximate its first and second order differentials with respect to Υ and Φ by

$$
\begin{aligned}
F_{\Upsilon} &= (F_{i+1,j} - F_{i-1,j})/(2.d\Upsilon) \\
F_{\Phi} &= (F_{i,j+1} - F_{i,j-1})/(2.d\Phi) \\
F_{\Upsilon\Upsilon} &= (F_{i+1,j} - 2.F_{i,j} + F_{i-1,j})/(d\Upsilon^2) \\
F_{\Phi\Phi} &= (F_{i,j+1} - 2.F_{i,j} + F_{i,j-1})/(d\Phi^2)
\end{aligned}
\qquad [3.8]
$$

Substituting the forms [3.8] into equation [3.2] gives

$$(r_{i+1,j} - 2.r_{i,j} + r_{i-1,j})/(d\Phi^2) + (R_{i,j+1} - 2.R_{i,j} + R_{i,j-1})/(d\Phi^2) = 0 \qquad [3.8a]$$

where $R_{i,j} = \ln (r_{i,j})$.

Solving for $r_{i,j}$ or $R_{i,j}$ yields two equations either of which may form the basis of an iterative routine to calculate $r_{i,j}$ (or $R_{i,j}$) at a given mesh point.

Thus making $r_{i,j}$ or $R_{i,j}$ the subject of [3.8a] gives

$$r_{i,j} = [r_{i+1,j} + r_{i-1,j} + D_1.\ln(r_{i,j+1} .r_{i,j-1} /r^2_{i,j})]/2 \qquad [3.9]$$

or

$$R_{i,j} = (R_{i,j+1} + R_{i,j-1} + D_2[\exp(R_{i+1,j}) - 2.\exp(R_{i,j}) + \exp(R_{i-1,j})] \qquad [3.10]$$

where

$$D_1 = (d\Upsilon/d\Phi)^2 \; ; \; D_2 = (d\Phi/d\Upsilon)^2 \; ; \; D_3 = d\Upsilon/d\Phi \; ; \; D_4 = d\Phi/d\Upsilon \qquad [3.10a]$$

Hence denoting $r_{i,j}(k)$ as the k^{th} iterated value $r_{i,j}$ we have from [3.9] as a possible iterative routine

$$r^{(k+1)}_{i,j} = .5.[r^{(k)}_{i+1,j} + r^{(k)}_{i-1,j} + D_1.\ln\{ r^{(k)}_{i,j+1} .r^{(k)}_{i,j-1} /r^{(k)2}_{i,j} \}] \qquad [3.11]$$

with a similar expression being available for [3.10].

An alternative to [3.11] based on Newton's method for finding a root of r = f(r) is

$$r^{(k)}_{i,j} = .5.[r^{(k)}_{i+1,j} + r^{(k)}_{i-1,j} + D_1.\{ 2 + \ln(r^{(k)}_{i,j+1}.r^{(k)}_{i,j-1} \}]/[1+ D_1/r^{(k)}_{i,j}] \qquad [3.12]$$

and similarly for [3.10]. For the most part [3.11] will form the basis of the iterative calculations.

The application of the prescribed speed distribution on the walls is made via one of the forms of equation [3.3]. The most convenient representation is that involving only r and its derivatives, thus

$$r^2_{\psi} + r^2_{\Phi}/r = q^2 \qquad [3.2]$$

may be approximated by

$$(r_{i+1,j} - r_{i-1,j})^2/(2.d\Psi)^2 + (1/r_{i,j})(r_{i,j+1} - r_{i,j-1})^2/(2.d\Phi)^2 = q^2_{i,j} \qquad [3.13]$$

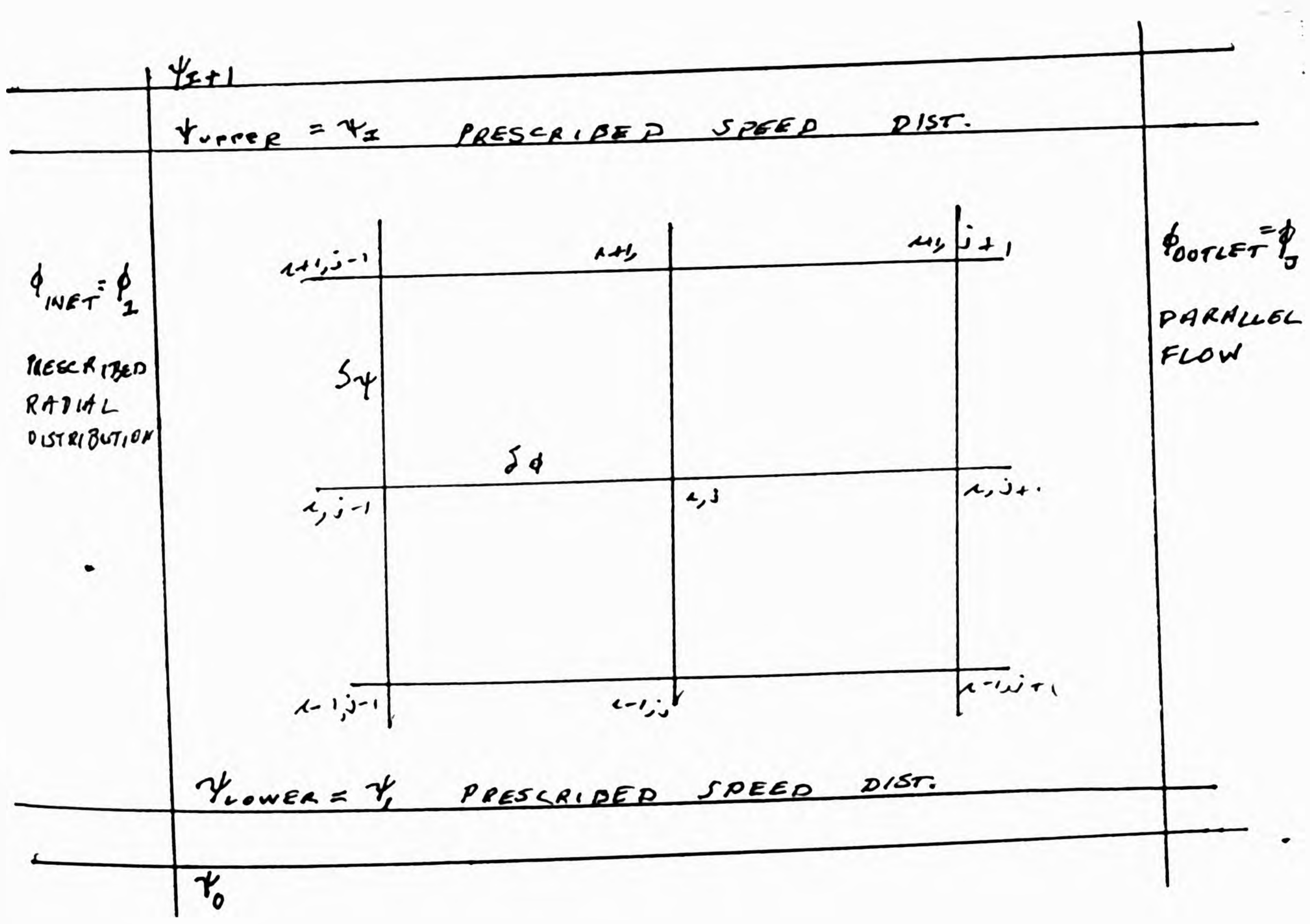


Fig 3.2.

For use at the upper and lower wall boundaries we solve successively for $r_{i+1,j}$ and $r_{i-1,j}$ and letting i = I,1 for upper an lower boundaries respectively (see Fig 3.2) we have

$$r^{(k+1)}_{i+1,j} = r^{(k)}_{i-1,j} + [(2d\Upsilon q_{i,j})^2 - D_1.(r^{(k)}_{i,j+1} - r^{(k)}_{i,j-1})^2/r^{(k)}_{i,j}]^{.5} \qquad [3.14a]$$

$$r^{(k+1)}_{i-1,j} = r^{(k)}_{i+1,j} - [(2d\Upsilon q_{i,j})^2 - D_1.(r^{(k)}_{i,j+1} - r^{(k)}_{i,j-1})^2/r^{(k)}_{i,j}]^{.5} \qquad [3.14b]$$

Boundary Conditions In Finite Difference Form

Inlet. The inlet conditions are known from the exact solution and remain fixed throughout the iteration. Thus $r_{i,1}$ is known along the inlet Φ characteristic,

$r_{i,1}$ = (Known) i= 1 to I.

Outlet As for inlet. $r_{i,J}$ = (Known) i= 1 to I.

Inner and Outer Duct Walls. The speed, q, is prescribed at equal delta Φ intervals along the inner and outer walls represented by

$\Upsilon = \Upsilon_1$ and $\Upsilon = \Upsilon_I$.

Hence $q_{i,j}$ is known for i=1,I and j=1,2...,J. on AB and CD in Fig 3.2. These speed distributions are invariant throughout the iteration but the corresponding $r_{i,j}$ are not constant on these stream lines. Equation [3.11] is used to calculate sucessive approximations for $r_{i,j}$ for i=1,2...I; j = 2,3..(J-1).

In calculating $r_{i,j}$ on the upper and lower boundaries, Υ_1 and Υ_n, the values of $r_{I+1,j}$ and $r_{0,j}$ on the 'false' boundaries are required. These are calculated via equations [3.14a] and [3.14.b] which involve the application of the prescribed speed distribution on the walls.

Calculation of the x Coordinate.

Method 1

From [3.5] we have $x_{\Phi} = r_{\Psi}$ (i) and $x_{\Psi} = -(\ln r)_{\Phi}$ (ii) [3.5]

A discrete forward difference representation of [3.5] (i) & (ii) is

$$(x_{i,j+1} - x_{i,j})/d\Phi = (r_{i+1,j} - r_{i,j})/d\Psi$$

and

$$(x_{i+1,j} - x_{i,j})/d\Psi = (-1/r_{i,j}).(r_{i,j+1} - r_{i,j})/d\Phi$$

Solving for $x_{i,j+1}$ and $x_{i+1,j}$ gives

$$x_{i,j+1} = x_{i,j} + D_4.(r_{i+1,j} - r_{i,j}) \qquad [3.15a]$$

$$x_{i+1,j} = x_{i,j} - D_3.(r_{i,j+1}/r_{i,j} - 1) \qquad [3.15b]$$

Since $x_{1,1}$ is known at inlet, then with i=1, [3.15a] may be used to calculate $x_{1,j}$ for j = 2,3...J along the characteristic Ψ_1. Then for any given j = a (say) equation [3.15b] yields the values of $x_{i,a}$ across the duct along the characteristic Φ_a for i = 2 to I. In this manner, the x-coordinates are calculated over the whole (Φ,Ψ) domain.

Method 2.

From [3.5(i)] $\qquad r_{\Psi} = x_{\Phi}$

differentiating w.r.t Φ we have $\qquad r_{\Psi\Phi} = x_{\Phi\Phi} \qquad$ [3.5(iii)]

With alternative finite difference forms for x_{Φ}, x_{Ψ}, r_{Φ}, r_{Ψ} [3.5(i)] & [3.5(iii)] may be approximated as

$$(r_{i+1,j} - r_{i-1,j})/(2.d\Psi) = (x_{i,j+1} - x_{i,j-1})/(2.d\Phi) \qquad [3.16a]$$

$$(1/(4.d\Psi.d\Phi)).(r_{i+1,j+1} + r_{i-1,j-1} - r_{i+1,j-1} - r_{i-1,j-1}) = (x_{i,j+1} - 2.x_{i,j} + x_{i,j-1})/d\Phi \qquad [3.16b]$$

Solving [3.16a] and [3.16b] for $x_{i,j+1}$ and $x_{i,j}$ respectively gives

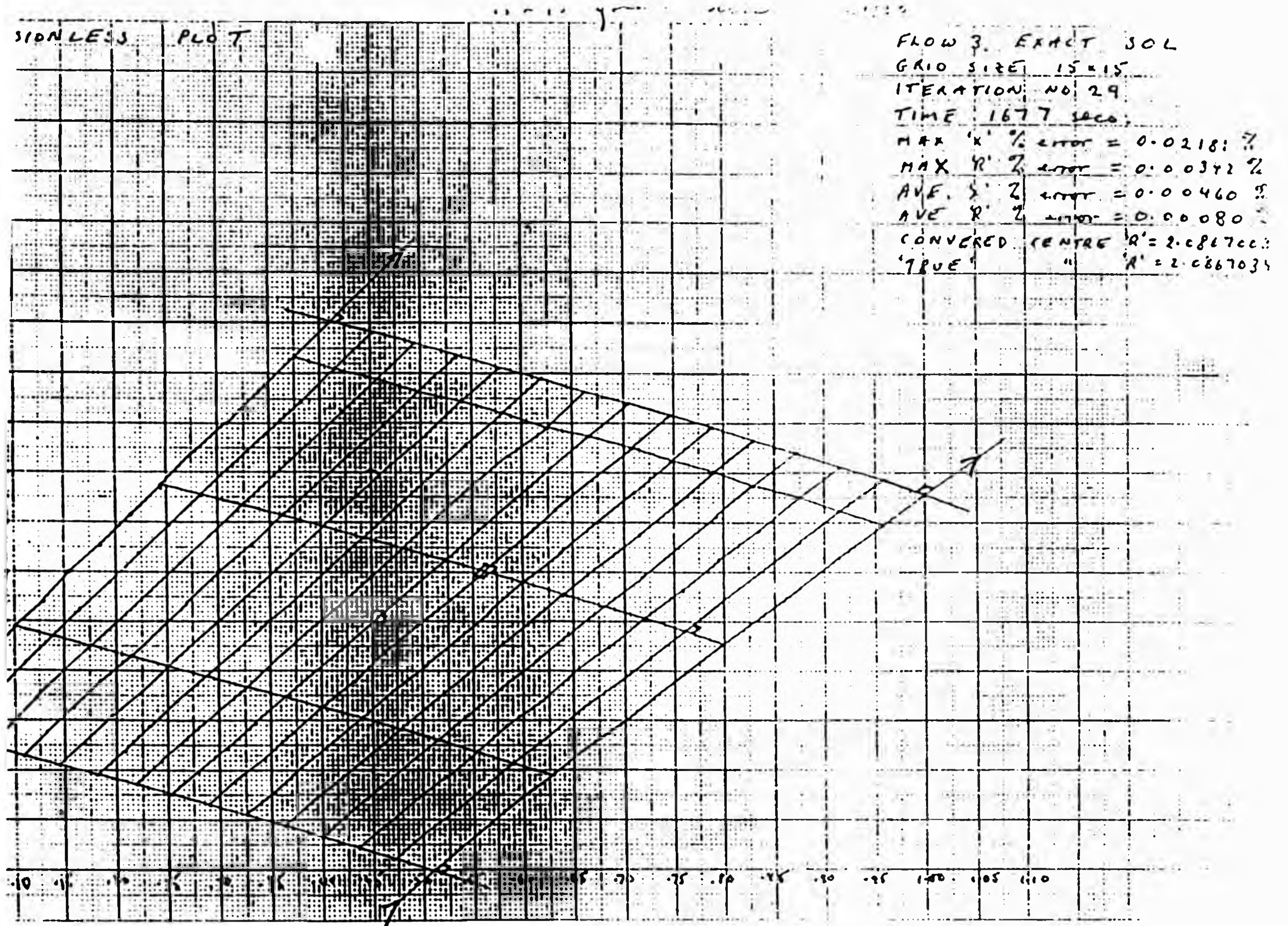


Fig. 3.3 (i)

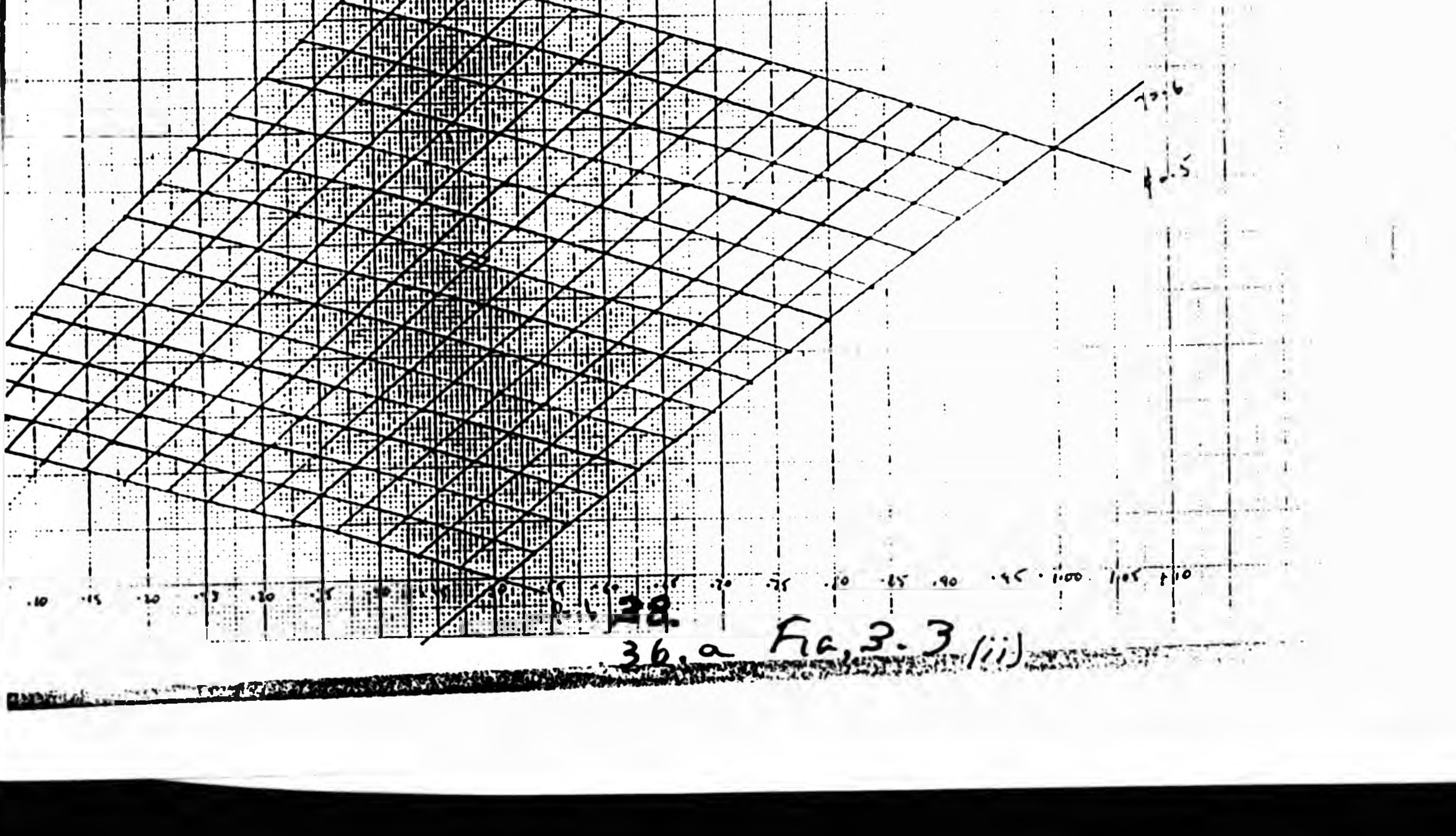


Fig. 3.3 (ii)

$$x_{i,j+1} = x_{i,j-1} + D_4.(r_{i+1,j} - r_{i-1,j}) \qquad [3.16c]$$

$$x_{i,j} = (x_{i,j+1} + x_{i,j})/2-(1/8).D_4.(r_{i+1,j+1} + r_{i-1,j-1} - r_{i+1,j-1} + r_{i-1,j+1}) \qquad [3.16d]$$

By setting $j = 2k$ in [3.16c] the ('odd') values of $x_{i,2k+1}$ can be determined and hence with $j = 2k$ in [3.16d] the intervening ('even') $x_{i,2k}$ are calculated.

The values of the x-coordinates calculated by methods 1 & 2 above give x's whose average % deviation from the exact solution is about five times greater than that for the 'r' coordinate. These errors, although small of the order of 10^{-2}% are cumulative and can 'build up' with increasing mesh size. An indication of this can be seen in the plot of a solution in Fig 3.3 (i).

To improve the accuracy of the 'x'-coordinate solution a secondary iteration routine for 'x' may be used. By forming a second order PDE in 'x' and using the values of 'x' obtained from the 'r' solution as the initial x-distribution, the acccuracy of the solution may be improved as shown in Fig 3.3 (ii).

An Iterative Routine for 'x'

From $\quad x_{\Phi} = r_{\Psi}$ and $x_{\Psi} = -(\ln r)_{\Phi}$

we have $\quad x_{\Phi\Phi} + x_{\Psi\Psi} = r_{\Phi\Psi} - (\ln r)_{\Psi\Phi} = (r - \ln r)_{\Psi\Phi} \qquad [3.17]$

Let $F = r - \ln r$, then [3.17] may be approximated by the finite difference equation

$$(x_{i,j+1} - 2.x_{i,j} + x_{i,j-1})/(d\Phi^2) + (x_{i+1,j} - 2.x_{i,j} + x_{i-1,j})/(d\Psi^2) =$$

$$(F_{i+1,j+1} + F_{i-1,j-1} - F_{i+1,j-1} - F_{i-1,j+1})/(4.d\Phi.d\Psi)$$

Solving for $x_{i,j}$ gives

$$x_{i,j} = A1.(x_{i,j+1} + x_{i,j-1}) + A2.(x_{i+1,j} + x_{i-1,j}) + A3.F^*_{i,j} \quad [3.18]$$

where $A1= .5.(1 + D_2)^{-1}$; $A2 = A1.D_2$; $A3 = -A1.D_4/4$;

$$F^* = F_{i+1,j+1} + F_{i-1,j-1} - F_{i+1,j-1} - F_{i-1,j+1}.$$

Equation [3.18] forms the basis of an iteration routine together with boundary conditions on 'x' furnished by [3.5] above since 'r' is known over the whole flow field. Hence

$$x^{(k+1)}_{i,j} = [A_1.(x_{i,j+1} + x_{i,j-1}) + A2.(x_{i+1,j} + x_{i-1,j}) + A3]^{(k)}$$

Values of 'x' on the lower and upper boundaries are derived from the numerical equivalents of [3.5(ii)] and are given by

$$x_{i+1,j} = x_{i-1,j} - D_3.\ln(r_{i,j+1}/r_{i,j-1})$$

$$x_{i-1,j} = x_{i+1,j} + D_3.\ln(r_{i,j+1}/r_{i,j-1})$$

for the upper and lower boundaries respectively. This correction improved the accuracy of the calculation of the x coordinate and reduced the errors to the same order as that of the r-coordinate. At outlet the boundary condition for x is obtained from [3.5(i)] giving $x_{i,j+1} = x_{i,j-1} + D_4.(r_{i+1,j} - r_{i-1,j})$

In the context of the present 'test case' this last condition is redundant since the outlet values of r and x can be calculated from the exact solutions available to us. However in cases in which this data is not available the above procedure provides a means of applying an outlet condition on x.

Test programs to determine the distributions of x,r,q over the (Φ,Ψ) plane for all the exact flow solutions were written for use on micro-computers.

Convergence.

The iteration was continued until the relative difference between successive approximations for some specified test value of r, R_T (= $r_{a,b}$ say) was less than some assigned quantity 'e'. This value was taken as a fraction of the average difference between the radial coordinates of the 'middle' Φ-line.

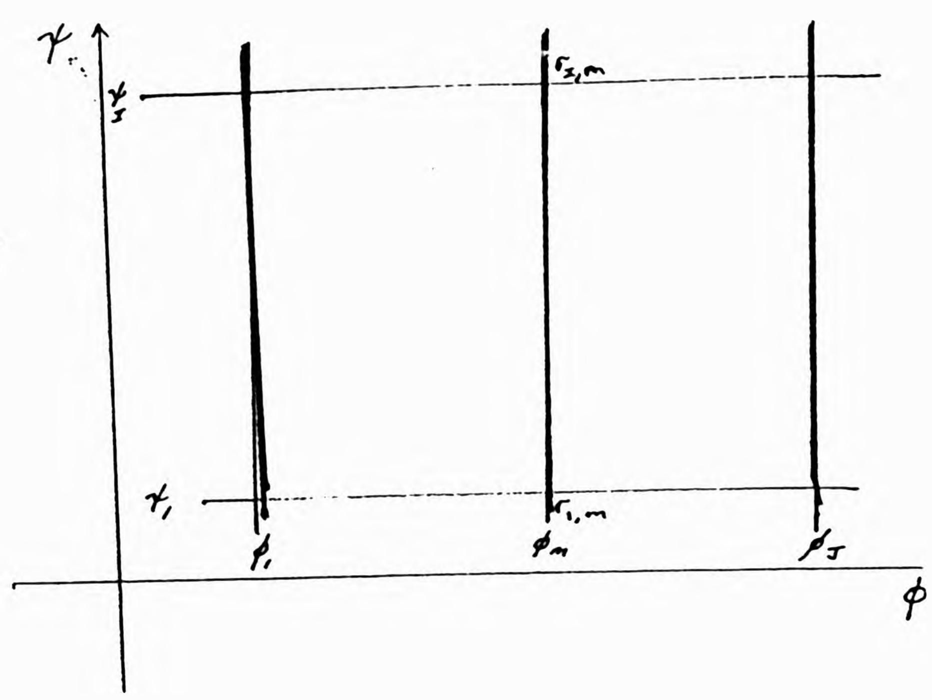


Fig. 3.5

Hence

$$e = (r_{I,m} - r_{1,m})/(5000.m) \quad = o(10^{-5}) \text{ with } m = \mathrm{Int}(J/2).$$

and the iteration was deemed to have converged when

$$\left| \; r_{a,b}(k+1)/r_{a,b}(k) - 1 \; \right| \;<\; e$$ (where 'k' is the iteration no.

The values of x and r derived from the iteration were compared with those of the exact solution and the maximum and average relative errors calculated. These results were checked for consistancy by back substitution into the finite difference forms of the PDE and their variation with increasing mesh size noted. These comparisons are detailed in Tables 3.6 and 3.7 and shown graphically in Fig 3.8 (i) and (ii).

An Acceleration Procedure.

It was noted that in the course of an iteration, the ratio of successive differences of 'r' was approximately constant. Based upon this, the following procedure was deduced to accelerate the convergence of the iteration routine. Suppose that, for some 'r',

$$\frac{(r(k+2) - r(k+1))}{(r(k+1) - r(k))} = A \text{ (say)}$$

where $r(k)$ is the k^{th} iterate of an 'r'. Rearrangement gives

$$r(k+2) - (1 + A).r(k+1) + A.r(k) = 0 \qquad [3.22]$$

Letting $r(k) = X^k$ and substituting into [3.22] we have (after dividing through by $A.X^k$)

$$X^2 - (1 + A).X + A = 0$$

$\Rightarrow$ $X = 1$ or A.

$\Rightarrow$ $r(k) = a_1.(1)^k + a_2.A^k$

For $n = 0,1$ we have $r(0) = a_1 + a_2$

$r(1) = a_1 + a_2.A$

Solving for a_1 and a_2 gives

$a_1 = (r(1) - A.r(0))/(1 - A)$; $a_2 = (r(0) - r(1))/(1 - A)$

Hence the general form of solution for $r(k)$ is

$$r(k) = (r(1) - A.r(0))/B + (r(0) - r(1)).A^k/B \qquad [3.23]$$

where $B = 1 - A$.

With the best available estimate for A given by

$$A = (r(2) - r(1))/(r(1) - r(0))$$

Thus the k^{th} iterate of r , i.e $r(k)$, can be considered as the k^{th} term in the sequence given by [3.23].

Now providing $|A| < 1$ then the limit of the sequence in [3.23] is $r^{(L)}$ where

$$r^{(L)} = (r^{(1)} - A.r^{(0)})/(1 - A) =$$
$$= r^{(2)} - \{(r^{(2)} - r^{(1)})^2\}/\{ r^{(0)} - 2.r^{(1)} + r^{(2)} \}$$

which can be recognized as Aitken's delta squared process.

It was found that if the limiting form of [3.23] was used to increase the rate of convergence, the predicted values of 'r' tended to 'overshoot' the required value. In practice the full form of [3.23], was used (at every third iteration) with 'k' taken as some suitable function of the mesh size to give 'smoother' convergence.

Further it was found that, for some choices of the Φ-Υ range in which a solution was sought to the 'exact' flows, the iteration did not always converge. The application of the following condition was found to remedy this difficulty.

If $H(r)$ is some function of r such that $H(r)$, $H_r(r)$ [= dH/dr] are defined and continuous in some range $r_1 <= r <= r_2$ (say), and if $| H_r(r) | <= K < 1$ in $r_1 < r < r_2$, then the iteration $r^{(n+1)} = H(r^{(n)})$ will converge to a root of $r = H(r)$ in (r_1, r_2).

Extending this principle to the system $r_{i,j} = H(r_{i,j} , r_{p,q})$

then the iteration $r^{(k+1)}_{i,j} = H(r^{(k)}_{i,j} , r^{(k)}_{p,q})$

would converge if

$$| \partial H(r_{i,j} , r_{p,q})/\partial r_{i,j} | <= K < 1 \quad \text{for all } i,j \qquad [3.25]$$

The particular iteration used in this chapter is based on

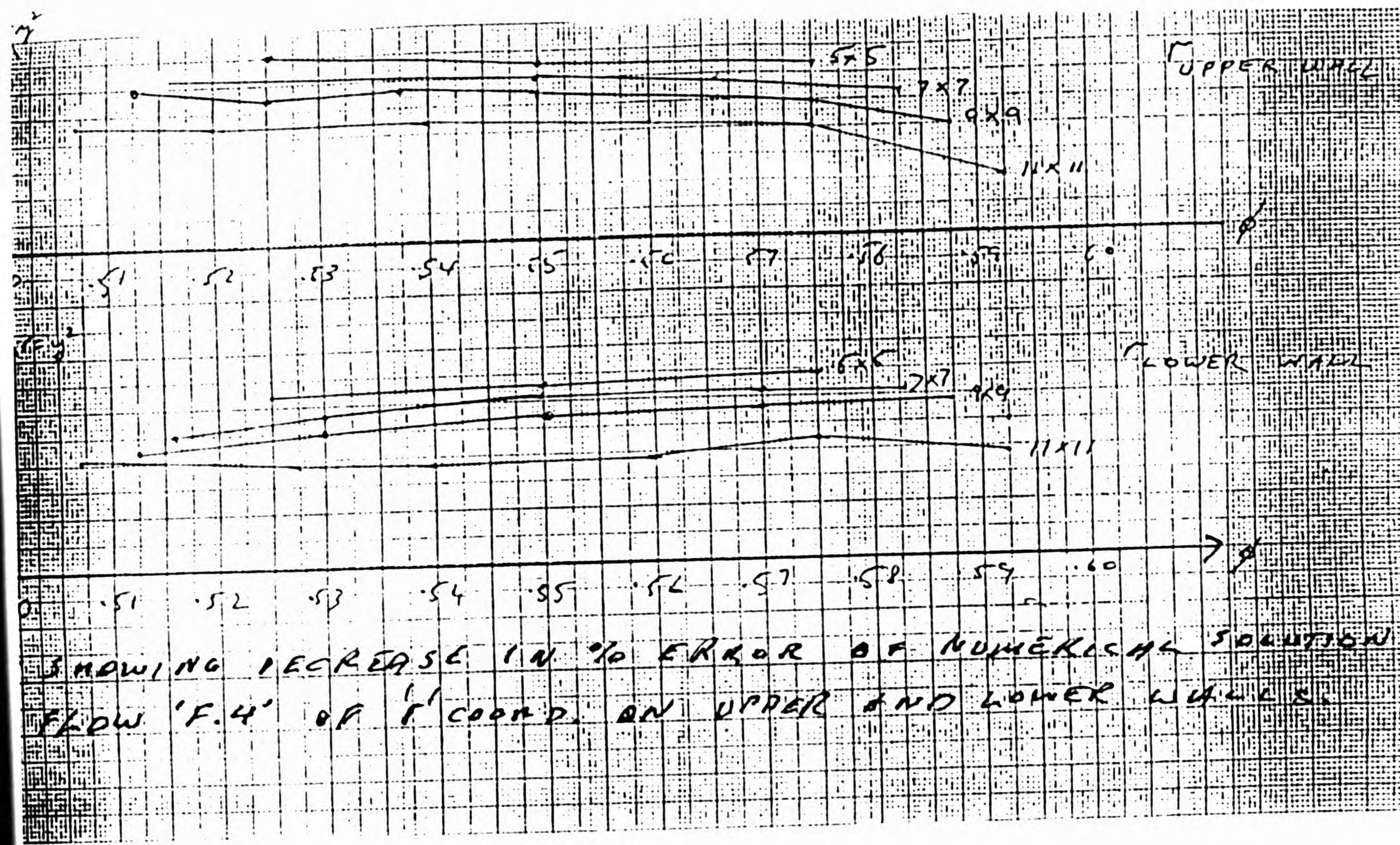


Fig 3.8(i)

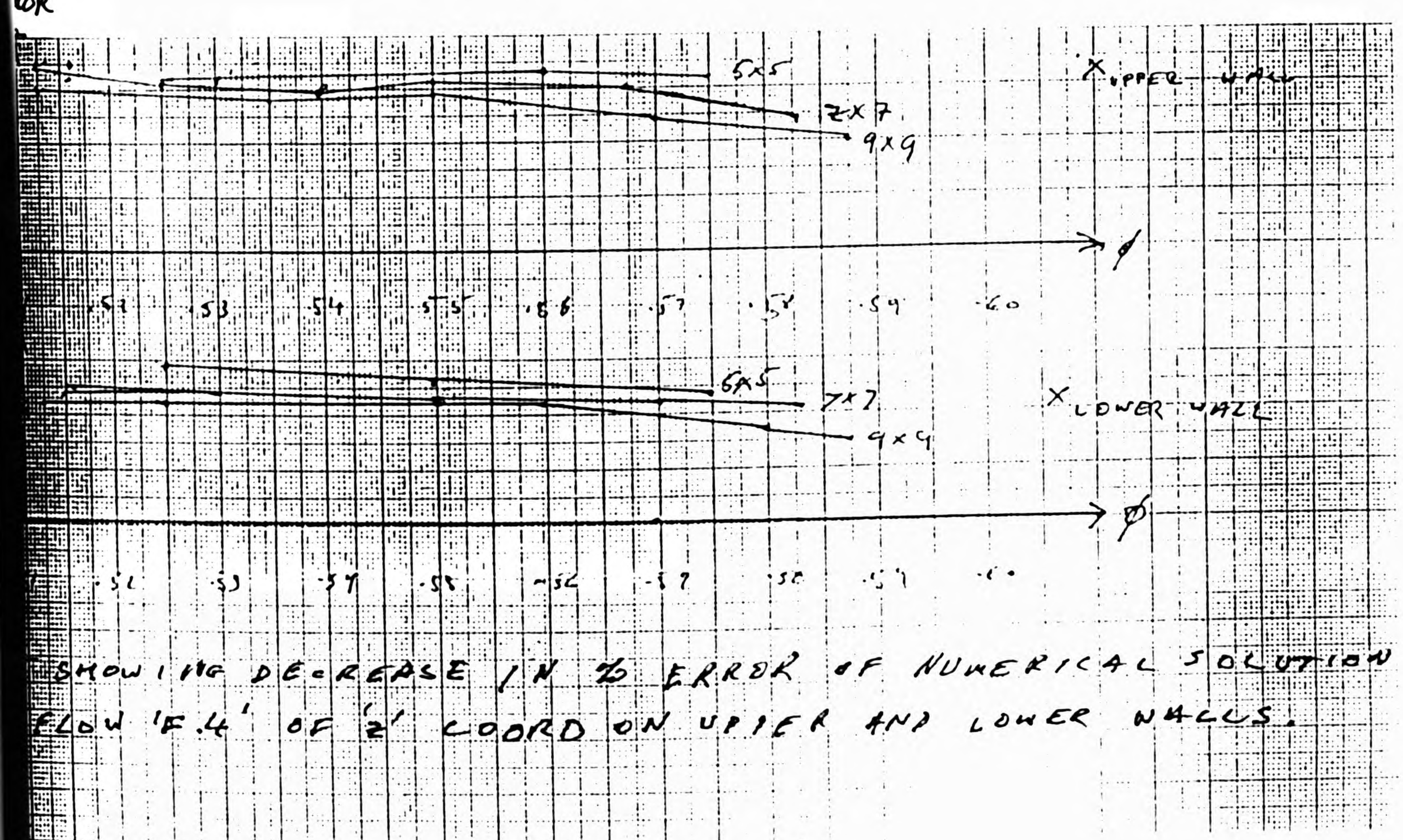


Fig. 3.8 (ii)

$$r_{i,j} = [\, (r_{i+1,j} + r_{i-1,j}) + D_1 . \ln(r_{i,j+1} . r_{i,j-1} / r^2_{i,j}) \,]/2 \qquad [3.9]$$

$$\Rightarrow \quad r_{i,j} = F^* - D_1 . \ln(r_{i,j}) = H(r_{i,j}\,,\, r_{p,q})$$

where F^* is a function independent of $r_{i,j}$. Differentiating this expression with respect to $r_{i,j}$ and applying [3.25] gives

$$\left| \partial H(r_{i,j}\,, r_{p,q}) / \partial r_{i,j} \right| = \left| -D_1 / r_{i,j} \right| = D_1 / r_{i,j} < 1$$

$$\Rightarrow \quad r_{i,j} >= D_1 \text{ for all } i,j. \qquad [3.26]$$

In the course of many test runs of the programmes, it was found that the iteration invariably converged when this condition was satisfied and diverged or oscillated otherwise. A suitable choice of $d\Upsilon$ and $d\Phi$ can be made to ensure that [3.26] is satisfied.

<u>RESULTS</u>

The program provides numerical results for all the exact flow solutions and Table 3.6 and 3.7 below gives details of the numerical values obtained for the exact solution for flow F4. The graphs in Fig 3.8(i) , (ii) below show the improvement in acccuracy of the numerical routines with increasing grid size.

[Epsilon = 10^{-5}]

(1) Grid Size	(2) Num. of Pts.	(3) Iteration Number	(4) (ACC)	(5) Time(Secs)	(6) (ACC)	(13) Converged Val.of R_T [2.08670349 True]
5*5	15	19	(13)	144	(114)	2.08863442
7*7	35	22	(15)	306	(231)	2.08668037
9*9	63	32	(18)	685	(425)	2.08669332
11*11	99	43	(22)	1329	(750)	2.08669820
13*13	143	58	(22)	2438	(1020)	2.08670060
15*15	195	73	(26)	4011	(1557)	2.08670245
17*17	255	91	(34)	6327	(2582)	2.08670376
19*19	323	112	(67)	9640	(6262)	2.08670481
21*21	399	133	(63)	13860	(7109)	2.08669848

Table 3.6

Relative% Errors.

Grid Size	(7) Max x%	(8) Max r%	(9) Ave. x%	(10) Ave. r%	(11) Max r% 10^{-5}	(12) Ave r% 10^{-6}
5*5	.2830	.0499	.0700	.0096	2.390	2.010
7*7	.1220	.0215	.0279	.00438	.726	.674
9*9	.0680	.0116	.0149	.00247	.460	.580
11*11	.0430	.0072	.0093	.00158	.380	.520
13*13	.0290	.0048	.0063	.00109	.290	.430
15*15	.0219	.0034	.0046	.00080	.300	.470
17*17	.0167	.0025	.0035	.00060	.280	.470
19*19	.0131	.0019	.0027	.00048	.260	.450
21*21	.0107	.0016	.0021	.00039	.250	.215

Table 3.7

Column Key.

1 : Grid Size,
2 : Number of variable points calculated
3 : " " iterations to convergence.
4 : " " " " " (Accelerated)
5 : Time in secs to converge.
6 : " " " " " (Accelerated)
7 : Max Relative % error in x-coord from exact solution.
8 : " " " " " r " " " "
9 : Ave " " " " x " " " "
10 : " " " " " r " " " "
11 : Max " " " " r " when backsub. into PDE
12 : Ave " " " " r " " " " "
13 : Converged value of r_{test} point.

The number of iterations required to satisfy the convergence criteria is reduced by up to 60% when the ‘accelerator’ is applied to the iteration. Within the range of mesh size considered the errors decrease steadily to a small fraction of a percent.

Solution Scheme In Terms Of Matrices

In this section the basic equations for incompressible, irrotional flow are expressed in an alternative finite difference form and the set of finite difference equations so obtained are expressed as matrices. A method of solution based on this formulation is presented and numerical results for the 'exact' solutions obtained from a computer program using this approach are given. The degree of accuracy of the results and the rate of convergence of the iterative routine is similar to that of the point iteration method.

Matrix Form of the Finite Difference Equations.

The fundamental equation [3.2] is

$$r_{\psi\psi} + (\ln r)_{\Phi\Phi} = 0$$

We may rearrange this as

$$r_{\psi\psi} + r_{\Phi\Phi} = (r - \ln r)_{\Phi\Phi} = F_{\Phi\Phi} \qquad [3.27]$$

Similarly we may obtain

$$R_{\psi\psi} + R_{\Phi\Phi} = (R - e^{R})_{\psi\psi} = -F_{\psi\psi} \qquad [3.27a]$$

where $R = \ln r$ and $F = r - \ln r = e^{R} - R$.

Either of [3.27] or [3.27a] may be used in the subsequent derivation. Expressing [3.27] in finite difference form we have

$$(r_{i+1,j} - 2r_{i,j} + r_{i-1,j})/d\Upsilon^2 + (r_{i,j+1} - 2r_{i,j} + r_{i,j-1})/d\Phi^2 =$$

$$(F_{i,j+1} - 2.F_{i,j} + F_{i,j-1})/d\Phi^2$$

Putting $D = D_2$

$$r_{i,j-1} + (D.r_{i-1,j} - 2.(1+D).r_{i,j} + D.r_{i+1,j}) + r_{i,j+1} =$$

$$= F_{i,j+1} - 2.F_{i,j} + F_{i,j-1}$$

for i= 1 to I ; j =2 to J-1: (The values j = 1,J being excluded since these are the known fixed inlet and outlet values).
It follows that the complete set of equations may be written with j = 2,..(J-1)

$$\begin{bmatrix} -2(1+D) & D & & & & & \\ D & -2(1+D) & D & & & 0 & \\ & D & -2(1+D) & D & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & D & -2(1+D) & D & \\ & 0 & & & D & -2(1+D) & D \\ & & & & & D & -2(1+D) \end{bmatrix} \begin{bmatrix} r_{1,j} \\ r_{2,j} \\ r_{3,j} \\ \cdots \\ r_{I-2,j} \\ r_{I-1,j} \\ r_{I,j} \end{bmatrix} =$$

$$\begin{bmatrix} r_{1,j-1} \\ r_{2,j-1} \\ r_{3,j-1} \\ . \\ r_{I-2,j-1} \\ r_{I-1,j-1} \\ r_{I,j-1} \end{bmatrix} + \begin{bmatrix} r_{1,j+1} \\ r_{2,j+1} \\ r_{3,j+1} \\ . \\ r_{I-2,j+1} \\ r_{I-1,j+1} \\ r_{I,j+1} \end{bmatrix} + \begin{bmatrix} D.r_{0,j} \\ 0 \\ 0 \\ . \\ 0 \\ 0 \\ D.r_{I+1,j} \end{bmatrix} = \begin{bmatrix} F_{1,j-1} \\ F_{2,j-1} \\ F_{3,j-1} \\ \\ F_{I-2,j-1} \\ F_{I-1,j-1} \\ F_{I,j-1} \end{bmatrix} -2 \begin{bmatrix} F_{1,j} \\ F_{2,j} \\ F_{3,j} \\ \\ F_{I-2,j} \\ F_{I-1,j} \\ F_{I,j} \end{bmatrix} + \begin{bmatrix} F_{1,j+1} \\ F_{2,j+1} \\ F_{3,j+1} \\ \\ F_{I-2,j+1} \\ F_{I-1,j+1} \\ F_{I,j+1} \end{bmatrix}$$

Defining the column vectors L_j and H_j as

$$L_j = \begin{bmatrix} r_{1,j} \\ r_{2,j} \\ . \\ r_{i,j} \\ . \\ r_{I,j} \end{bmatrix} \quad ; \quad H_j = \begin{bmatrix} F_{1,j-1} \\ F_{2,j-1} \\ . \\ F_{i,j-1} \\ . \\ F_{I,j-1} \end{bmatrix} -2. \begin{bmatrix} F_{1,j} \\ F_{2,j} \\ . \\ F_{i,j} \\ . \\ F_{I,j} \end{bmatrix} + \begin{bmatrix} F_{1,j+1} \\ F_{2,j+1} \\ . \\ F_{i,j+1} \\ . \\ F_{i,j+1} \end{bmatrix} - \begin{bmatrix} D.r_{0,j} \\ 0 \\ . \\ 0 \\ . \\ D.r_{I,j} \end{bmatrix}$$

and the 'D' matrix by A then the set may be written as

$$L_{j-1} + A.L_j + L_{j+1} = H_j \quad ; \quad j = 2 \text{ to } J-1 \qquad [3.29]$$

Form of soution for Equation [3.29]

Scheme 'A'

Suppose that the vectors L_j satisfy a relation of the form

$$L_j = B.L_{j+1} + C_j \qquad [3.30]$$

where the C_j are column vectors and B is a constant matrix.

From [3.30] we have (for 'j = j-1')

$$L_{j-1} = B.L_j + C_{j-1} \qquad [3.31]$$

Substituting from [3.31] into [3.29] for L_{j-1}

$$\Rightarrow \quad B.L_j + C_{j-1} + A.L_j + L_{j+1} = H_j$$

Solving this equation for L_j ;

$$L_j = -(A + B)^{-1}.L_{j+1} + (A+B)^{-1}.(H_j - C_{j-1}) \qquad [3.32]$$

Comparing this expression for L_j with the original one in [3.30]

i.e $L_j = B.L_{j+1} + C_j$

shows that the B matrix and C_j vectors satisfy the equations

$$\begin{aligned} B &= -(A + B)^{-1} && \text{(a)} \\ C_j &= (A + B)^{-1}.(H_j - C_{j-1}) = - B.(H_j - C_{j-1}) && \text{(b)} \end{aligned} \qquad [3.33]$$

<u>Scheme 'B'</u>

Alternatively let

$$L_j = M.L_{j-1} + E_j$$ where M is a constant matrix.

A similar calculation to the above will lead to the corresponding set of relations for M and E.

$$\begin{aligned} M &= -(A + M)^{-1} && \text{(a)} \\ E_j &= (A + M)^{-1}.(H_j - E_{j+1}) = - M.(H_j - E_{j+1}) && \text{(b)} \end{aligned} \qquad [3.34]$$

Since the matrices M and B satisfy the same equation we may set M = B. Either of [3.33] or [3.34] may be used as the basis for an iterative routine to solve for the L_j vectors. Thus with 'k' denoting the iteration number we may formulate the iteration schemes

$$L_j^{(k+1)} = B.L_{j+1}^{(k)} + C_j^{(k+1)} \quad \text{(a)} : C_j^{(k+1)} = -B.(H_j^{(k)} - C_{j-1}^{(k)}) \quad \text{(b)} \quad \text{('A',}$$

$$L_j^{(k+1)} = B.L_{j-1}^{(k)} - E_j^{(k+1)} \quad \text{(a)} : E_j^{(k+1)} = -B.(H_j^{(k)} - E_{j+1}^{(k)}) \quad \text{(b)} \quad \text{('B',} \qquad [3.35]$$

$$B = - (A + B)^{-1} \qquad \text{(c)}$$

For scheme 'A' we recall that from the definitions, the vector H_j is a function of the current $r^{(k)}_{i,j}$. Thus given some initial vector, $C^{(k)}_1$, we may calculate the $C^{(k+1)}_j$ vectors from [3.35] in a left to right sweep across the grid. The L_j vectors are then derived via [3.35(a)] by sweeping back across the grid in the opposite sense. This iteration cycle is repeated until some convergence criterion is satisfied by the set of L_j vectors. Scheme 'B' differs only in so far as the direction of the sweep is reversed.

The matrix B, once calculated, is constant throughout the iteration, however by the nature of its definition it must be derived iteratively by solving [3.35 (c)]

using $$B^{(k+1)} = -\left(A + B^{(k)} \right)^{-1}$$

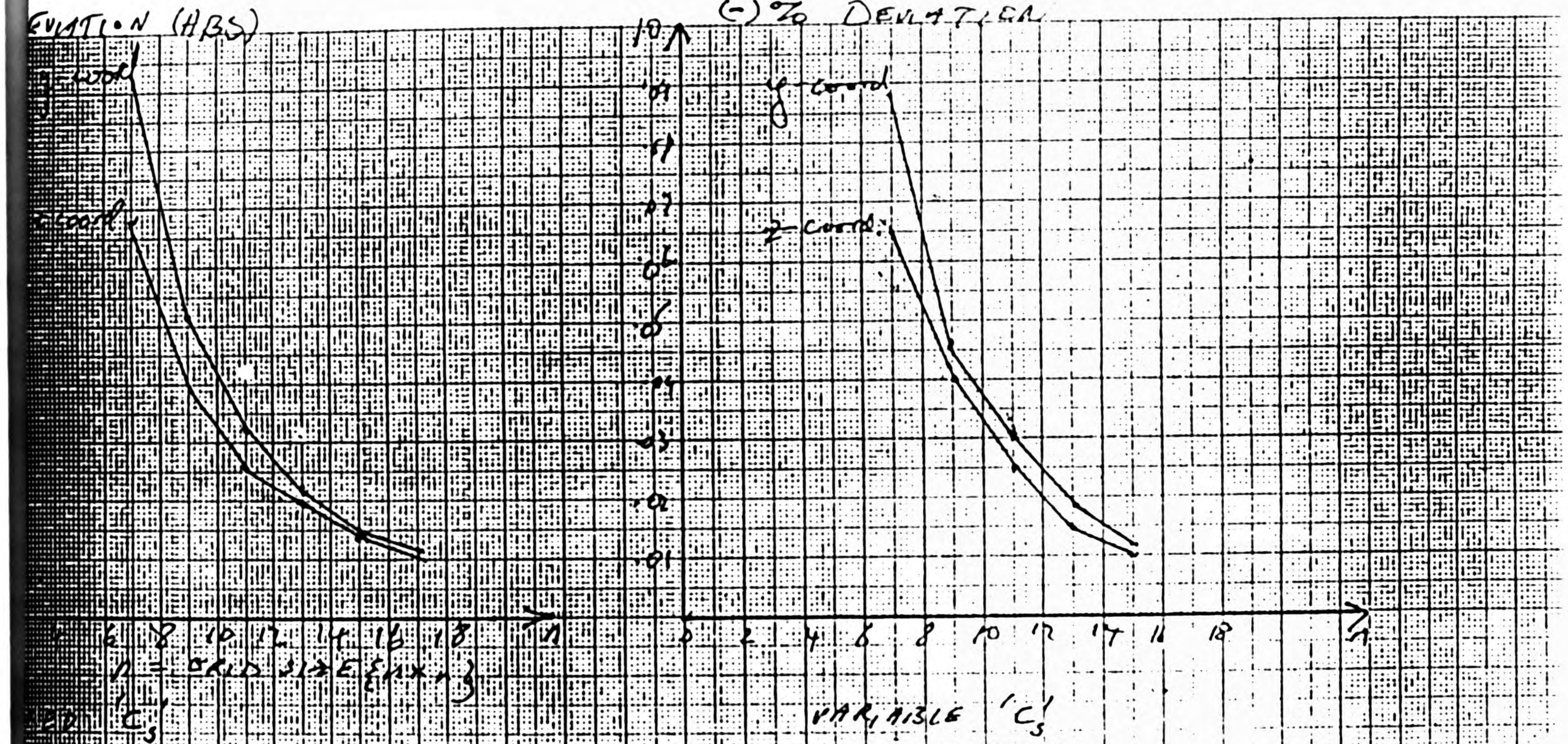
At each iteration the matrix is inverted using Gaussian elimination. The computing time taken to calculate the converged 'B' matrix was of the same order of magnitude as that required to solve for the L_j vectors. The B matrix was found to be centro-symmetric.

In order to calculate the $C^{(k+1)}$ vectors, some initial vector $C^{(k)}_s$ is required.

It is posible to define at least two distinct $C^{(k)}_s$ vectors for an iteration, corresponding to the situations in which the boundary conditions across the duct at the inlet and outlet station are known

(i) only at the inlet and outlet stations for i=1 to I ; j =1,J:

(ii) at and upstream of inlet and at and downstream of outlet at j= Ø,1 and j = J,J+1.



...XIMUM % DEVIATION OF CONVERGED SOLUTION FROM ...ACT SOLUTION.

Thus for (i) at j=1 (or j= J) $L_j = B.L_{j+1} + C_j$

$$\Rightarrow \quad L_1 = B.L_2 + C_1$$

Hence for the start of the k^{th} iteration the initial 'C' vector, C_s, is given by

$$C^{(k)}_s = C^{(k)}_1 = L^{(k)}_1 - B.L^{(k)}_2$$

In this case since L_1 is fixed and L_2 varies then $C^{(n)}_s$ changes with each iteration.

(ii) If information is available upstream of j= 1 (i.e at j=Ø) then from

$$L_j = B.L_{j+1} + C_j$$

with j = Ø we have

$$L_\emptyset = B.L_1 + C_\emptyset$$

$$\Rightarrow \quad C_\emptyset = L_\emptyset - B.L_1$$

In this case $C^{(n)}_s$ is constant through out the iteration and is given by

$$C^{(k)}_s = C^{(\emptyset)}_s = L_\emptyset - B.L_1 = C_\emptyset$$

Programs were written to allow for the application of both of these types of inlet and outlet conditions and produced identical solutions of similar accuracy.

It is posssible to combine the two schemes 'A' & 'B' but it was found that an iteration based jointly on 'A' and 'B' would not satisfy equivalent convergence criteria when applied separately and oscillated between the two solutions associated with the schemes. However the numerical difference between the two solutions yielded by 'A' and 'B' is very small and a slight relaxation of the convergence condition when employing the two schemes jointly would produce convergence. However in the present context there is no obvious advantage to such an approach.

In Fig.3.10 below 'A' represents the 'path' of a typical test point R_T from its initial value $R^{(0)}_T$ to its converged value $R^{(c)}_T$ when using scheme 'A' and similarly for 'B' while 'C' represents the path when 'A' and 'B' are used jointly. The relative difference defining convergence for 'A' and 'B' used separately was of the order of 10^{-7}, hence the separation between the two solutions is at most 10^{-6} (See Fig 3.10).

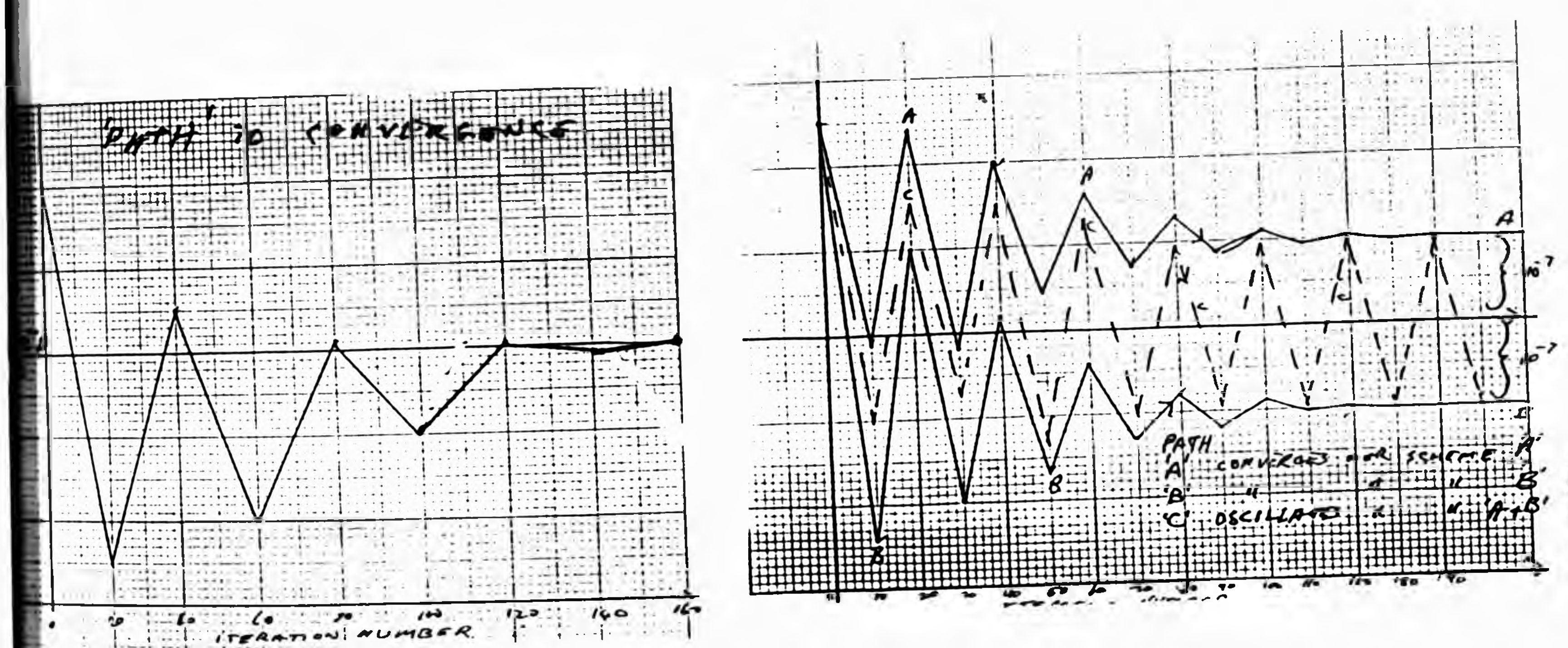
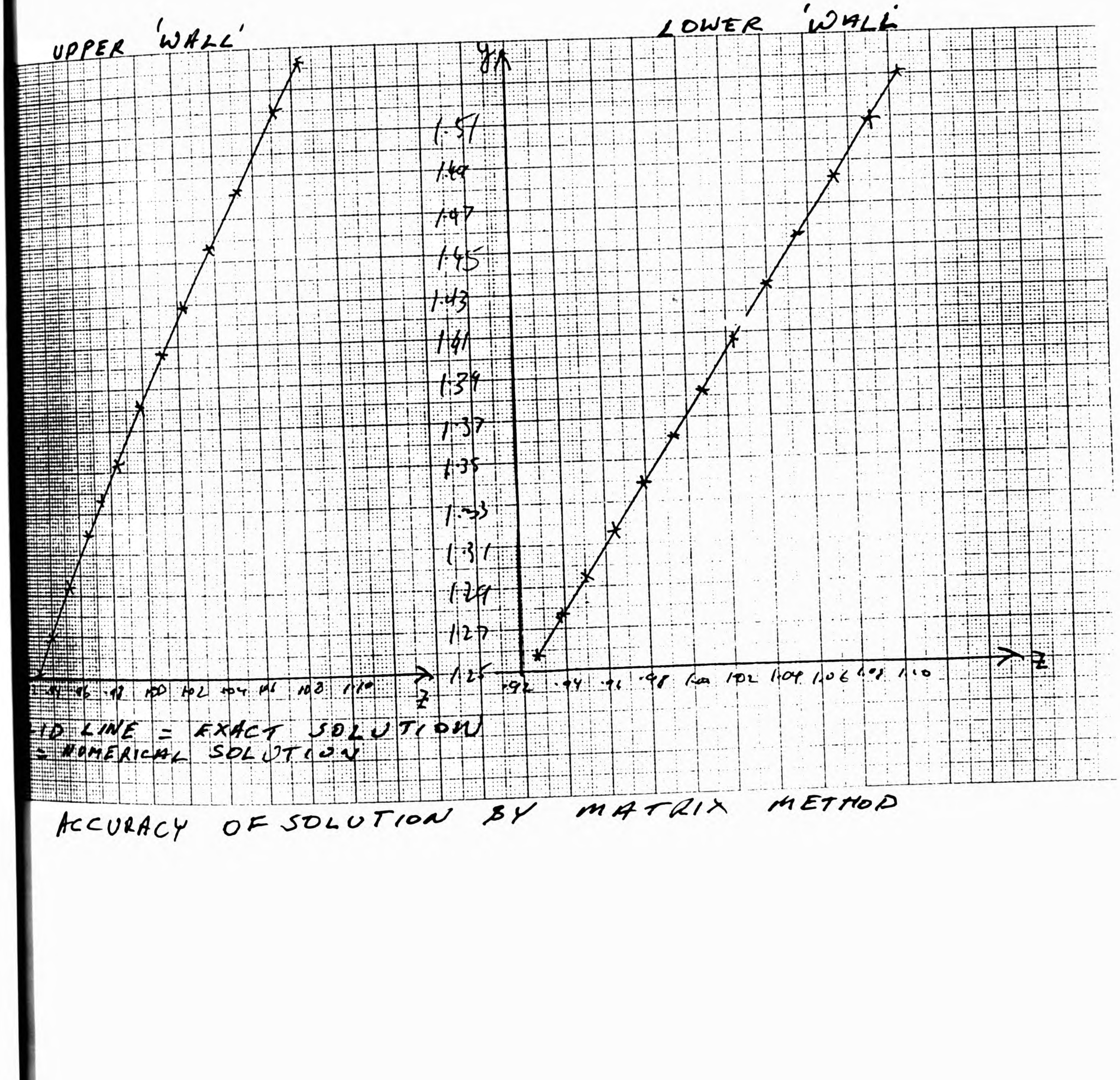


Fig 3.10

The table below gives the numerical results obtained for the solution to flow F4 and may be compared with the results in tables 3.6 and 3.7 derived by the point iterative method. Since the degree of accuracy achieved by both methods is the same only columns 1,3,5,7,8,13 are listed (below) for comparison.



ACCURACY OF SOLUTION BY MATRIX METHOD

Numerical Results for flow F4 By Matrix Method.

1	3	5	7	8	13
Grid Size	Iter. Numb.	Time Secs	Max x% Error	Max r% Error	Converged R_T
5*5	47	976	.1226	.0857	2.08671750
7*7	66	1349	.0935	.0681	2.08670732
9*9	66	2769	.0518	.0405	2.08670829
11*11	39	2725	.0324	.0270	2.08670520
13*13	46	5176	.0221	.0194	2.08670471
15*15	27	4508	.0155	.0146	2.08670331
17*17	36	8647	.0115	.0107	2.08669981
19*19	31	10123	.0098	.0092	2.08670662
21*21	35	15390	.0084	.0075	2.08671137
					(2.08670349 True)

Table 3.11

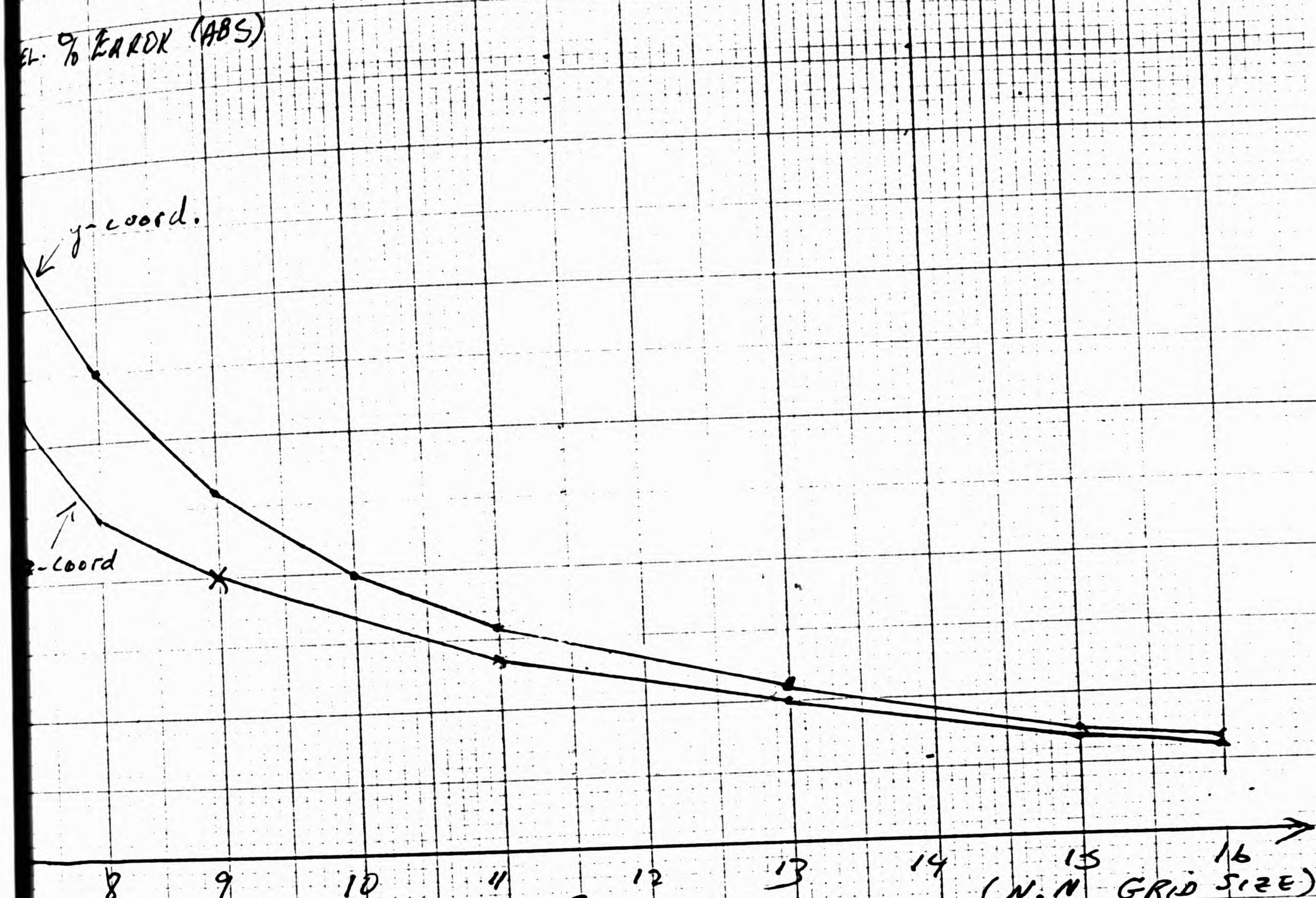
In Table 3.13, below, the comparison between the results obtained for fixed and variable inlet C_s vectors is given. The values listed are the ratios of corresponding results of the two schemes; e.g; Col 2 = (conv. R_T for fixed C_s/conv. R_T for var. C_s}.

Grid Size	Converged R	Ratios of Max X% Error	Max R% Error
7*7	1.000013078	1.02252	1.07471
9*9	1.000007318	1.02015	1.12854
11*11	1.000004040	1.02009	1.13998
13*13	1.000002943	1.04301	1.16931
15*15	1.000002233	1.02816	1.19230

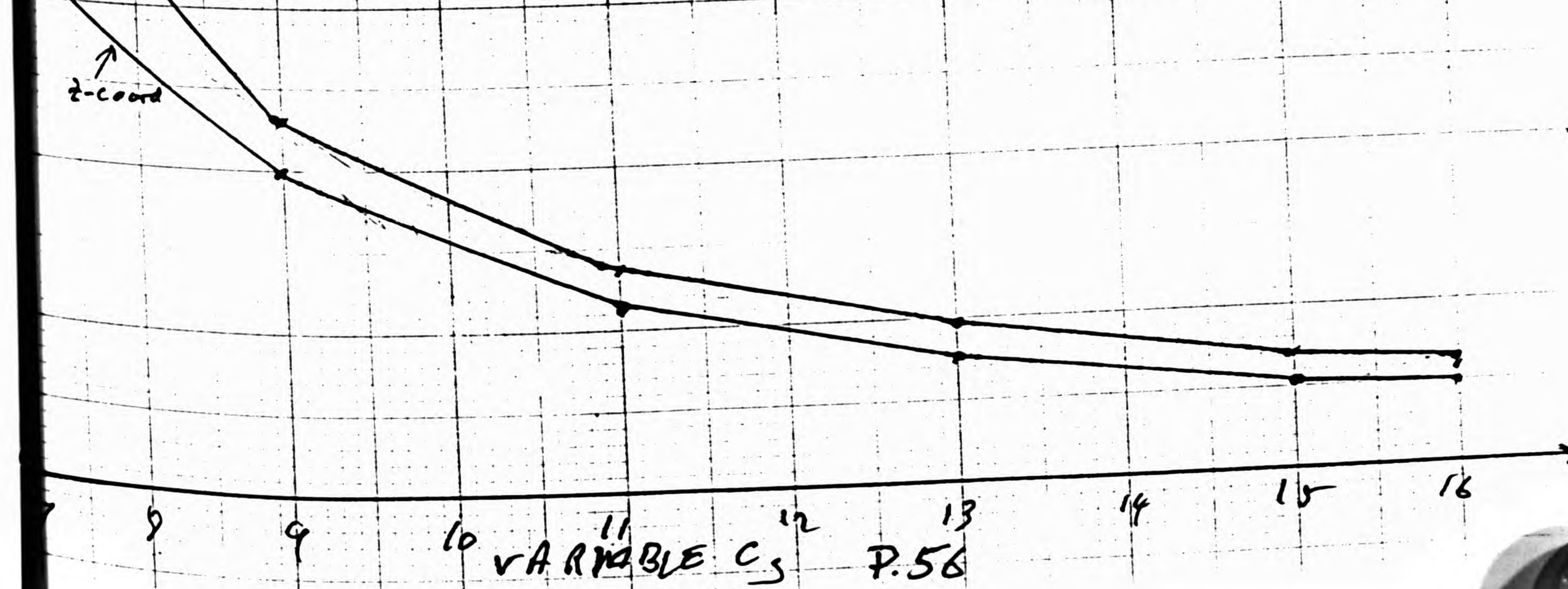
Table 3.13

Bearing in mind that maximum errors are of the order of 10^{-2} of a percent, the agreement between the two methods of solution is good. The number of iterations required for convergence is much less for the matrix method but the time required for convergence is comparable. This apparent contradiction is due to the fact that the matrix method involves substantially greater amount of manipulation of the variables (in the form of matrix arithmatic etc.).

OMPARISON of % DEVIATION of CONVERGED SOLUTION FROM
XACT SOLUTION FOR (a) FIXED C_s, (b) VARIABLE C_s

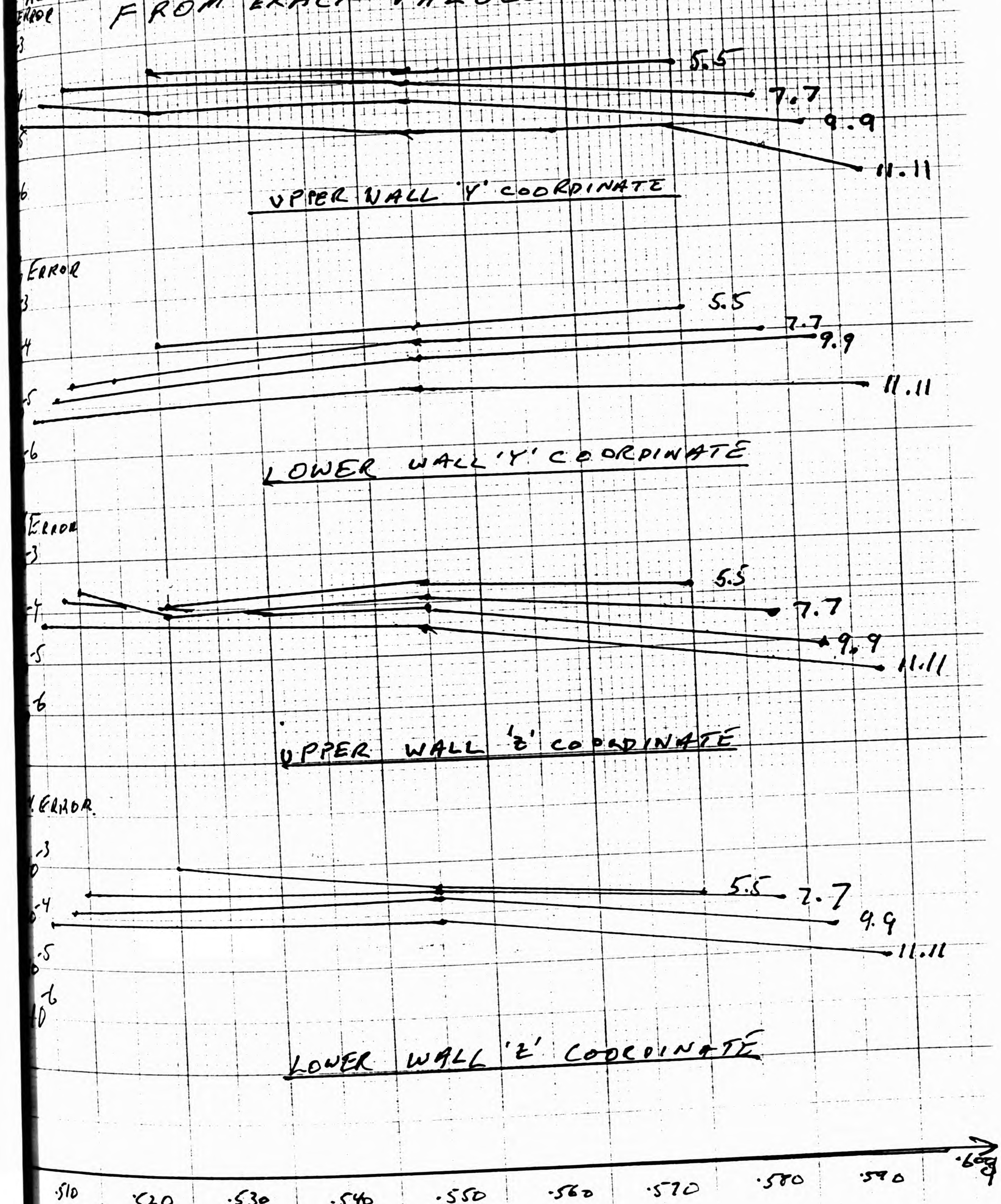


FIXED C_s



VARIABLE C_s P.56

ABSOLUTE % DEVIATION OF CALCULATED CONTOUR FROM 'EXACT' VALUES



n.n denotes grid size.

Fig 3.12 compares the convergence for R_T for fixed and variable C_s; maximum deviation of 'y' and 'z'-coordinates from the exact solution for increasing grid size.

The conclusion is that both these methods of solution yield very accurate results for the flow fields calculated and may be safely extended to obtain solutions to the partial differential equations for alternative boundary conditions.

The programs for solving the flow equations for both the point and matrix iteration methods were written for micro-computers with a clock speed of the order of 1MHz. In the interest of reducing the time to convergence the 'kernel' of the routine for the point iteration method was rewritten in assembly code and accessed outside the normal 'Basic'. This reduced the time required for the programme to converge by a factor of 3 (somewhat dissappointingly). However given that current micros have clock speeds of the order of 20+MHz and that Basic compilers are now available for use on them, the run times listed above may be reduced by up to 2 to 3 orders of magnitude giving times of approx 60 secs for a 21.21 matrix on 'stand alone' micros. On larger computer systems the time to needed for the iteration to converge would be reduced to a fracion of a second.

Chapter 4

In this chapter the solution to the equation of flow is derived in terms of a function of a complex variable and expressed as a contour and field integral in the (Φ,Υ) domain. A two point Lidstone expansion is used to approximate variations of the flow quantities across the duct as a power series in Υ, an alternative expansion is also available for this purpose. The coefficients of this series are functions of the dependent variables, r, x and their derivatives with repect to Φ evaluated at the wall boundaries and are therefore independent of any cross-stream variations and are functions of Φ alone.

This permits the integration of the field term with respect to Υ thereby removing the cross stream (Υ) dependency from the field integral. The result may be expressed in closed form thus reducing the field term to a line integral.

The values of the dependent variable pair (x,r) at any point on the contour are then given as the sum of a contour integral and line integral of a function of the complex variable $z = \Phi + i.\Upsilon$.

4.(I) Solution as an Integral Equation of a Complex Variable .

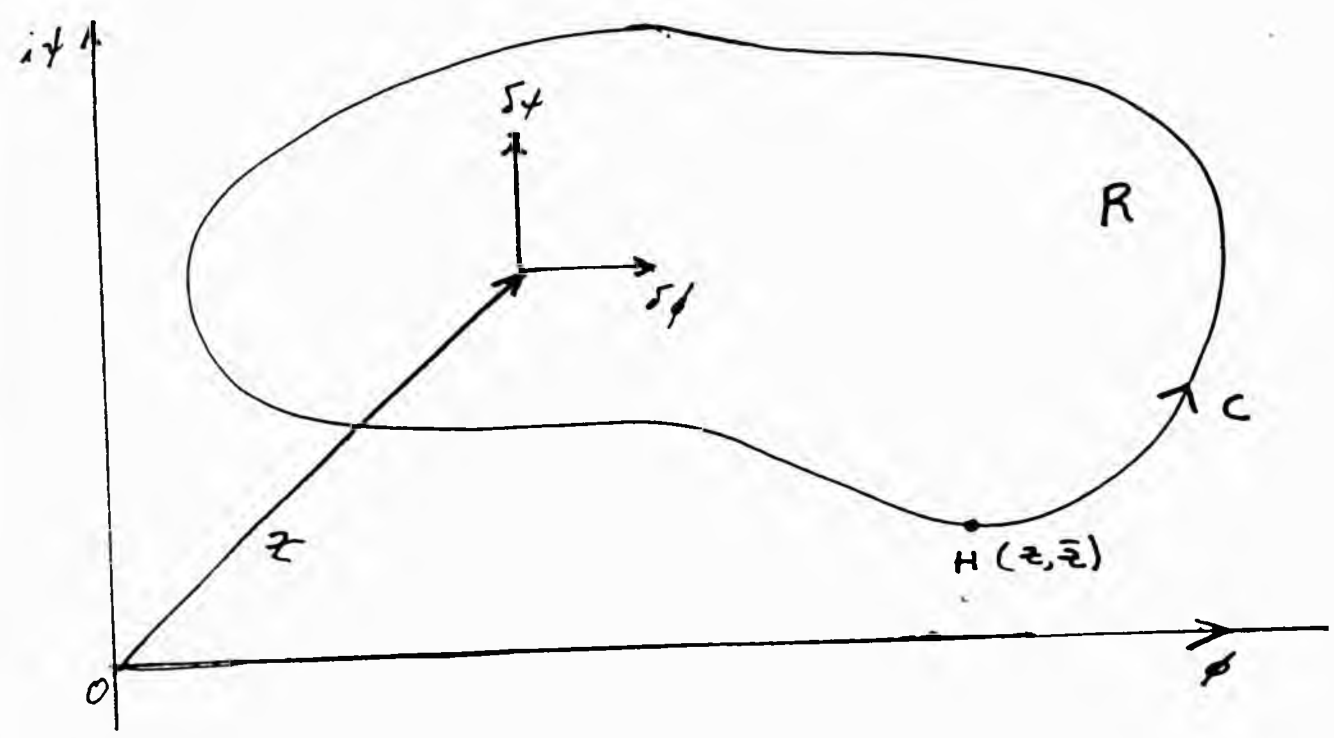


Fig 4.1

Consider a function, $H(z,z^*)$ which is continuous and has continuous partial derivatives over the region R enclosed by the contour C (See Fig 4.1). Then Green's theorem may be written in complex variable form as

$$\int_C H(z,z^*).dz \quad = \quad 2.i. \iint_R \frac{\partial H}{\partial z^*}.d\Phi.dY \qquad [4.1]$$

where $z = \Phi + i.Y$; $z^* = \Phi - i.Y$.

$$\text{Let} \qquad H(z,z^*) = F(z,z^*).G(z,a) \qquad [4.2]$$

where a is a given point. Substituting [4.2] into [4.1] gives

$$\int_C F(z,z^*).G(z,a).dz = 2.i.\iint_R \frac{\partial\{F(z,z^*).G(z,a)\}}{\partial z^*}.d\Phi.dY$$

Now $\frac{\partial\{F.G\}}{\partial z^*} = F.\frac{\partial G}{\partial z^*} + G.\frac{\partial F}{\partial z^*}$

But since $G(z,a)$ is independent of z^* then $\frac{\partial G}{\partial z^*} = 0$

$$\Rightarrow \int_C F(z,z^*).G(z,a).dz = 2.i \iint_R [G(z,a).\frac{\partial F}{\partial z^*}(z,z^*)].d\Phi.d\Upsilon \quad [4.3]$$

Suppose, now, that z = a is a point on the contour C, and define a new contour C* being the contour C indented by a circular arc of radius c centre z = a this path now enclosing a region R*. (See Fig 4.2)

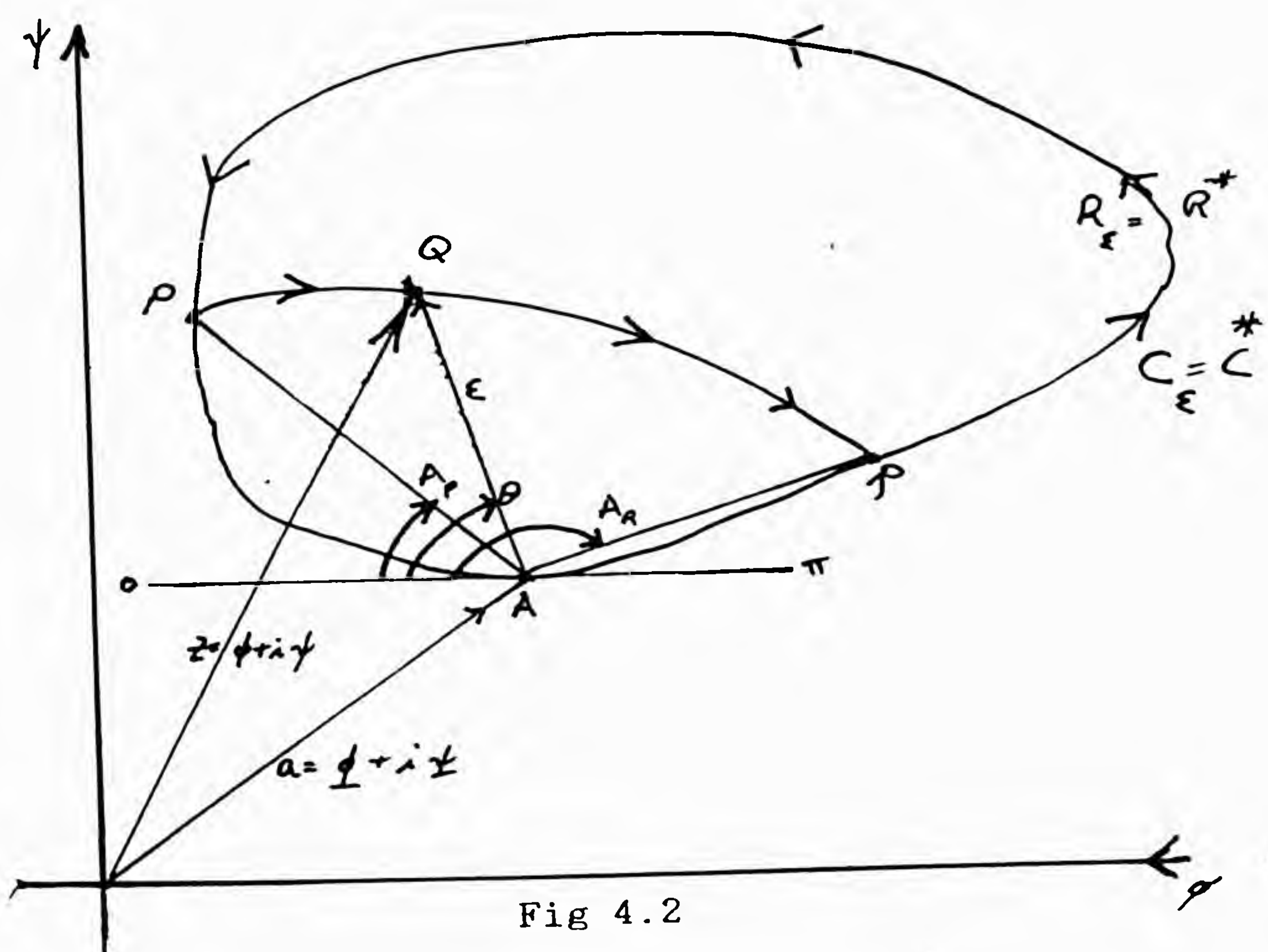


Fig 4.2

Identifying C and R in [4.3] with C* and R* defined above we have

$$\int_{C^*} F(z,z^*).G(z,a).dz = 2.i.\iint_{R^*} G(z,a).\frac{\partial F(z,z^*)}{\partial z^*}.d\Phi.d\Upsilon$$

$$\Rightarrow \int_{C^*} F.G.dz = \int_{RSP} F.G.dz + \int_{PQR} F.G.dz = 2.i \iint_{R^*} G.\frac{\partial F}{\partial z^*}.d\Phi.d\Upsilon$$

$$\Rightarrow \int_{PQR} F.G.dz = -\int_{RSP} F.G.dz + 2.i.\iint_{R^*} G.\frac{\partial F}{\partial z^*}.d\Phi.d\Upsilon \quad [4.4]$$

The Integral 'PQR'

The integral around the arc 'PQR' can now be expressed in terms of the value of the function $F(z,z^*)$ at $z = a$ together with a power series in 'c' the arc radius of PQR by

(i) Expanding F as a double Taylor series in z and z* about z = a and (ii) making a suitable choice of the function G.

(i) On the contour PQR we have

$$z = a + c.e^{i\theta} \; ; \; dz = i.ce^{i\theta}.d\theta \; ; \; z - a = c.e^{i\theta}$$

$$z^* = a^* + c.e^{-i\theta} \; ; \; dz^* = -i.ce^{-i\theta}.d\theta \; ; \; z^* - a^* = c.e^{-i\theta} \qquad [4.5]$$

Expanding $F(z,z^*)$ as a Taylor series about z = a gives

$$F(z,z^*) =$$

$$F(a,a^*) +$$

$$[(z-a).(\partial F/\partial z) + (z^* - a^*).(\partial F/\partial z^*)] +$$

$$[(z-a)^2(\partial^2 F/\partial z^2)+2(z-a)(z^*-a^*)(\partial^2 F/\partial z\partial z^*)+(z^*-a^*)^2.(\partial^2 F/\partial z^{*2})]/2!+..$$

.....

where the derivatives are evaluated at z = a.

Substituting the expressions for z, z*, a, a*, z-a, z*-a* gives

$$F(z,z^*) = F(a,a^*) + c[e^{i\theta}.\partial F/\partial z + e^{-i\theta}.\partial F/\partial z^*]$$

$$+ c^2[e^{2i\theta}\partial^2 F/\partial z^2 + 2\partial^2 F/\partial z\partial z^* + e^{-2i\theta}\partial^2 F/\partial z^{*2}]/2!$$

$$+...........$$

$$\Rightarrow \quad F(z,z^*) = F(a,a^*) + c.D_1 + c^2.D_2 + ... + c^k.D_k + ...$$

where $D_k = D_k\{\theta, \partial^k F/\partial z^{(k-p)}\partial z^{*(p)}\}$ for $k=1,2,...; \; p = 0,1...k$.

the D_k being functions of θ and the values of the derivatives of order k evaluated at z = a.

Thus on PQR we can write

$$F(z,z^*) = F(a,a^*) + \sum_{k=1}^{\infty} c^k.D_k \qquad [4.7]$$

(ii) Choice of G(z,a)

Choosing the function $G(z,a) = (z-a)^{-1}$ [4.8]

Then by virtue of [4.7] and [4.8] we may write the integral on the L.H.S of [4.4] as

$$\int_{PQR} F.G.dz = \int_{PQR} F(z,z^*).dz/(z-a) = \int_{PQR} [F(a,a^*) + \sum_{k=1}^{\infty} c^k.D_k]/(z-a).dz$$

$$= \int_{PQR} F(a,a^*)dz/(z-a) + \sum_{k=1}^{\infty} c^k.\int \{D_k.dz/(z-a)\} \qquad [4.9]$$

Consider the integrals on the right hand side of [4.9]

(a)

On the arc PQR, $z = a + c.e^{i\theta}$; $dz = ic.e^{i\theta}.d\theta$

$$\int_{PQR} \frac{F(a,a^*)}{(z-a)}.dz$$

$$= \int_{Z_P}^{Z_R} F(a,a^*).dz/(z-a) = \int_{\theta=A_P}^{\theta=A_R} F(a,a^*).i.c.e^{i\theta}.d\theta/c.e^{i\theta}$$

$$= i.\int_{\theta=A_P}^{\theta=A_R} F(a.a^*).d\theta$$

$$= i.F(a,a^*).[A_R - A_P] = i.A.F(a,a^*) \qquad [4.10]$$

where $A_R - A_P = A$.

(b) If the derivatives of F with respect to z and z* to all orders are bounded on the arc PQR then it follows that the functions D_k in [4.9] are also bounded

i.e $|D_k| < M$ (say) for $c < c_0 < 1$ for all k. Therefore

$$\sum_{k=1}^{\infty} \{ c^k.\int_{PQR} D_k.dz/(z-a) \} < \sum_{k=1}^{\infty} \{ c^k.\int_{PQR} M.dz/(z-a) \}$$

$$= \sum_{k=1}^{\infty} \{M.c^k . \int_{PQR} i.e^{i\theta}.d\theta/e^{i\theta} \}$$

$$= i.M.A. \sum_{k=1}^{\infty} \{ c^k \} = i.M.A.c/(1-c) .$$

Therefore

$$\sum_{k=1}^{\infty} \{ c^k \int_{PQR} D_k.dz/(z-a) \} = i.A.c.M_1/(1-c) \text{ for some } M_1 . \quad [4.11]$$

Substituting from [4.10] and [4.11] into [4.9] and applying this result to the LHS of [4.4]

$$\int_{PQR} F.G.dz = \int_{PQR} F(z,z^*).dz/.(z-a) = i.A.F(a,a^*) + i.A.c.M_1/(1-c)$$

Further substituting into [4.4] with $G(z,a) = (z-a)^{-1}$ yields

$$i.A.F(a,a^*) + i.A.c.M_1/(1-c) = - \int_{C^*} F(z,z^*).dz/(z-a)$$

$$+ \iint_{R^*} 2.i.\partial F/\partial z^*.d\Phi.d\Upsilon/(z-a)$$

Solving for $F(a,a^*)$ gives

$$F(a,a^*) = (i/A). \int_{C^*} F(z,z^*).dz/(z-a)$$

$$+ (2/A). \iint_{R^*} (\partial F/\partial z^*).d\Phi.d\Upsilon/(z-a) - c.M_1/(1-c) \quad [4.14]$$

If the radius of the arc, c, is allowed to tend to zero then in the limit we have $A_R \to A^L_R$; $A_P \to A^L_P$; $A \to A^L = A^L_R - A^L_P = -\pi$.

and

$F(a,a^*) =$

$$= -(i/\pi). \int_{C^*} F(z,z^*).dz/(z-a) - (2/\pi) \iint_{R^*} (\partial F/\partial z^*)d\Phi.d\Upsilon/(z-a) \quad [4.15]$$

Thus equation [4.15] expresses the value of a function $F(z,z^*)$at some point, $z=a$, on a contour in terms of a contour and field integral over the domain enclosed by the contour.

If $F(z,z^*)$ is a known function of z ($= \Phi + i.\Upsilon$) and provided that the integrals may be expressed in closed form then [4.15] would give the exact solution to the problem.

However for computational purposes the limiting form of [4.15] would not be appropriate and will be replaced by a numerical equivalent of [4.14] since the term '$c.M_1/(1-c)$' will make a contribution to the value of $F(a,a^*)$ for non zero radius c and this is necessarily the case for a discrete representation of a physical system.

Now if, in [4.15], the integration with respect to Υ (say) in the integral

$$\iint_{R^*} (\partial F/\partial z^*).d\Phi.d\Upsilon/(z-a)$$

were to be accomplished then [4.14] (or [4.15]) would be reduced to a contour integral and a line integral giving the value of $F(z,z^*)$ at any point on C^* in terms of its values on C^* alone.

This reduction of the field integral to a line integral is achieved by

(i) Approximating $\partial F/\partial z^*$ by a polynomial in Υ whose coefficients are functions of Φ only;

(ii) Performing the integration with respect to Υ (across the duct) and expressing the result in closed form.

Thus let $\partial F/\partial z^* = \sum_{k=0} F_k(\Phi).(\Upsilon)^k$ where $F_k(\Phi)$ are functions of Φ only. Then

$$\iint_{R^*} (\partial F/\partial z^*).d\Phi.d\Upsilon/(z-a) = \iint_{R^*} \sum_{k=0} F_k(\Phi).(\Upsilon)^k.d\Phi.d\Upsilon/(z-a)$$

$$= \sum_{k=0} \iint_{R*} F_k(\Phi).(Y)^k.d\Phi.dY/(z-a)$$

$$= \sum_{k=0} \int_{\Phi(in)}^{\Phi(out)} F_k(\Phi).\{ \int_{Y=Y_1}^{Y=Y_u} (Y)^k.dY/(z-a) \}.d\Phi$$

$$= \sum_{k=0} \int_{\Phi(in)}^{\Phi(out)} F_k(\Phi).F^*_k(\Phi).d\Phi \qquad [4.16]$$

where

$$F^*_k(\Phi) = \int_{Y=Y_1}^{Y=Y_u} (Y)^k.dY/(z-a) \quad ; \; z= \Phi + i.Y \; ; \; a= \Phi^* + i.Y^*$$

4.(II). Identification of $F(z,z^*)$ and $\partial F/\partial z^*$ with Flow variables.

Generally if $F(z,z^*) = A(z,z^*) + i.B(z,z^*)$

where $z = s + i.t$; $z^* = s - i.t$ then it can be shown that

$$\frac{\partial F}{\partial z^*} = (1/2).(A_s - B_t) + i.(1/2).(B_s + A_t)$$

Letting $A = x$; $B = r$; $s = \Phi$; $t = Y$;

$$\text{Then} \quad F = x + i.r \qquad \text{(i)}$$

$$\partial F/\partial z^* = (1/2).\{ [x_\phi - r_\psi] + i.[r_\phi + x_\psi] \} \qquad \text{(ii)} \qquad [4.18]$$

But from chapter 3, equation [3.5] we have

$$x_\phi = r_\psi \; ; \; x_\psi = -(\ln r)_\phi \qquad [3.5]$$

Substituting into [4.18] gives

$$\partial F/\partial z^* = (1/2).i.[r - \ln(r)]_\phi = (i/2).f(r) \quad \text{(say)} \qquad [4.19]$$

With these expression for F and $\partial F/\partial z^*$ equations [4.14] and [4.15] become

$$F(a,a^*) = x + i.r = \underset{c \to \emptyset}{\text{Lt}} -(i/\pi).\int_{C^*} F(z,z^*).dz/(z-a) +$$

$$- (i/\pi) \iint_{R^*} [(r-\ln(r))_{\Phi}/(z-a)]d\Phi.dY \qquad [4.15a]$$

or $F(a,a^*) = x + i.r =$

$$=-(i/\pi)\int_{C^*} F(z,z^*).dz/(z-a) - (i/\pi)\iint_{R^*} [(r-\ln r)_{\Phi}/(z-a)].d\Phi.dY$$

$$- M_1.c/(1-c) \qquad [4.14a]$$

where $F(a,a^*)$ is a given point on the contour and $F(z,z^*)$ is a variable point with $z = \Phi + i.Y$; $a = \underline{\Phi} + i.\underline{Y}$.

4.(III) An Expansion for $f(r) = (r - \ln(r))_{\Phi}$ as a Power Series in Y.

Let $f(r) = f(\Phi,Y) = (r - \ln(r))_{\Phi}$

Suppose that the function $f(r)$ and its derivatives to all orders exist in the domain $a_1 <= Y <= b_1$, $c_1 <= \Phi <= d_1$

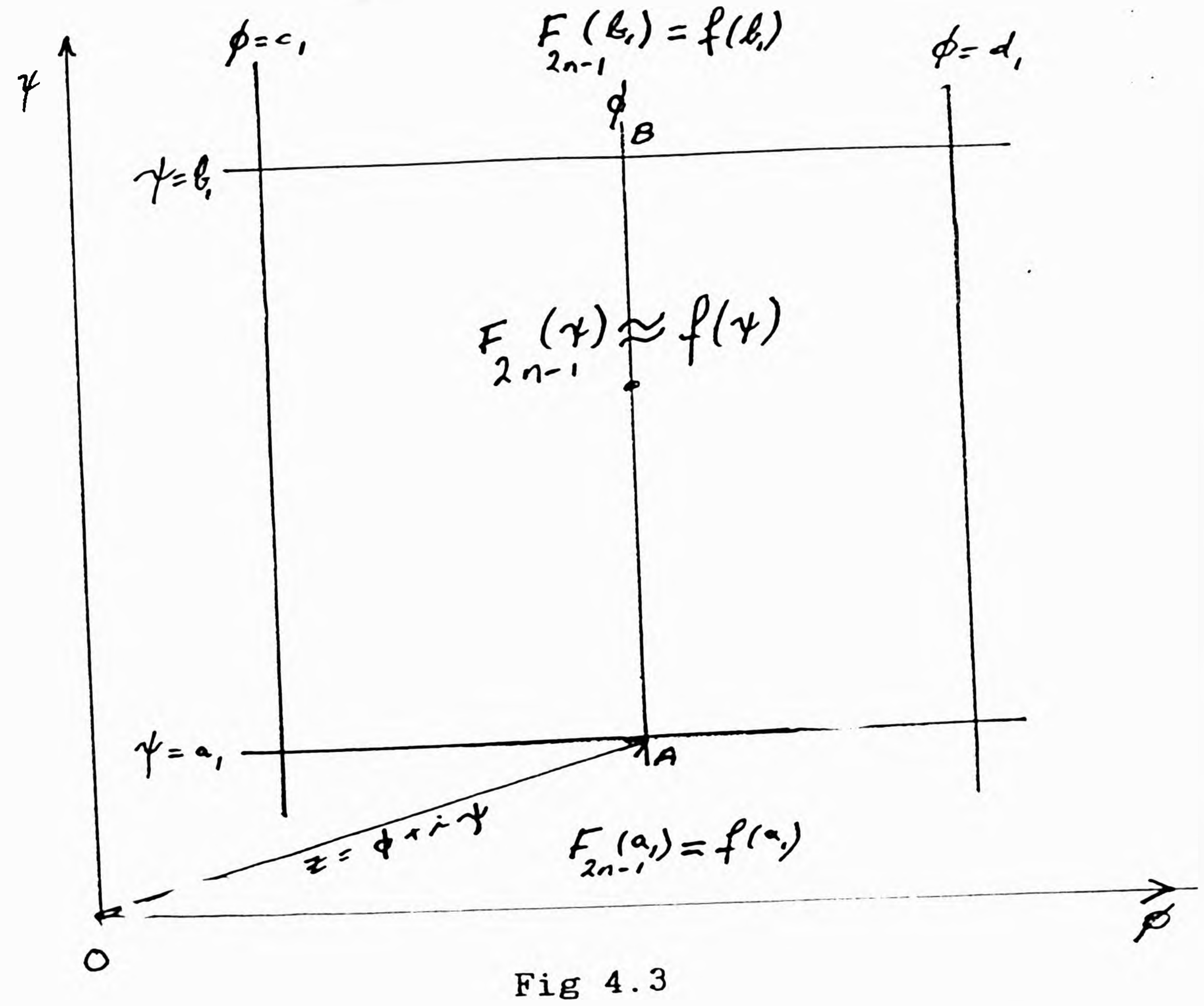


Fig 4.3

For some given value of Φ, let the value of $f(r)$ at the point $r = a = (\Phi + i\Upsilon)$ be denoted by $f(a)$. The term, $f(r) = (r - \ln(r))_{\Phi}$, in the double integral is now replaced by an approximating polynomial F_i which is a power series in Υ whose coefficients are functions of Φ alone. One choice of polynomial is the two point Lidstone expansion of degree 2n-1 defined by

$$F_{2n-1}(\Phi,\Upsilon) =$$

$$= (\Upsilon - b_1)^n \sum_{k=\emptyset}^{k=n-1} \{A_k(\Phi).\frac{(\Upsilon - a_1)^K}{k!}\} + (\Upsilon - a_1)^n \sum_{k=\emptyset}^{k=n-1} \{ B_k(\Phi).\frac{(\Upsilon - b_1)^k}{k!} \}$$

where the coefficients A_k and B_k are given by

$$A_k(\Phi) = [d^k \{ f(\Upsilon)/(\Upsilon - b^n) \}/d\Upsilon^k]_{\Psi = a1}$$

$$B_k(\Phi) = [d^k \{ f(\Upsilon)/(\Upsilon - a^n) \}/d\Upsilon^k]_{\Psi = b1}$$

This expansion is such that derivatives up to and including those of order k of $F(\Upsilon)_{2n-1}$ are equal to those of $f(\Upsilon)$ at $\Upsilon = a_1$ and $\Upsilon = b_1$.

Thus

(i) $F^k_{2n-1}(a_1) = f^k_{2n-1}(a_1)$; (ii) $F^k(b_1) = f^k(b_1)$ for $k = \emptyset$ to n.

where the superscripts refer to the order of the differential.

Applying Leibnitz's formula for repeated differentiation to the definitions of the coefficients $A_k(\phi)$ and $B_k(\Phi)$ it can be shown that

$$A_k(\Phi) = [(a_1 - b_1)^{-n}/(n-1)!]. \sum_{r=1}^{r=k} \{ {}^kC_r. (n-r+1)!(b_1 - a_1)^{-r}.f(a_1)^{k-r} \} \quad [4.19.a]$$

$$B_k(\Phi) = \{(b_1 - a_1)^{-n}/(n-1)!]. \sum_{r=1}^{r=k} \{ {}^kC_r. (n-r+1)!(a_1 - b_1)^{-r}.f(b_1)^{k-r} \} \quad [4.19.b]$$

Defining $A'_k = A_k/k!$ and $B'_k = B_k/k!$ then the approximating polynomial of degree 2n-1 can be written as

$$F(\Phi,Y)_{2n-1} = \sum_{k=\emptyset}^{k=n-1} \{ A'_k.(Y-b_1)^n (Y-a_1)^k + B'_k.(Y-a_1)^n (Y-b_1)^k \} \quad [4.19.c]$$

Replacing f(r) by its approximation F_{2n-1} in equations [4.14a] and [4.15a]

$F(a,a^*) = x_1 + i.r_1 =$

$$= -(i/\pi)\{ \underset{Lt\ c\to\emptyset}{} \int_{C^*} F(z,z^*).dz/(z-a) + \iint_{R^*} F_{2n-1}.d\Phi.dY/(z-a) \} \quad [4.12b]$$

or

$$= -(i/\pi)\{ \int_{C^*} F(z,z^*).dz/(z-a) + \iint_{R^*} F_{2n-1}.d\Phi.dY/(z-a) \} -$$

$$- M_1.c/(1-c) \quad [4.14b]$$

Let $I^*_1 = \int_{C^*} F(z,z^*).dz/(z-a)$; $I^*_2 = \iint_{R^*} F_{2n-1}.d\Phi.dY/(z-a)$

then the solution may be written in compact form as

$$i.\pi.F(a,a^*) = \underset{C^*\to C}{Lt} \{ I^*_1 + I^*_2 \} \quad [4.15c]$$

where $F(z,z^*) = x + i.r$; $z = \Phi + i.Y$; $a = \underline{\Phi} + i.\underline{Y}$ and $F(a,a^*) = x_1 + i.r_1$ (a given point on the contour).

Evaluation of the integrals I^*_1 and I^*_2 will give the value of F(a,a*) at any point on the contour C.

<u>4(IV). Determination of the Coefficients A'k , B'k</u>

Besides factorials upto order n and powers of $(a_1 - b_1)$ which are known, the coefficients A'_k and B'_k in [4.19a & b] depend on the quantities $f^k(t)$ which are the k^{th} derivatives of $f[r(\Phi,Y)]$ with respect to Y evaluated on the duct walls $\Phi = a_1, b_1$. In order to remove this dependency on Ψ, these functions are expressed as derivatives with respect to Φ on the wall boundaries by repeated application of [3.5 (i) & (ii)]. Thus from equation [3.5] we have

$r_{\psi} = x_{\phi}$; $x_{\psi} = -(\ln r)_{\psi} = -r^{-1}.r_{\phi}$ and $f = (r - \ln r)_{\phi}$

denoting $\frac{\partial^k r}{\partial \Phi^k} = r_k$ and $\frac{\partial^k x}{\partial \phi^k} = x_k$ and

differentiating f with respect to Ψ and replacing r_Ψ and x_Ψ when they occur with their equivalent forms involving derivatives with respect to Φ only then we have

$$f = r^{-1}.(-r_1) + r_1$$

$$f = r^{-2}.(r_1.x_1) + r^{-1}.(-x_2) + x_2$$

$$f_{\Psi\Psi} = r^{-4}.(3.r_1^3) + r^{-3}.(-2.r_1^3 - 4r_1.r_2 - 2.r_1.x_1^2) + r^{-2}.(r_3 + 3.r_1.r_2 + 2.x_1.x_2) + r^{-1}(r_3)$$

$$f_{\Psi\Psi\Psi} = r^{-5}.(-20r_1^3.x_1) + r^{-4}(6x_1.r_1^3 + 11.x_1^2 + 6r_1.x_1^3 + 22x_1.r_1.r_2) + r^{-3}(-6r_1^2.x_2 - 6x_1^2.x_2 - 4x_1.r_3 - 6x_2.r_2 - 4.r_1.x_3 - 6x_1.r_1.r_2) + r^{-2}(x_4 + 3x_2.r_2 + 3.r_1.x_3 + r_3.x_1) + r^{-1}(-x_4)$$

The derivatives upto order three of the function $f(\Phi,\Psi)$ in the cross-stream direction Ψ, are now expressed as derivatives of 'x' and 'r' in the Φ direction (along the boundary) and the stream-wise dependency is removed. With n = 4 we can now express $f(\Phi,\Psi)$ as a polynomial of degree seven across the duct along the Φ characteristic $\Phi = \Phi_i$ (say) between $\Psi = a_1$ and $\Psi = a_2$.

4(V): REDUCTION OF THE FIELD INTEGRAL TO A LINE INTEGRAL.

The value of the function $F(a,a^*) = x_i + i.r_i$, (a particular point) on the contour C is given by equation [4.15c] i.e

$$i.\pi.F(a,a^*) = I_1^* + I_2^* \qquad [4.15c]$$

Now the field integral is of the form

$$I_2^* = \iint (z-a)^{-1}\{\sum_{k=\emptyset}^{k=n-1} A'_k.(\Upsilon-b_1)^n (\Upsilon-a_1)^k + B'_k.(\Upsilon-a_1)^n (\Upsilon-b_1)^k\}d\Phi.d\Upsilon \qquad [4.17]$$

Since n is finite we may rearrange the order of the integral and summation signs and write

$$I_2^* = \int_{\phi=c_1}^{\phi=d_1} [\sum_{k=\emptyset}^{k=n-1} \{ A'_k \int_{\Upsilon=a_1}^{\Upsilon=b_1} (\Upsilon-b_1)^n (\Upsilon-a_1)^k (z-a)^{-1}.d\Upsilon \; +$$

$$+ B'_K \int_{\Upsilon=a_1}^{\Upsilon=b_1} (\Upsilon-a_1)^n (\Upsilon-b_1)^k (z-a)^{-1}.d\Upsilon \}].d\Phi$$

where $z = \Phi + i.\Upsilon$ is a variable point on the contour and $a = \Phi^* + i.\Upsilon^*$ is a point in the (Φ,Υ) plane at which $F(a,a^*)$ is to be evaluated.

Now $z - a = \Phi + i.\Upsilon - \Phi^* - i.\Upsilon^*$

$= i.\{ \Upsilon - [(\Upsilon^* + i(\Phi - \Phi^*)] \}$

Let $P = \Upsilon - [(\Upsilon^* + i(\Phi - \Phi^*)] = - i.(z - a)$

Then $dP = d\Upsilon$ and $P = -i(z-a)$; $(z-a)^{-1} = -i/P = 1/iP$.

$$\begin{aligned}
&\text{when } \Upsilon = b_1 \quad P = U \text{ where } U = b_1 - [\Upsilon^* + i.(\Phi - \Phi^*)] && \text{(i)}\\
&\qquad \Upsilon = a_1 \quad P = L \text{ where } L = a_1 - [\Upsilon^* + i.(\Phi - \Phi^*)] && \text{(ii)}\\
&\text{hence } \Upsilon - a_1 = P + C \text{ where } C = \Upsilon^* - a_1 + i.(\Phi - \Phi^*) && \text{(iii)}\\
&\text{and } \Upsilon - b_1 = P + D \text{ where } D = \Upsilon^* - b_1 + i.(\Phi - \Phi^*) && \text{(iv)}
\end{aligned} \qquad [4.17a]$$

Then the integral I_2^* has the form

$$I_2^* = \int_{\Phi=\Phi(in)}^{\Phi=\Phi(out)} \{ \sum_{k=\emptyset}^{k=n-1} [A'_k . \int_{P=L}^{P=U} \frac{(P+D)^n (P+C)^k}{i.P}.dP + B'_k . \int_{P=L}^{P=U} \frac{(P+C)^n (P+D)^k}{i.P}.dP] \}d\Phi \qquad [4.18]$$

Define

$$I^{A,B,L,U}_{p,q} = \int_L^U \frac{(P+A)^p (P+B)^q}{P}.dP \qquad [4.19a]$$

$$J^{A,B}_{p,q} = \int \frac{(P+A)^p (P+B)^q}{P}.dP \quad ; \text{ For } p>= 1; \; ; q =\emptyset,1, \ldots(p-1) \qquad [4.19b]$$

$$K^{A,B}_{i,j} = \int (P+A)^i (P+B)^j.dP \qquad [4.19c]$$

$$H^{A,B}_{i,j} = (P+A)^i (P+B)^j \qquad [4.19d]$$

Hence

$$I^{A,B,L,U}_{p,q} = [J^{A,B}_{p,q}]^{P=U}_{P=L} \qquad [4.19e]$$

$$dK^{A,B}_{i,j} = H^{A,B}_{i,j}.dP \quad ; \quad \frac{d}{dP}\{ K^{A,B}_{i,j} \}= H^{A,B}_{i,j} \qquad [4.19f]$$

The functions H satisfy the diffferential relation

$$\frac{d}{dP}[H_{i,j}] = i.H_{i-1,j} + j.H_{i,j-1} \qquad [4.19g]$$

Hence we may write I2* in the form

$$i.I^*_2 = \int_{\Phi(in)}^{\Phi(out)} \{ \sum_{k=\emptyset}^{k=n-1} [A'_k.I^{B,A,L,U}_{n,k} + B'_k.I^{A,B,L,U}_{n,k}] \} d\Phi \qquad [4.2\emptyset]$$

After expressing the definite integrals $I^{A,B,L,U}_{p,q}$ in closed form a numerical integration from Φ_{in} to Φ_{out} will determine the value of I2*. The evaluation of the definite integrals $I^{\#,\#,L,U}_{p,q}$ is obtained by

(i) Finding a reduction formula for the indefinite integral $J^{A,B}_{p,q}$.

(ii) Similarly for $K^{A,B}_{i,j}$.

(iii) Deriving the explicit form for $J^{A,B}_{i,j}$ from (i) and (ii)

(iv) Evaluating $I^{A,B,L,U}_{p,q}$ via equation [4.19e].

<u>(i) Reduction Formula for $J^{A,B}_{p,q}$</u> For $p > q >= 1$

$$J^{A,B}_{p,q} = \int \frac{(P+A)^p (P+B)^q}{P} . dP$$

$$= \int (P+A)^p (P+B)^{q-1} . dP \quad + \quad B . \int \frac{(P+A)^p (P+B)^{q-1}}{P} . dP$$

i.e $$J^{A,B}_{p,q} = K^{A,B}_{p,q-1} + B . J^{A,B}_{p,q-1} \qquad ; q = 1, 2, \ldots\ldots, p.$$

From this reduction relation it can be shown that

$$J^{A,B}_{p,q} = B^q . J^{A,B}_{p,\emptyset} + B^{q-1} . K^{A,B}_{p,\emptyset} + \sum_{s=1}^{s=q-1} \{ B^{q-s-1} . K^{A,B}_{p,s} \} \qquad [4.21]$$

<u>(ii) Reduction formula for $K^{A,B}_{i,j}$</u>

Integrating by parts gives

$$K^{A,B}_{i,j} = \int (P+A)^i (P+B)^j \, dP =$$

$$= \frac{(P+A)^{i+1} . (P+B)^j}{(i+1)} - \frac{j}{(i+1)} \int (P+A)^{i+1} (P+B)^{j-1} \, dP$$

$$K^{A,B}_{i,j} = \frac{1}{(i+1)} H^{A,B}_{i+1,j} - \frac{j}{(i+1)} K^{A,B}_{i+1,j-1}$$

From this relation it follows that the explicit form for $K^{A,B}_{p,s}$ is given by

$$K^{A,B}_{p,s} = \frac{(-1)^s . p! . s!}{(p+s)! . \emptyset!} . K^{A,B}_{p+s,\emptyset} + \sum_{t=\emptyset}^{t=s-1} \frac{(-1)^t . p! . s! . H^{A,B}_{p+1+t,s-t}}{(p+1+t)!(s-t)!} \qquad [4.22]$$

(iii) The Explicite form for $J^{A,B}_{p,q}$

Substituting [4.22] into [4.21] gives

$$J^{A,B}_{p,q} = \sum_{s=1}^{s=q-1} \{ B^{q-1-s} .\{ \sum_{t=0}^{t=s-1} [\frac{(-1) .p!.s! .H^{A,B}_{p+1+t,s-t}}{(p+1+t)!(s-t)!}] \}$$

$$+ \sum_{s=0}^{s=q-1} \{ \frac{(-1)^{s} .B^{q-1-s} p!s!.K^{A,B}_{p+s,0}}{(p+s)!} \} + B^{q} .J^{A,B}_{p,0} \qquad [4.23]$$

where $K^{A,B}_{p,0}$ has been incorporated into the 2nd summation and the lower limit set equal to zero. After finding expressions for the integrals $K^{A,B}_{p+s,0}$ and $J^{A,B}_{p,0}$, [4.23] will give the value of $J^{A,B}_{p,q}$ explicitly. Thus

$$\text{(a) } K^{A,B}_{p+s,0} = \int (P+A)^{p+s} .dP = \frac{(P+A)^{p+s+1}}{(p+s+1)} = \frac{H^{A,B}_{p+s+1,0}.}{(p+s+1)} \qquad [4.24a]$$

$$\text{(b) } J^{A,B}_{p,0} = \int \frac{(P+A)^{p}}{P} .dP = \int \{ (P+A)^{p-1} .\frac{(P+A)}{P}.dP \}$$

$$= \int \{ (P+A)^{p-1} + A.\frac{(P+A)^{p-1}}{P} \}.dP$$

$$= \frac{(P+A)^{p}}{P} + A. \int \frac{(P+A)^{p-1}}{P} dP = (1/P).H^{A,B}_{p,0} + A.J^{A,B}_{p-1,0} \qquad [4.24b]$$

$$\text{and} \quad J^{A,B}_{0,0} = \int \frac{dP}{P} = \ln(P) \qquad [4.24.c]$$

This reduction relation [4.24b] gives $J^{A,B}_{p,0}$ as

$$J^{A,B}_{p,0} = A^{p} .\ln(P) + \sum_{u=1}^{u=p} \frac{A^{p-u}}{u} .H^{A,B}_{u,0} \qquad [4.25]$$

Substitution of [4.24a] and [4.25] into [4.23] gives the expression for $J^{A,B}_{p,q}$ explicitly in terms of the algebraic functions H defined in [4.19d]. Thus

$$J(P)^{A,B}_{p,q} = A^{p}.B^{q} \{ \ln(P) + \sum_{u=1}^{u=p} [\frac{A^{-u}}{u} .H^{A,B}_{u,0}] \quad \}$$

$$+ p!.H^{A,B}_{p+1,0} .B^{q-1} \{ \sum_{s=0}^{s=q-1} [s!.B^{-s} . \sum_{t=0}^{t=s} ((-1)^{t} H^{A,B}_{t,s-t})] \} \qquad [4.26]$$

It may verified by direct diferentiation that this expression for $J^{A,B}_{p,q}$ in [4.26] satisfies $\frac{d}{dP} (J^{A,B}_{p,q}) = \frac{(P+A)^{p}.(P+B)^{q}}{P}$

Briefly define $M_{u} = \frac{A^{-u}}{u}$; $L_{s,t} = \frac{(-1)^{t}.p!.B^{q-1-s} s!}{(p+1+t)!(s-t)!}$

$$T_{1} = \sum_{u=1}^{u=p} (u.M_{u} .H^{A,B}_{u-1,0})$$

and

$$T_{2} = \sum_{s=0}^{s=q-1} \{ \sum_{t=0}^{t=s} [(p+t+1).L_{s,t} .H^{A,B}_{p+t,s-t} + (s-t)L_{s,t} .H^{A,B}_{p+t+1,s-t-1}] \}$$

With these definitions [4.26] becomes

$$J^{A,B}_{p,q} = A^{p}. B^{q} [\ln(P) + \sum_{u=1}^{u=p} (M_{u} .H_{u,0}) + \sum_{s=0}^{s=q-1} \{ \sum_{t=0}^{t=s} L_{s,t} .H^{A,B}_{p+t+1,s-t} \}$$

Differentiating with respect to P gives, after some rearrangement,

$$\frac{d(J^{A,B}_{p,q})}{dP} = A^{p}.B^{q}.(P^{-1} + T_{1}) + T_{2} \qquad [4.27]$$

Now $T_{1} = \sum_{u=1}^{u=p} M_{u}.u.H^{A,B}_{u-1,0} = (P+A)^{-1} . \sum_{u=1}^{u=p} ((P+A)/A)^{u} = \frac{((P+A)/A)^{p} -1}{P}$

and examination of the coefficients of $H^{A,B}_{i,j}(P)$ in the expressions for T_2 will show that the only non-zero ones are those for which $t=0$.

Thus with $t=\emptyset$ we have

$$T_2 = \sum_{s=\emptyset}^{s=q-1} (p+1).L_{s,\emptyset}.H^{A,B}_{p,s} = B^{q-1}.H^{A,B}_{p,\emptyset}.\sum_{s=\emptyset}^{s=q-1}((P+B)/P)^s$$

Summing this series gives

$$T_2 = B^q.\frac{(P+A)^p}{P}.[((P+B)/B)^q - 1)$$

Substitution of these values into [4.27] gives

$$\frac{d}{dP}(J^{A,B}_{p,q}) = A^p.B^q.[P^{-1} + P^{-1}.((P+A)/A)^p - P^{-1}] +$$

$$+ B^q(P+A)^p.P^{-1}.[((P+B)/B)^q - 1]$$

$$= \frac{(P+A)^p.(P+B)^q}{P}$$

which verifies the expression for the indefinite integral $J^{A,B}_{p,q}(P)$ in [4.26].

The values of the definite integrals $I^{B,A,L,U}_{n,k}$ and $I^{A,B,L,U}_{n,k}$ needed to complete the reduction of $I^{*}{}_2$ to a line integral is given in the next section.

4(VI) Evaluation of the Integral $I^{A,B,L,U}_{p,q}$

Generally

$$I^{A,B,L,U}_{p,q} = [J^{A,B}_{p,q}(P)]^{P=U}_{P=L} = J^{A,B}_{p,q}(U) - J^{A,B}_{p,q}(L) \qquad [4.28]$$

In particular we require the values of

(1) $I^{C,D,L,U}_{n,k}$ and (2) $I^{D,C,L,U}_{n,k}$

(1) The Integral $I^{C,D,L,U}_{n,k}$

If $A = C$, $B = D$, $p = n$, $q = k$, then [4.28] gives

$$I^{A,B,L,U}_{n,k} = J^{A,B}_{n,k}(U) - J^{A,B}_{n,k}(L)$$

(i) The evaluation of $J(U)^{A,B}_{n,k}$

With A = C , B = D , P = U then from [4.19d]

$$H(U)^{C,D}_{i,j} = (U+C)^{i}.(U+D)^{j}$$

But from [4.17a]

$$U = b_1 - (Y^* + i.(\Phi-\Phi^*)) \;;\; C = Y^* - a_1 + i.(\Phi-\Phi^*) \;;$$

$$D = Y^* - b_1 + i.(\Phi-\Phi^*)$$

$$\Rightarrow U + C = b_1 - a_1 \;;\quad U + D = \emptyset \Rightarrow H(U)^{C,D}_{i,j} = (b_1 - a_1)^{i}.(\emptyset)^{j}$$

Thus the only non zero terms in the expression for $J(U)^{C,D}_{n,k}$ are those involving the functions $H(U)^{C,D}_{i,\emptyset}$ (i.e for j=Ø)

This implies that we must have t=s in the double summation of equation [4.26]. Hence

$$J(U)^{C,D}_{n,k} = C^{n}.D^{k}\{ \ln(U) + \sum_{u=1}^{u=n} [\frac{C^{-u}}{u} .H(U)^{C,D}_{u,\emptyset}] \} + n!.H(U)^{C,D}_{n+1,\emptyset}.D^{k-1} \{ \sum_{s=\emptyset}^{s=k-1} [\frac{(D^{-s}.s!H^{C,D}_{s,\emptyset}(U).(-1)^{s})}{(n+1+s)!.(\emptyset)!}] \}$$

$$= C^{n}.D^{k}\{ \ln(U) + \sum_{u=1}^{u=n} [\frac{C^{-u}}{u} .H(U)^{C,D}_{u,\emptyset}] \} + \sum_{s=\emptyset}^{s=k-1} \{ \frac{(-1)^{s}.n!.s!.D^{k-1-s}}{(n+1+s)!} .H(U)^{C,D}_{n+1+s,\emptyset} \}$$

where $H(U)^{C,D}_{i,\emptyset} = (b_1 - a_1)^{i}$ and $U = b_1 - Y^* + i.(\Phi - \Phi^*))$ making $J(U)^{C,D}_{n,k}$ a function of Φ only.

(ii) Evaluation of $J(L)^{C,D}_{n,k}$

For A = C , B = D , P = L then from [4.19d] $H(L)^{C,D}_{i,j} = (L+C)^{i} (L+D)^{j}$

But from [4.17a]

$$L = a_1 - (\Upsilon + i.(\Phi - \Phi^*)) \;;\; C = \Upsilon^* - a_1 +- i(\Phi - \Phi^*)$$

$$D = \Upsilon - b_1 + i.(\Phi - \Phi^*)$$

$$\Rightarrow L + C = 0 \;;\; L + D = a_1 + b_1 \Rightarrow \overset{C,D}{H(L)}_{i,j} = (0)^i\,(a_1 - b_1)^j .$$

Thus only those $H^{C,D}_{i,j}$ with $i = 0$ give a non-zero contribution to the value of $\overset{C,D}{J(L)}_{n,k}$. Hence both summations are zero in equation [4.26] since (a) there are no terms in the single summation and (b) all $H^{A,B}_{p+1,0} = 0$ in the double summation.

Hence
$$\overset{C,D}{J(L)}_{n,k} = C^n.B^k.\ln(L) \qquad [4.30]$$

Substituting from [4.29] and [4.30] into [4.28] gives

$$\overset{C,D,L,U}{I(\Phi)}_{n,k} = \overset{C,D}{J(U)}_{n,k} - \overset{C,D}{J(L)}_{n,k}$$

$$= C^n.D^k.\{\ln(U/L) + \sum_{u=1}^{u=n}[\frac{C^{-u}}{u}\overset{C,D}{H(U)}_{u,0}]\} + \sum_{s=0}^{s=k-1}\frac{[(-1)^s n!s!D^{k-1-s}\overset{C,D}{H(U)}_{n+1+s,0}}{(n+1+s)!}]$$

where $\overset{C,D}{H(U)}_{i,0} = (b_1 - a_1)^i$ [4.31]

A similar evaluation for $\overset{D,C,L,U}{I(\Phi)}_{n,k}$ gives

$$\overset{D,C,L,U}{I(\Phi)}_{n,k} =$$

$$= C^k D^n.\{\ln(U/L) - \sum_{u=1}^{u=n}[\frac{D^{-u}}{u}\overset{D,C}{H(L)}_{u,0}]\} + \sum_{s=0}^{s=k-1}\frac{[(-1)^s n!s!C^{k-1-s}\overset{D,C}{H(L)}_{n+1+s,0}}{(n+1+s)!}]$$

where $\overset{D,C}{H(L)}_{i,0} = (a_1 - b_1)^i$ [4.32]

4(VII) Summary of the Solution

The value of the function $F(\Phi,Y)$ at the point (Φ^*,Y^*) is given by

$$i.\pi.F(\Phi^*,Y^*) = I^*_1 + I^*_2 \qquad [4.31a]$$

where

(i) $$I^*_1 = \int_{C^*} \frac{F(\Phi,Y).dz}{(z - a)} \qquad [4.31.b]$$

(ii) $$I^*_2 = \iint_{R^*} \frac{F_{2n-1}(\Phi,Y).d\Phi.dY}{(z - a)} \qquad [4.31.c]$$

(iii) $z = \Phi + i.Y$; $a = \Phi^* + i.Y^*$; $F(\Phi,Y) = x(\Phi,Y) + i.r(\Phi,Y)$ [4.31.d]

(iv) $$F_{2n-1}(\Phi,Y) = \sum_{k=\emptyset}^{k=n-1} \{A'_k(\Phi).(Y-b_1)^n.(Y-a_1)^k + B'_k(\phi)(Y-a_1)^n.(Y-b_1)^k\}$$

(v) $$A'_k(\Phi) = (a_1 - b_1)^{-n} . \sum_{r=\emptyset}^{r=k} [(n+r-1)!(b_1 - a_1)^{-r}.f(a_1)^{(k-r)}/((r!.(k-r)!)]$$ [4.31.e]

$$B'_k(\Phi) = (b_1 - a_1)^{-n} . \sum_{r=\emptyset}^{r=k} [(n+r-1)!(a_1 - b_1)^{-r}.f(b_1)^{(k-r)}/((r!.(k-r)!)]$$ [4.31.f]

$$f(X)^{(i)} = [d^i f(\Phi,Y)/dY^i] \; ; \; Y^*_{inner} = a_1 \; ; \; Y^*_{outer} = b_1$$ [4.31.g]

(vi) Defining

$$I^{A,B,L,U}_{p,q} = \sum \int_{P=L}^{P=U} \{ \frac{(P+A)^p.(P+B)^q}{P}.dp \} \; ; \; p>= 1 \; , \; q = \emptyset,1,2..,p-1$$ [4.31.h]

Then

$$i.I^*_2 = \int_{\Phi(in)}^{\Phi(out)} \{ \sum_{k=\emptyset}^{k=n-1} [A_k(\Phi).I(\Phi)^{D,C,L,U}_{n,k} + B_k(\Phi)I(\Phi)^{C,D,L,U}_{n,k}] \}.d\Phi$$ [4.31.i]

with

$P = Y - (Y^* + i(\Phi-\Phi^*))$; $Y - a_1 = P+C$; $C = (Y-a_1) + i(\Phi-\Phi^*)$

$U = D - (Y^* + i(\Phi-\Phi^*))$; $Y - b_1 = P+D$; $D = (Y-b_1) + i(\Phi-\Phi^*)$

$L = C - (Y^* + i(\Phi - \Phi^*))$

and where the values of the integrals $I(\Phi)^{D,C,L,U}_{n,k}$ and $I(\Phi)^{C,D,L,U}_{n,k}$ are given by equations [4.31] and [4.32].

With these formulations for the integrals I^*_1 and I^*_2 a numerical integration around the contour C* in (i) together with the line integral in (vi) will give the value of $F(\Phi,\Upsilon)$ on the contour.

The precision to which the function $f(r) = (r - \ln(r))_\Phi$ may be approximated across the duct depends on the value of n in the approximating polynomial F_{2n-1} and in principle this can be increased without limit although there is a likelyhood of over prescription on the boundary in the limit as n-> to infinity.

A polynomial of degree 2n-1 will involve derivatives of f(r) w.r.t Υ upto order (n-1). The expression of these derivatives in terms of derivatives with respect to Φ become progressively more cumbersome with increasing 'n' (See 4(VI).1). However it would be possible to incorporate a routine in the programme code to automatically generate the expressions for these derivatives of higher order if required.

For this reason, n is taken as four thus allowing the crossstream variation of f(r) to be represented by a polynomial of degree seven and requiring derivatives of order three for f(r).

If the boundary conditions $\underline{B}(F(\underline{\Phi},\underline{\Upsilon}))$ of the flow were invariant then one application of the technique summarized above would provide the solution. In the case of varying boundary conditions in the type of problem being considered an iterative procedure is required and the general form of the solution would be

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CHAPTER 5

Finite Difference Forms For The Integral Equation Solution

The numerical equivalent to the integral equation solution obtained in Chapter 4 is derived from equation [4.4].

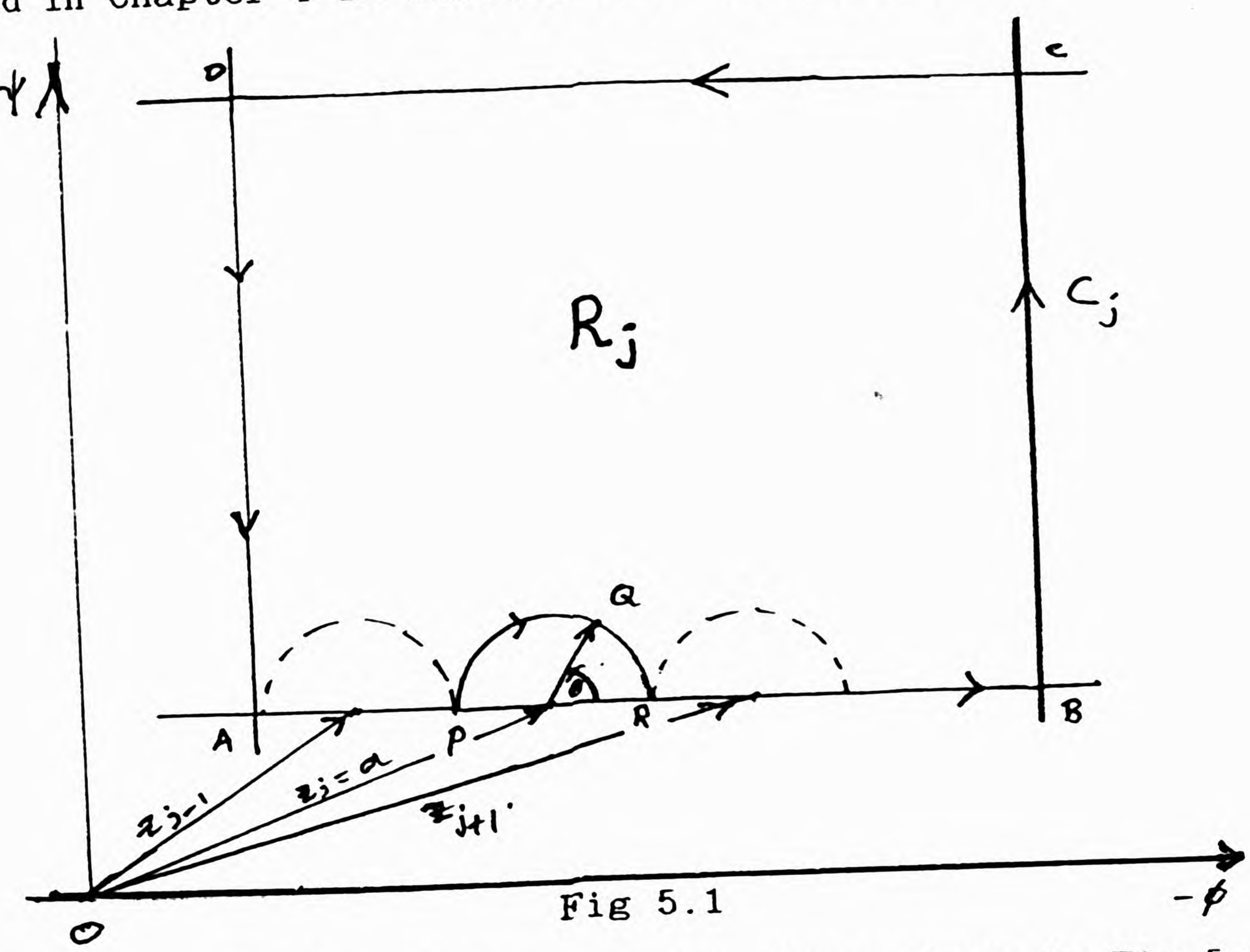

Fig 5.1

Applying this equation to the contour shown above in Fig 5.1

$$\int_{PQR} F.G.dz = - \int_{RBCDAP} F.G.dz + \iint_{R_j} 2.i.G.\partial F/\partial z^*.d\Phi.dY \qquad [5.\emptyset]$$

where
(i) $F(z) = F(\Phi + i.Y) = x(\Phi,Y) + i.r(\Phi,Y);$

(ii) $G(z,a) = (z - a)^{-1}$

(iii) $\partial F/\partial z^* = (i/2).f\{r(\Phi,Y)\}$

(iv) The contour D_j = PQRBCDAP

(v) " " S_j = PQR

(vi) " " C_j = RBCDAP i.e $S_j + C_j = D_j$

(vii) R_j is the region enclosed by D_j.

Therefore $\int_{S_j} F.dz/(z-a) = -\int_{C_j} F.dz/(z-a) - \iint_{R_j} G.f(r).d\Phi.d\Upsilon$

Define

$IS_j = \int_{S_j} F.dz/(z-a)$; $IC_j = \int_{C_j} F.dz/(z-a)$; $IR_j = \iint_{R_j} G.f(r).d\Phi.d\Upsilon$ [5.1]

Then $IS_j = -IC_j - IR_j$ [5.2]

The Integral IS_j

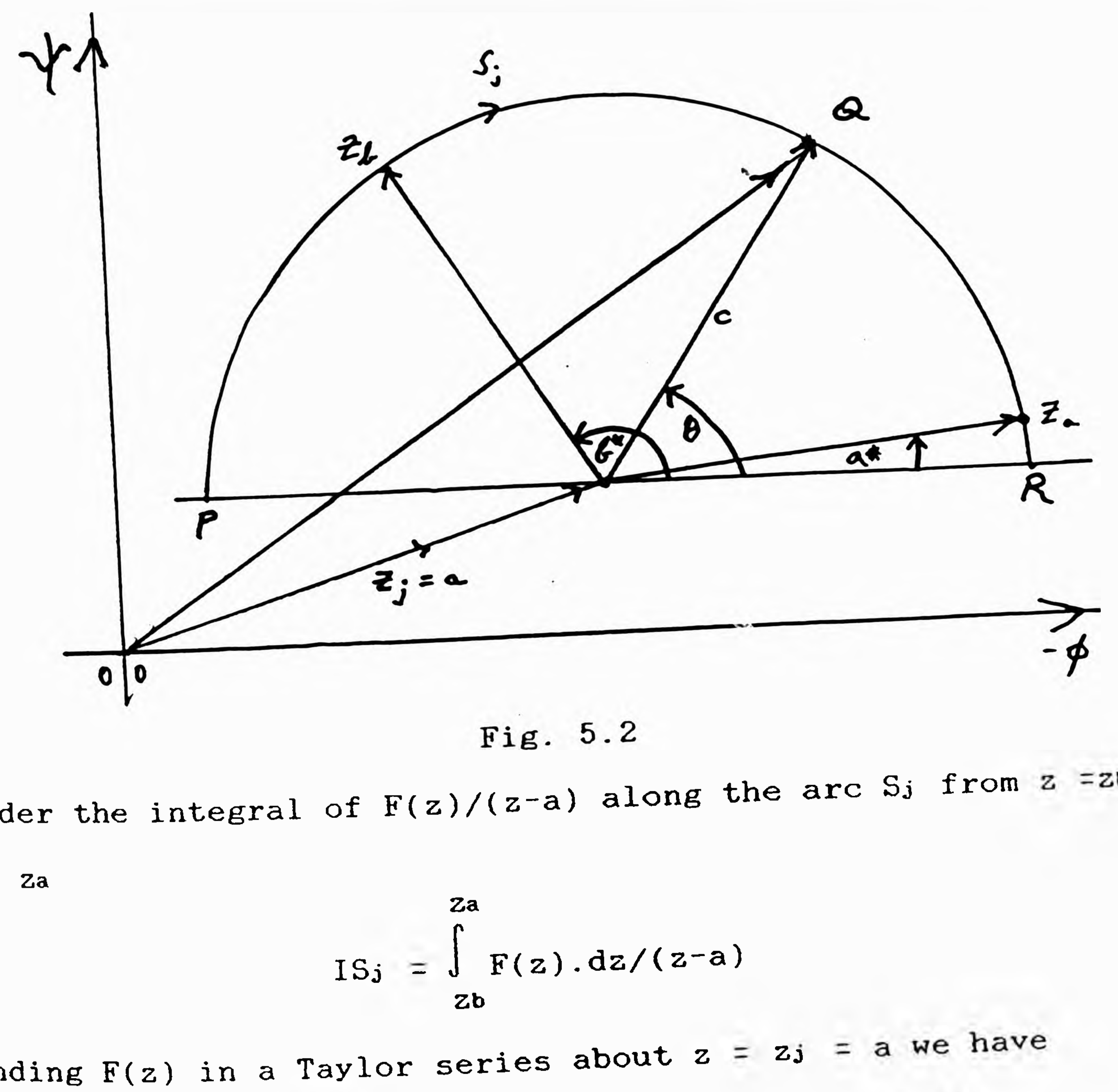

Fig. 5.2

Consider the integral of $F(z)/(z-a)$ along the arc S_j from $z = z_b$ to $z = z_a$

$$IS_j = \int_{z_b}^{z_a} F(z).dz/(z-a)$$

Expanding $F(z)$ in a Taylor series about $z = z_j = a$ we have

$$F(z) = F(a) + (z-a).F^{(1)}(a)/1! + (z-a)^2.F^{(2)}(a)/2! + (z-a)^3.F^{(3)}(a)/3! + \ldots$$

$$= F(a) + \sum_{q=1}^{\infty} \left[\frac{(z-a)^q}{q!}.F^{(q)}(a) \right] \quad [5.3]$$

Hence

$$IS_j = \int_{z_b}^{z_a} \{F(a) + \sum_{q=1}^{\infty} [\, \frac{(z-a)^q . F^{(q)}(a)}{q!}]\} . \frac{dz}{(z-a)} \qquad [5.4]$$

On the arc S_j,

$z = a + h.(\, \text{Cos}\theta + i.\text{Sin}\theta) = a + h.e^{i\theta}$; $dz = i.h.e^{i\theta}.d\theta$;

when $z = z_b$, z_a then $\theta = b^*$, a^* , hence

$$IS_j = \int_{b^*}^{a^*} \{F(a) + \sum_{q=1}^{\infty} \frac{h^q . e^{iq\theta} . F^{(q)}(a)}{q!} \} . \frac{i.h.e^{i\theta}.d\theta}{h.e^{i\theta}}$$

$$= i \int_{b^*}^{a^*} \{F(a) + \sum_{q=1}^{\infty} \frac{h^q . e^{iq\theta} . F^{(q)}(a)}{q!} \}.d\theta$$

$$= i \, [\, F(a).\theta + \sum_{q=1}^{\infty} \frac{h^q . e^{iq\theta} . F^{(q)}(a)}{i.q.q!} \,]_{\theta=b^*}^{\theta=a^*}$$

$$= i.F(a).(a^*-b^*) + \sum_{q=1}^{\infty} \{ \frac{h^q . F^{(q)}(a)}{q.q!} .(e^{iqa^*} - e^{iqb^*})\} \qquad [5.6]$$

On AB, $b^* = \emptyset$, $a^* = \pi$; Hence $a^* - b^* = \pi - \emptyset = \pi$;

$e^{iqa^*} - e^{iqb^*} = e^{iq.\pi} - e^{\emptyset} = (-1)^q - 1$

On CD $b^* = \pi$, $a^* = 2\pi$; Hence $a^* - b^* = 2\pi - \pi = \pi$;

$e^{iqa^*} - e^{iqb^*} = e^{i2q.\pi} - e^{iq.\pi}$

$= \text{Cos } 2q.\pi + i.\sin 2q.\pi - \cos q.\pi - i.\text{Sin } q.\pi$

$= 1 - (-1)^q$

Thus

$$IS_j = i.\pi.F(a) + \sum_{q=1}^{\infty} \{ \frac{h^q . F^{(q)}(a)}{q.q!} .((-1)^q - 1)\} \qquad [5.7]$$

<u>The Integral IC_i</u>

Let the contour C_j be partitioned by the points z_i for $i = \emptyset$ to T. (See Fig 5.3 below) and let $dz_{i,0}$ and $dz_{i,1}$ be intervals to the left and right of the point z_i respectively.

Then the integral IC_j may be written as

$$IC_j = \int_{RBCDA} \frac{F(z).dz}{(z-z_j)} = \int_{C_j} \frac{F(z).dz}{(z-a)} = \sum_{\substack{i=0 \\ i \neq j}}^{i=T} \int_{z_i - dz_{i,0}}^{z_i + dz_{i,1}} \left\{ \frac{F(z).dz}{(z-z_j)} \right\}$$

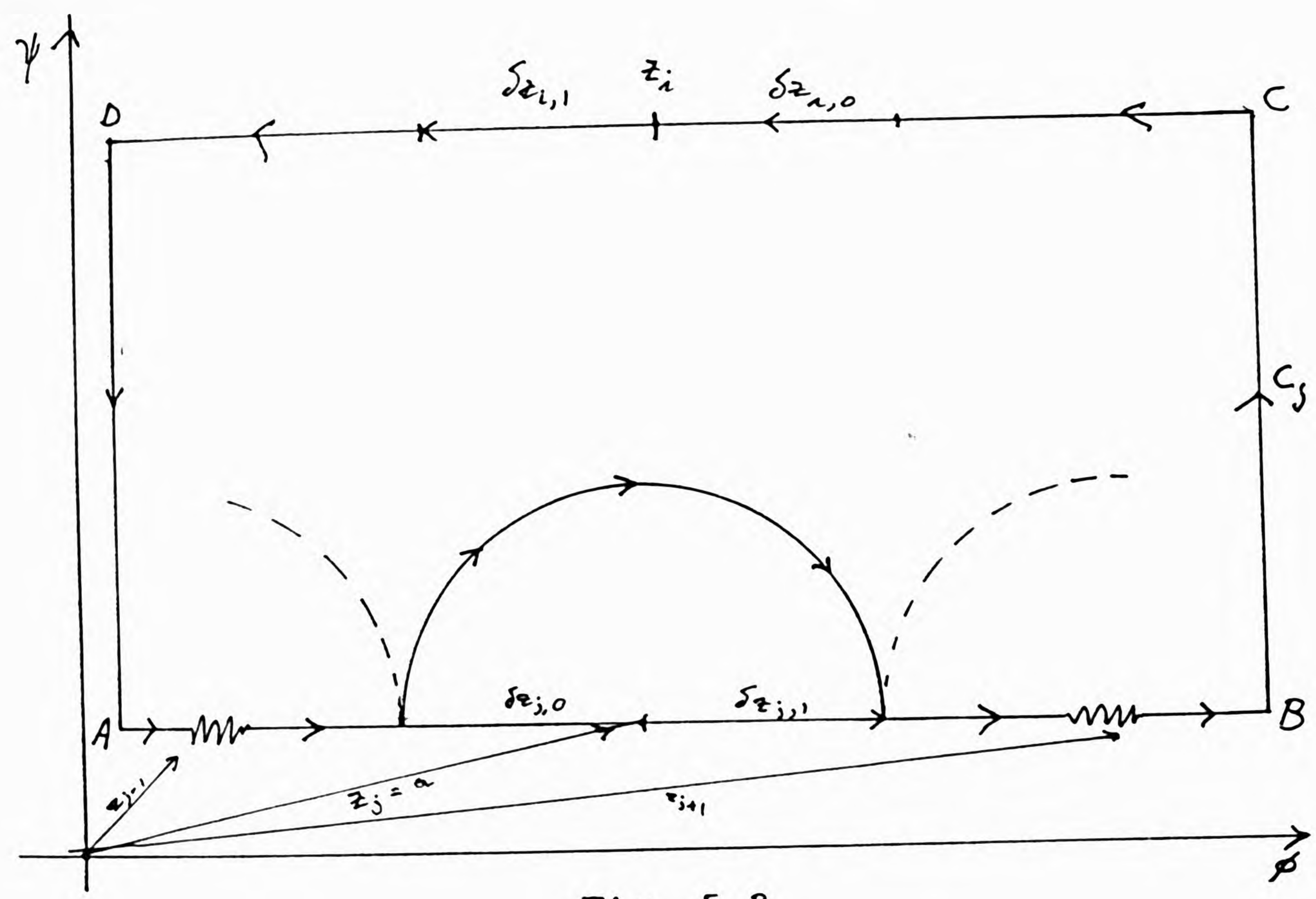

Fig. 5.3

Define $h_j(z) = h(z,z_j) = h(z,a) = (z-a)^{-1}.F(z) = (z-z_j)^{-1}.F(z)$.

Then IC_j may be written as

$$IC_j = \sum_{\substack{i=0 \\ i \neq j}}^{i=T} \int_{z_i - dz_{i,0}}^{z_i + dz_{i,1}} h_j(z).dz \qquad [5.9]$$

Expanding the function $h_j(z)$ about the point $z = z_i$ gives

$$h_j(z) = h_j(z_i) + h_j^{(1)}(z_i).\frac{(z-z_i)}{1!} + h_j^{(2)}(z_i).\frac{(z-z_i)^2}{2!} + .. + h_j^{(q)}(z_i).\frac{(z-z_i)^q}{q!} + ..$$

$$h_j(z) = h_j(z_i) + \sum_{q=1}^{\infty} \frac{h_j^{(q)}(z_i).(z-z_i)^q}{q!} = \sum_{q=0}^{\infty} \frac{h_j^{(q)}(z_i).(z-z_i)^q}{q!}$$

where $h_j^{(q)}(z_i) = [d^q\{h_j(z)\}/dz^q]$ at $z = z_i$.

Hence substituting this expansion for $h_j(z)$ into [5.9] the integral

IC_j may be written as

$$IC_j = \sum_{i=0,\, i\neq j} \{ \int_{z_i - dz_{i,o}}^{z_i + dz_{i,1}} [\, h_j(z_i) + \sum_{q=1}^{\infty} \frac{h_j^{(q)}(z_i).(z-z_i)^{q}}{q!}].dz \} \qquad [5.10]$$

Integrating with respect to z, [5.10] gives

$$IC_j = \sum_{i=0,\, i\neq j}^{i=T} \{ [h_j(z_i).z + \sum_{q=1}^{\infty} \frac{h_j^{(q)}(z_i).(z-z_i)^{q+1}}{(q+1)!}]_{z=z_i-dz_{i,o}}^{z=z_i+dz_{i+1}} \}$$

and substituting in the limits for z gives

$$IC_j = \sum_{i=0,\, i\neq j}^{i=T} \{ h_j(z_i).(dz_{i,1}+dz_{i,o}) +$$

$$\sum_{q=1}^{\infty} [\, \frac{h_i^{(q)}(z_i)}{(q+1)!}.(\, dz_{i,1}^{q+1} + (-1)^{q+2}.dz_{i,o}^{q+1})]\} \qquad [5.11]$$

The Integral IR_j

From equation [4.20] we have

$$i.I^*_2 = \int_{\Phi(in)}^{\Phi(out)} \{ \sum_{k=0}^{k=n-1} [\, A'_k(\Phi).I^{B,A,L,U}_{n,k}(\Phi) + B'_k I^{A,B,L,U}_{n,k}(\Phi) \,] \}.d\Phi$$

The line AB in Fig 5.1 is partitioned into m sections by the m+1 points z_t ; t= 0,1,...m at intervals of dz_t. Then I^*_2 may be approximated by the expression

$$iI^*_2 = \sum_{t=0}^{t=m-1} d\Phi_t \,[\, \sum_{k=0}^{k=n-1} \{ A'_k(\Phi_t).I^{D,C,L,U}_{n,k}(\Phi_t) + B'_k(\Phi_t).I^{C,D,L,U}_{n,k}(\Phi_t) \} \,]$$

Along AB $dz_t = dz_{t,1} + dz_{t,o} = d\Phi_t$. If $dz_t = dz'$ (say) for t= 1 to m, then $d\Phi_t = d\Phi'$ and hence identifying IR_j with I^*_2 gives

$$i.IR_j = d\Phi.\sum_{t=0}^{t=m-1} \sum_{k=0}^{k=n-1} \{ A'_k(\Phi_t).I^{D,C,L,U}_{n,k}(\Phi_t) + B'_k(\Phi_t).I^{C,D,L,U}_{n,k}(\Phi_t) \}$$

This expression gives the approximate value of the line integral used to replace the field integral in [5.0], however both the expressions for the integrals IS_j and IC_j are 'exact' in the sense that their summation is taken to infinity. In the numerical context they would naturally be truncated but are given in this form to allow the option of improving the accuracy and determining the error of any computational solution.

Substituting the expressions for IS_j, IC_j and IR_j (= I^*_2) given by [5.7], [5.11] and [4.31.i] into [5.2] and solving for F(a) gives

$$IS_j = - IC_j - IR_j \qquad [5.2]$$

$$\pi.F(a) = i. \sum_{p=1}^{\infty} \left[\frac{h^p.F^{(p)}(a)}{p.p!}.((-1)^p - 1) \right] \qquad [a]$$

$$+ i \left\{ \sum_{i=\emptyset, i \neq j}^{i=T} \left(\sum_{q=\emptyset}^{\infty} \left[\frac{h_i^{(q)}(z_i)}{(q+1)!}.(dz_{i,1}^{q+1} + (-1)^{q+2}.dz_{i,\emptyset}^{q+1}) \right] \right) \right\} \qquad [b] \quad [5.12]$$

$$+ \quad d\Phi. \sum_{t=1}^{t=m-1} \sum_{k=\emptyset}^{k=n-1} [A'_k(\Phi_t).I_{n,k}^{D,C,L,U}(\Phi_t) + B'_k(\Phi_t).I_{n,k}^{C,D,L,U}(\Phi_t)] \qquad [c]$$

<u>The Computed Solution</u>

A computer program was developed to use formula [5.12] to evaluate F(z) on a contour. Initially a trial program was constructed to evaluate the regular function $F(z) = z^2 = (\Phi + i.\Upsilon)^2$ on the perimeter of a unit square. For test purposes the upper values of the summations were taken as p=2, q=2, n=4. In this example the term [c] in [5.12] is zero since F(z) is independent of z* (i.e [5.12.c] represents $\partial F(z)/\partial z^*$:

Thus

$$\pi.F(z_j) = \sum_{p=1}^{p=2} \left[\frac{h^p.F^{(p)}(z_i)}{p.p!}.((-1)^p - 1) \right] +$$

$$+ \sum_{i=\emptyset, i \neq j}^{i=T} \left[\sum_{q=\emptyset}^{q=2} \left\{ \frac{h^{(q)}(z_i)}{(n+1)!}.(z_{i,1}^{q+1} + (-1)^{q+2}. z_{i,o}^{q+1}) \right\} \right]$$

$$= T_1 + T_2 + \sum_{i=\emptyset,\, i\neq j}^{i=T} [\, T_3 + T_4 + T_5 \,]$$

where

$$T_1 = \frac{h.F^{(1)}(z_j)}{1.1!}.(-2) \quad ; \quad T_2 = \frac{h^2.F^{(2)}(z_j)}{2.2!}.(\emptyset) = \emptyset$$

$$T_3 = h_j(z_i).(dz_{i,1} + dz_{i,o}) \quad ; \quad T_4 = \frac{h_j^{(1)}(z_i)}{2!}.(dz_{i+1}^2 - dz_{i,o}^2)$$

$$T_4 = \frac{h_j^{(2)}(z_i)}{3!}.(dz_{i,1}^3 + dz_{i,o}^3) \qquad [5.13]$$

$h_j(z_i)=(z_i-z_j)^{-1}.F(z_i)$

$h_j^{(1)}(z_i)=-(z_i-z_j)^{-2}.F(z_i)+(z_i-z_j)^{-1}.(\partial F/\partial z)$

$h_j^{(2)}(z_i)=2.(z_i-z_j)^{-3}.F(z_i)^{-2}(z_i-z_j)-2(\partial F/\partial z)+(z_i-z_j)^{-1}(\partial^2 F/\partial z^2)$

where the derivatives are evaluated at $z = z_i$.

The terms T_1 and T_2 are the 1st and 2nd order contributions to the value of $F(z_j)$ obtained by integrating around the semicircle centre $z=z_j$ radius 'h' and it can be seen that T_2 is zero. Further the term $T_4 = \emptyset$ for all i except i = n(a) where n(a) are 'corner points' on the contour.

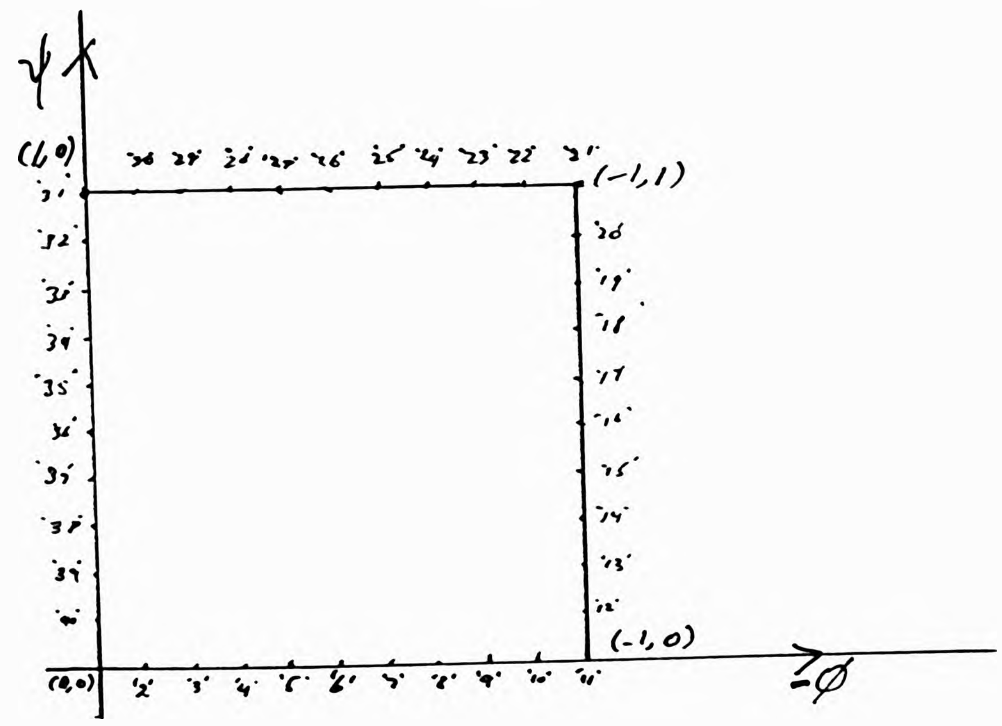

Fig 5.4

The proramme was written to manipulate complex variable arithmatic and a typical result for evaluating the regular function $F(z)=z^2$ on the boundary of a unit square, partitioned by 11 points on each side is given in table 5.5 where (C) and (E) represent the calculated and exact values of $z = x + i.r$ respectively.

Pt.	x	r
2	.00998776	-.0000136
	.01	0
3	.03999991	-.0000013
	.04	0
4	.08999999	$-2.3*10^{-7}$
	.09	0
5	.16000000	$-6.7*10^{-8}$
	.16	0
6	.250000000	$7.4*10^{-11}$
	.25	0
7	.360000001	$.53*10^{-7}$
	.36	0
8	.490000000	$1.29*10^{-6}$
	.49	0
9	.63999856	$2.11*10^{-5}$
	.64	0
10	.80900862	.0011016
	.81	0

Pt.	x	r	
22	-.19222156	-1.79753	(C)
	-.19	-1.8	(E)
23	-.3600520	-1.59998	(C)
	-.36	-1.6	(E)
24	-.5100037	-1.39999	(C)
	-.51	-1.4	(E)
25	-.6400005	-1.19999	(C)
	-.64	-1.2	(E)
26	-.7500000	-1	(C)
	-.75	-1	(E)
27	-.8399996	-.800000	(C)
	-.84	-.8	(E)
28	-.9099984	-.600000	(C)
	-.91	-.6	(E)
29	-.9599846	-.400030	(C)
	-.96	-.4	(E)
30	-.988516293	-.201101	(C)
	-.99	-.2	(E)

Table 5.5

The error in evaluating this function is very small but can be seen to grow as towards the 'corners'. However increasing the number of points on the contour allows this error to be localized and reduced 'indefinitely'.

The function $F(z) = z^2$ was then replaced by the function for the flow solution F_4 obtained in Chapter 2. From Table 2.2 we have for solution four (with a = 1, b = 0)

$F(z) = x + i.r = (\Upsilon.\mathrm{Coth}(\Phi)) + i.(1 - \Upsilon^2).\mathrm{Cosech}^2(\Phi)$

with $z = \Phi + i.\Upsilon$.

Expressions for $\partial F/\partial z$ and $\partial^2 F/\partial z^2$ (either 'exact' or numerical) are required for the evaluation of [5.13].

Denoting Coth and Cosech by CH and CC respectively we have

$$x = 2.\Upsilon.CH(\Phi) \; ; \; r = (1 - \Upsilon^2).CC^2(\Phi) \; ; \; F = x + i.r \; ;$$

$$x_\Psi = 2.CH(\Phi) \; ; \; r_\Psi = -2.\Upsilon.CC^2(\Phi) \; ; \; x_{\Psi\Psi} = \varnothing \; ; \; r_{\Psi\Psi} = -2.CC^2(\Phi)$$

$$x_{\Psi\Phi} = -2.CC^2(\Phi) \; ; \; r_{\Psi\Phi} = 4.\Upsilon.CC^2(\Phi).CH(\Phi) \; ; \; x_\Phi = -2.\Upsilon CC^2(\Phi)$$

$$r_\Phi = -2.(1 - \Upsilon^2).CC^2(\Phi).CH(\Phi) \; ; \; x_{\Phi\Phi} = 4.\Upsilon.CC^2(\Phi).CH(\Phi)$$

$$r_{\Phi\Phi} = 2.(1 - \Upsilon^2).CC^2(\Phi).[\, 2.CH^2(\Phi) + CC(\Phi) \,] \qquad [5.14]$$

$$F_z = (1/2).(\, F_\Phi - i.F_\Psi \,) \; ; \; F_\Phi = x_\Phi + i.r_\Phi \; ; \; F_\Psi = x_\Psi + i.r_\Psi$$

$$F_z = (1/2).(\, x_\Phi + r_\Psi \,) + (1/2)(\, r_\Phi - x_\Psi \,) .$$

$$F_{zz} = (1/4)(\, x_{\Phi\Phi} - x_{\Psi\Psi} + 2.r_{\Phi\Psi} \,) + (1/4)(r_{\Phi\Phi} - r_{\Psi\Psi} - 2.x_{\Phi\Psi} \,) \qquad [5.15]$$

Then [5.14] and [5.15] yield the following forms for F_z and F_{zz}

$$F_z = -2.\Upsilon.CC^2(\Phi) - i.CH(\Phi).[(1-\Upsilon^2).CH^2(\Phi)+1]$$

$$F_{zz} = 3.\Upsilon.CC^2(\Phi).CH(\Phi) + \\ + i(1/2)CC(\Phi)^2.[(1-\Upsilon^2).\{\, 2.CH^2(\Phi)+CC^2(\Phi) \,\} + 1]$$

The computed solution (with and without the crossstream correction) for the region of the flow bounded by the characteristics Φ=.5, Φ=.6 , Υ =.5, Υ=.51 are given in Table 5.6.

Inspection of this table shows that the accccuracy of the solution increases when the effect of the cross stream variation is taken into account and continues to improve when the number of boundary points is increased as well as the order of the approximating polynomial for $\partial F/\partial z^*$. The average percentage error in the calculation of the 'r' and 'x' coordinates were reduced from Ø.76324% and 1.93679% to Ø.10737% and 1.2684% respectively validating the use of the contour integral method together with the cross-stream approximation.

Pt	z = x	+ i.r		z = x	+ i.r	
2	1.84789350	1.95751522	22	2.20231038	2.58186034	(Ø)
	1.86260935	1.92918213		2.19447270	2.62495066	(1)
	1.88716416	1.92104143		2.17046584	2.61036063	(E)
3	1.87730099	2.01778493	23	2.17821561	2.48256022	(Ø)
	1.89394513	1.99451100		2.16368386	2.50371707	(1)
	1.91326984	1.99545110		2.13523126	2.50246963	(E)
4	1.90160022	2.08532163	24	2.14296687	2.39206462	(Ø)
	1.91973648	2.07223156		2.12753977	2.40003464	(1)
	1.94039390	2.07384637		2.10144157	2.40066164	(E)
5	1.92656502	2.15981072	25	2.10767028	2.30531884	(Ø)
	1.94602241	2.15471150		2.09274951	2.30312449	(1)
	1.96859139	2.15651404		2.06901602	2.30449075	(E)
6	1.95258649	2.23853883	26	2.07338958	2.21999765	(Ø)
	1.97322499	2.24201230		2.05947145	2.21186703	(1)
	1.99792132	2.24376720		2.03787975	2.21355113	(E)
7	1.97979986	2.32108268	27	2.04028628	2.13944796	(Ø)
	2.00139970	2.33448139		2.02759368	2.12571198	(1)
	2.02844708	2.33594816		2.00796322	2.12747299	(E)
8	2.00845649	2.40884311	28	2.00832603	2.06281532	(Ø)
	2.03053182	2.43257579		1.99699811	2.04923225	(1)
	2.06023683	2.43343185		1.97920178	2.04591857	(E)
9	2.03948668	2.49432142	29	1.99752161	1.98990908	(Ø)
	2.06060991	2.53665767		1.96774998	1.96672443	(1)
	2.09336398	2.53662957		1.95153523	1.96857902	(E)
10	2.08534236	2.57747062	30	1.95441169	1.91502148	(Ø)
	2.09987522	2.64175344		1.94664607	1.88671005	(1)
	2.12790768	2.64599334		1.92490745	1.89517141	(E)

(Ø) = Results with no cross stream approximation.
(1) = Results with 1st order cross stream approximation.
(E) = Results derived from exact solution.

	x	r	
Average % errors	1.93679966%	.763242026%	(Ø)
	1.26840004%	.107375194%	(1)

<u>Table 5.6</u>

Chapter 6

Boundary Layer Considerations, Swirl and Boundary Conditions (I).

In the application of the numerical methods used to solve the partial differential equations for irrotational, incompressible flow discussed so far, the boundary conditions (B.Cs) were of two kinds namely;

(1) At inlet and outlet the distribution of the dependent variables (x,r) with respect to Φ and Υ were known from the exact solutions and remained fixed throughout the iterative computation;

(2) On the inner and outer duct walls neither x nor r were explictly defined. Instead, a velocity distribution, again calculated from the exact solutions, was used to define r (or x) implicitly on the duct walls. Specifically this condition had the form

$$r_{\Psi} + (1/r).r_{\Phi} = Q(\Phi,\Upsilon) \qquad \text{(a)}$$

where $Q(\Phi,\Upsilon)$ was some known function of the wall boundary speeds. Now although the distributions of Q are invariant throughout the iteration, the corresponding distributions of r along the duct wall implied by (a) above vary continuously, a new boundary value of r being calculated at each iterative step until both the field and boundary distributions in 'r' satisfy some convergence criteria. Another aspect of these prescribed velocity distributions is that they are applied irrespective of any boundary layer (B.L) effects, advantageous or otherwise, and take no account of the associated B.L behaviour implied by them.

In designing annular ducts it would be desirable to produce a duct geometry which would, in some sense, control the B.L on the duct walls and also the character of the outlet velocity profile. In particular, avoidance of boundary layer separation and the possible onset of reverse flow in the presence of adverse pressure gradients leading to significant disturbance in the character of the primary flow would be a useful design feature. The achievement of this aim naturally depends on the type of boundary conditions to be applied and it is not obvious how an invariant wall condition might be defined which would satisfy this requirement. Earlier work of Stratford (Ref:11,12) on the prediction of the separation of the two-dimensional laminar and turbulent B.Ls is here extended to the axisymmetric case to yield feasable wall B.Cs. For a given point this new 'mixed' B.C depends on the wall geometry and velocity distribution up-stream of the point at which the condition is to be applied. Thus, the velocity distributions used to define this 'mixed' wall B.C are themselves varying (unlike those for the exact solutions given above) and hence all flow variables u,v,w,x and r (on the boundary) change with iteration number until convergence is established. Further by considering the derivation of the B.L equations on a body of revolution, the above condition may be extended to a class of swirling flows for the laminar B.L. These B.Cs together with the numerical techniques described above enable us to generate duct wall shapes implying specific prescribed B.L behaviour. Given the freedom available in applying arbitrary wall velocity distributions there is no necessity to restrict this application to 'Stratford' type distributions, and examples of duct shapes can be generated for flows with combinations of accelerating, deccelerating and constant velocity distributions applied piece-wise on the duct walls in conjunction (if desired) with 'patches' of constant radii.

Sections of constant radius or speed may be especially appropriate at inlet where the application of a sudden adverse pressure gradient can yield an abrupt change in the duct radius. The flexibility of the technique is such that it can be used to determine duct geometries subject to quite random and arbitrary boundary conditions.

(II) Summary of Stratford's Results For The Two Dimensional B.L.

In his paper, Stratford examined the effect of an adverse pressure gradient, incident at $x = x_o$, Fig. 6.1 on a Blasius type (zero pressure gradient) boundary layer which had developed

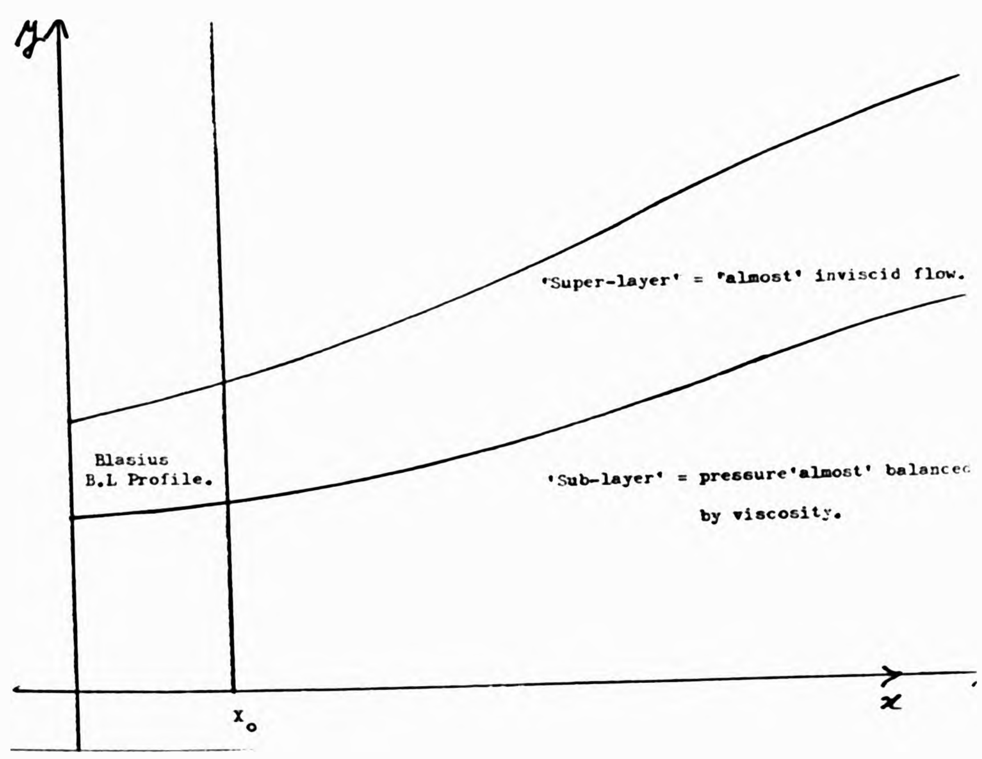

Fig. 6.1

up-stream of the point $x = x_o$ with a view to determining the conditions defining the separation of the B.L.

Stratford's analysis was based on the conceptual device of dividing the flow within the B.L for $x>x_o$ into two parts; a sub-layer and a super-layer. The main feature of the flows in these two layers is that

(1) In the super-layer the flow is 'almost' inviscid and satisfies an approximate form of Bernouilli's equation incorporating a term to allow for the small viscous effects present in the upper part of the B.L.

(2) The flow in the sub-layer is one in which the inertia forces are negligible and the pressure effects are 'almost' entirely balanced by those due to viscosity.

Having established equation sets representative of these two distinct flow regimes, Stratford derived solutions for the inner and outer flows for both the laminar and turbulent B.Ls. A compatability condition applied at the interface, 'J', of these two flows imposing continuity in Y, u, u_y and u_{yy} suffices to determine the solution for various pressure/velocity distributions of the free-stream. Stratford's conclusion was that downstream of x_o at $x = x_s$, the point of separation of the B.L, the following conditions hold

(1) In the laminar case

$$C_p.(xdC_p/dx)^2 = k_1 \text{ at } x = x_s \qquad [6.1]$$

(2) In the turbulent case

$$C_p.(x.dC_p/dx)^{1/2}.(10^{-6}.R_x)^{-1/10} = k_2 \text{ at } x = x_s \qquad [6.2]$$

where C_p is the pressure coefficient defined by

$$C_p = (p-p_o)/[(1/2)\,\rho\, U_o{}^2] = 1-(U/U_o)^2 \qquad \text{(i)} \qquad [6.3]$$

and $$R_x = x.U/\mu \qquad \text{(ii)}$$

where both k_1 and k_2 are constants and ρ_o , p_o. U_o are the values of the density pressure and speed in the free-stream (edge of B.L) at the station $x=x_o$. The constants k_1 and k_2 depend on the nature of the pressure gradient encountered at and downstream of $x=x_o$. In particular if k (=k_1 , k_2) is the constant for that flow the pressure gradient of which is such that u_y = Ø when y=Ø (implying that the shear stress at the wall is zero) and the flow is always on the point of separating then equations [6.1] and [6.2] represent an implicit definition of the pressure distributions and may be integrated with respect to arc length (x) to give the distribution of the pressure coefficient explicitly as

$$C_p = k.(\ \ln\ (\ x/x_o\ )\ )^{2/3} \qquad \text{(Lam. B.L.)} \qquad [6.4]$$

$$C_p = k.(10^{-6}.R_o{}^{1/15}).(\ (\ x/x_o\ )^{1/5} - 1\ )^{1/3} \qquad \text{(Turb. B.L.)} \qquad [6.5]$$

where $R_o = x_o U_o/\mu$

Since C_p may be expressed directly in terms of the speed, U, at the edge of the B.L via equation [6.3(i)] then [6.4] and [6.5] give the speed distribution with respct to arc length of a flow which is continuously on the point of separating for the laminar and turbulent B.Ls respectively. Stratford and Curle (Ref.9) have presented methods for improving the accuracy of the prediction of the point of separation of the laminar B.L by replacing the constant k_1 by a function depending on two parameters D* and G* given by

$$D^* = C_p/(x.dC_p/dx) \qquad [6.6]$$

$$G^* = (C_p.d^2C_p/dx^2)/(dC_p/dx)^2 \qquad [6.7]$$

The separation value of k_1, D^* and G^* are quoted from Curle for flows with various free-stream pressure/velocity distributions (identified by author);

Author		Separation values of k_1	D^*	G^*
1. Stratford, 1954	:	Ø.Ø74514	Ø	Ø
2. Curle, 1976	:	Ø.59Ø77	Ø	-Ø.5
3. Howarth,	:	1.ØØ211	1.Ø681	-Ø.1454
4. Tani, 1949	:	1.Ø4Ø61	Ø.5198	Ø.4376
5. Banks, 1967	:	1.Ø5137	Ø.94Ø	Ø.1331
6. Riley/Stewartson,	:	Ø.46367.G*	Ø	--> ∞
7. Williams, 1976(a)	:	Ø.74276	-2.3113	-2.1899
8. Williams, 1976(b)	:	Ø.56412	3.9223	-4.4Ø72
9. Curle, 1977	:	Ø.91373	Ø	Ø.5

Table 6.2

It can be shown that D^* and G^* satisfy the relation

$$dD^*/dx = 1 - (D^* + G^*) = 1 - X \text{ where } X = D^* + G^*$$

A plot of the separation values of k_1 against X shows that for $D^* = Ø$ the data points for results 1,2,9 above are almost collinear. Curle has shown that for $D^* = Ø$ the separation values of K_1 satisfy a relation of the form

$$K_1 = S(X,Ø) = (a_0 + a_1.X + a_2.X^2).e^{-a_4X} + a_3.X \qquad [6.8]$$

where the exponential term accomodates the result for X--> ∞ in result 6 above. Alternative to Curle, a least squares fit for these data point (D^*=Ø) gives the values of the constants as

a_0 = Ø.74514; a_1 = Ø.36224; a_2 = Ø.1Ø16Ø6747; a_3 = Ø.46367; a_4 =2/3.

Assuming further that the data points for $D^* \neq Ø$ satisfy a relation of the form

$$k_1 = S(X,D^*) = S(x,Ø).(1 + (b_0+b_1.X+b_2X^2)(1- e^{-D^*}).D^{*b_3}) \qquad [6.9]$$

A further least squares fit gives the values of the constants b_i as b_0 =-0.044663068; b_1=-0.024227219; b_2= -0.01424262023; b_3 = 0.2770

Equation [6.9] has a maximum relative error of 10^{-2}% for all data points and may be used to replace the k_1 in [6.1] to improve the accuracy of prediction of the point of separation of the boundary layer, thus

$$C_p.(x.dC_p/dx)^2 = S(X,D^*) \qquad [6.10]$$

A suitable finite difference form for [6.10] would enable the corresponding wall velocity distributions to be calculated for flows whose B.Ls are continuously on the point of separation. Given the availability of data, a similar calculation would yield the corresponding results for the turbulent B.L.

By virtue of their definition, B.Ls corresponding to duct geometries calculated in this way are likely to be unstable and easily 'tripped' into separation, however the resulting contours will represent the limiting cases for flows derived from separation parameters below the critical ones. It is useful to examine the variation of the pressure coefficients and speed with respect to arc length for the laminar and turbulent B.Ls in two dimensional flow with a view for later comparison with the axisymmetric case. [See fig 6.2]

Thus for the laminar and turbulent B.L we have from equations [6.4] and [6.5]

$$C_p = k_L.(\ \ln(x/x_0)\ )^{2/3}$$

$$U = U_0.(\ 1 - k_L.(\ \ln(x/x_0)\ )^{2/3}\ )^{1/2} \qquad \text{(Lam.)}$$

$$C_p = k_T.(\ (x/x_0)^{1/5} - 1\ )^{1/3}$$

$$U = U_0.(1 - k_T.(\ (x/x_0)^{1/5} - 1\ )^{1/3}\ )^{1/2} \qquad \text{(Tur.)}$$

where k_L = 0.223 , k_T = 1.230.

Letting $X = x/x_o$ then

$$dC_p/dX = F(X).(\ \ln(X)\ )^{-1/3} \qquad \text{(Lam.)}$$

$$dC_p/dX = F(X).(X^{1/5} - 1\ )^{-2/3} \qquad \text{(Tur.)}$$

where $F(X)$ represents some function of X and $F(1) \neq \emptyset$.

Further, the velocity distributions for both the laminar and turbulent B.L are related to the pressure coefficient by

$$U = U_o.(\ 1 - C_p\ )^{1/2}$$

hence $$dU/dx = U_o.(\ 1 - C_p\ )^{-1/2}(-1).dC_p/dx.$$

When X =1 i.e when $x = x_o$ then $dC_p/dx = \infty$ showing that both the pressure and velocity gradients are discontinuous at $x = x_o$.

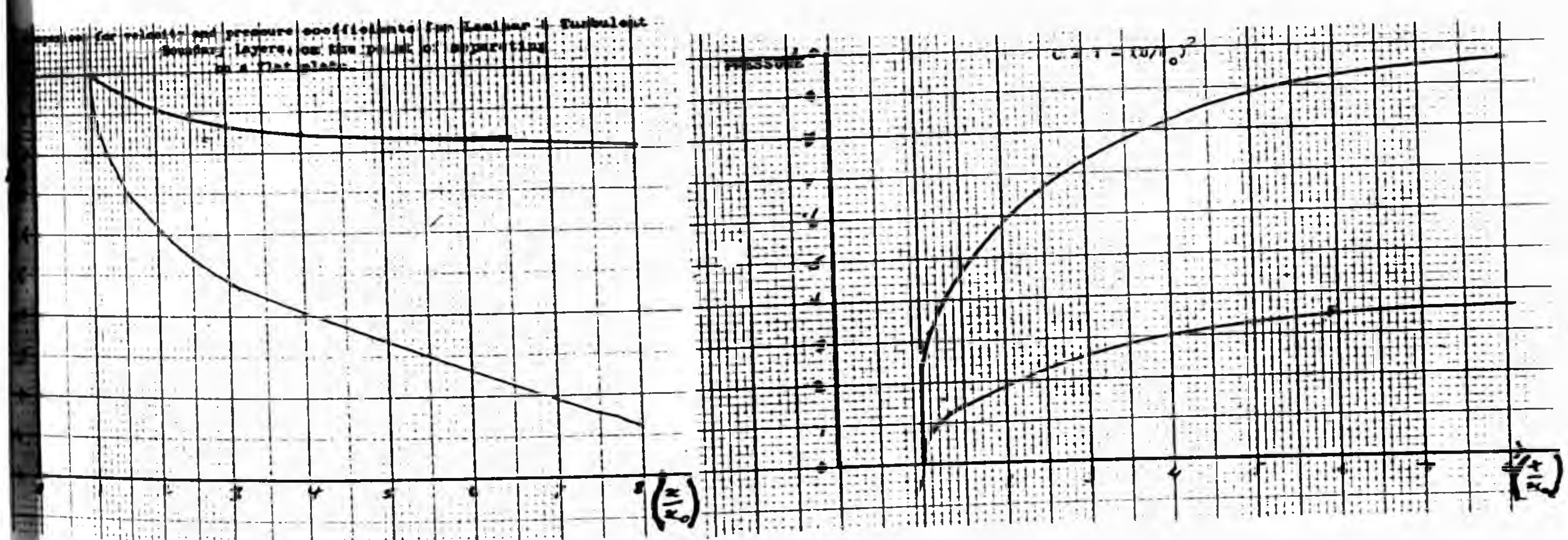

Fig 6.3

<u>(III) Mangler's Transform</u> In order to apply the results of the previous section to axisymmetric flow, use is made of Mangler's transform (Ref; 5) which maps the boundary layer equations (B.L.E) for axisymmetric flow into those for two dimensional

plane flow. Thus let x,y be coordinates along and perpendicular to the surface OX and u,v be the corresponding velocity components with p and U denoting the pressure and speed in the free stream for 2D plane flow (Fig. 6.3.(i)) and let z,r,w,q,P,W be the corresponding quantities for flow over a body of revolution (Fig. 6.3(ii)).

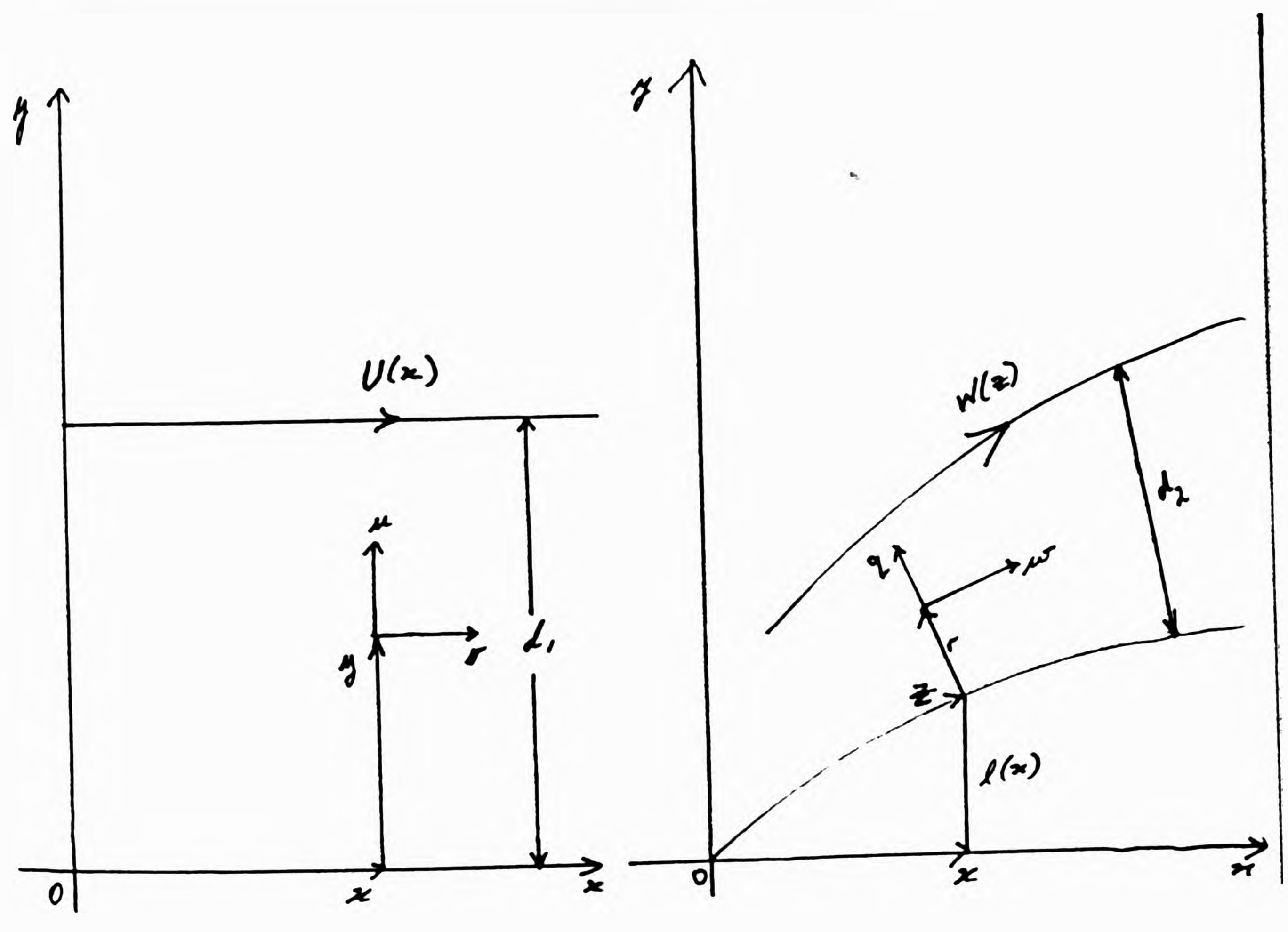

Fig. 6.3.(i) & (ii)

Then the B.L.E for plane flow are given by

$$u.u_x + v.u_y = U.U_x + \mu.u_{yy} \quad (a)$$
$$u_x + v_y = 0 \quad (b) \qquad [6.11]$$
$$p(x) = U.U_x \; ; \; u=U(x) \quad (c)$$

For axisymmetric (non-swirling) flows the B.L.E on a body of revolution, as derived by Boltze (Ref; 13) are

$$w.w_z + q.w_r = W.W_z + \mu.w_{rr} \quad \text{(a)}$$

$$(l(x).w)_z + (l(x).q)_r = \emptyset \quad \text{(b)} \qquad [6.12]$$

$$P(z) = W.W_z \; ; \; W = W(z) \quad \text{(c)}$$

where l(x) is a function describing the axisymmetric body contour. It can be seen that the B.L.Es are similar in form for the two regimes differing only where the contour function l(x) appears explicitly in [6.12(b)], the continuity equation for axisymmetric flow.

The transform which maps the set [6.12] into [6.11] is given by

$$x = L^{-2} . \int_{\emptyset}^{z} l^{2}(z).dz \quad \text{(a)}$$

$$y = l(z).L^{-1}.r \quad \text{(b)}$$

$$u = w \quad \text{(c)} \qquad [6.13]$$

$$v = L.l^{-1}(z).(q + r.w.l^{-1}(z).l(z)_z) \quad \text{(d)}$$

$$U(x) = W(z) \quad \text{(e)}$$

where L is some length representative of the dimension of the body of revolution.

Since the arc length, 'z', along the contour is a function of x alone (i.e z=z(x) and x=x(z)), then l(x) may be considered as a function of z only thus

$$l = l(x) = l\{x(z)\} = l(z)$$

From [6.13] it can be shown that and if F is an arbitrary function then the differential operators of the transform are

$$z_x = L^2 l^{-2}(z); \quad z_y = \emptyset; \quad r_y = L.l^{-1}(z); \quad r_x = -L^3.y.l^{-4}(z).l(z)_z \qquad (a)$$

$$F_x = (\ L^2.l^{-2}(z)\ ).F_z - (\ L^2.l^{-3}(z).r.l(z)_z\ ).F_r \qquad (b)$$

$$F_y = (\ L.l^{-1}(z)\ ).F_r \qquad (c) \quad [6.14]$$

$$F_z = (\ l^2(z).L^{-2}\ ).F_x + (\ y.l^{-1}(z).l(z)_z\ ).F_y \qquad (d)$$

$$F_r = (\ l(z).L^{-1}\ ).F_y \qquad (e)$$

(IV) Derivation of Condition of Non-Separation of A Boundary Layer on a Body of Revolution in Axisymmetric Flow.

Consider the behaviour of the boundary layers shown in Figs. 6.4 (i), (ii) for plane and axisymmetric flow respectively.

For both flows

(i) 'o' refers to the station at which the external flow U(x) or W(z) encounters a sharp pressure change;

(ii) 'p' refers to some general point ;

(iii) 's' refers to the point(s) or region at which the B.L is about to separate.

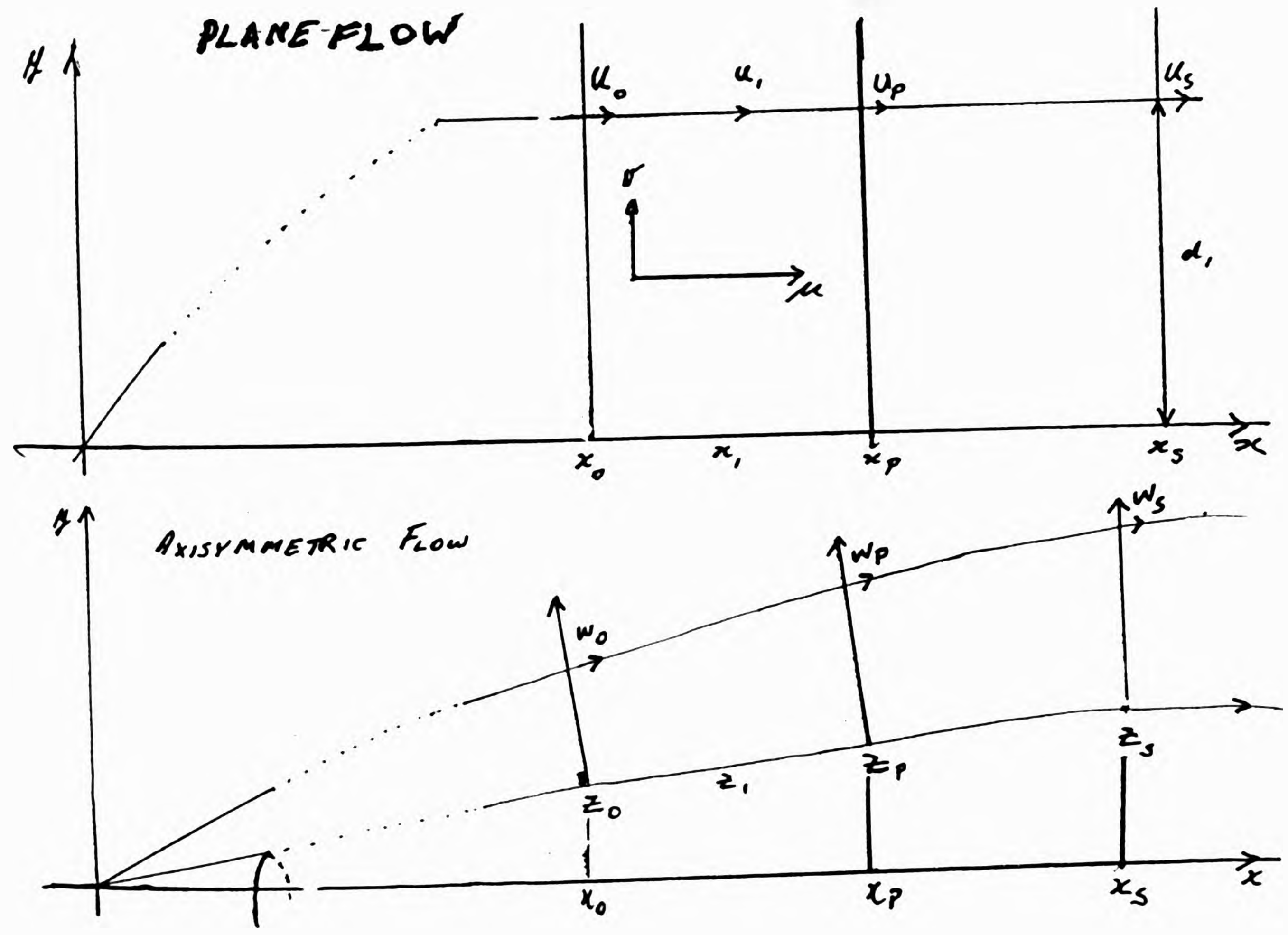

Fig 6.4.(i) & (ii)

For the plane flow, it is assumed that the B.L has commenced its developement upstream of $x=x_o$ at $x = 0$. To determine the corresponding point for the axisymmetric case we have from [6.13.a]

$$x = L^{-2} \int_{z=\emptyset}^{z=z'} l^2(z).dz = \emptyset$$

Since the integrand is positive definite the upper limit must be zero hence the B.L in the axisymmetric case commences at $z = \emptyset$.

<u>(A) Laminar B.L.</u>

The separation of the laminar boundary layer (L.B.L) for plane flow depends upon the parameter S_L defined by

$$S_L = C_p.(x.dC_p/dx)^2 \qquad [6.1]$$

When S_L reaches some critical value S_{Lc} (say) then separation is said to have occured. By virtue of [6.13e]

$$U(x) = W(z)$$

implying that the pressure coeficient, C_p

$$C_p = 1 - (U_p/U_o)^2 = 1 - (W_p/W_o)^2 = K_p \quad \text{(say)} \qquad [6.15]$$

Hence we may write the separation condition, [6.1], as

$$S_L = K_p.(x.dK_p/dx)^2 \qquad [6.1a]$$

Let $\quad Z = L^{-2}. \int l^2(z).dz \; (= x)$

Then $\quad dZ/dz = l^2(z)/L^2$ and $d\;/dZ = (L^2/l^{-2}).d\;/dz$

Hence from [6.14.b]

$$(K_p)_x = (L^2.l^{-2}).(K_p)_z - (L^2.l^{-3}.r.l_z).(K_p)_r$$

$$= (L^2.l^{-2}).(K_p)_Z.(dZ/dz) - (L^2.l^{-3}.r.l_z).(K_p)_r$$

$$= (K_p)_Z - (L^2.l^{-3}.r.l_z).(K_p)_r$$

Substituting into [6.1a] gives

$$S_L = K_p.(\,Z.(K_p)_Z - Z.(L^2.l^{-3}.r.l_z).(K_p)_r\,)^2$$

$$= K_p.(\,Z.(K_p)_Z - Z.(\,L^2.l^{-3}.r.L^{-2}.l^2.l_Z\,).(K_p)_r\,)^2$$

$$= K_p.[\,Z.(\,Z.K_p)_Z - (r.l^{-1}.Z.l_Z\,).(K_p)_r\,]^2 \qquad [6.1b]$$

Since Kp ($= 1 - (W_p(z)/W_o)^2$) is a function of z alone then

$$(K_p)_r = \emptyset$$

Hence for axisymmetric flow the separation condition may be written

as $$S_L = K_p . (Z.(K_p)_Z)^2 \qquad [6.16]$$

In support of this formal derivation it can be argued that if d_2 (the thickness of the B.L) is very much smaller than the width of the body characterized by the function l(z) then

$$d_2 << l(z)$$

also $$r <= d_2 << l(z) \quad \text{(within the B.L)}$$

=> $$r.l^{-1} = o(d_2) \quad \text{(say)}$$

Further at the 'edge' of the B.L, the normal velocity, w(z,r), varies very slowly with r hence we may take

$$w_r = o(d_2)$$

Since $l(z) = o(L)$ then $Z = L^{-2} . \int_0^z l^2(z).dz = o(1)$

then

$$o(l_Z) = o(l_z . z_Z) = o(l_z).o(z_Z) = o(\text{Sin } \theta).o(L^2.l^{-2}) = o(\text{Sin}\theta)$$

and

$$(K_p)_r = (1 - (w(z,r)/W_o)^2)_r = -2.(w/W_o^2).w_r = o(d_2)$$

Hence the term

$$((r.l^{-1}).Z.(l_Z).(K_p)_r) = o(d_2).o(1).o(\sin \theta).o(d_2) = o(d^2)$$

and is negligible compared with the 1st term in [6.1b].

Let $Z^* = \ln(Z)$; => $dZ^*/dZ = 1/Z$;

Hence for any function F

$$dF/dZ = (dF/dZ^*).(dZ^*/dZ) = Z^{-1}.dF/dZ^*$$

$$Z.dF/dZ = dF/dZ^* \qquad [6.17]$$

Hence [6.16] may be written as

$$S_L = K_p.(\ (K_p)_{Z*}\ )^2 \qquad -[6.18]$$

where

$$Z* = \ln(\ Z\ ) = \ln[\ L^{-2}\ .\int_0^z l^2(z).dz\ )\ ] \qquad [6.18a]$$

Separation occurs when S_L, defined by [6.18], attains some critical value S_{Lc}.

(B) Turbulent B.L.

The separation criteria for the turbulent boundary layer on the flat plate is given by

$$S_T = C_p.(\ x.dC_p/dx)^{1/2}).(10^{-6}.R_x)^{-1/10} \qquad [6.2]$$

A derivation, similar to that for the laminar case, transforms [6.2] to

$$S_T = K_p.(\ Z.(dK_p/dZ)\ )^{1/2}.(10^{-6}.R_Z)^{-1/10}\ ;\quad R_Z = W_o.Z/\mu \qquad [6.19a]$$

With the substition of [6.17] this leads to

$$S_T = K_p.(\ K_p\ )^{1/2}.(10^{-6}.R_{Z*})^{-1/10} \quad \text{where } R_{Z*} = W_o.Z/\mu \qquad [6.19]$$

Again separation is said to be occuring when S_T takes some critical value S_{Tc}.

As in the two dimensional case, the axisymmetric forms of the separation criteria may be viewed as defining pressure distributions (and consequently velocity distributions) corresponding to various choices of the separation parameters S_L and S_T. In particular if S_L (or S_T) are taken as the separation values S_{Lc} or (S_{Tc}), not necessarily constant , then we may deduce the equations defining the velocity distributions corresponding to those flows which are continuously on the point of separation for both the laminar and turbulent B.L on a body of revolution.

Velocity/Pressure distributions for Axisymmetric Flows

(A) Laminar B.L.

From [6.18] $K_p.((K_p)_{Z*})^2 = S_L$

=> $K_p^{1/2}.dK_p = S_L^{1/2}.dZ*$

Integrating with respect to z from $z = z_1$ to $z = z_p$ where z_1 is arbitrary gives

$$[(2/3).K_p^{3/2}]_{z=z1}^{z=zp} = \int_{z=z1}^{z=zp} S_L^{1/2}.dZ* = I_{1,p} \text{ (say)} \qquad [6.2Ø]$$

$$\Rightarrow \quad K_p^{3/2}(z_p) - K_p^{3/2}(z_1) = (3/2).I_{1,p}$$

$$\Rightarrow \quad K_p(z_p) = (K_p^{3/2}(z_1) + (3/2).I_{1,p})^{2/3} \qquad [6.21]$$

Since z_1 is arbitrary let $z_1 = z_o$ (the point at the commencement of the pressure change) then

$K_p(z_p) = 1 - (W_p/W_o)^2$; $K_p(z_1) = K_p(z_o) = 1 - (W_o/W_o)^2 = Ø$.

$$\text{and } I_{1,p} = I_{o,p} = \int_{z=zo}^{z=zp} S_L^{1/2}.dZ* \qquad [6.21a]$$

Substituting these values into [6.21] we have after rearrangement the expression for the speed distribution ot the edge of the B.L

$$W_p = W_o.(1-((3/2).I_{o,p})^{2/3})^{1/2} \qquad [6.22]$$

which is of the same form as that for the plane flow case.

A specific choice of the function S_L in [6.21a] will determine the integral $I_{o,p}$ and hence the precise form of the velocity distribution given by [6.22]. In general S_L might be chosen arbitrarily to produce a variety of velocity distributions. However in this context it is taken as the function S(X,D*) of [6.1Ø] and is, in the first instance, set equal to one of the constants from the values for separation listed in Table 6.2.

If S_L is constant, then [6.21a] may be integrated directly to give

$$I_{o,p} = \int_{z=zo}^{z=zp} S_L^{1/2}.dZ^* = \int_{z=zo}^{z=zp} S_L^{1/2} dZ^* = S_L^{1/2}.[\ Z^*\ ]_{z=zo}^{z=zp} \quad [6.23]$$

Now from [6.18a]

$$Z^* = \ln(\ Z\ ) = \ln(\ L^{-2}.\int_{z=\emptyset}^{z} l^2(z).dz\ ) = \ln(l^{-2}.J_{\emptyset,z}\ )$$

where we define the integral $J_{\emptyset,z} = \int_{\emptyset}^{z} l^2(z).dz$

Hence

$$[\ Z^*\ ]_{z=zo}^{z=zp} = \ln(\ L^{-2}.J_{\emptyset,zp}\ ) - \ln(\ L^{-2}.J_{\emptyset,zo}\ ) =$$

$$= \ln(J_{\emptyset,zp}/J_{\emptyset,zo}) = \mathrm{Ln}(\ (J_{\emptyset,zo} + J_{zo,zp})/J_{\emptyset,zo})$$

$$= \ln(\ 1 + (\ J_{zo,zp}/J_{\emptyset,zo}\ )\ ) \quad \text{where } J_{a,b} = \int_{z=a}^{z=b} l^2(z).dz \quad [6.24]$$

Substituting [6.24] (via [6.23]) into [6.22] gives the relation defining the velocity distribution corresponding to the separation parameter S_L(constant).

Thus

$$W_p = W_o.\{1 - [\ (3/2).S_L^{1/2}.\ln(\ 1 + (\ J_{zo,zp}/J_{\emptyset,zo}\ )\ )\ ]^{2/3}\}^{1/2} \quad [6.25]$$

The integral $J_{\emptyset,zo}$ remains constant throughout any iteration once z_o has been chosen. However $J_{zo,zp}$, depending as it does on the current values of $l^2(z)$ defining the wall contours in the range $[z_o, z_p]$ will vary thus producing continuously changing velocity distributions on the duct boundaries.

The Turbulent B.L

From equation [6.19a] the separation criterion for the turbulent B.L is

$$K_p.(Z.(dK_p/dZ))^{1/2}.(10^{-6}.R_z)^{1/10} = S_T \qquad [6.19a]$$

with $R_z = W_o.z/\mu$; $Z = L^{-2}.\int_0^z l^2(z).dz$; $K_p = 1-(W_p(z)/W_o)^2$

Substituting R_z into [6.19a], and rearanging gives

$$Z^{4/5}.K_p^2.dK_p/dz = S_T^2.(10^{-6}.W_o/\mu)^{1/5} = A_1 \quad \text{(say)}$$

$$\Rightarrow \quad K_p^2.dK_p = A_1.Z^{-4/5}.dZ \qquad [6.19b]$$

Integrating from $z = z_o$ to $z = z_p$ gives (assuming A_1 is constant)

$$\left[(1/3).K_p^3 = 5.A_1.Z^{1/5} \right]_{z=z_o}^{z=z_p}$$

$$\Rightarrow K_p^3(z_p) - K_p^3(z_o) = 15.A_1.[Z^{1/5}(z_p) - Z^{1/5}(z_o)]$$

$$= [15.A_1.Z^{1/5}(z_o)].[\{Z(z_p)/Z(z_o)\}^{1/5} - 1]$$

When $z = z_o$; $W(z) = W(z_o) = W_o$; Hence $K_p(z_o) = 1 - (W_o/W_o)^2 = 0$

Let $A_2 = (15.A_1.Z^{1/5}(z_o))^{1/3}$, then the pressure coefficient may be written as

$$K_p(z_p) = A_2.[\{Z(z_p)/Z(z_o)\}^{1/5} - 1]^{1/3} \qquad [6.28]$$

Substituting for K_p in terms of the speeds from [6.19a] gives

$$W(z_p) = W_o.\{1 - A_2.[(Z(z_p)/Z(z_o))^{1/5} - 1]^{1/3}\}^{1/2} \qquad [6.29]$$

Now $A_2 = (15.A_1.Z^{1/5}(z_o))^{1/3}$

$$= [15.\{S_T^2.[10^{-6}.W_o/\mu]^{1/5}\}.Z^{1/5}(z_o)]^{1/3} \qquad \{ \text{[See 6.19b]} \}$$

$$= (15.S_T^2)^{1/3}.[(10^{-6}).(W_o.Z(z_o)/\mu)]^{1/15}$$

Now if $R_{z(z_o)} = W_o.Z(z_o)/\mu$ is of the order of 10^6 then the constant A_2 is given by $A_2 = (15.S_T^2)^{1/3}$.

Using Stratford's value for S_T for the separation constant for the turbulent B.L in plane flow we have

$$A_2 = (15.(.39^2))^{1/3} = 1.31645744$$

Thus the expressions [6.25] and [6.29] define boundary velocity distributions (for constant separation parameters S_L, S_T) for B.Ls at the point of separation at each point of the boundary for laminar and turbulent B.Ls respectively. Velocity distributions may be generated for values of S_L and S_T below the critical ones.

Variable Separation Parameters. In arriving at the results for both the turbulent and laminar B.Ls it was assumed that the separation parameter defining the flows was constant. However it is feasible to allow for variable separation parameters as for example S(X,D*) defined in [6.10]. Thus noting that S(X,D*) is a function of arc length, z, we can write the pressure coefficient, K, as

$$K = F\{ \int_{z=0}^{z=a} S.Z^n.dZ \}$$

where F is some function and n= -1, -4/5 for laminar and turbulent B.Ls respectively with S being some function of the 'local' value of the separation parameter.

Since $K = 1 - (W(z)/W_o)^2$

then the boundary velocity distributions are of the form

$$W(z) = W_o.[1 - F\{ \int_{z=0}^{z=a} S(z).Z^n(z).dZ(z) \}]^{1/2} \qquad [6.30]$$

From the computational aspect, this more general form of the velocity distributions involves no special numerical difficulties since even in their simplest forms ([6.25] and [6.29]) need to be integrated numerically.

(V) Extension to a Class Of Swirling Flows.

(1) An extension to the axisymetric condition to cater for a class of swirling flows is obtained by examining the derivation of the B.L approximation for such flows which, for the sake of completeness, is given below (See Fig 6.5).

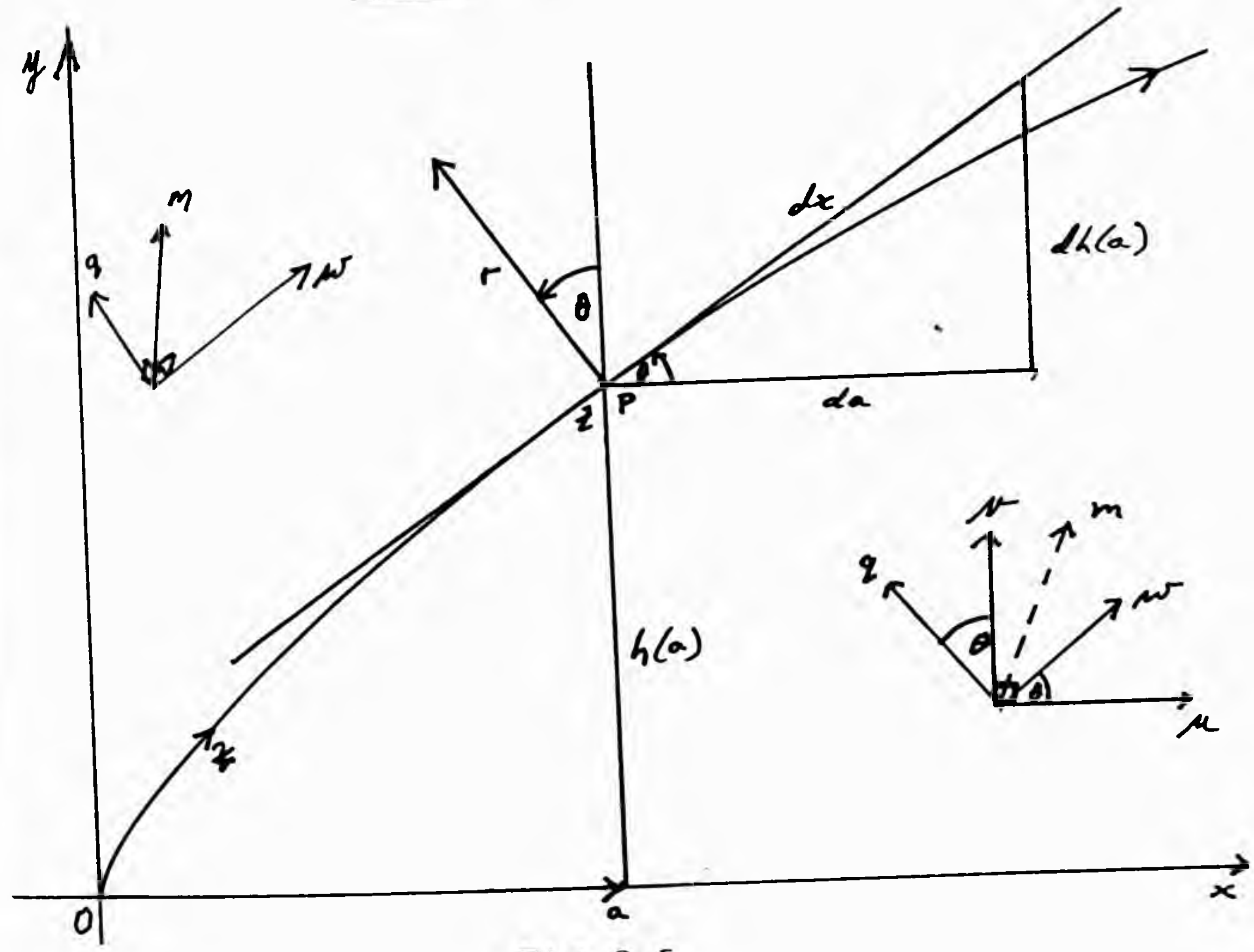

Fig 6.5

Let z be the arc length measured along the contour OP and r the coordinate normal to this contour at P with w and q the corresponding velocity components and m the velocity component perpendicular to the plane of w and q while x,y,u,v are the corresponding quantities in the XOY plane. Let 'a' be a metric along the x-axis and let h(a) be the length of the perpendicular PA from P to the x-axis where h(a) is a known function describing the contour.

Then $$dz^2 = da^2 + dh^2 = da^2.(1 + (dh/da)^2)$$

=> $$dz = da.[1 + (dh/da)^2]^{1/2}$$

=> $$z = \int_{0}^{z=a} [1 + (dh/da)^2]^{1/2}.da = G(a) \quad \text{(say)}$$

Thus 'z' is, in principle, a known function of 'a' and vice-versa, and hence h(a) may also be considered a known function of 'z'.

Thus let $a = H(z)$ where $(H) = (G)^{-1}$ (The inverse of G)

(2) Since h is a function of 'z' only, then $h = h(z)$ and the angle $\theta = \theta(z)$, thus

$\tan\theta = dh/da = h_a$; $\sin\theta = dh/dz = h_z$; $\cos\theta = da/dz = dH/dz = H_z$.

and $h_z^2 = 1 - H_z^2$; $H_z^2 = 1 - h_z^2$ [6.31]

(3) Coordinate Relationships.

From Fig. 6.6 we see that if (x,y) and (z,r) the coordinates of a point in the two frames of reference then

$$x = a - r.\sin\theta = a - r.h_z \quad \text{(i)}$$
$$y = h + r.\cos\theta = h + r.H_z \quad \text{(ii)} \qquad [6.32]$$

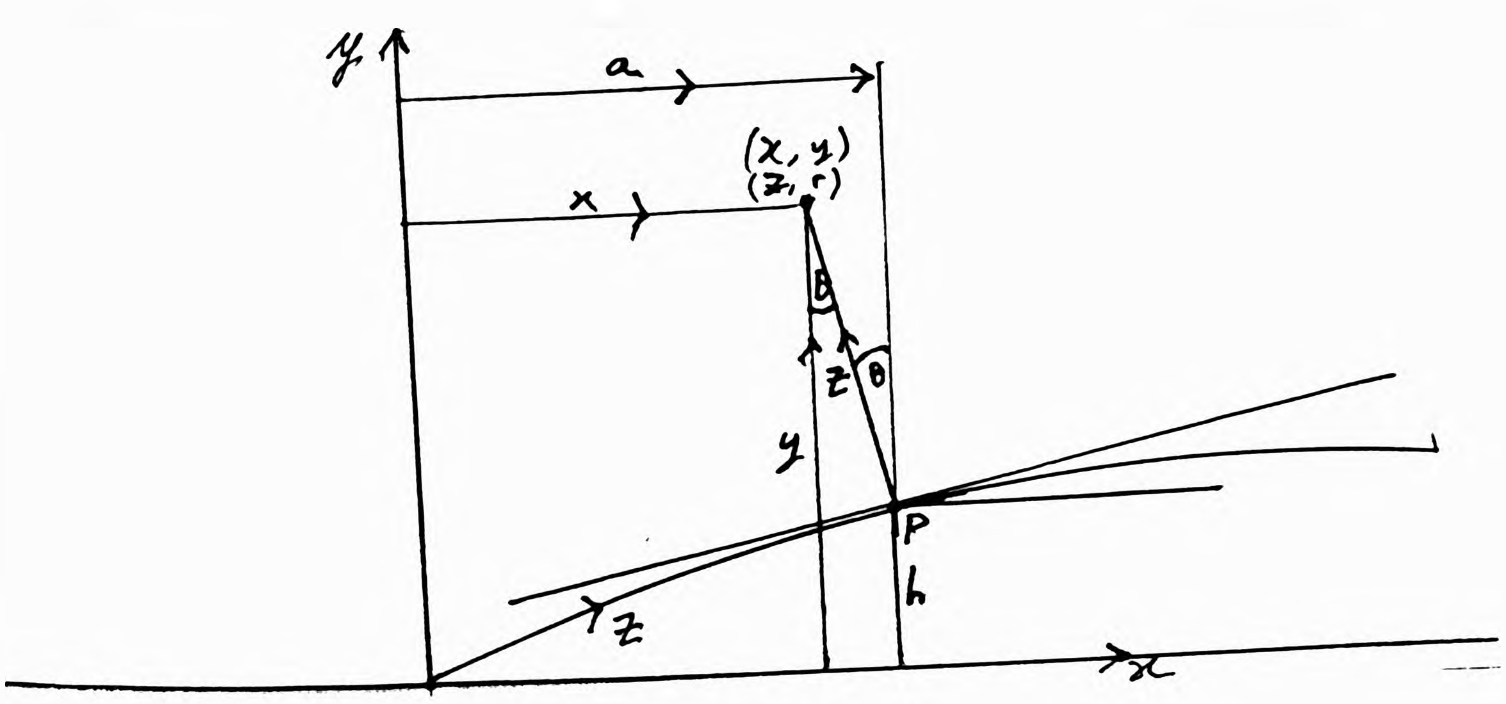

Fig. 6.6

(4) From equations [6.32]

$$dx = da - h_z.dr - r.h_{zz}.dz = H_z.dz - h_z.dr - r.h_{zz}.dz$$

$$dy = h_z.dz + H_z.dr + r.H_{zz}.dz$$

Hence

$$dx = (H_z - r.h_{zz}).dz + (-h_z).dr \;;\; dy = (h_z + r.H_{zz}).dz + (H_z).dr \quad (i)$$

$$x_z = H_z - r.h_{zz} \;;\; y_z = h_z + r.H_{zz} \quad (ii) \quad [6.33]$$

$$x_r = -h_z \;;\; y_r = H_z \quad (iii)$$

Also since $H_z = \text{Cos}\,\Theta$; $h_z = \text{Sin}\,\Theta$,

$$H_{zz} = -\text{Sin}\,\Theta.\Theta_z = -h_z.\Theta_z \;;\; h_{zz} = \text{Cos}\,\Theta.\Theta_z = H_z.\Theta_z \quad (iv)$$

and

$$h_z.h_{zz} + H_z.H_{zz} = h_z.(H_z.\Theta_z) + H_z.(-h_z.\Theta_z) = \emptyset$$

where Θ_z is the curvature of the countour.

(5) Differential Operators.

Generally for any function F we have

$$F_z = F_x.x_z + F_y.y_z \quad (i)$$

$$F_r = F_x.x_r + F_y.y_r \quad (ii)$$

Letting $J = x_z.y_r - x_r.y_z$ (iii) [6.34]

Then $F_x = J^{-1}.(y_r.F_z - y_z.F_r)$ (iv)

$$F_y = J^{-1}.(x_z.F_r - x_r.F_z) \quad (v)$$

From [6.34(iii)] and [6.33]

$$J = x_z.y_r - x_r.y_z = (H_z - r.h_{zz}).H_z - (-h_z).(h_z + r.H_{zz})$$

$$= h_z^2 + h_z^2 - r.(H_z.h_{zz} - h_z.H_{zz}) = 1 - r.(H_z^2 + h_z^2).\Theta_z$$

$$J = 1 - r.\Theta_z \quad (vi)$$

Also

$$x_z = H_z - r.h_{zz} = H_z - r.(H_z.\Theta_z) = (1 - r.\Theta_z).H_z = J.H_z \qquad \text{(vii)}$$

$$y_z = h_z - r.H_{zz} = h_z - r.(-h_z.\Theta_z) = (1 - r.\Theta_z).h_z = J.h_z \qquad \text{(viii)}$$

and $x_r = -h_z$; $y_r = H_z$

It follows that [6.34] (iv) and (v) can be written as

$$F_x = J^{-1}.H_z.F_z - h_z.F_r \qquad \text{(ix)}$$

$$F_y = J^{-1}.h_z.F_z + H_z.F_r \qquad \text{(x)}$$

$$F_{xx} = (J^{-2}.H_z^{\,2}).F_{zz} + (h_z^{\,2}).F_{rr} + (-2J^{-1}.h_z.H_z).F_{rz} + $$
$$+ (J^{-2}.H_z).(H_{zz} - J^{-1}.J_z.H_z + h_z.J_r).F_z + (-h_{zz}.H_z.J^{-1}).F_r \qquad \text{(xi)}$$

$$F_{yy} = (J^{-2}.H_z^{\,2}).F_{zz} + (H_z^{\,2}).F_{rr} + (-2J^{-1}.H_z.h_z).F_{rz} + $$
$$+ (j^{-2}.h_z).(h_{zz} - J^{-1}.J_z.h_z - H_z.J_r).F_z + (H_{zz}.h_z.J^{-1}).F_r \qquad \text{(xii)}$$

$$F_{xx} + F_{yy} = J^{-2}.F_{zz} + F_{rr} + J^{-3}r.\Theta_{zz}.F_z - J^{-1}.\Theta_z.F_r \qquad \text{(xiii)}$$

Also $h_z.h_{zz} - h_z.H_{zz} = \Theta_z$; $h_{zzz} = -h_z.\Theta_z^{\,2} + H_z.\Theta_{zz}$

$$H_{zzz} = -H_z.\Theta_z^2 - h_z.\Theta_{zz} \;;\; J_z = -r.\Theta_{zz} \;;\; J_r = -\Theta_z \qquad \text{(xiv)}$$

<u>(6) Velocitv relations</u> From Fig 6.5 we have

$$u = w.\text{Cos}\ \Theta - q.\text{Sin}\ \Theta = H_z.w - h_z.q \qquad \text{(i)}$$

$$v = w.\text{Sin}\ \Theta + q.\text{Cos}\ \Theta = h_z.w + H_z.q \qquad \text{(ii)}$$

$$w = u.\text{Cos}\ \Theta + v.\text{Sin}\ \Theta = H_z.u + h_z.v \qquad \text{(iii)} \quad [6.35]$$

$$q = -u.\text{Sin}\ \Theta + v.\text{Cos}\ \Theta = -h_z.u + H_z.v \qquad \text{(iv)}$$

Differentiating u and v with respect to z and r gives

$$u_z = H_{zz}.w - h_{zz}.q + H_z.w_z - h_z.q_z \qquad \text{(v)}$$

$$u_{zz} = H_{zzz}.w + 2H_{zz}.w_z + H_z.w_{zz} - h_{zzz}.q - 2h_{zz}.q_z - h_z.q_{zz} \qquad \text{(vi)}$$

$$u_r = H_z.w_r - h_z.q_r \qquad \text{(vii)}$$

$$u_{rr} = H_z.w_{rr} - h_z.q_{rr} \qquad \text{(viii)}$$

$$v_z = h_{zz}.w + h_z.w_z + H_{zz}.q + H_z.q_z \qquad \text{(ix)}$$

$$v_{zz} = h_{zzz}.w + 2.h_{zz}.w_z + h_z.w_{zz} + H_{zzz}.q + 2H_{zz}.q_z + H_z.q_{zz} \qquad \text{(x)}$$

$$v_r = h_z.w_r + H_z.q_r \qquad \text{(xi)}$$

$$v_{rr} = h_z.w_r + H_z.q_r \qquad \text{(xii)}$$

It follows from [6.3.(i) & (ii)] and [6.3.4 (i)]

$$u.F_x + v.F_y = J^{-1}.w.F_z + q.F_r \qquad \text{(xiii)}$$

<u>(7)</u>
The equations for axisymmetric flow are now expressed, using the above relations, in terms of the surface coordinates of a body of revolution and the corresponding velocity components and a B.L approximation derived. The full axisymmetric flow equations in cylindrical coordinates are

$$u.u_x + v.u_y = -p_x + S(u_{xx} + u_{yy} + u_y/y) \qquad \text{(Axi.) (i)}$$

$$u.v_x + v.v_y - m^2/y = -p_y + S(v_{xx} + v_{yy} + v_y/y - v/y^2) \qquad \text{(Rad.)(ii)}$$

[6.36]

$$u.m_x + v.m_y + v.m/y = S(m_{xx} + m_{yy} + m_y/y - m/y^2) \qquad \text{(Ang)(iii)}$$

$$(yu)_x + (y.v)_y = Ø \qquad \text{(Con) (iv)}$$

where all quantities are dimensionless and S =(Reynolds no.)$^{-1}$

Referred to the new coordinate system (z,r), the set [6.36] becomes

$$wu_z + Jqu_r = -(H_z p_z - Jh_z p_r) + S(J^{-1}u_{zz} + Ju_{rr} + J^{-2} r\, \Theta_{zz} u_z - \Theta_z u_r + (h_z u_z + JH_z u_r)/(h + rH_z)) \qquad \text{(Axi) (i)}$$

$$wv_z + Jqv_r - Jm^2/y = -(h_z p_z + JH_z p_r) + S(J^{-1}v_{zz} + Jv_{rr} + J^{-2} r\Theta_{zz} v_z - \Theta_z.v_r + (h_z v_z + JH_z v_r)/(h+r.H_z) - Jv/(h+rH_z)^2) \qquad \text{(Rad)(ii)}$$

$$wm_z + Jqm_r + J(h_z w + H_z q)m/y = S(J^{-1}m_{zz} + Jm_{rr} + J^{-2}\Theta_{zz} m_z - \Theta_z m_r + (h_z m_z + Jh_z m_r)/(h+rH_z) - Jm/(h+rH_z)^2 \qquad \text{(Ang)(iii)}$$

$$w_z + Jq_r + (Jh_z/(h+rH_z))w + ((JH_z)/(h+rH_z) - \Theta_z)q = Ø \qquad \text{(Con)(iv)}$$

[6.37]

where u_z, u_{zz} etc are given in terms of w and q by [6.35 (v)] et seq..

In common with other B.L approximations, it is assumed that the thickness, 't' of the B.L is small compared with the axial and transverse dimensions of the body and thus, for equations made dimensionless with respect to some characteristic length we have

$$t << 1.$$

Supposing that the Reynold's number of the flow is proportional to t^{-2} [i.e $t = o(R^{-1/2})$ while the B.L approx. is valid] then

$$S = R^{-1} = o(t^2).$$

If w_z and q_r are of the same order of magnitude within the B.L and bearing in mind that q varies from zero at the wall through non-zero values and decays towards the edge of the B.L within a distance 't' then we may assume that $q = o(t)$ within the B.L.

Taking quantities in the axial and transverse directions to be of the order of unity then the following order of magnitude assumptions are applied to the equation set [6.37] above.

Order of mag: Terms of the order of mag. of

$o(1)$: w ; w_z ; w_{zz} ; m ; m_z ; m_{zz} ; q_r ; z ; $h(z)$; h_z ; H_z

$o(t)$: q ; r ;

$o(t^{-1})$: w_r ; m_r ; q_{rr} ;

$o(t^{-2})$: w_{rr} ; m_{rr} ;

$o(\theta_z)$: h_{zz} ; H_{zz} ;

$o(\theta_{zz})=o(\theta_z{}^2)$: h_{zzz} ; H_{zzz} ;

If further the curvature of the surface , θ_z, is not large then

$$\theta_z << 1 \text{ and } \theta_z = o(t) \quad \text{(say)} ;$$

then to a first approximation equations [6.37] may be written as

$$ww_z + Jqw_r = -p_z + Jh_z(H_z)^{-1}p_r + SJw_{rr} \quad \text{(i)}$$

$$ww_z + Jqw_r - J(h_z)^{-1}(h+rH_z)^{-1}m^2 = -p_z - JH_z(h_z)^{-1}p_r + SJW_{rr} \quad \text{(ii)}$$

[6.37.A]

$$wm_z + Jqm_r + Jh_z(h+rH_z)^{-1}w.m = SJm_{rr} \quad \text{(iii)}$$

$$w_z + Jq_r + Jh_z(h+rH_z)^{-1}.w = \emptyset \quad \text{(iv)}$$

Now subtracting the radial equation (ii) from the axial (i) gives

$$ww_z + Jqw_r = -p_z + (Jh_z/H_z)p_r + SJw_{rr} \quad \text{(i)}$$

$$m^2/(h+rH_z) = p_r/H_z \quad \text{(ii)}$$

[6.37.B]

$$wm_z + Jqm_r + Jh_z wm/(h+rH_z) = SJm_{rr} \quad \text{(iii)}$$

$$w_z + Jq_r + Jh_z w/(h+rH_z) = \emptyset \quad \text{(iv)}$$

Further within the B.L, $r = o(t)$, hence from [6.34(vi)]

$$J = 1 - r.\Theta_z = 1 - o(t^2) = o(1)$$

and $$h + rH_z = h + o(t).1 = h + o(t) = h$$

Hence

$$ww_z + qw_r = -p_z + (h_z/H_z)p_r + S.w_{rr} \quad \text{(i)}$$

$$m^2/h = p_r/H_z \quad \text{(ii)}$$

[6.37.C]

$$wm_z + qm_r + h_z wm/h = Sm_{rr} \quad \text{(iii)}$$

$$w_z + q_r + h_z w/h = \emptyset \quad \text{(iv)}$$

Rearrangement gives

$$ww_z + qw_r = -p_z + (h_z/H_z)p_r + Sw_{rr} \quad \text{(Axial)} \quad \text{(i)}$$

$$m^2/h = p_r/H_z \quad \text{(Radial)} \quad \text{(ii)}$$

[6.38]

$$w(hm)_z + q(hm)_r = S(hm)_{rr} \quad \text{(Azim.)} \quad \text{(iii)}$$

$$(hw)_z + (hq)_r = \emptyset \quad \text{(Cont.)} \quad \text{(iv)}$$

The set [6.38] represents a B.L approximation of the flow equations on a body of revolution for swirling flows. If the swirl velocity is zero i.e $m = \emptyset$ then [6.38] reduce to [6.12], the equations for zero swirl.

Free Stream Conditions

Let W,U,Q,P represent the corresponding flow quantities in the free stream just ouside the B.L. The inviscid form of the flow equations governing the flow in this region is obtained by setting S=0 in [6.37]. Hence

$$WU_z + JQU_r = -(H_z P_z - Jh_z P_r) \qquad \text{(i) Axi.}$$

$$WV_z + JQV_r - Jy^{-1}M^2 = -(h_z P_z + JH_z P_r) \qquad \text{(ii) Rad.} \qquad [6.40]$$

$$WM_z + JQM_r + Jy^{-1}(h_z W + H_z Q)M = 0 \qquad \text{(iii) Ang.}$$

$$W_z + JQ_r + ((Jh_z)/(h+rH_z))W + ((JH_z)/(h+rH_z) - \Theta_z)Q = 0 \qquad \text{(iv) Cont.}$$

where U_z, U_r, V_z and V_r are given by [6.35] in terms of W,Q etc., free stream quantities replacing the corresponding B.L quantities and $y = h + r.H_z$.

Substituting for U_z, U_r, V_z, V_r and rearranging we have

$$H_z(WW_z + JQW_r - \Theta_z WQ) - h_z(JQQ_r + WQ_z + \Theta_z W^2) = -H_z P_z + Jh_z P_r \qquad \text{(i)}$$

$$h_z(WW_z + JQW_r - \Theta_z WQ) + H_z(JQQ_r + WQ_z + \Theta_z W^2) - JM^2(h+rH_z)^{-1} = -h_z P_z - JH_z P_r \qquad \text{(ii)}$$

[6.40a]

$$WM_z + JQM_r + J(h_z W + H_z Q)M(h+rH_z)^{-1} = 0 \qquad \text{(iii)}$$

$$W_z + JQ_r + Jh_z W(h+rH_z) + [(JH_z)/(h+rH_z) - \Theta_z].Q = 0 \qquad \text{(iv)}$$

By forming (a) H_z*(i) + h_z*(ii) and (b) H_z*(ii) - h_z*(i) we may write [6.40a] as

$$WW_z + JQW_r - \Theta_z WQ - Jh_z M^2(h+rH_z)^{-1} = -P_z \qquad \text{(i) Axi.}$$

[6.40b]

$$JQQ_r + WQ_z + \Theta_z W^2 - JH_z M^2(h+rH_z)^{-1} = -JP_r \qquad \text{(ii) Rad.}$$

with the angular and continuity equations unchanged.

At the edge of the B.L, 'r' will be of the order of the B.L thickness and assuming that the curvature, Θ_z, is also of the same order, then $r = o(t)$; $\Theta_z = o(t)$;

Hence $J = 1 - r.\Theta_z = 1 - o(t).o(t) = 1$ and $h + rH_z = h + o(t) = h$

Hence set [6.40b] become

$$
\begin{aligned}
&WW_z + QW_r - h_z M^2/h = -P_z && \text{(i)} \quad \text{(Axi)}\\
&QQ_r + WQ_z - H_z M^2/h = -P_r && \text{(ii)} \quad \text{(Rad)}\\
&WM_z + QM_r + (h_z W + H_z Q)M/h = 0 && \text{(iii)} \quad \text{(Ang)}\\
&W_z + Q_r + h_z W/h + H_z .Q/h = 0 && \text{(iv)} \quad \text{(Con)}
\end{aligned}
\qquad [6.40c]
$$

If it is assumed that the axial and circumferential velocity components W and M are functions of 'z' only (as is the case at the edge of the B.L.),

then $W = W(z)$; $M = M(z)$ and [6.40c] reduces to

$$
\begin{aligned}
&WW_z - h_z M^2/h = -P_z && \text{(i)} \quad \text{(Axi)}\\
&WQ_z - H_z M^2/h = -P_r - QQ_r && \text{(ii)} \quad \text{(Rad)}\\
&WM_z + (h_z W + H_z Q)M/h = 0 && \text{(iii)} \quad \text{(Ang)}\\
&W_z + h_z W/h + H_z Q/h = 0 && \text{(iv)} \quad \text{(Con)}
\end{aligned}
\qquad [6.40d]
$$

A specific expresssion can be derived for Q from [6.40.d.(iii)]

$Q = - hW(H_z)^{-1}(\ln(hM))_z = Q(z)$ showing that Q is a function of z only.

Substituting for Q into the continuity equation gives

$$hW_z + h_z W + (-hW)(\ln(hM))_z = 0$$

$$(\ln(hW))_z - (\ln(hM))_z = 0$$

$$(\ln(W/M))_z = 0$$

Hence $M(z) = kW(z)$ implying that all streamlines are parallel in the freestream this flow being comparable with that derived for flow over a yawed wing (Ref.5, p.240).

Since Q= Q(z) set [6.40d] reduce to

$$WW_z - h_z M^2/h = - P_z \qquad \text{(i) (Axi)}$$

$$WQ_z - H_z M^2/h = -P_r \qquad \text{(ii) (Rad)} \qquad [6.40e]$$

$$Q(z) = - hW(Hz)^{-1} (\ln(hM))_z \qquad \text{(iii) ('Ang')}$$

$$M(z) = kW(z) \; ; \; k \neq 0 \qquad \text{(iv) ('Con')}$$

An arbitrary choice of the axial component of speed, W(z), will define the flow field completely by virtue of [6.40e] (iii) & (iv), while (i) & (ii) define the pressure gradient.

Pressure Change Across The B.L.

From the radial B.L equation $H_z . m^2/h = p_r$ [6.38(ii)]

Assuming that the swirl velocity, m, is bounded within the B.L and that the radius of the body of revolution is large compared to the thickness of the B.L ('t')

then $t << h$, and $m^2 < M^{*2}$ (say).

Hence integrating w.r.t 'r' across the B.L (width 't') we have

$$\int_{r=0}^{r=t} |p_r . dr| = \int_{r=0}^{r=t} (H_z/h).m^2.dr < (H_z/h).\int_{r=0}^{r=t} M^{*2}.dr = [(H_z/h.M^{*2}.t]_{r=0}^{r=t} = o(t)$$

[6.40f]

Hence the presssure difference across the B.L is of the order of the B.L thickness and it can be assumed that the free-stream pressure distribution is 'impressed' upon the B.L.

Axisymmetric Flows With $P_z = 0$.

If $P_z = 0$ it follows [6.40e.(i)] defines the axial velocity component in terms of the contour function h(z).

Thus
$$W.W_z - h_z.M^2/h = - P_z = 0.$$

Also since $M(z) = kW(z)$ we have

$$W.W_z - h_z.k^2W^2/h = 0 \Rightarrow W^{-1}.W_z - k^2.h^{-1}.h_z = 0$$

$$\Rightarrow \quad [\ln(W)-k^2.\ln(h)]_z = 0$$

Hence
$$W = k_1.h^K \; ; \; M = k.W \; ; \; Q = -H_z^{-1}.(h.W)_z$$

It follows that for this particular zero pressure gradient distribution that if one of the functions H,W,M,Q are prescribed the others are defined once k and k_1 are chosen.

6.(V) A Mapping For A Class Of Swirling Flows

Consider the set of streamlines passing through a given normal at a point of a body of revolution and suppose that the angle that the projection of the flow direction of the streamline on the tangent plane perpendicular to the specified normal is the same for each stream line (i.e the streamlines in the B.L are parallel).

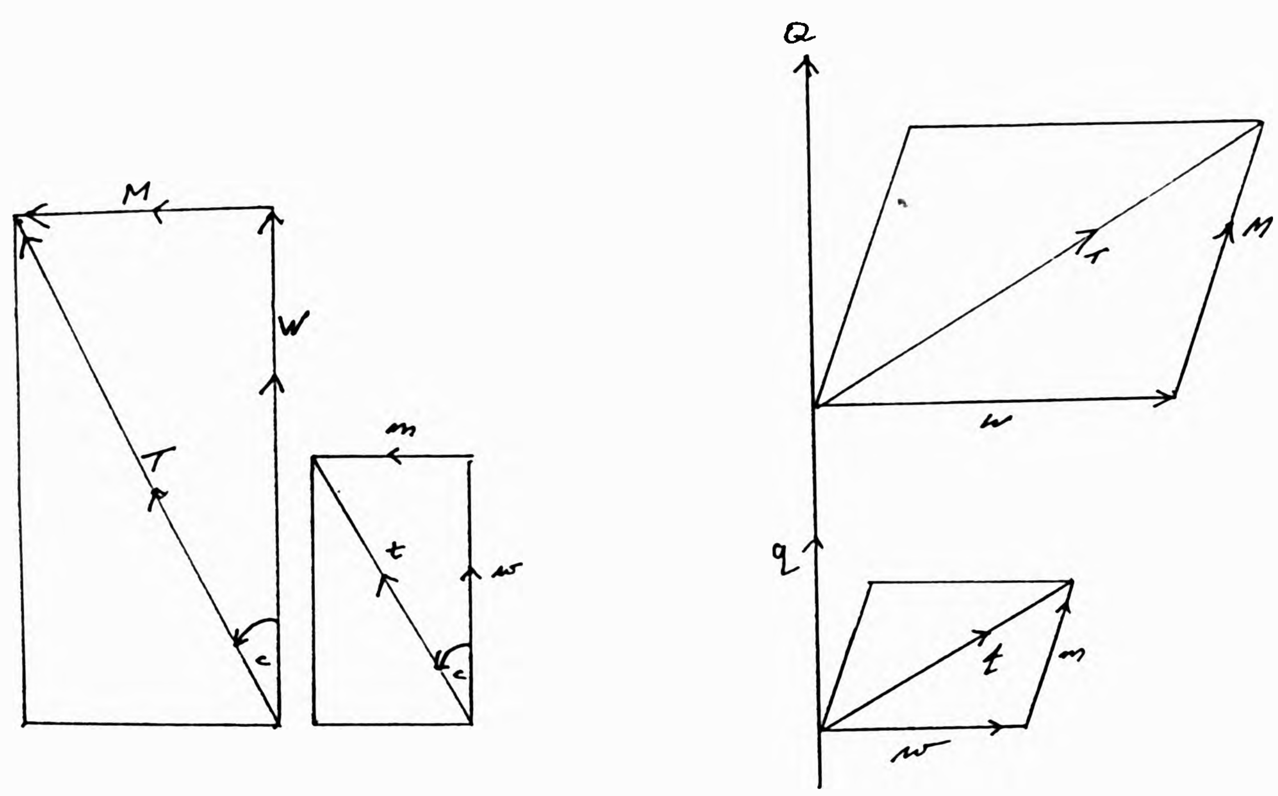

Fig. 6.7

Then with this assumption that the flow in the B.L is not skewed

$$w : m : t = W : M : T ;$$

where $w^2 + m^2 = t^2$, $W^2 + M^2 = T^2$ [6.42.a]

where 'w', 'm' are the axial and circumferential components of velocity within the B.L and W, M the corresponding quantities at the edge of the B.L. From Fig 6.7 we have

$\text{Sin } c = m/t = M/T$; $\text{Cos } c = w/t = W/T$; $\text{Tan } c = m/w = M/W$ [6.42.b]

and $T = T(z)$; $c = c(z)$ [6.42.c]

the quantities T and c being functions of z only since M and W are assumed to be functions of z alone.

Thus

$$w = t.\mathrm{Cos}\,c \quad : \quad m = t.\mathrm{Sin}\,c \qquad \text{(i)}$$

$$w_z = t_z.\mathrm{Cos}\,c - t.\mathrm{Sin}c.c_z \quad : \quad m_z = t_z.\mathrm{Sin}c + t.\mathrm{Cos}c.c_z \qquad \text{(ii)} \qquad [6.43]$$

$$w_r = t_r.\mathrm{Cos}\,c \quad : \quad m_r = t_r.\mathrm{Sin}\,c \qquad \text{(iii)}$$

$$w_{rr} = t_{rr}.\mathrm{Cos}\,c \quad : \quad m_{rr} = t_{rr}.\mathrm{Sin}c \qquad \text{(iv)}$$

Writing [6.38a] in the form

$$ww_z + qw_r = -p_z + (h_z/h)m^2 + Sw_{rr}$$

$$w(hm)_z + q(hm)_r = S(hm)_{rr} \qquad [6.38a]$$

$$(hw)_z + (hq)_r = 0$$

and substituting from [6.43] gives

$$\mathrm{Cos}\,c.t.t_z + q.t_r = (-1/\mathrm{Cos}\,c).p_z + At^2 + St_{rr} \qquad (1)$$

$$\mathrm{Cos}\,c.t.t_z + q.t_r = B.t^2.S.t_{rr} \qquad (2) \quad [6.38b]$$

$$(ht.\mathrm{Cos}\,c)_z + (h.q)_r = 0 \qquad (3)$$

where $A = (\, h_z.\mathrm{Sin}^2 c\,)/(\, h.\mathrm{Cos}\,c\,) + \mathrm{Sin}\,c.c_z$

and $B = -\,\mathrm{Cos}\,c.(\, h.\mathrm{Sin}\,c\,)_z/(\, h.\mathrm{Sin}\,c\,)$

From [6.38b], forming (i) 'B*(1) - A*(2)' and (ii) '(1) - (2)' gives

$$\mathrm{Cos}\,c.t.t_z + q.t_r = -B/(\,(B-A).\mathrm{Cos}\,c\,).p_z + S.t_{rr} \qquad (1)$$

$$p_z = (A - B).\mathrm{Cos}c.t^2 \qquad (2) \quad [6.38c]$$

$$(h.t.\mathrm{Cos}\,c)_z + (h.q)_r = 0 \qquad (3)$$

From the definitions of A and B above it can be shown that

$$(B - A).\mathrm{Cos}\,c = -(\,\ln(h.\mathrm{Sin}\,c)\,)_z$$

$$B = -\,\mathrm{Cos}\,c.(\,\ln(h.\mathrm{Sin}\,c)\,)_z = (B - A).\mathrm{Cos}^2 c$$

$$A = -\,B.\tan^2 c = (\,\mathrm{Sin}^2 c/\mathrm{Cos}\,c).(\,\ln(h.\mathrm{Sin}\,c)\,)_z$$

Hence (in [6.38c (1)]) the coefficient of p_z is Cos c.

Then set [6.38c] becomes

$(\mathrm{Cos}\ c).t.t_z + q.t_r = -(\mathrm{Cos}\ c).p_z + S.t_{rr}$ (1)

$p_z = (\ln(h.\mathrm{Sin}\ c))_z .t^2$ (2) [6.38d]

$(h.\mathrm{Cos}c.t)_z + (h.q)_r = 0$ (3)

Defining a new independent variable Z = Z(z) such that

$dZ/dz = (\mathrm{Cos}\ c)^{-1} = (W(z)/T(z))^{-1} = T(z)/W(z)$

Hence $Z = \int [\,T(z)/W(z)\,].dz$

and define $h^* = h^*(z) = h(z).\mathrm{Cos}[c(z)] = h.\mathrm{Cos}\ c$

Then for any function F(z)

$F_z = F_z .dZ/dz = (\mathrm{Cos}\ c)^{-1}.F_z \Rightarrow F_Z = (\mathrm{Cos}\ c).F_z$

Making these substitutions into [6.38d] gives

$t.t_Z + q.t_r = -p_Z + S.t_{rr}$ (1)

$p_Z = (\ln(h^*.\mathrm{Tan}\ c))_Z .t^2$ (2) [6.44]

$(h^*.t)_Z + (h^*.q)_r = 0$ (3)

Comparing [6.44] with the B.L equations for zero-swirl flows [6.12]

[where '#' represents 2-D flow quantities]

where $w^{\#}.w^{\#}_z + q^{\#}.w^{\#}_r = -P^{\#}_z + \mu.w^{\#}_{rr}$ (a)

$(l^{\#}.w^{\#})_z + (l^{\#}.q^{\#})_r = 0$ (b) [6.12]

$P^{\#}(z) = W^{\#}.W^{\#}_z \ ; \ W^{\#} = W^{\#}(z)$ (c)

Since the pressure change across the B.L is o(t) (See [6.40f])

then we may replace the pressure term in [6.44] (1) by its

free-stream value and hence

$t.t_Z + q.t_r = -P_Z + S.t_{rr}$ (1)

$(h^*.t)_Z + h^*.q)_r = 0$ (3) [6.44a]

$-P_Z = W.W_Z - h^*_Z .M^2/h^*$ (2)

[Note from 6.40e (iv) M = k.W. therefore the angle c is constant.
$k = \mathrm{Cos}\ c; \Rightarrow d/dZ_1 = K.d/dz$; But $h^* = k.h$, $h = h^*/k$
$(-1/K).dP/dz_1 = W(1/k_1).dW/dz_1 - M^2/(h^*.k_1).dh^*/dz_1$]

Comparing [6.44a] with [6.12] we can make the following identifications $w^{\#} = t$; $q^{\#} = q$; $P^{\#}_z = P_z$; $l^{\#} = h^{*}$ which will map [6.44a] into [6.12].

Thus the swirling flow with free stream components W(z) and M(z) at the B.L edge on a body of revolution defined by h(z) may be replaced by an equivalent ' axial' flow with freestream speed T, ($T^2 = W^2 + M^2$) over a body whose contour is defined by

$$h^{*}(z) = h(z).\mathrm{Cos}\ c = h(z).W(z)/T(z).$$

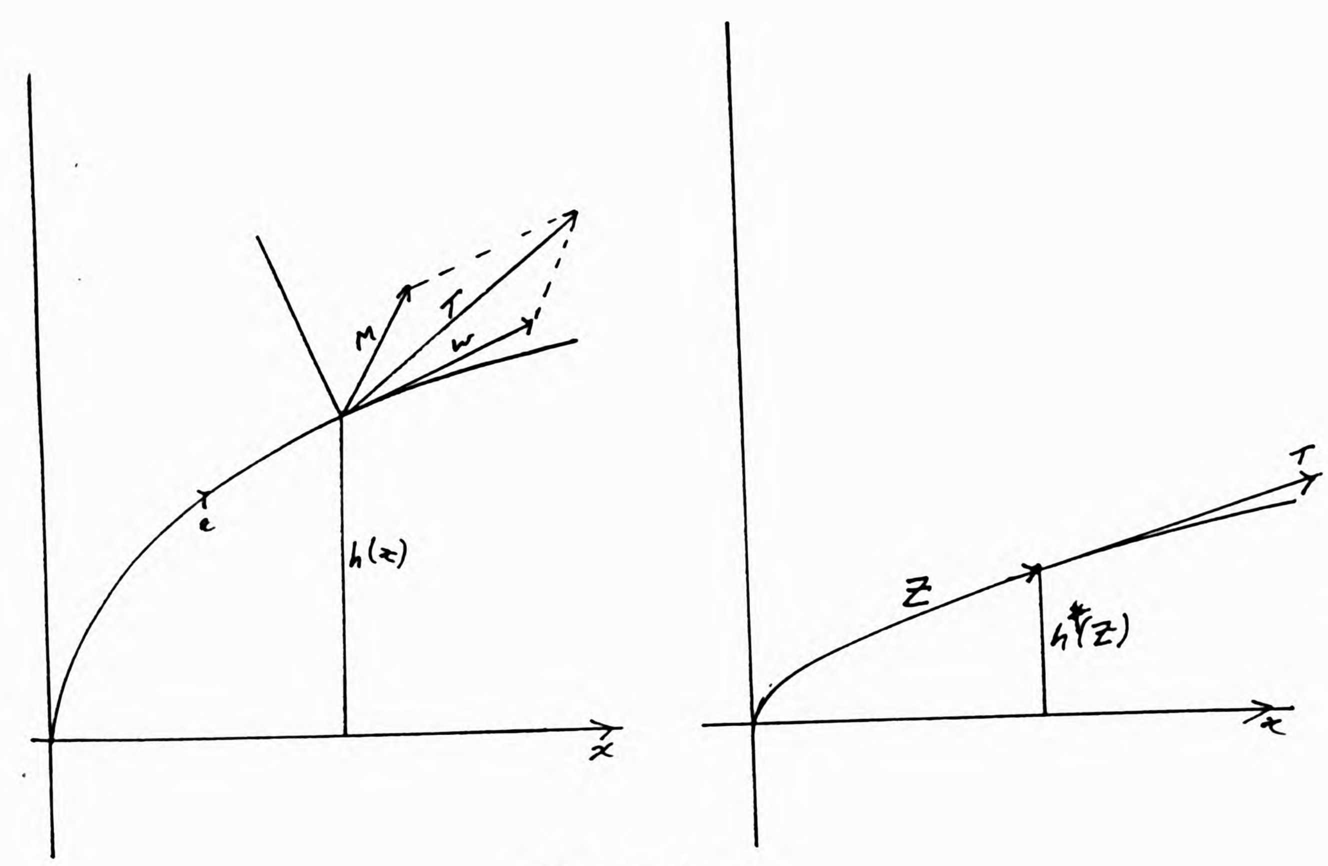

Fig 6.8

From [6.40e(iv)], since the angle 'c' is constant throughout this flow then

$$Z = \int T/W.dz = \int dz/\mathrm{Cos}\ c = (1/\mathrm{Cos}\ c).\int dz = z/\mathrm{Cos}\ c.$$

Hence the equivalent axial flow effectively is one with increased speed over a longer narrower body.

It should be noted that this mapping is not unique but there is no apparent advantage in using any of the alternatives.

Further, flow quantities normal to the surface such as radial speed 'q' and the coordinate 'r' are unaffected by the transform since the contour function h(z) has been reduced by a factor of Cos c and the 'z' coordinate has been magnified by a factor of (1/Cos c) implying a relative thickening of the B.L with respect to the dimensions of the body compared with the non-swirl case. The above calculation refers to the inner wall where 'r' is positive in the sense of the outward normal to the wall. To deduce the equivalent wall conditions for the outer wall where 'r' and 'q' are directed inwards let

$q = - q^* \; ; \; r = - r^*$

$\Rightarrow w_r = - w_r \; ; \; w_{rr} = w_{r^*r^*} \; ; \; q_r = -(-q^*_{r^*}) = q^*_{r^*}$

With this substitution equation [6.38] becomes

$ww_z + q^*.w_{r^*} = -p_z - (h_z/H_z).p_{r^*} + Sw_{r^*r^*}$

$-m^2/h = p_{r^*}/H_z.$

$w(hm)_z + q^*(hm)_{r^*} = S(hm)_{r^*r^*}$

$(hw)_z + (hq^*)_{r^*} = 0$

Eliminating p_{r^*}

$ww_z + q^*w_{r^*} = -p_z + (h_z/h)m^2 + Sw_{r^*r^*}$

$w(hm)_z + q^*(hm)_{r^*} = S(hm)_{r^*r^*}$

$(hw)_z + (hq^*)_{r^*} = 0$

which is identical in form to [6.38a].

Computer programmes were developed to incorporate the various flow parameters in these B.C's to generate duct geometries.

The effect on duct shape was was examined by varying the values of these parameters the results being outlined in the next section. The specific form of the velocity distributions used for swirling flows is

$$T_p = T_o.[1 - ((3/2).S_L^{1/2}.\ln(1 + J_{zo,zp}/J_{\emptyset,zo}))^{2/3}]^{1/2} \quad \text{(Laminar)} \quad [6.25a]$$

and

$$T_p = T_o.[1 - A_2.((J_{zo,zp}/J_{\emptyset,zo})^{1/5} - 1)^{1/3}]^{1/2} \quad \text{(Turbulent)} \quad [6.29a]$$

Where $J_{a^*.b^*} = \int_{z=a^*}^{z=b^*} h^{*2}(Z).dZ = \int_{z=a^*}^{z=b^*} (hW/T)^2 T.dz/W =$

$$= \int_{z=a^*}^{z=b^*} h^2 W.dz/T = \int_{z=a^*}^{z=b^*} h^2 W.d\Phi/(TW) = \int_{z=a^*}^{z=b^*} h^2.d\Phi/T$$

since $h^*(Z) = h(z).\text{Cos } c = h(z).W(z)/T(z)$; $dZ = T(z).dz/W(z)$ and $T^2 = W^2 + M^2$ with W and M being the axial and swirl speed respectively. More complex functional relationships governing the variation of velocity within the boundary layer could be used to simulate the behaviour of skewed boundary layers.

6.VI The Numerical Solution.

In this numerical results are derived for a set of irrotational, incompressible flows for a variety of boundary conditions.

The duct is considered to be devided into three distinct sections

(i) an upstream section consisting of two coaxial cylinders ;

(ii) a transition region ;

(iii) a down stream region bounded by two coaxial cylinders ;

Because of the multiplicity of boundary conditions that can be applied, Fig.6.9 below represents (qualitatively) only one of a set of duct geometries that may be created.

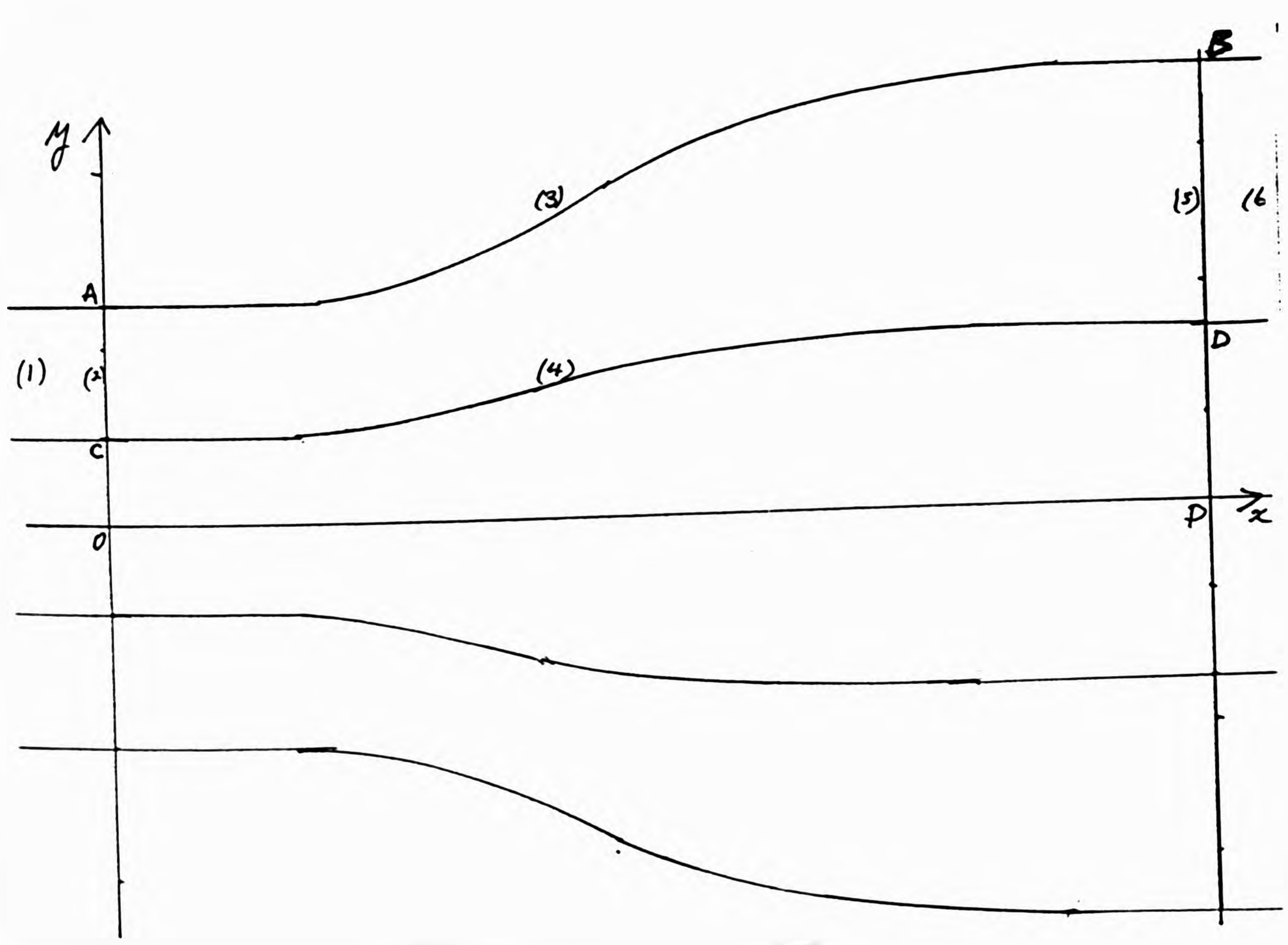

Fig 6.9

Boundary Conditions and Initial Values.

(1) Upstream Region. In the case of the 'exact' solutions derived in Chapter 2, all upstream flow quantities were known but were not required in the determination of the numerical solution. In the present case the upstream region consists of a pair of coaxial cylinders containing a prescribed velocity distribution, (U,V,W), consistant with an irrotational flow field. The velocity components chosen are

$$W = W_o \quad (W_o = \text{constant})$$

$$V = 0$$

$$M = A/y \ (A = \text{constant})$$

Also associated with the upstream region is a parameter indicative of the relative size of the B.L which is assumed to have developed in this region.

(2) Inlet Station. The inlet radii of the hub and the casing are chosen arbitrarily and, on the basis of the flow presented at inlet being that of the upstream region, the values of 'y' are calculated at equal $d\Upsilon$ across the duct along some arbitrary Φ characteristic.

Now from equations [1.7.6/7/8] and [2.9] we have

$d\Phi = (q/B).ds$; $d\Upsilon = (q/A).dn$ and $A = 1/y$, $B = 1$ for irrotational incompresssible flow.

At inlet $dn = dy$; $q = W_o$

hence $d\Phi = W_o.ds$ (a) ; $d\Upsilon = W_o.y.dy$ (b) [6.50]

Integrating [6.50b] gives

$$\Upsilon = \Upsilon_{hub} + (1/2).[\ y^2 - y^2_{hub}\ ].W_o$$

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Let W^*_o be an estimate of q throughout the transition region and hence

$$d\Phi = W^*_o . ds$$

Integrating from inlet to outlet

$$\Phi_{out} - \Phi_{in} = W^*_o . (s_{out} - s_{in})$$

$$\Rightarrow \quad \Phi_{out} = \Phi_{in} + W^*_o . (s_{out} - s_{in}) \qquad [6.50d]$$

Now $(s_{out} - s_{in})$ may be taken as an indication of the axial length of the duct, 'L' (say). Thus if L is prescribed, and if Φ_{in} is arbitrary, then [6.50d] defines the Φ range. On the Φ_{out} characteristic, a parallel (in contrast to uniform) flow condition is imposed to complete the set of B.C's required for a numerical solution.

Using the new 'mixed' prescribed velocity distributions defined in [6.25a] and [6.29a], an initial outlet speed is calculated on the basis of the duct length 'L' but this speed is no more than a starting estimate for the outlet velocity from which to derive some initial values of the oulet radii. In defining the velocity distributions to be applied on the walls, it is necessary to define (arbitrarily) some upstream length in which a Blasius (zero pressure gradient) B.L has developed. This length is defined as a fraction of the inner inlet radius and is another flow paramater which may be varied for comparison. The precise form of the quantity defining the upstream B.L developement is the integral J_{0,z_o} used in the definition of the integral $I_{o,p}$ of equation [6.22] & [6.25] which give the wall velocity distributions. Now if $z = 0$ ($\Phi = \Phi_0$) is the point at which the B.L is assumed to have started its developement (upstream) and $z = z_o$ ($\Phi = \Phi_o$)

is the point at which the 'sharp' pressure gradient is encountered at inlet

$$J_{0,zo} = \int_{z=0}^{z=zo} l^2(z).dz = \int_{\Phi=\Phi 0}^{\Phi=\Phi(zo)} l^2(\Phi).d\Phi/q = \int_{\Phi=\Phi 0}^{\Phi=\Phi(zo)} y^2.d\Phi/q \qquad [6.50e]$$

To determine Φ_0, noting that $q = W_o$ upstream of z_o we have

$$\int_{\Phi=\Phi 0}^{\Phi=\Phi(zo)} d\Phi = \int_{z=o}^{z=zo} q.dz = W_o.\int_{z=0}^{z=zo} dz = W_o.(z_o - 0) = W_o.z_o \qquad [6.50f]$$

Hence $\Phi_{zo} - \Phi_0 = W_o.z_o \Rightarrow \Phi_0 = \Phi_{zo} - W_o.z_o$.

Since $\Phi(zo) = \Phi_{zo}$ is arbitrary, then the upstream value of Φ_0 at the commencement of the upstream B.L developement is known and hence determines $J_{0,zo} = \int_{\Phi(0)}^{\Phi(zo)} y^2.d\Phi/q$ [6.50g]

For computational purposes the integrals $J_{a,b}$ are approximated by the summations $J^*_{a,b} = \Sigma\ [(y^2)^*.(1/q)^*.d\Phi]$

where $(F)^*$ represents a mean value of (F) in the interval $[a,b]$.

The finite difference form of the fundamental equation set

$r_{\psi\psi} + (\ln r)_{\phi\phi} = 0$; $x_{\phi} = r_{\psi}$; $x_{\psi} = -(\ln r)_{\phi}$ where $r = y^2$

is given by equations [3.5], [3.9], [3.15a] and [3.15b].

<u>Error and Consistancy Checks.</u> Unlike the solutions of Chapter 2, we have no exact values against which to test numerical results. However the following checks for error and consistancy are made

(i) 'r' coordinate.
(ii) 'x' coordinate.
(iii) Orthoganal Test on Φ, Υ lines.
(iv) Mass flow.
(v) Vorticity.

(i) The 'r' coordinate satisfies $r_{\Psi\Psi} + (\ln r)_{\phi\phi} = 0$ [3.5]

replacing [3.5] with its numerical equivalent and solving for $r_{i,j}$ we have [3.10]

$$r_{i,j} = (1/2)(r_{i+1,j} + r_{i-1,j} + (d\Phi/d\Upsilon)^2(\ln(r_{i,j+1}.r_{i,j-1}/r^2_{i,j})))$$

The values of 'r' obtained from the iterative routine are compared with those calculated from [3.10] and the maximum and average relative errors evaluated.

(ii) Similarly 'x' satisfies the equation

$$y_{\phi\phi} + x_{\psi\psi} = (r - \ln(r))_{\phi\phi} \qquad [6.51]$$

The finite difference form of [6.51] becomes (after rearrangement)

$$x_{i,j} = K_1.(x_{i,j+1} + x_{i,j-1} + K_2.(x_{i+1,j} + x_{i-1,j}) + K_3.K_{i,j}) \qquad [6.52]$$

where $K_1 = (1 + (d\Phi/d\Upsilon)^2)^{-1}$; $K_2 = (d\Phi/d\Upsilon)^2$; $K_3 = -(d\Phi/d\Upsilon)/4$

and $K_{i,j} = F_{i+1,j+1} + F_{i-1,j-1} - F_{i-1,j+1} - F_{i+1,j-1}$

with $F_{a,b} = r_{a,b} - \ln(r_{a,b})$.

A similar comparison is made for 'x' as for 'r'.

(iii) <u>Orthogonality Test</u> By definition the Φ, Υ lines should be orthogonal throughout the flow field.

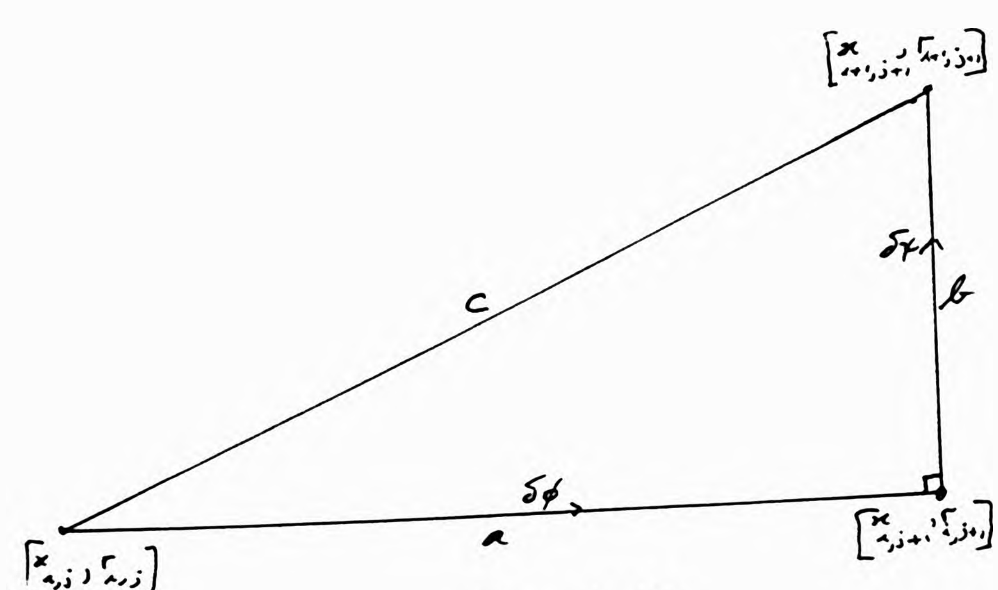

Fig. 6.10

(i) The 'r' coordinate satisfies $r_{\psi\psi} + (\ln r)_{\phi\phi} = 0$ [3.5]

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where $K_1 = (1 + (d\Phi/d\Upsilon)^2)^{-1}$; $K_2 = (d\Phi/d\Upsilon)^2$; $K_3 = -(d\Phi/d\Upsilon)/4$

and $K_{i,j} = F_{i+1,j+1} + F_{i-1,j-1} - F_{i-1,j+1} - F_{i+1,j-1}$

with $F_{a,b} = r_{a,b} - \ln(r_{a,b})$.

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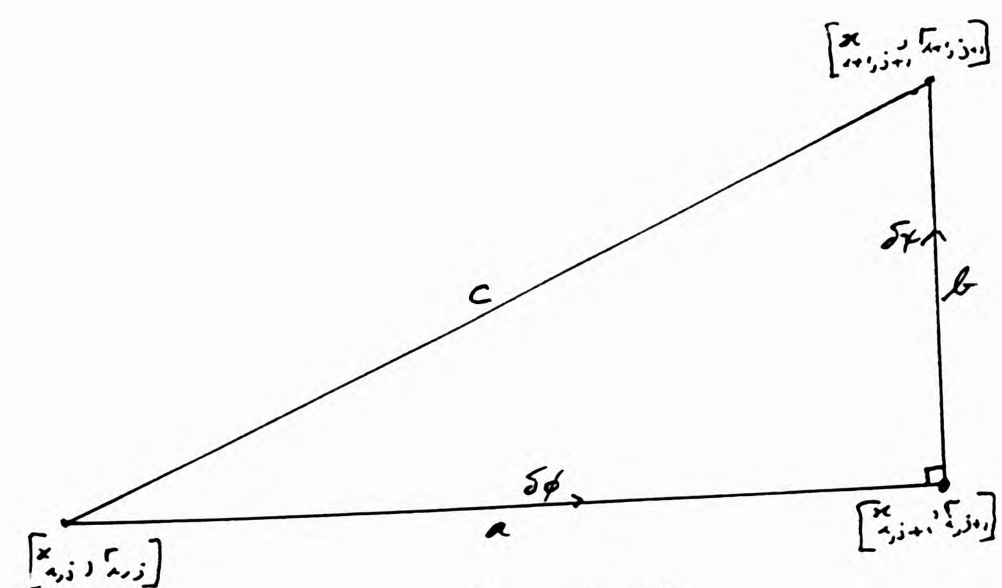

Fig. 6.10

To estimate the deviation of the (Φ,Y) characteristics from orthogonality the relative % error in the diagonal was determined by calculating the quantity $[c/(a^2 + b^2)^{1/2} - 1]$ for each grid cell.

(iv) <u>Mass Flow and Vorticity Check</u> Further checks for the self consistancy of the numerical solution is obtained by calculating the mass flow and circulation through and around each grid cell defined by the coordinates of four adjacent points in the flow.

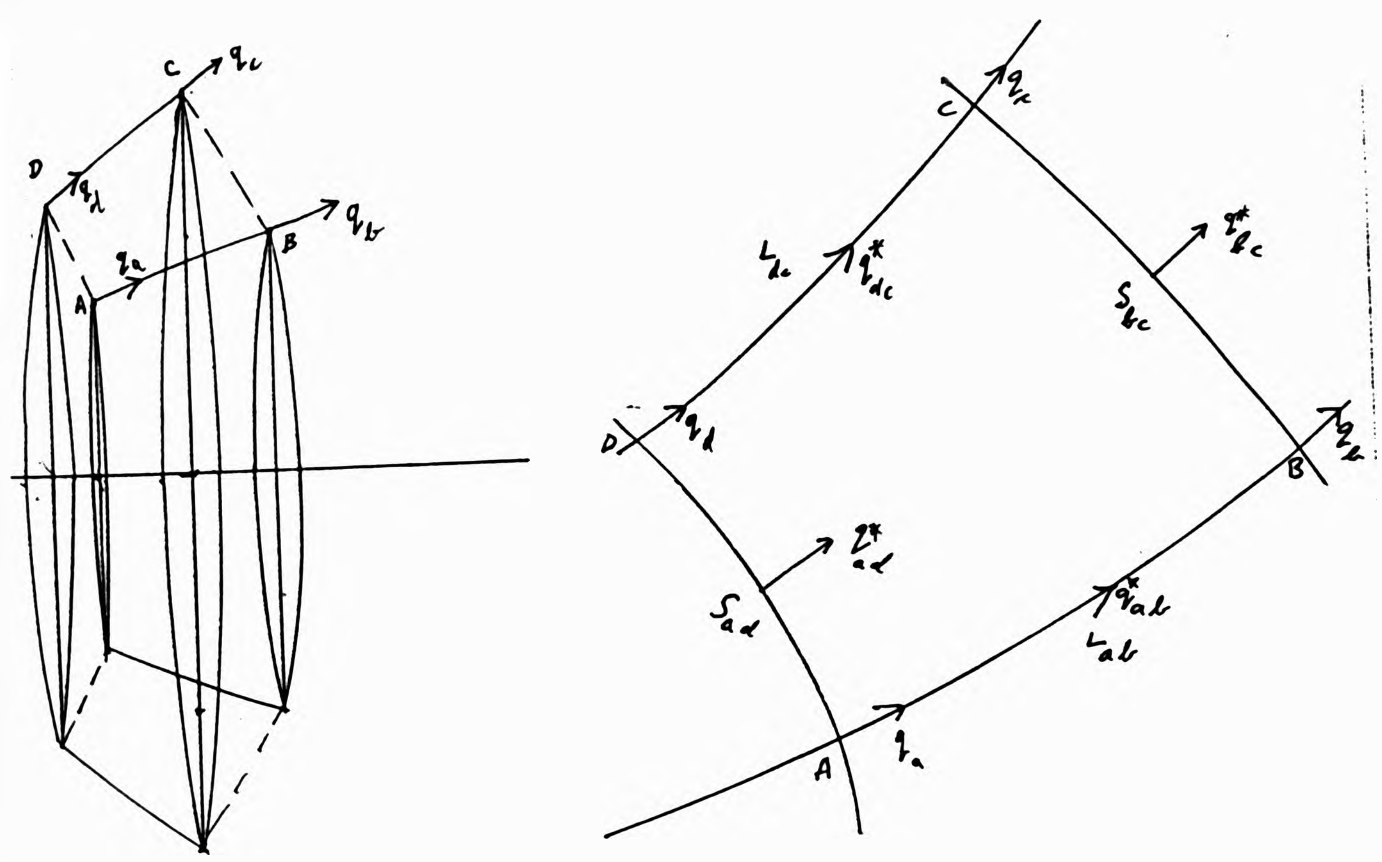

Fig 6.11

In order to allow for the curvature of the stream and potential lines the flow surface is approximated by a frustum of a cone. The mass flow through sections AD and BC is approximated to by M_{ad} and M_{bc} where $M_{ad} = q^*_{ad}.S_{ad}$; $M_{bc} = q^*_{bc}.S_{bc}$

$q^*_{ad} = (q_a + q_b)/2$; $q^*_{bc} = (q_b + q_c)/2$

where S_{ad} and S_{bc} are the curved surfaces of the frustums 'AD' and 'BC'. For continuity we should have $M_{ad} = M_{bc}$.

The relative error, defined as $(M_{ad}/M_{bc} - 1)$, was found to be of the order of 1% throughout the grid. Similarly in calculating the circulation, the quantities C_{ab} and C_{dc} are evaluated where

$$C_{ab} = L_{ab}.q^{*}_{ab} \quad ; \quad C_{dc} = L_{dc}.q^{*}_{dc}$$

For irrotational flow $C_{ab} = C_{dc}$ and the relative error in circulation on adjacent streamlines defined as $(C_{ab}/C_{dc} - 1)$.

Table 6.12 below lists a typical set of errors for a sequence of various grid sizes.

	Average % Error in	
Grid Size	Mass Flow	Circulation
7.7	1.357	2.49
9.9	1.149	0.841
11.11	.662	1.171
13.13	.465	1.185
15.15	.374	.656
17.17	.311	.548
19.19	.243	.427
21.21	.190	.354
23.23	.172	.338
25.25	.161	.320

Table 6.9

It was found that there was a fairly large maximum % error in the mass flow and circulation of the order of 15% and 30% respectively occuring in the neighbourhood of the point at which the initial 'Stratford' velocity distributions are applied at the wall.

The error in mass flow and vorticity decays rapidly away from the point of application of the sharp pressure/velocity gradient and the size of this region can be reduced by increasing the number of grid points. A similar calculation may be done for the angular momentum in the case of swirling flows.

The parameters affecting the duct geometry are as follows;

(1) Inlet Axial Velocity Profile.

(2) Inlet Swirl Profile.

(3) Upstream Blasius B.L Developement Length.

(4) Wall Boundary Velocity/Radii Distributions

(5) Outlet Condition.

(6) Laminar or Turbulent B.L.

The program developed for this section allows all the parameters listed above to be varied. In order for the flow to be irrotational it must have a uniform inlet profile together with a swirl speed of the form m = k/y. The B.L presenting itself at inlet is assumed to have developed in some upstream region the length of which is a variable input parameter. The wall boundary conditions may be taken as 'Stratford' type distributions which contain a parameter allowing the velocity distributions to be 'wound up' to their full critical values independently of each other on either wall. There is no necessity to limit the choice of PVDs to the 'Stratford' types and a simple numerical device in the form of the velocity distributions will convert them to accelerating flows. A parallel flow condition is applied at outlet but this could be replaced by an alternative PVD across the duct linking the 'ends' of the two wall PVD at outlet. From Fig 6.2, which shows the distribution of the Stratford velocity/pressure distributions for plane flow laminar and turbulent B.Ls on the point of separation, it can be seen that at the onset of pressure rise the gradients of both the velocity and

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(1) 'Winding up' the Stratford PVDs on the duct walls to their separation values.

(2) Allowing sections of constant velopcity/radius at inlet.

(3) Increasing the upstream B.L developement length; i.e increasing the thickness of the B.L.

(4) Increasing the ratio of swirl to axial speed at inlet.

(5) Difference between laminar and turbulent B.L.

The limitation on the increase in the swirl speed (consistant with irrotation) is quite severe. From Fig xxx, showing the variation of duct geometry with increasing swirl, it can be seen that the change in the shape of the outer wall is steady and 'small'. For the hub, the initial rate of change of shape due to increasing swirl is similar to that on the casing, but when the swirl parameter reaches some critical value, there begins a sudden and rapid collapse of the hub towards the axis thus producing an infinite swirl component.

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(c) On the upper wall the swirl velocity varies only slowly with arc length having only a mild effect on duct geometry.

(d) From the 'swirl' plots, it can be seen (Fig. 6.13) that the increase in swirl with arc length is substantial even for swirl coefficients as small as 15% of hub inlet axial speed.

The results obtained thus far are for flows with PVDs accelerating and/or deccelerating on one or other or both walls and duct shapes consistant with these conditions are given below.

The imposition of parallel flow at outlet yields a 'smooth' transition to the constant radii outlet section.

In general, if the boundary velocity distributions are monitored then a variety of criteria can be used to trigger the application of a new type of B.C when some condition is satisfied. Possible examples are the restriction of the pressure coefficient to a prescribed range or limitations on the size of duct radii. For the purpose of the current calculation the transition region is divided into five sections for the application of B.C.

(1) Inlet region with constant radius.

(2) Inlet region with constant wall velocity.

(3) Transition region with 'Stratford' or other variable velocity distribution.

(4) Outlet region with constant velocity.

(5) Outlet region with constant radius.

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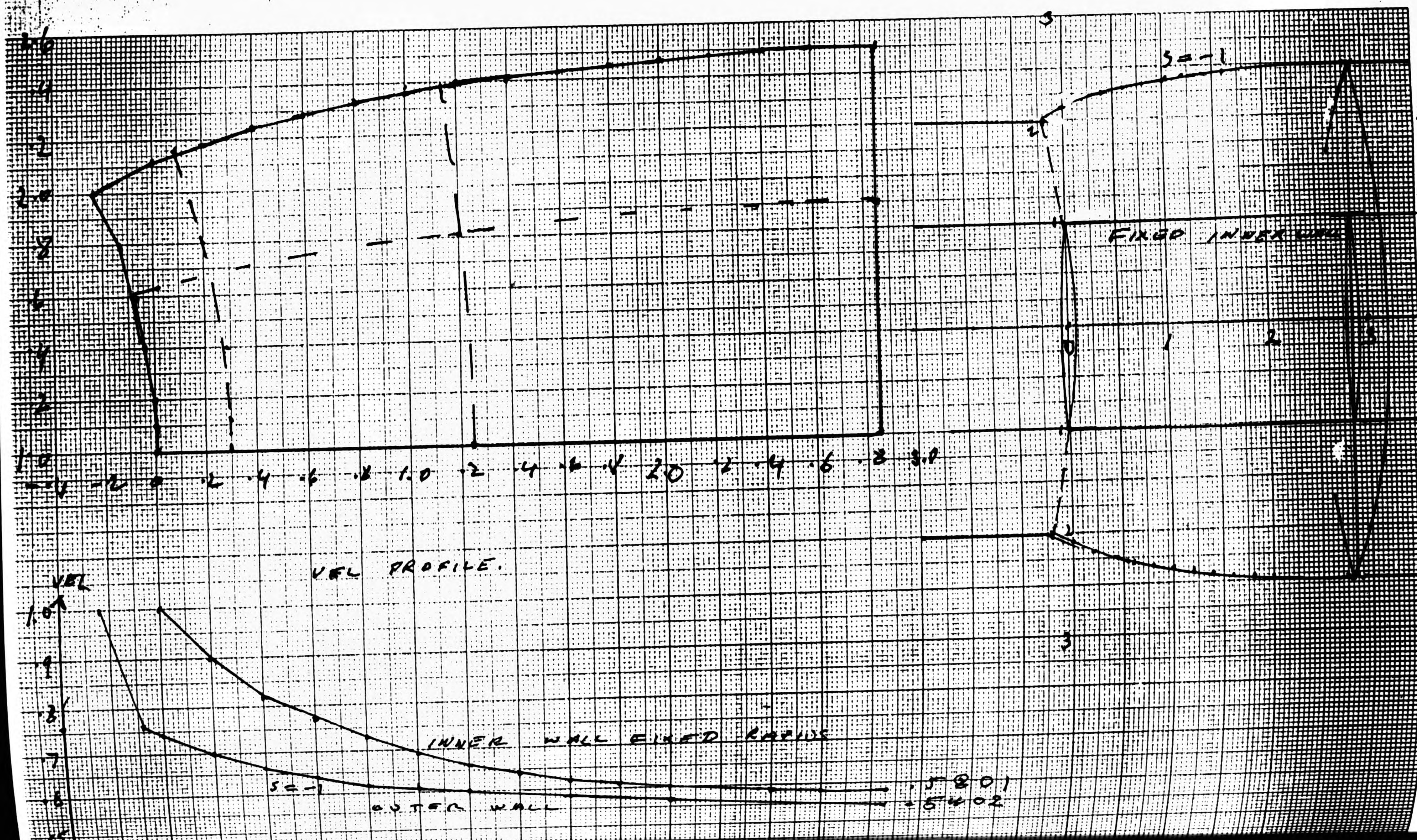
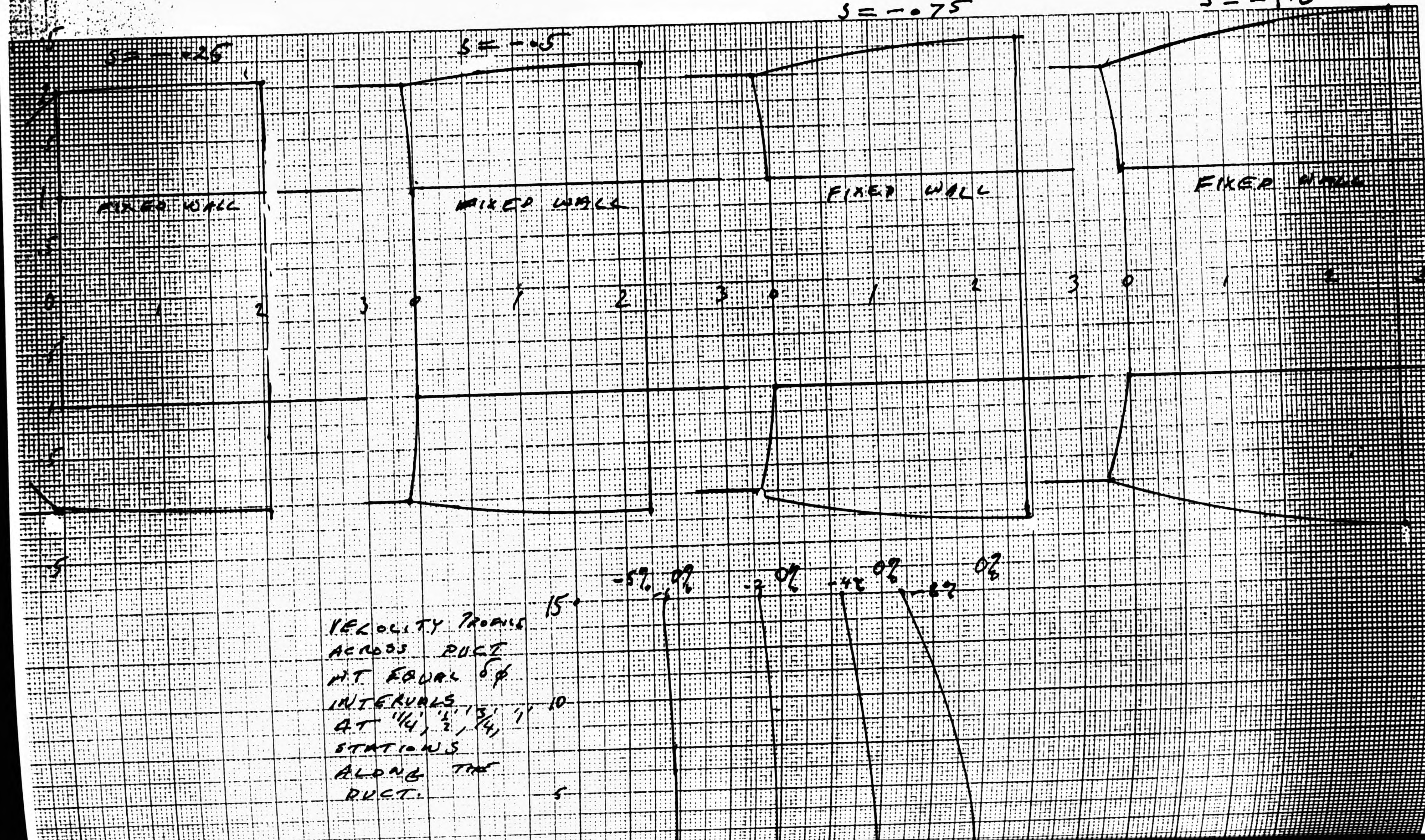
INL = U : OUT * P : INN = FR : UPP = S-1 : SW = 0 : GR = 15.15 : ACC = .9 : BL = L, .015625

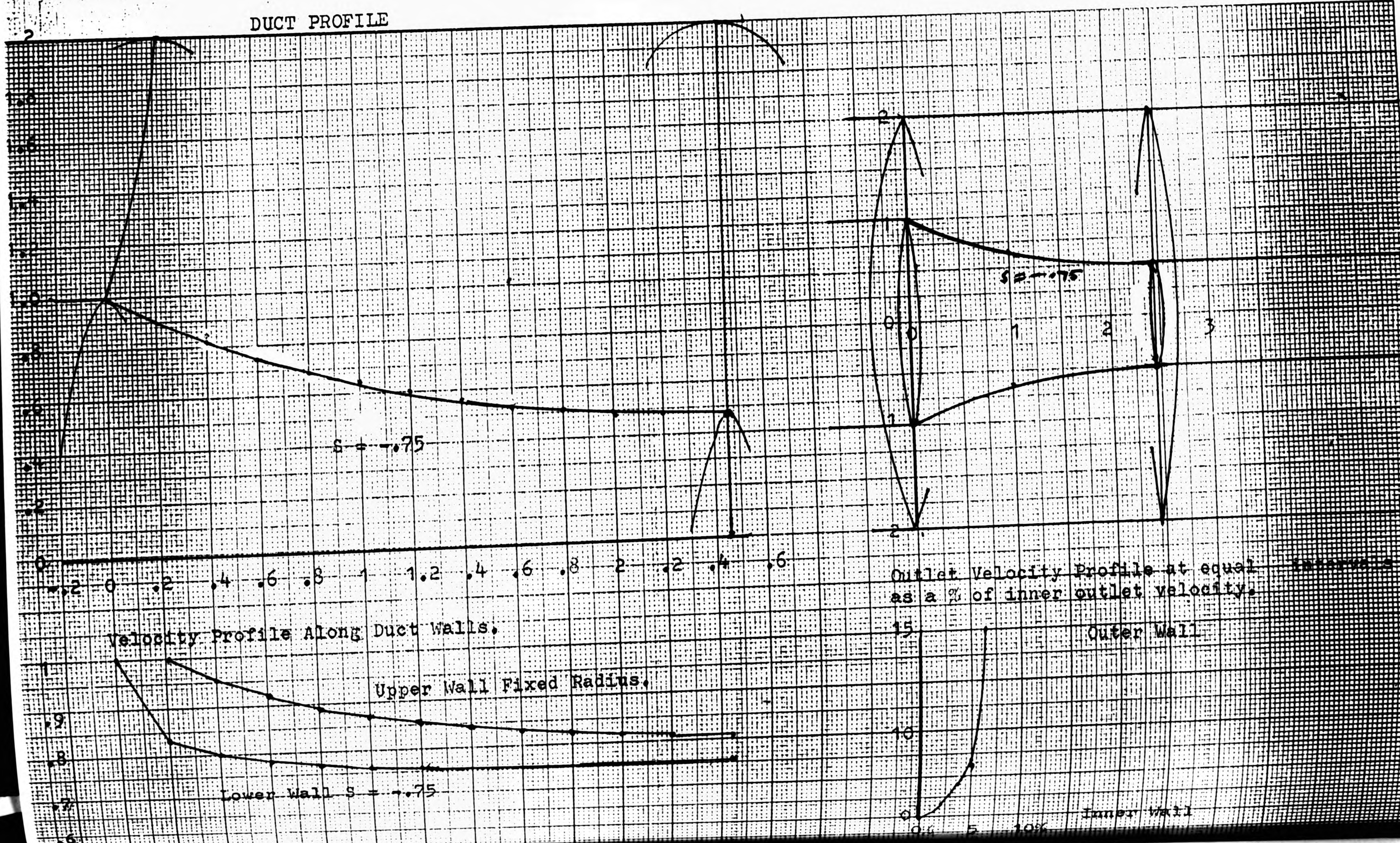
DL = 2 :

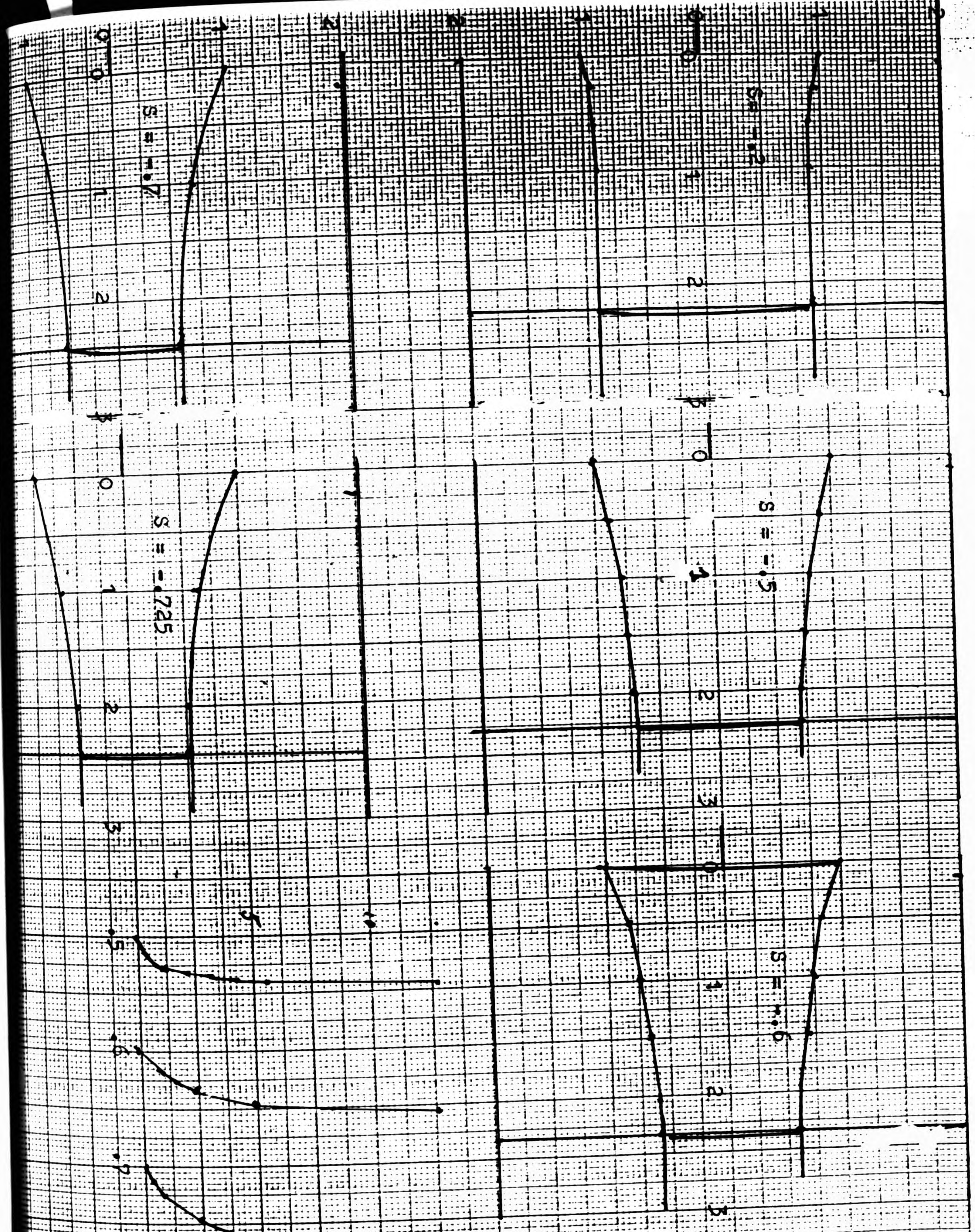



s = -.25
s = -.5
s = -.75
s = -1.0
FIXED WALL
FIXED WALL
FIXED WALL
FIXED WALL
-59% 0%
-.7 0%
-.48 0%
-.67 0%
VELOCITY PROFILES ACROSS DUCT AT EQUAL δφ INTERVALS AT 1/4, 1/2, 3/4, 1 STATIONS ALONG THE DUCT.

INL = U : OUT = P : INN = S-1 : UPP = FR : SW = 0 : GR = 15.15 : ACC = .9 : BL = L,.015625

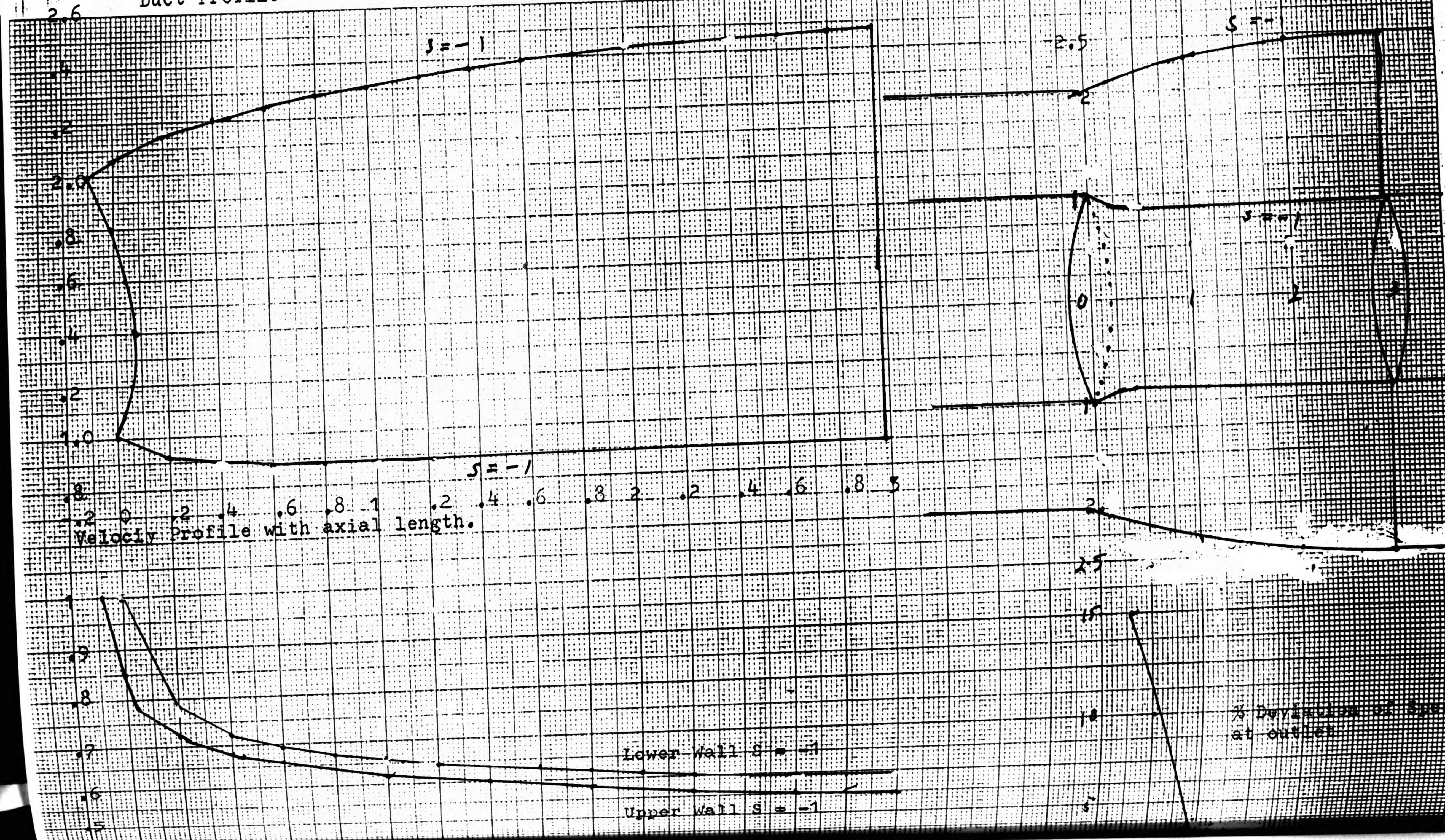
DL = 2

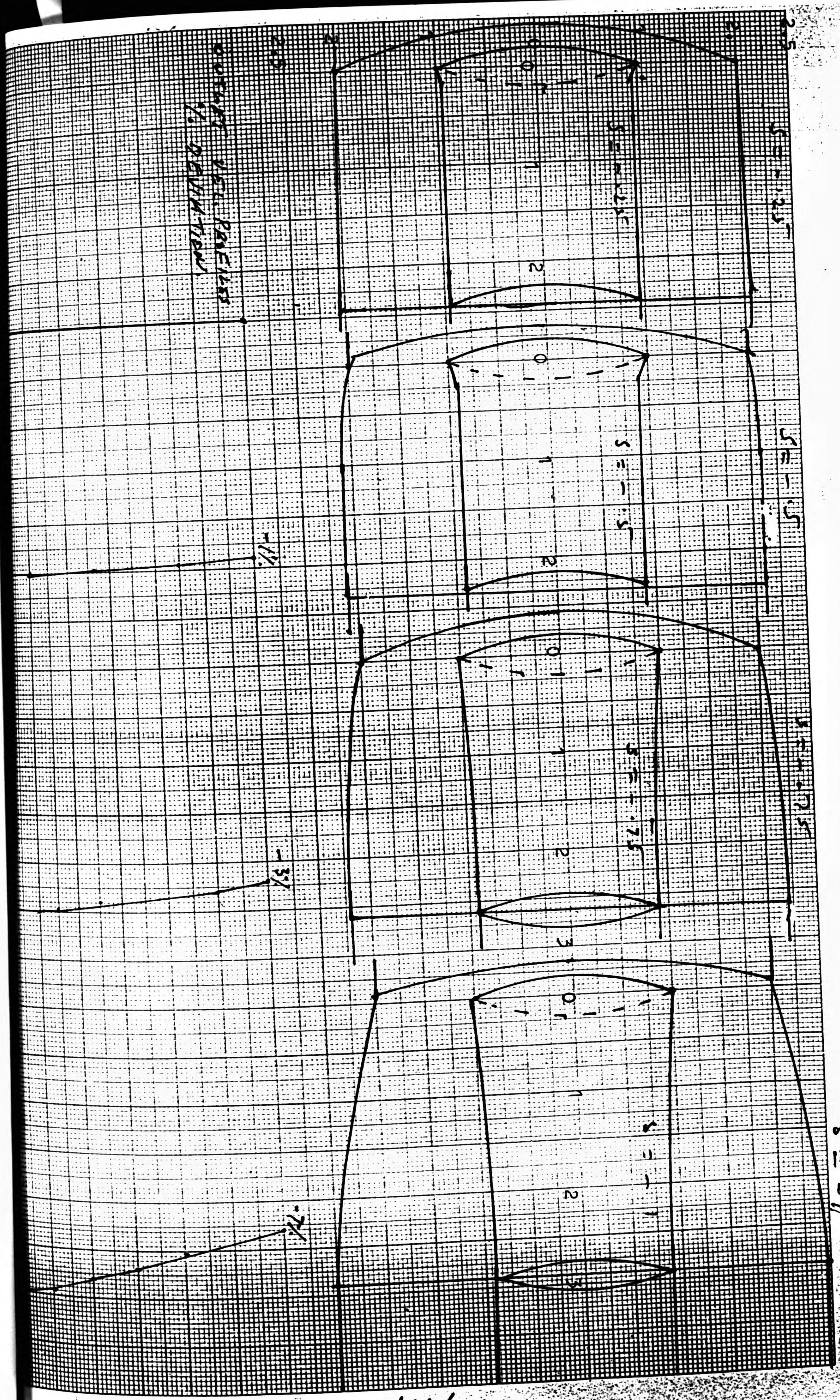



S = -.7
S = -.2
S = -.725
S = -.5
S = -.6

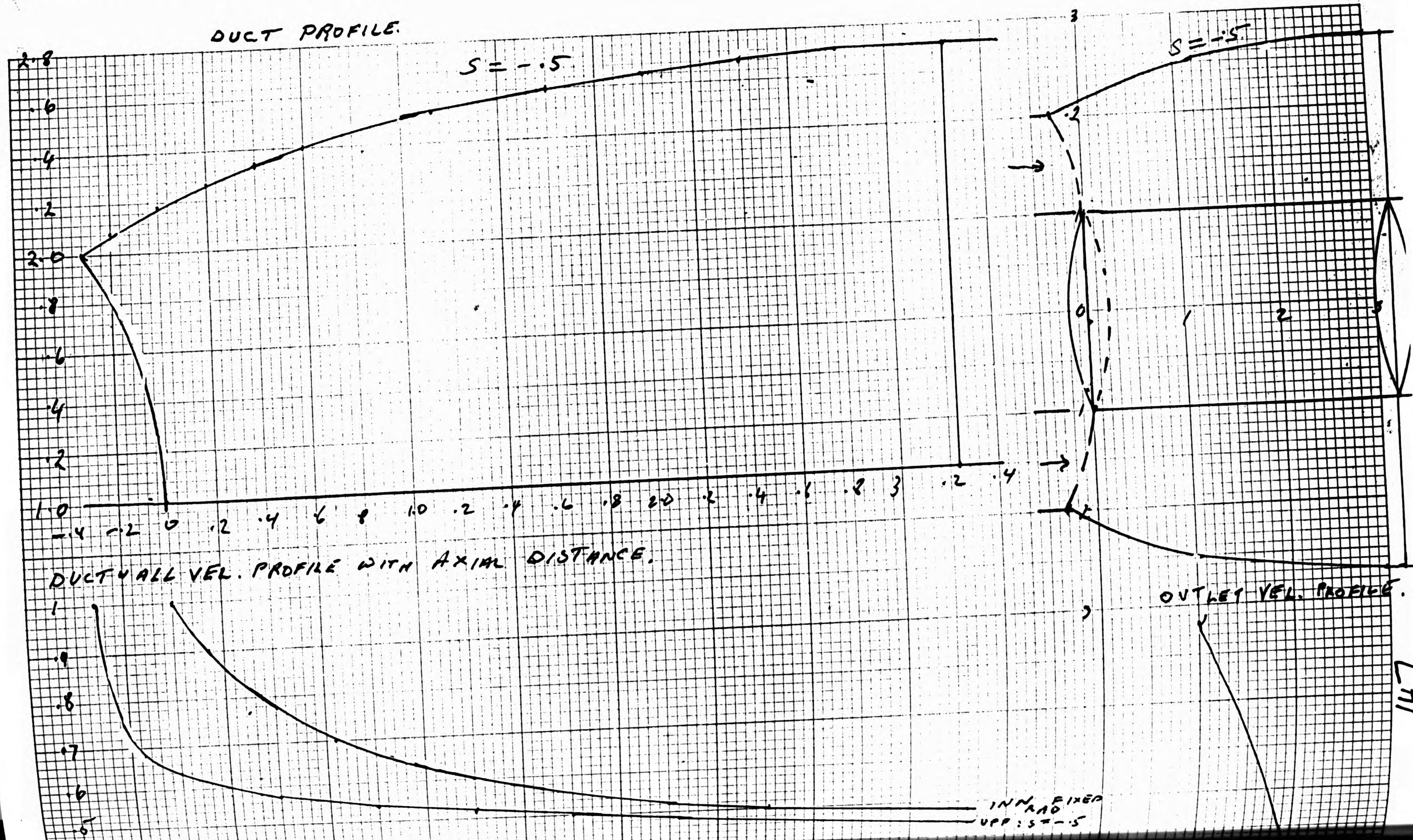
INL = U : OUT = P : INN = S -1 : UPP = S -1 : SW = 0 : GR = 15.15 : ACC = .9 : BL = L, .015625

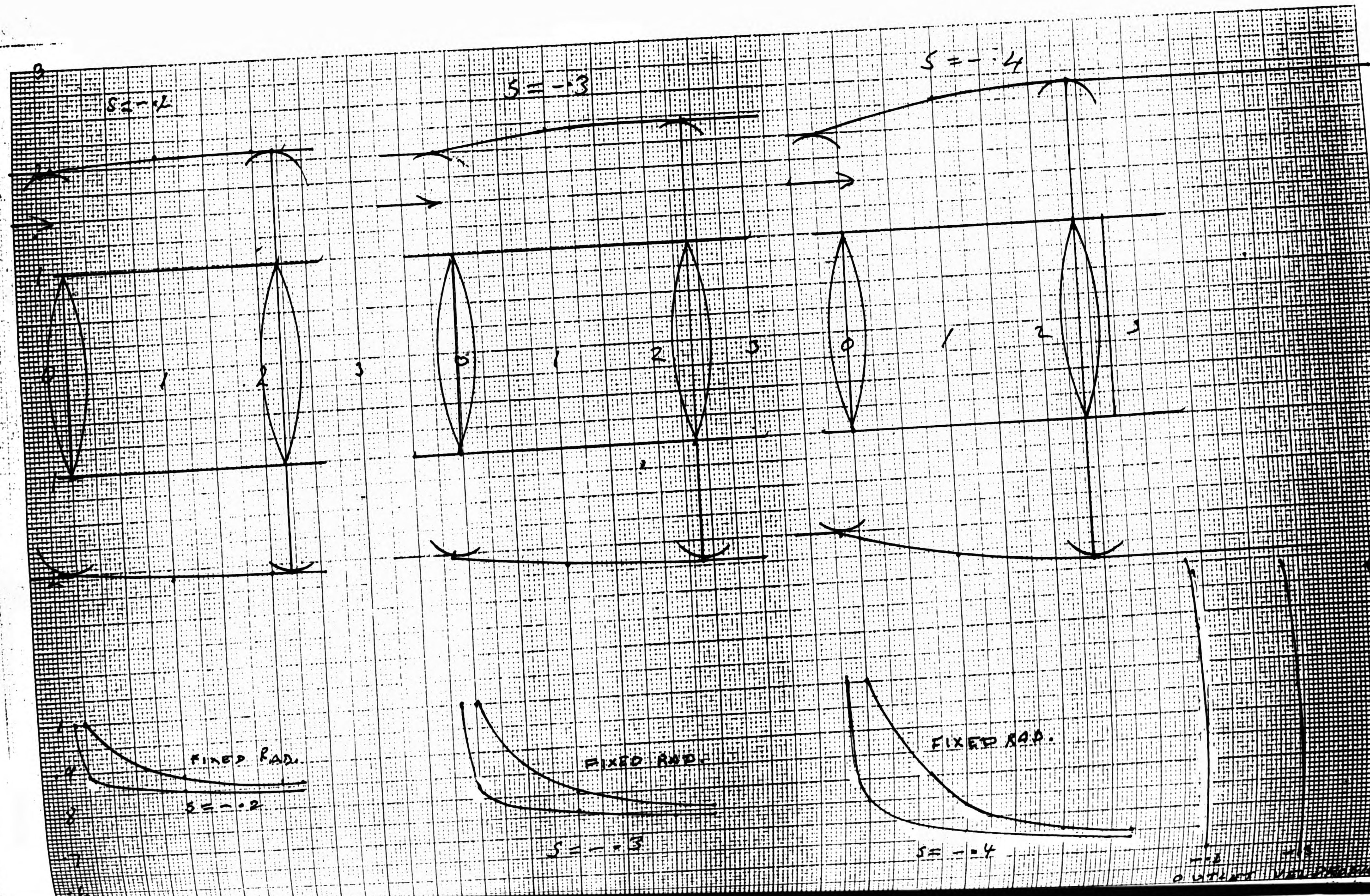
Duct Profile




OUTPUT VEL PROFILES
% DEVIATION
s = -.25
s = -.25
s = -.5
s = -.5
s = -.75
s = -.75
s = -1
s = -1
-1%
-3%
-7%
0
1
2
3

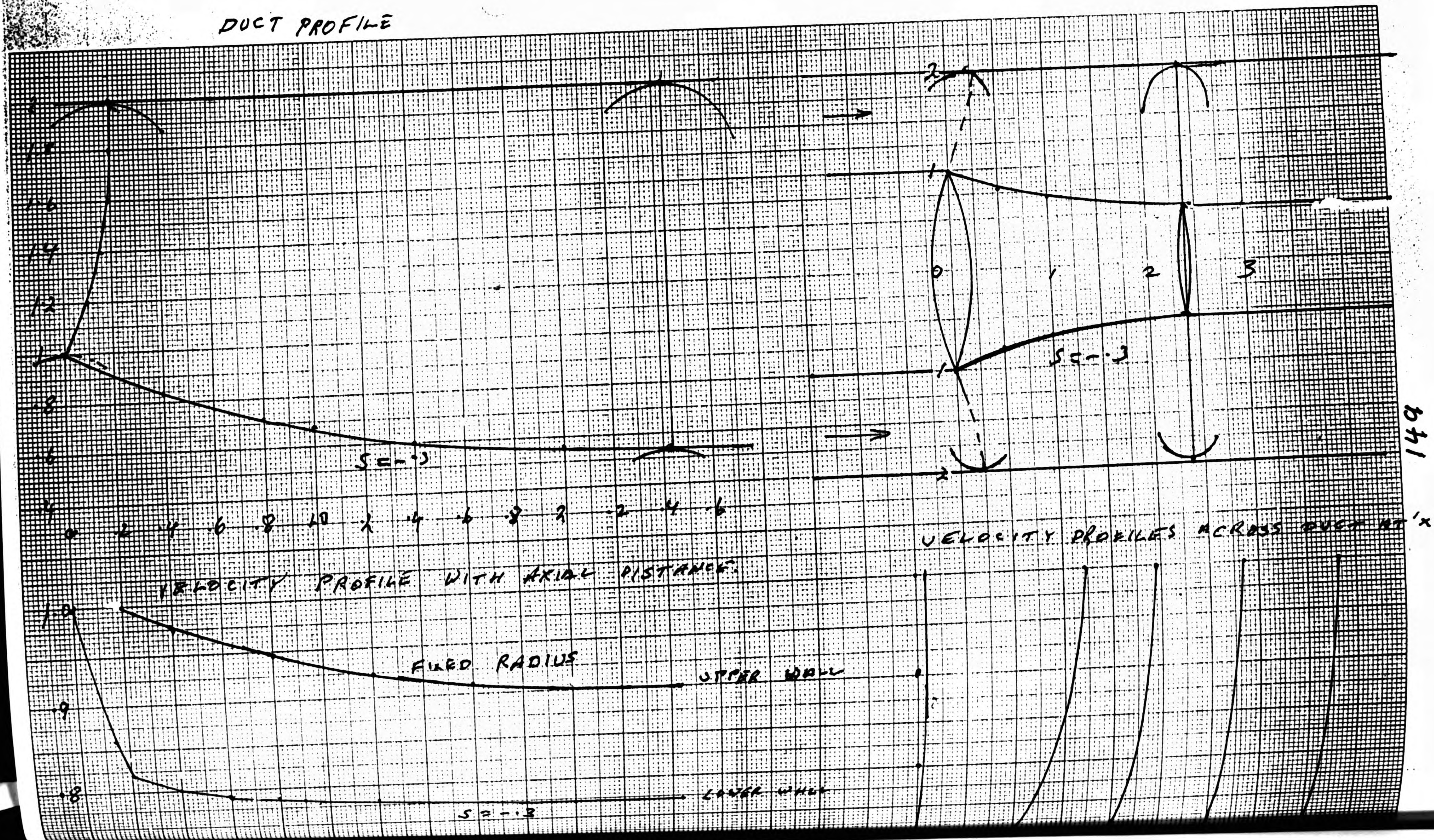
INL = 0 : OUT = P : INN = FR. : UPP = S - .5 : SW = 0 &R = 15.15 : ACC = .95 B.L = TU, .015625

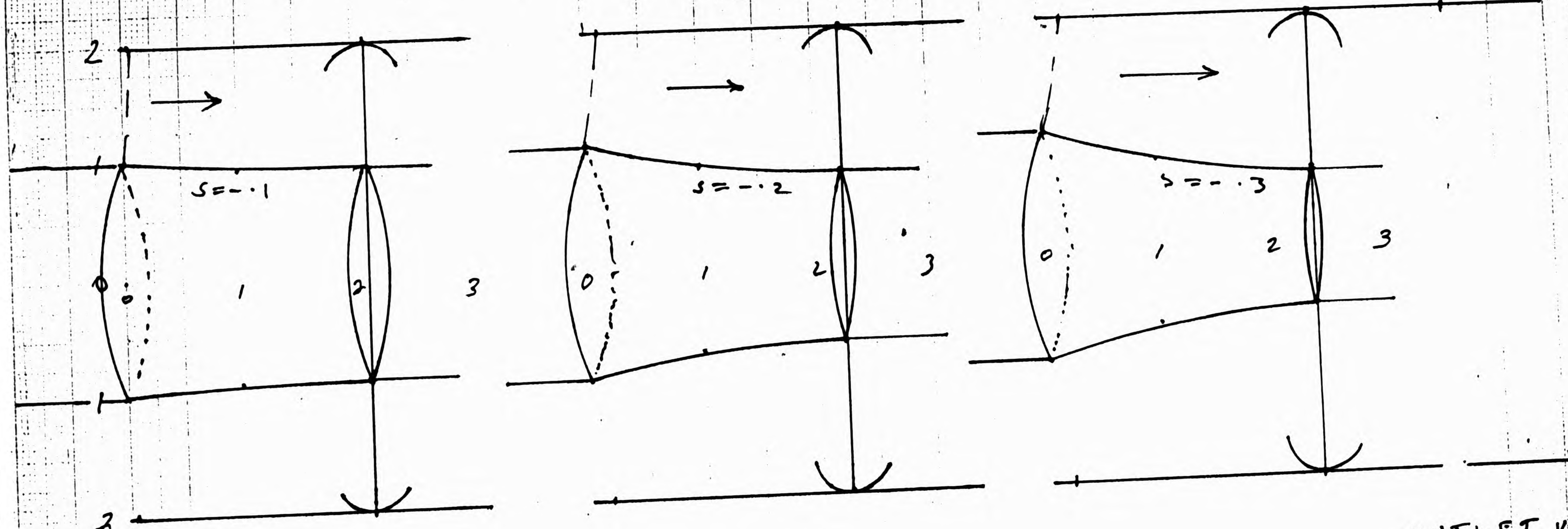



S = -.2
S = -.3
S = -.4
0
1
2
3
FIXED RAD.
S = -.2
FIXED RAD.
S = -.3
FIXED RAD.
S = -.4
OUTLET VELOCITIES

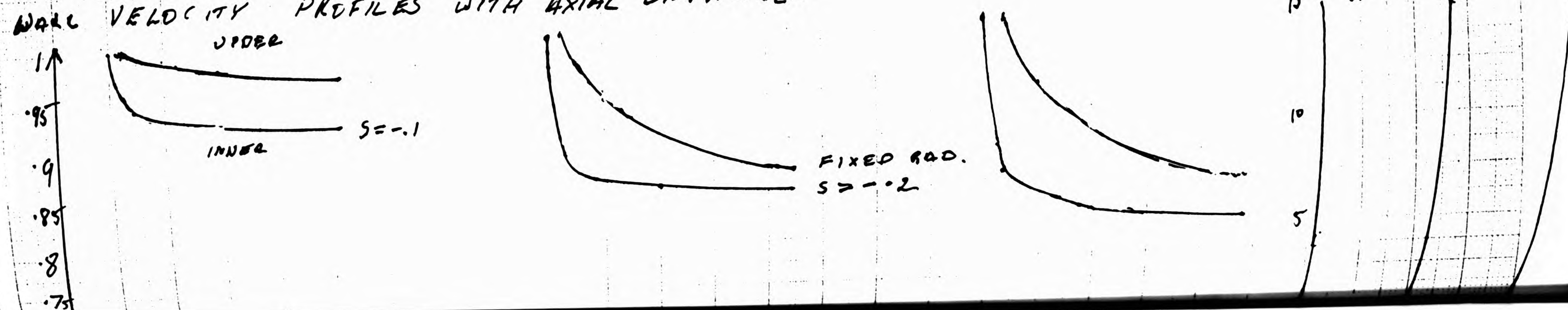
8.11

INL = U : OUT = P : INN = S - .3 : OPP = FR : JW = 0 : GR = 15.15 : ACC = .95 : B.L = T. .015625



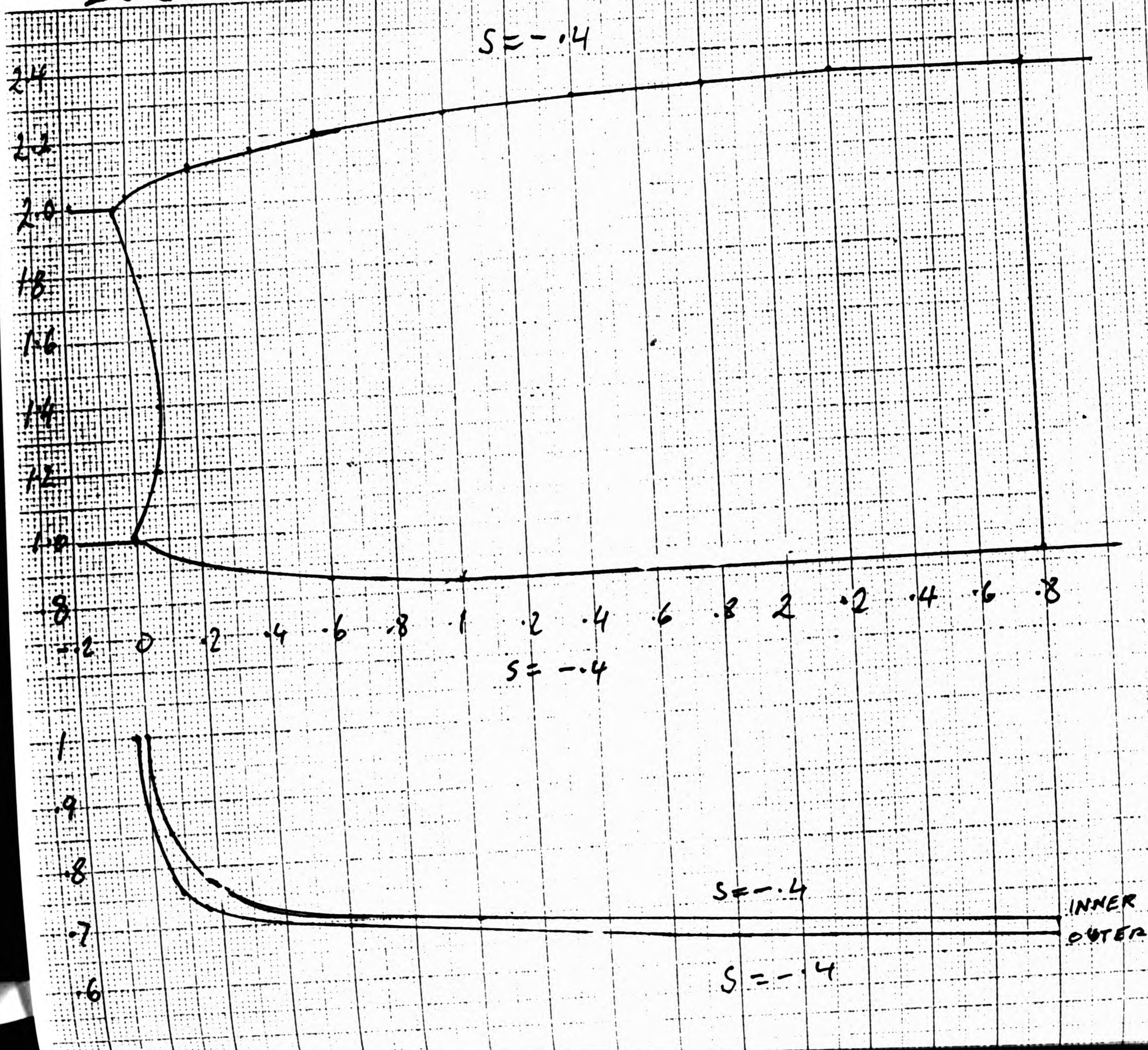


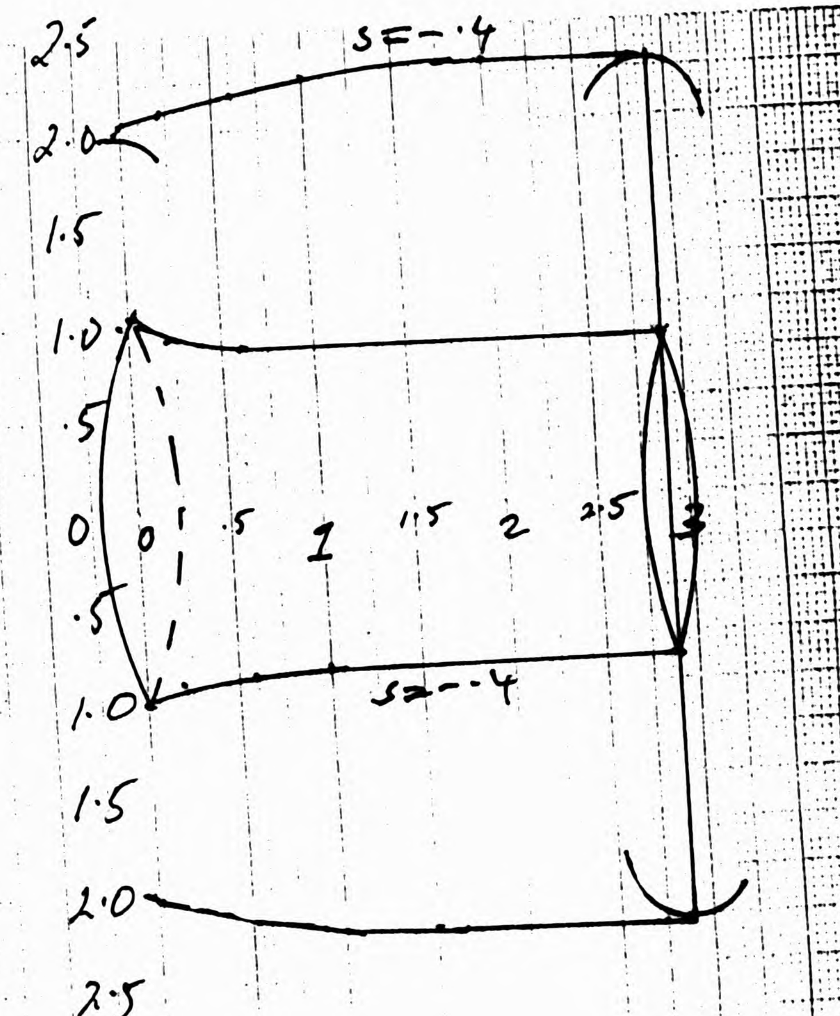
WALL VELOCITY PROFILES WITH AXIAL DISTANCE.

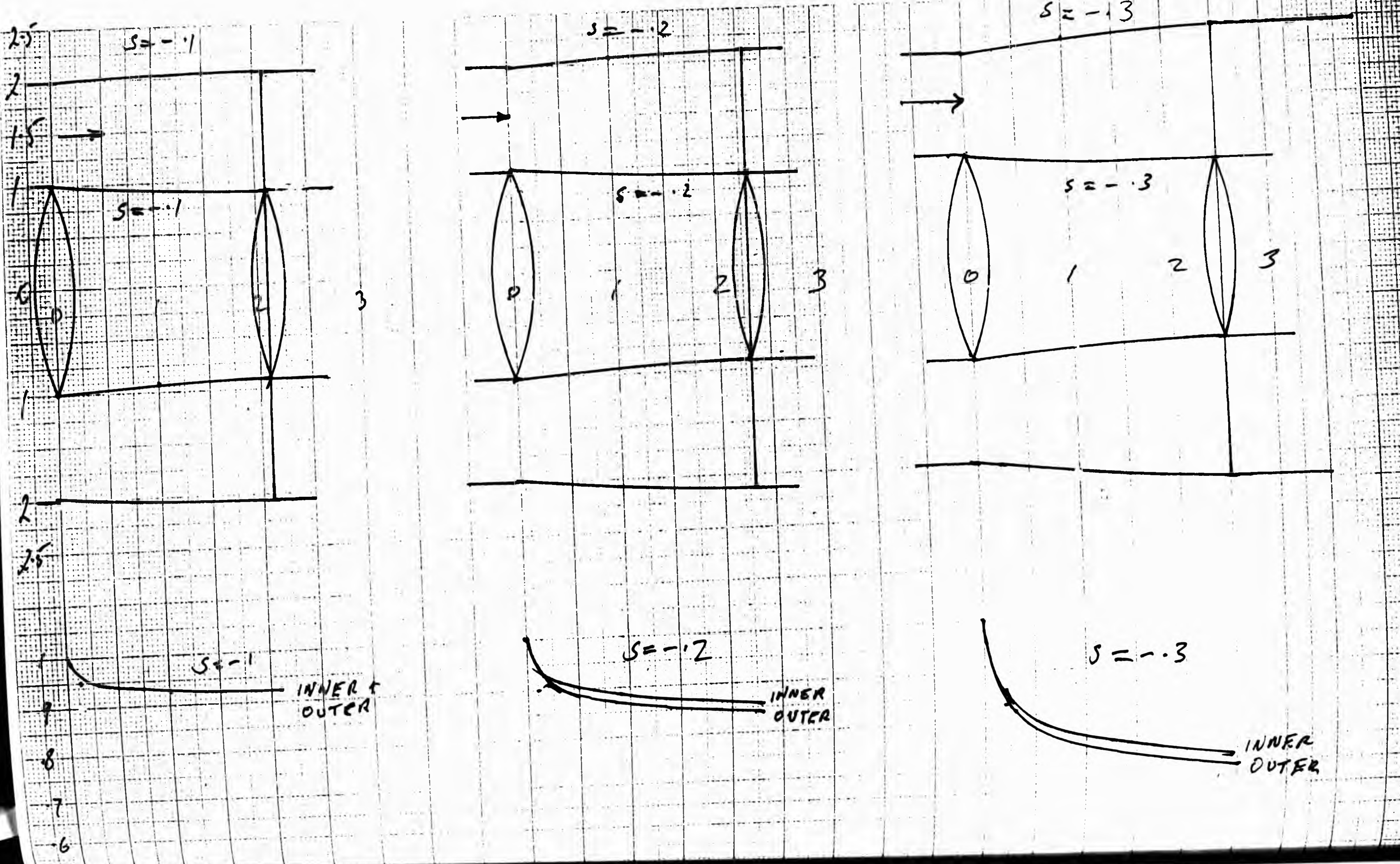


INL=V : OUTL=P : INN=S- ; UPP.=S- : SW=0 : DL=2 : GR=15x15 : ACC= .925 BL=TU

DUCT PROFILE

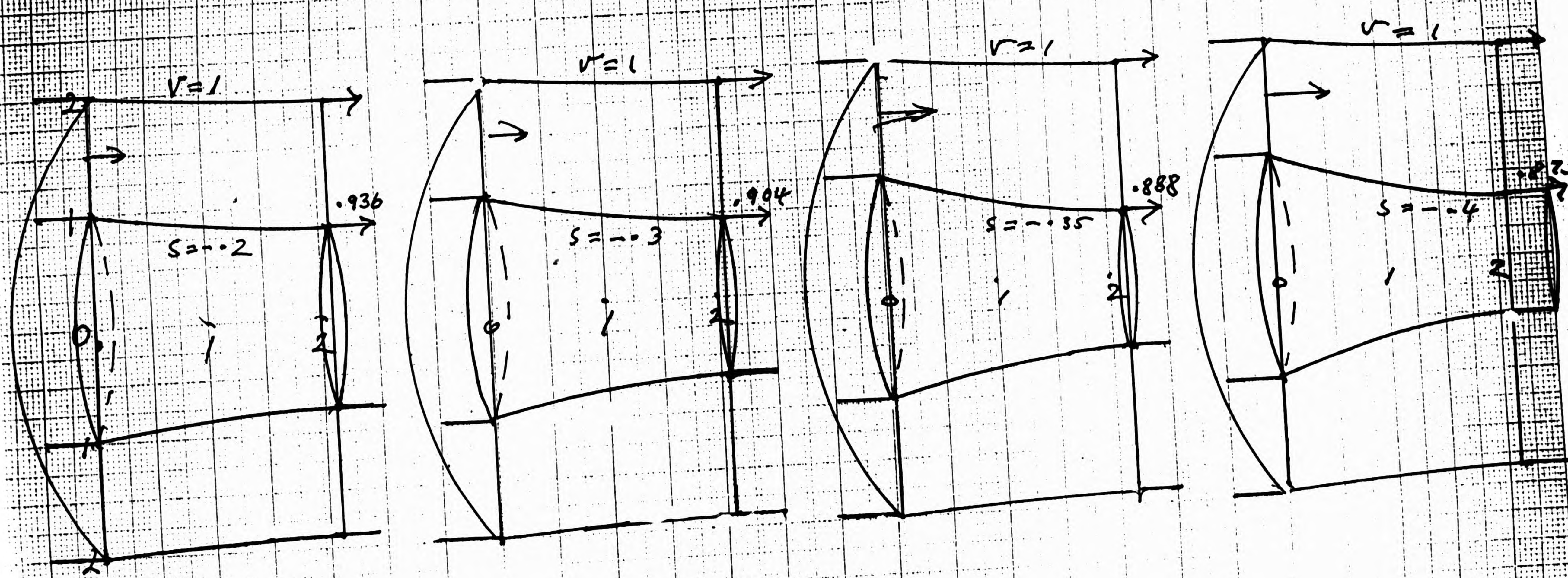





2·5
2
1·5
1
0
1
2
2·5
9
8
7
6
S = -·1
S = -·1
0
2
3
S = -1
INNER + OUTER
S = -·2
S = -·2
0
1
2
3
S = -·2
INNER
OUTER
S = -·3
S = -·3
0
1
2
3
S = -·3
INNER
OUTER

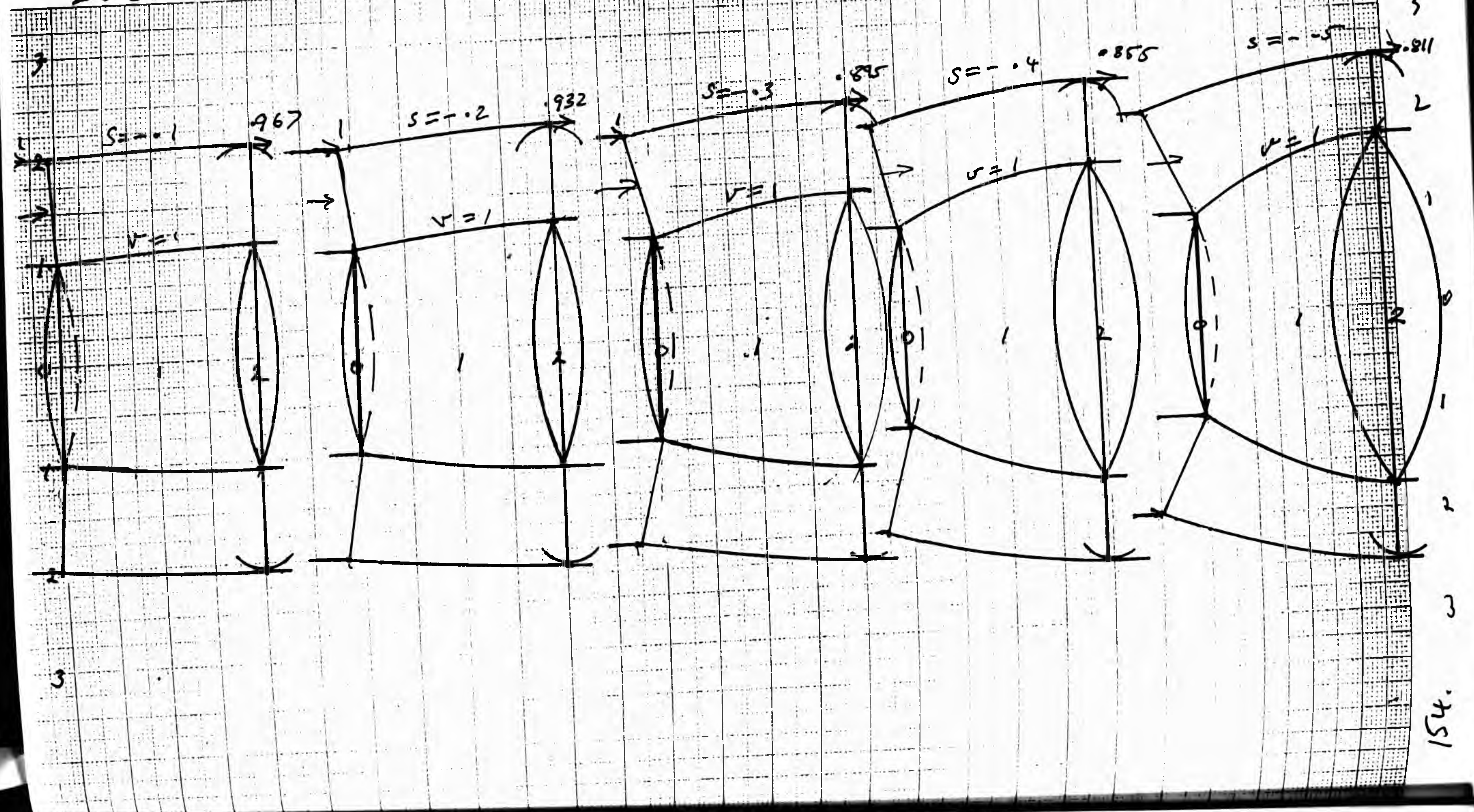
INL = U : OUTL = P : INN = S- : UPP. = CV : SW = 0 : DL = 2 : GR = 11 x 11 : ACC = 0

DUCT PROFILES



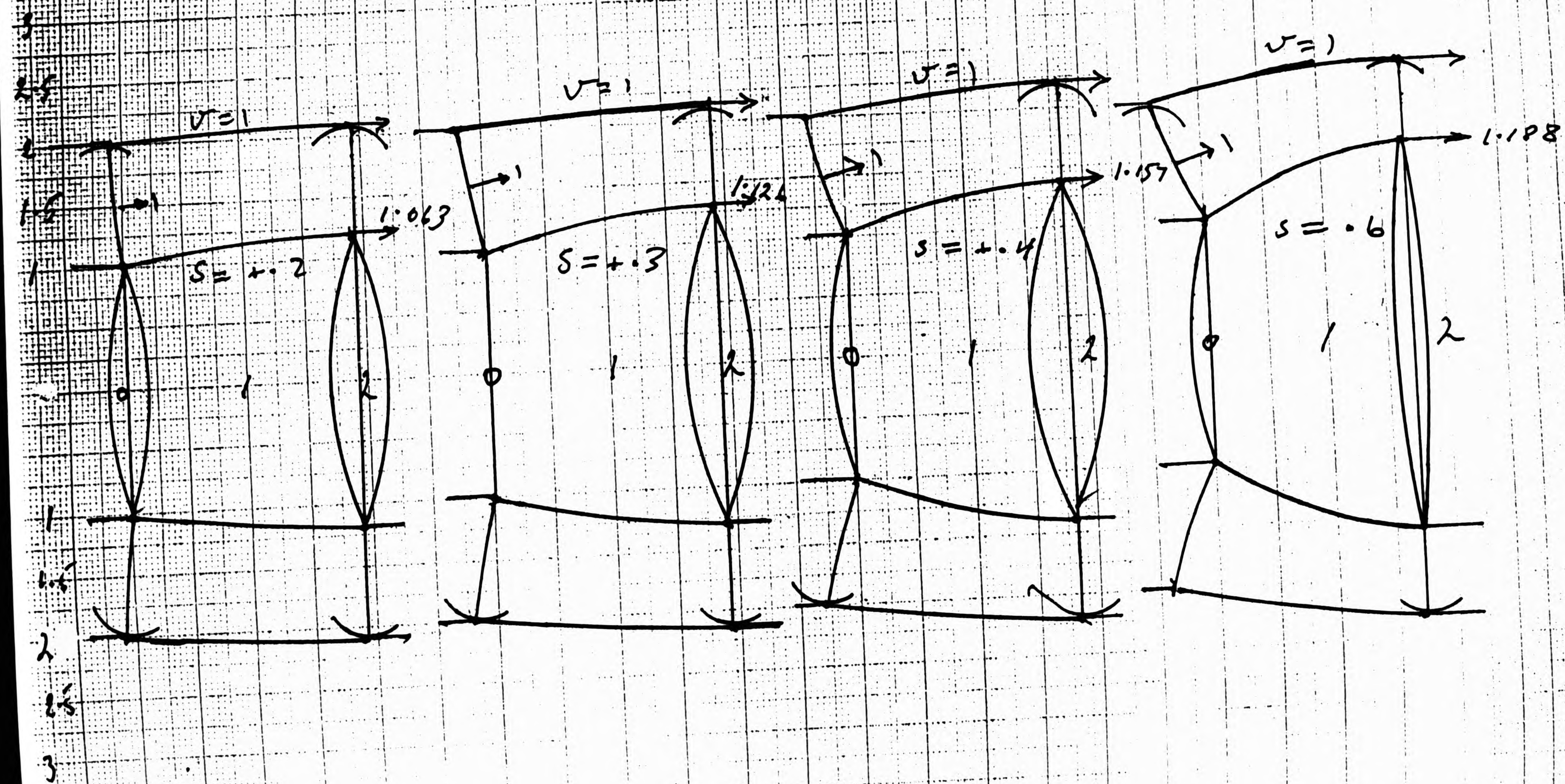
INL = U : OUTL = P : INN = CV; UPP = S- : SW = 0 : DL = 2; GR = 11×11 : ACC = 0

DUCT PROFILES



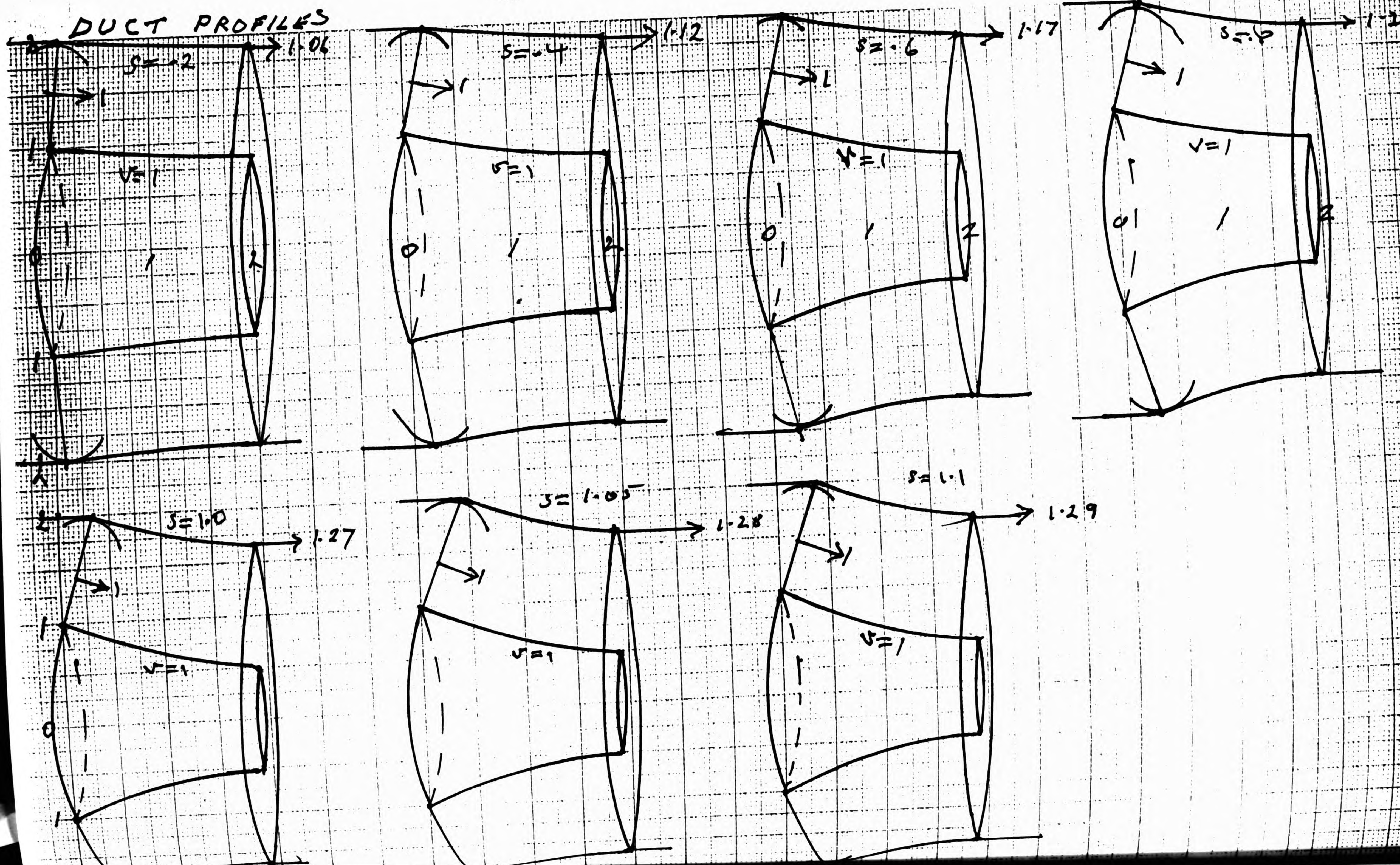
INL = U : OUTL = P : INN = St; UPP. = CV : SW = 0 : DL = 2 : GR = 11x11 : ACC = 0

DUCT PROFILES

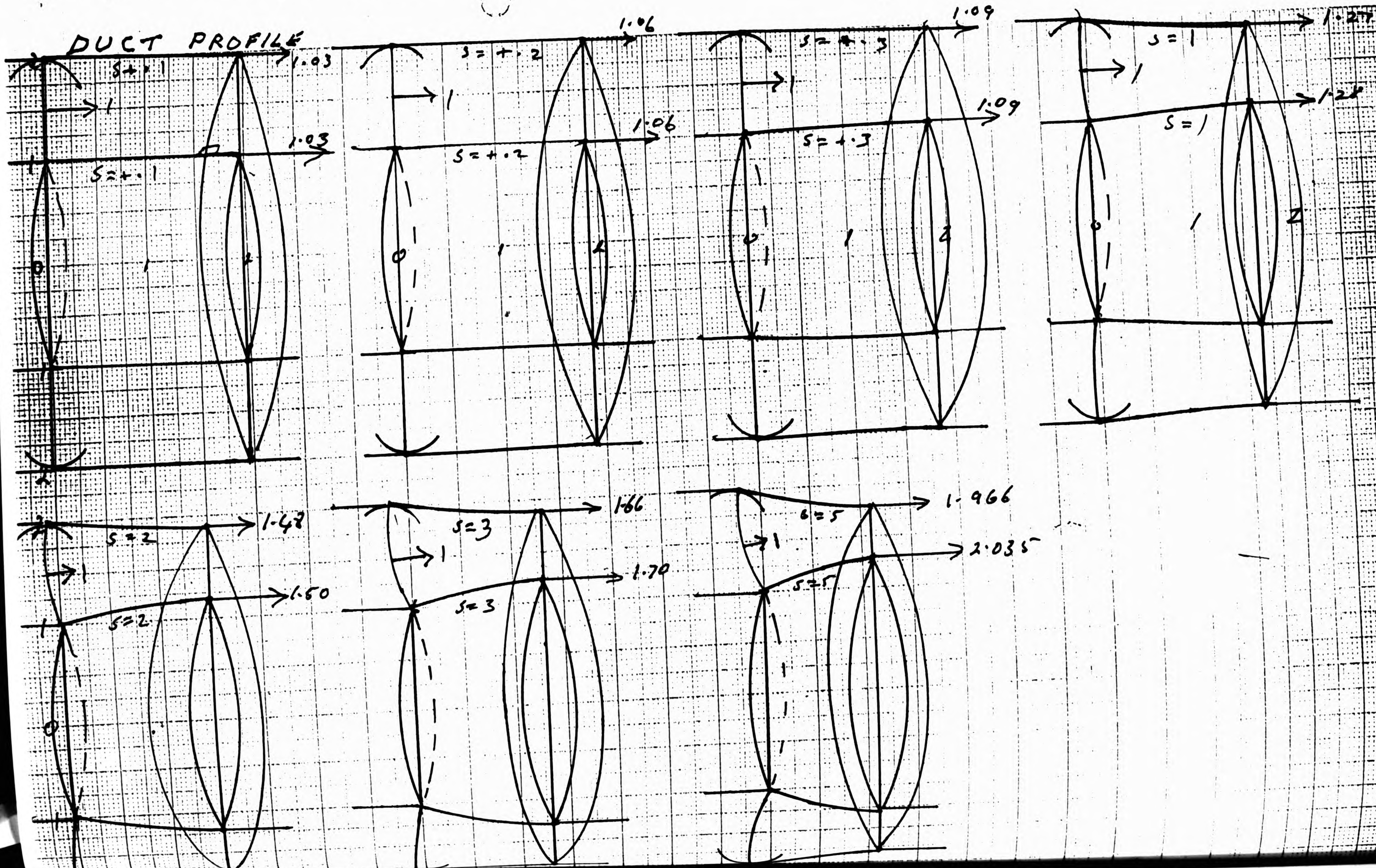


INL = U : OUTL = P : INN = CV; UPP. = St: SW = 0 : DL = 2; GR = 11.11 : ACC = 0

DUCT PROFILES

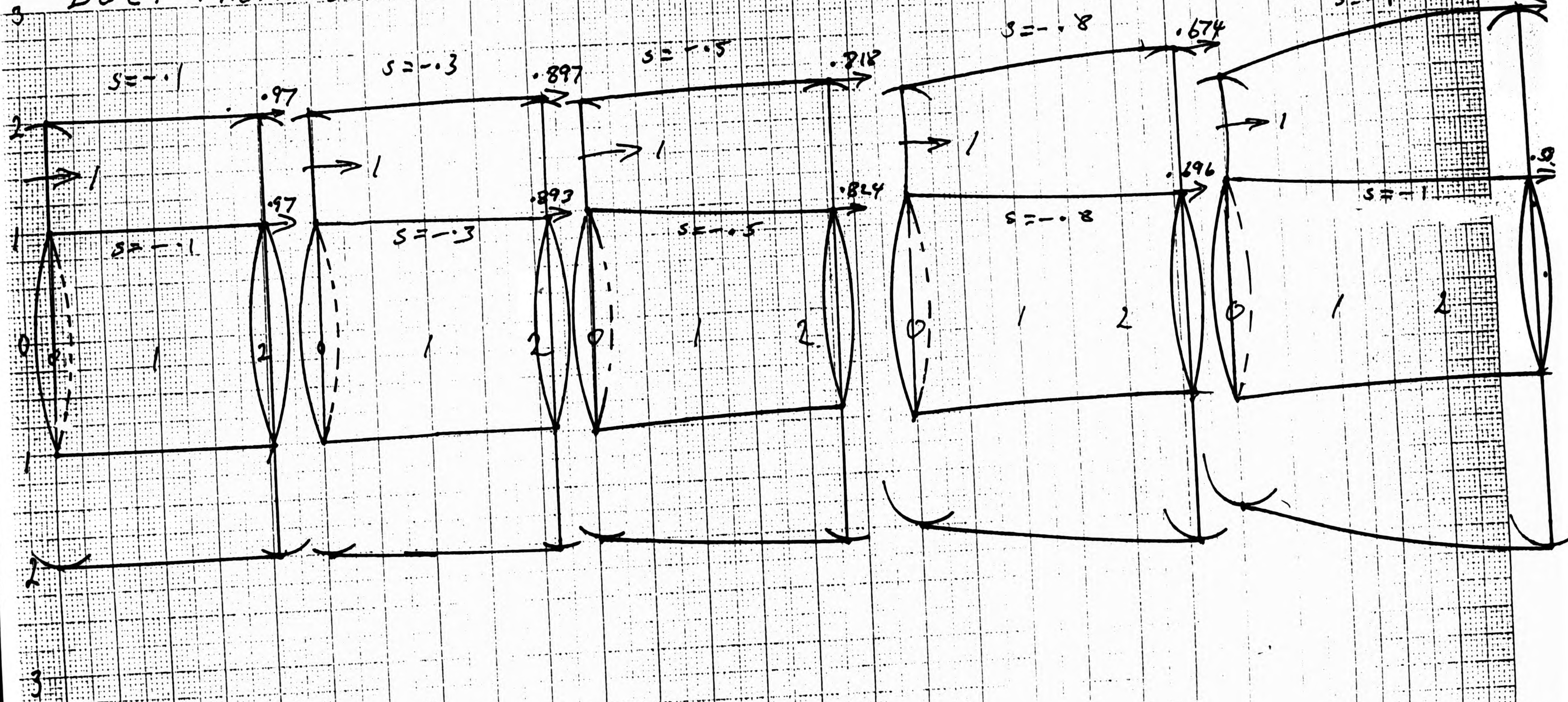


INL = U : OUTL = P : INN = St ; UPP. = St : SW = 0 : DL = 2 ; GR = 11.11 : ACC = 0



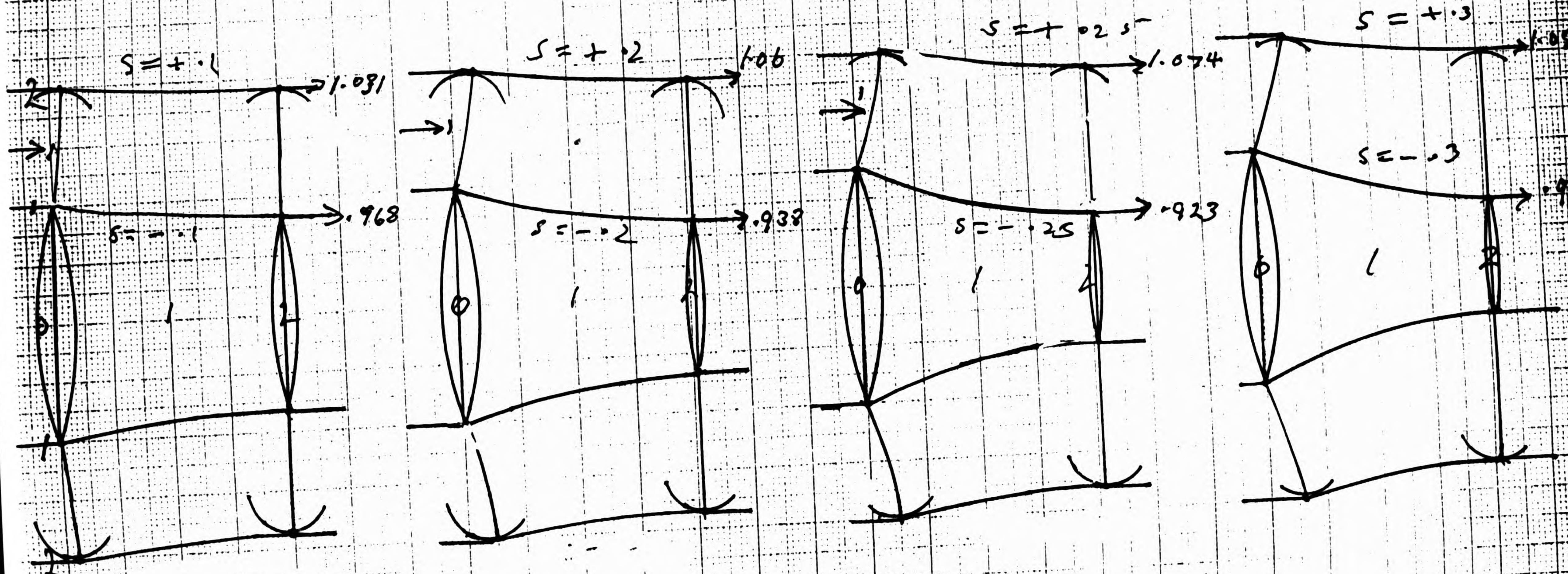
INL = U : OUTL = P : INN = S- ; UPP. = S- : SW = 0 : DL = 2 ; GR = 11.11 : ACC = 0

DUCT PROFILES



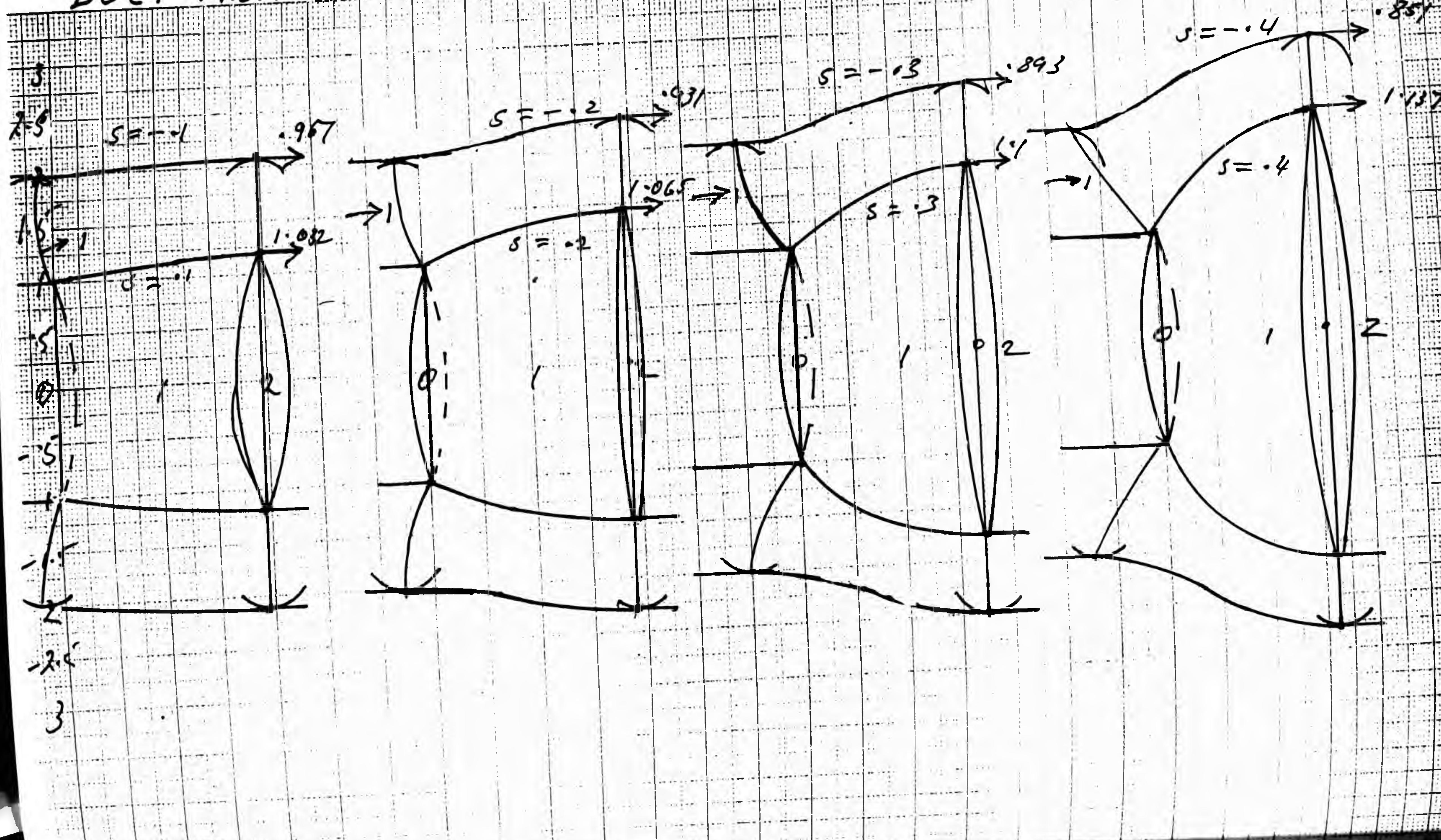
INL = U : OUTL = P : INN = S- ; UPP. = S+ : SW = 0 : DL = 2 ; GR = 11x11 : ACC = 0

DUCT PROFILES



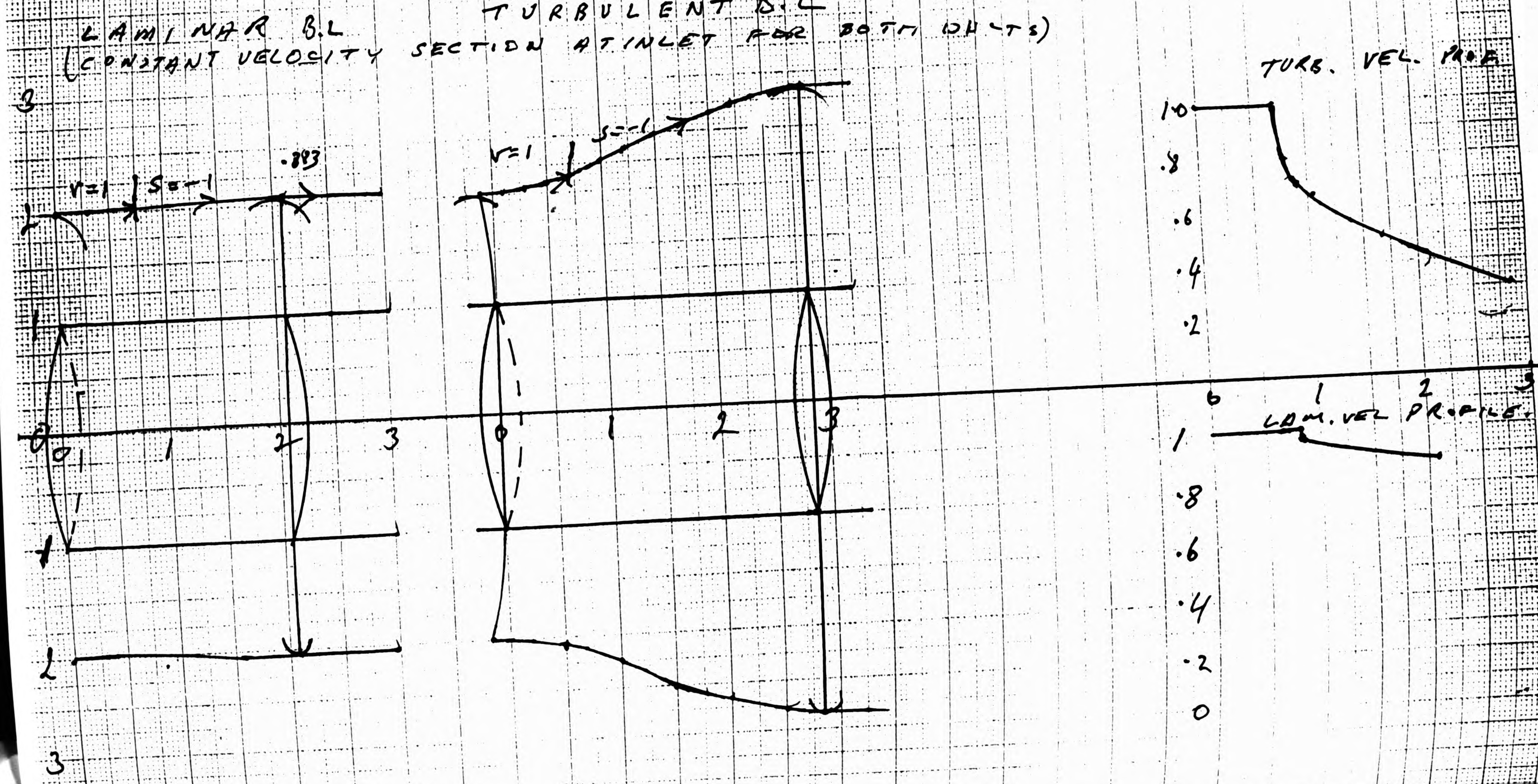
INL= U : OUTL = P : INN = 3+, UPP.=5- : SW= 0 : DL = 2 : GR= 11×11 : ACC= 0

DUCT PROFILES

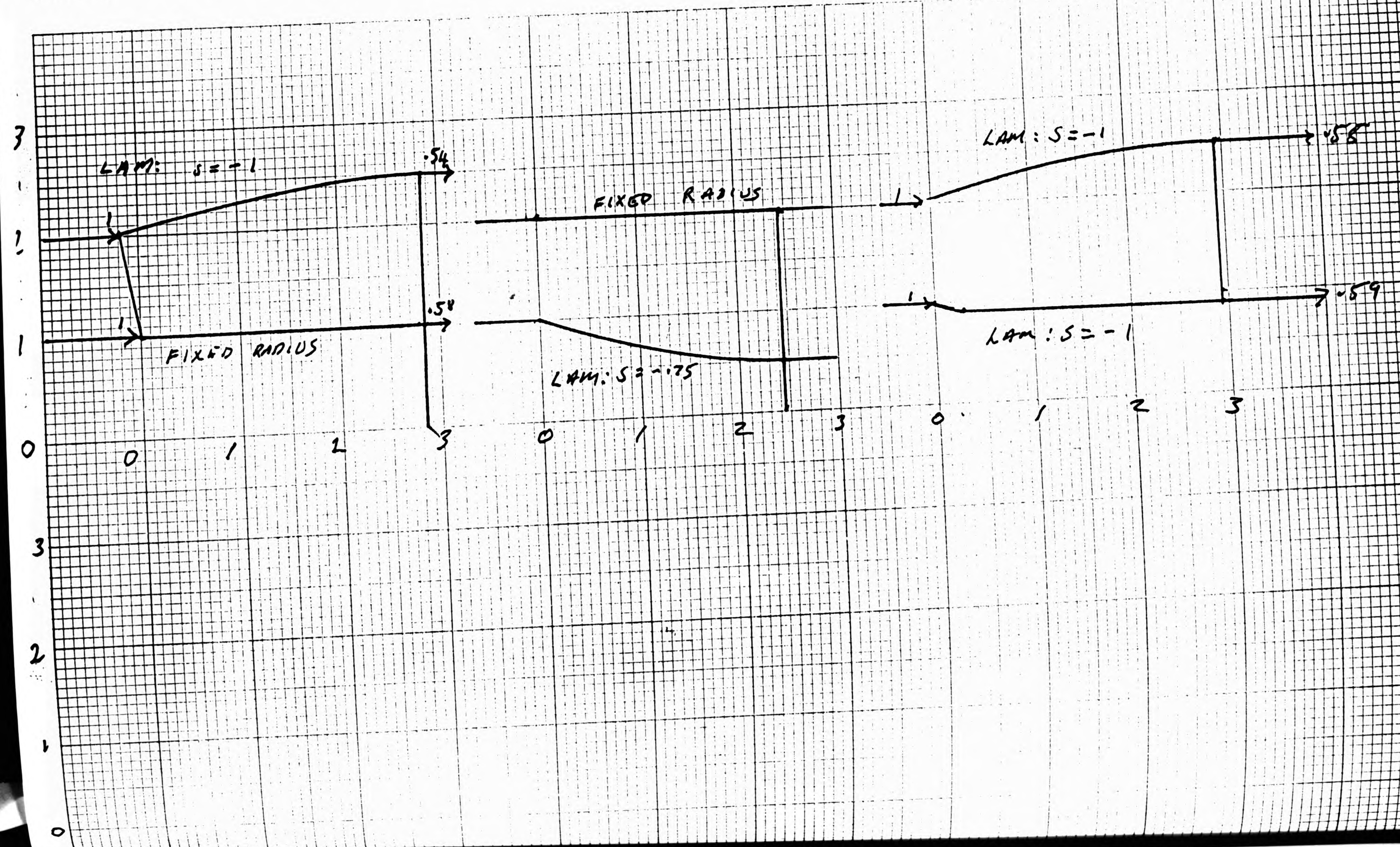


INL=U : OUTL=P : INN=F ; UPP.=S-1 : SW=0 : DL=2 ; GR=21x21 : ACC=0

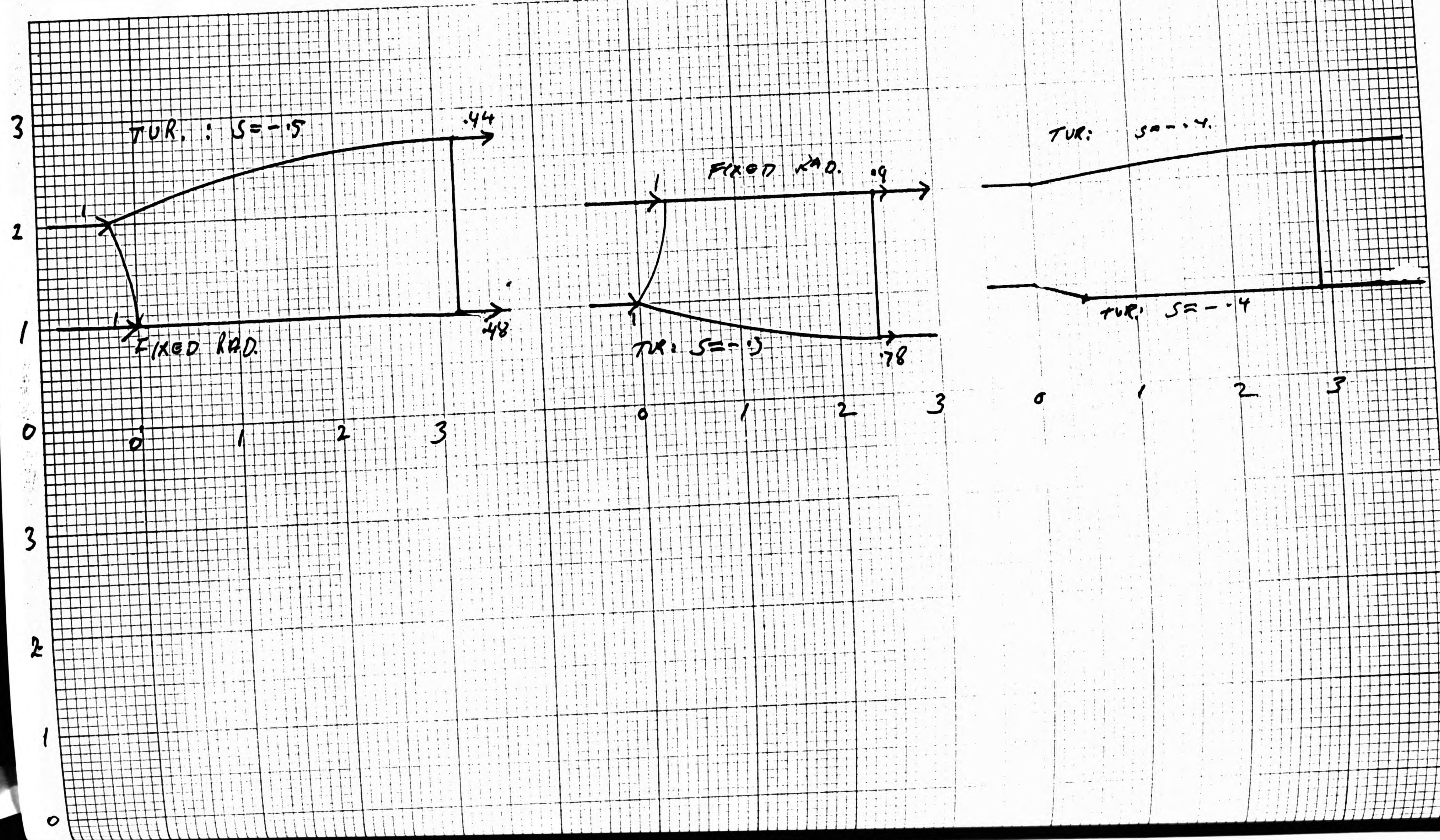
DUCT PROFILES : COMPARISON OF DUCT PROFILES GENERATED BY LAMINAR AND TURBULENT BOUNDARY LAYERS



EFFECT OF SECTION OF CONSTANT VELOCITY AT INLET : **LAMINAR . B.L**



EFFECT of CONSTANT VEL. INLET SECTION ON DUCT PROFILE. TURBULENT B.L



Extension of The Solution Having calculated the (x,r) distribution in the transition region it was attempted to extend the solution down stream (& upstream) of the outlet station. This was done by rewriting the PDE in (1) forward and (2) backward difference form and then 'stepping off' at outlet/inlet while assuming the flow to be contained betwen two coaxial cylinders.

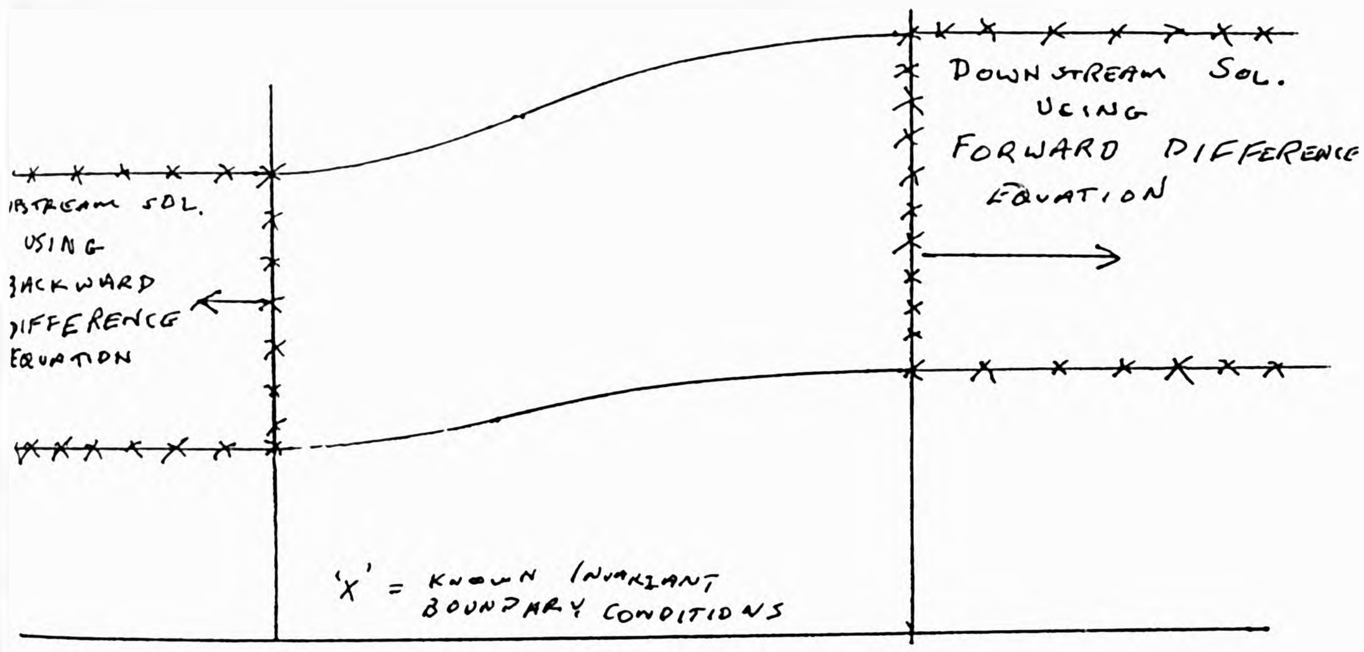


Fig 6.14

Thus the finite difference equation [6.6Ø] yields the forward difference equation

$$r_{i,j} = (r^2_{i,j} \;/r_{i,j-2}).ExP(2.r_{i,j-1} - r_{i-1,j-1} - r_{i+1,j-1})/(dY/d\Phi)^2$$

This process proved highly unstable and did not converge.

Chapter 7

(I) Introduction.

In this chapter the flow equations and their design plane counterparts are used to derive numerical solutions for the case of an inviscid axisymmetric flow with vorticity. Upstream, the axial and swirl components of velocity profiles are chosen to be of the form $q = a.y^2 + b.y + c$; $w = e.y + h/y$

In the general case for which both q and w are non zero, the vorticity vector will have three non zero components. The laws governing the behaviour of the vorticity through the transition region are incorporated into the general numerical scheme. Application of 'mixed' B.Cs on the walls and a parallel outlet flow condition suffice to define the solution completely. Calculation of the angular and axial momentum are used as a further numerical check on the consistancy of the computed solutions.

(II) Flow Equations and Design Plane Equivalents.

The equations for inviscid, axisymmetric, swirling flow are

$$u u_x + v u_y = -1/\rho . P_x \qquad \text{(Axi.)} \quad [7.1]$$

$$u v_x + v v_y - w^2/y = -1/\rho . P_y \qquad \text{(Rad.)} \quad [7.2]$$

$$(yu)_x + (yv)_y = 0 \qquad \text{(Cont)} \quad [7.3]$$

$$\Omega = (1/y)(yw)_y . \hat{x} + (-w)_x . \hat{y} + (v_x - u_y) . \hat{\theta} \qquad \text{(Vort)} \quad [7.4]$$

The design plane equations are

(a) (b)

$$(\ln(A))_\Phi = \epsilon^* . B/q^2 \quad ; \quad \epsilon^* = u_x + v_y \qquad [7.5]$$

$$(\ln(B))_\Psi = - \Omega_\theta . A/q^2 \quad ; \quad \Omega_\theta = v_x - u_y \qquad [7.6]$$

$$x_\Phi = (B/A) . y_\Psi \quad ; \quad \Omega_y = - w_x \qquad [7.7]$$

$-x_{\Psi} = (A/B).y_{\Phi}$; $\Omega_x = (1/y).(y.w)_y$ [7.8]

Elimination of the 'x' coordinate from [7.7] and [7.8] by differentiating with respect to Φ and Ψ gives

$$[(A/B).y_{\Phi}]_{\Phi} + [(B/A).y_{\Psi}]_{\Psi} = 0 \qquad [7.9]$$

From the continuity equation, [7.3] (see [2.1.2]),

$$\epsilon^* = u_x + v_y = (-q^2/B).(\ln(y))_{\Phi} \qquad [7.10]$$

Substituting into [7.5] (eliminating B.q) gives

$$(\ln(A))_{\Phi} = - (\ln(y))_{\Phi}$$

$$\Rightarrow \quad (\ln(A.y))_{\Phi} = 0$$

=> $A.y = g_1(\Psi)$ [say] where $g_1(\Psi)$ is an arbitrary function of Ψ.

$$\Rightarrow \quad A = g_1(\Psi)/y \qquad [7.11]$$

Substituting this form for the function A in [7.6] gives

$$(\ln(B))_{\Psi} = - \Omega_\theta.A/q^2 = - \Omega_\theta .g_1(\Psi)/(q^2.y) \qquad [7.12]$$

Since $g_1(\Psi)$ is arbitrary let $g_1(\Psi) = 1$ and hence from [7.11]

$$A = 1/y \qquad [7.13]$$

and $(\ln(B))_{\Psi} = - \Omega_\theta /(q^2.y)$ [7.14]

In the case of irrotational flow, $\underline{\Omega} = \underline{0}$ and hence $B = g_2(\Phi)$ where $g_2(\Phi)$ is arbitrary and set equal to unity making B = 1 everywhere. However in the case of non zero Ω_θ, [7.14] gives only the variation of B with respect to Ψ across the duct whilst the function A has the same form as for the irrotational case. However, if B is prescribed along one Ψ characteristic, then integrating [7.14] with respect to Ψ will enable the distribution of B to be determined throughout the (Φ,Ψ) plane. This could only be done in closed form for a restricted class of functions of Ω_θ, q, y. In practice [7.14] allows a numerical integration to determine the value of B along the Φ characteristics across the duct.

Substituting for A from [7.13] into [7.9] gives using [1.11.3]

$$[\, B.y.y_{\psi} \,]_{\psi} + [\, (y)/(B.y)_{\Phi} \,]_{\Phi} = 0 \qquad [7.14a.1]$$

$$[\, \ln(B) \,]_{\psi} = - \Omega\theta \,/(q^2.y) \qquad [7.14a.2]$$

$$(y_{\psi})^2/A^2 + (y_{\Phi})^2/B^2 = (y.y_{\psi})^2 + (y_{\Phi})^2/B^2 = 1/q^2 \qquad [7.14a.3]$$

Letting $r = y^2 \Rightarrow r_{\psi} = 2.y.y_{\psi}$; $r_{\Phi} = 2.y.y_{\Phi}$; $(F)_y = 2.r^{1/2}.(F)_r$

Then [7.14a.1,2,3] may be written as

$$[\, B.r_{\psi} \,]_{\psi} + [\, B^{-1}.(\ln(r))_{\Phi} \,]_{\Phi} = 0 \qquad [7.15]$$

$$[\, \ln(B) \,]_{\psi} = - \Omega\theta \,/(q^2.r^{1/2}) \qquad [7.16]$$

$$(r_{\psi})^2/4 + (r_{\Phi})^2/(4.r.B^2) = 1/q^2 \qquad [7.17]$$

Using the transform of Chapter 3 and denoting the transformed variables by '*' then equations [7.15,16,17] become

$$[\, B.r^*_{\psi^*} \,]_{\psi^*} + [\, B^{-1}.(\ln(r^*))_{\Phi^*} \,]_{\Phi^*} = 0 \qquad [7.15a]$$

$$[\, \ln(B) \,]_{\psi^*} = -[c_5^2/(4.c_7^3)].\Omega\theta.q^{*2}/r^{*1/2} \qquad [7.16a]$$

$$q^{*2} = (r^*_{\psi^*})^2 + (r^*_{\Phi^*})^2/(B^2.r^*) \qquad [7.17a]$$

Defining a dimensionless vorticity as $\Omega\theta^* = - (c_5^2/(4.c_7^3).\Omega\theta$

$$[\, B.r^*_{\psi^*} \,]_{\psi^*} + [\, B^{-1}.(\ln(r^*))_{\Phi^*} \,]_{\Phi^*} = 0 \qquad [7.15b]$$

$$[\, \ln(B) \,]_{\psi^*} = - \Omega\theta^*.q^{*2}/r^{*1/2} = -\Omega\theta^*.q^{*2}/y^* \qquad [7.16b]$$

$$q^{*2} = r^{*2}_{\psi^*} + r^{*2}_{\Phi^*}/(B^2.r^*) \qquad [7.17b]$$

If the distribution of $\Omega\theta^*$ were known throughout the flow field then [7.15b,16b,17b] are sufficient to determine the corresponding distributions of r^*, q^* and B. The form of the dependency of $\Omega\theta^*$ in the transition region of the flow and an outline of its derivation for this solution scheme is given below (Subscripts dropped).

(III) Vorticity Transport Through the duct.

For incompressible flow the total energy of a fluid element along a given stream line, Υ, is given by $H(\Upsilon)$ where

$$H(\Upsilon) = (u^2 + v^2 + w^2)/2 + p/\rho$$

and with $\Omega_x = (1/y).(y.w)_y$; $\Omega_y = -w_x$; $\Omega_\theta = v_x - u_y$

$$u = (1/y).\Upsilon_y \; ; \; v = (-1/y).\Upsilon_x \qquad [7.19]$$

it can be shown (Ref. 3)

$$v.\Omega_\theta - w.\Omega_y = H_x \qquad [7.20]$$

$$w.\Omega_x - u.\Omega_\theta = H_y \qquad [7.21]$$

$$u.\Omega_y - v.\Omega_x = H_\theta = 0 \qquad [7.22]$$

[Where that Ω_x denotes the component of Ω in the 'x' direction, whilst H_x denotes the der ivative of H with respect to x.]

Substituting Ω_x and Ω_y from [7.19] into [7.22] gives

$$u(-w_x) - v(1/y).(y.w)_y = 0 \Rightarrow u.(y.w)_x + v.(y.w)_y = 0 .$$

But $u = q.x_s$; $v = q.y_s$ where $ds^2 = dx^2 + dy^2$

Hence $q.(y.w)_x.x_s + q.(y.w)_y.y_s = 0 \Rightarrow (y.w)_s = 0$; $(q \neq 0)$

Thus the quantity (y.w) is constant along a given streamline and we may write $y.w = C(\Upsilon)$ or $w = C(\Upsilon)/y$ [7.23]

Therefore if $C(\Upsilon)$ is known at some point on a streamline (at inlet say) then the swirl speed, w, is determined along the whole streamline provided that the distribution of y ($= r^{1/2}$) is also known along this streamline.

Substituting for w (from [7.23]) into [7.19] gives

$$\Omega_x = (1/y).C_y = (1/y).C_\Upsilon.\Upsilon_y = u.C_\Upsilon$$

$$\Omega_y = (-1/y).C_x = (-1/y).C_\Upsilon.\Upsilon_x = v.C_\Upsilon$$

With these expressions for Ω_x and Ω_y we can obtain expressions for Ω_θ from [7.20] or [7.21]. Thus from [7.21]

$$w.\Omega_x - u.\Omega_\theta = H_y = H_\Psi.\Upsilon_y$$

$$(C/y).u.C_\Psi - u.\Omega_\theta = H_\Psi.y.u \quad \Rightarrow \quad \Omega_\theta = C.C_\Psi/y - y.H_\Psi$$

$$\Rightarrow \quad \Omega_\theta/y = (1/y^2).C.C_\Psi - H_\Psi = (1/y^2).(C^2)_\Psi/2 - H_\Psi = \Omega_\theta/r^{1/2} \qquad [7.24]$$

Now this expression for Ω_θ is just that which needs to be determined on the right hand side of [7.16b] and this will be possible since we have the freedom to prescribe the inlet (or upstream) conditions of the flow thus specifying the functions $H(\Upsilon)$ and $C(\Upsilon)$ at all points of the flow field.

<u>Dimensionless form of equations [7.7] and [7.8]</u>

$$x_\Phi = (B/A).y_\Psi \quad ; \quad A = 1/y$$

$$\Rightarrow \quad x_\Phi = B.y.y_\Psi = (1/2).B.(y^2)_\Psi = (1/2).B.r_\Psi$$

From the transform of Chapter 3 we have for any function F

$$r = c_1.r_1 \; ; \; x = c_2.x_1 \; ; \; F_\Phi = (1/c_7).F_{\Phi*} \; ; \; F_\Upsilon = (1/c_5).F_{\Psi*}$$

$$\Rightarrow \quad (1/c_7).(c_2.x_1)_{\Phi*} = (1/2).B.(1/c_5).(c_1.r_1)_{\Psi*}$$

$$\Rightarrow \quad (x_1)_{\Phi*} = B(c_1.c_7)/(2.c_5.c_2).(r_1)_{\Psi*}$$

But $(c_1.c_7)/(2.c_5.c_2) = 1$

Hence the dimensionless forms of [7.7] and [7.8] are

$$(x_1)_{\Phi*} = B.(r_1)_{\Psi*} \qquad [7.18b]$$

$$(x_1)_{\Psi*} = (-1/B).(\ln r_1)_{\Phi*} \qquad [7.19b]$$

Thus H and C being functions of Υ only, once defined, equations [7.23] and [7.24] determine the distribution of the swirl velocity and vorticity throughout the flow for a given distribution of y.

Upstream Conditions

Far upstream of the inlet station the flow is assumed to be cylindrical and all quantities are independent of x.

Hence v = Ø ; q = u; and from [7.1] and [7.2]

$$p_x = \emptyset \; ; \; p_y = w^2/y \; ; \; \Psi_x = \emptyset \; ; \; \Psi_y = y.u = y.q$$

Defining the total energy $\quad H = (1/2).(q^2 + w^2) + p/\rho$

Then
$$H_\psi = (1/2).(q^2 + w^2)_\psi + (1/\rho).p_\psi$$
$$= [(1/2).(q^2 + w^2)_y + (1/\rho).p_y\,].y_\psi$$
$$= [(1/2).(q^2 + w^2)_y + (w^2/y)_y\,].y_\psi \qquad [7.25]$$
$$= \{[(1/2).(q^2 + w^2) + w^2/y\,]_y\}/(q.y) \; : [\Psi_y = y.q].$$

Since both q and w are prescribed upstream in terms of y then [7.25] determines H_ψ along a given streamline throughout the flow.

Also since $C = C(\Psi)$ and $\Psi_y = y.q$ upstream

$$[(1/2).C^2\,]_\psi = [(1/2).C^2\,]_y\,.y_\psi = (1/qy).[(1/2).C^2\,]_y$$

Define $\quad G(\Psi) = (1/2).C^2(\Psi)$
$$G_\psi = (1/qy).[(1/2).C^2\,]_y \qquad [7.26]$$

Then [7.24] may be written as $\qquad [7.27]$

$$\Omega_\theta/y = \Omega_\theta/r^{1/2} = G_\psi/y^2 - H_\psi = G_\psi/r - H_\psi$$

where the quantities G_ψ and H_ψ are known from the upstream conditions as functions of Ψ or y along each stream line. The explicit expressions for G_ψ and H_ψ in terms of the inlet values of the flow parameters are

$$G_{\psi} = [w.w_{y} + w^2/y].(y/q) \qquad (a)$$

$$H_{\psi} = [w.w_{y} + w^2/y + q.q_{y}]/(q.y) \qquad (b) \qquad [7.28]$$

Let $$E_{1} = [w_{y} + w/y].(w/q) \qquad (c)$$

Then $$G_{\psi} = y.E_{1} \quad (d) \quad : \quad H_{\psi} = [E_{1} + q_{y}]/y \qquad (e)$$

If $$w = Ø \Rightarrow E_{1} = Ø \Rightarrow G_{1} = Ø \Rightarrow H_{1} = (q_{y})/y \qquad (f)$$

In terms of the variable $r = y^2$

$$G_{\psi} = [\, w + 2.r.w_{r} \,].w/q \; ; \; H_{1} = [\, 2.q.q_{r} + 2.w.w_{r} + w^2/2 \,]/q$$

Substituting into [7.27] $\Omega\theta / r^{1/2} = G_{\psi}/r - H_{\psi}$

Further substitution into [7.16] gives

$$[\ln(B)]_{\psi} = -[\, G_{\psi}/r - H_{\psi} \,]/q^2 \qquad [7.16b.a]$$

Using the transform of Chapter 3 to map onto a unit square gives

$$[\ln(B)]_{\psi*} = -[(c_{1}.c_{3}^2)).G_{\psi*}/r_{1} - (c_{3}^2).H_{\psi*}]$$

Defining $G^*_{\psi*} = (c_{1}.c_{3}^2).G_{\psi}$ and $H^*_{\psi*} = (1/c_{3}^2).H^*_{\psi*}$

then [7.16c] may be written as

$$[\ln(B)]_{\psi*} = -[\, G^*_{\psi}/r_{1} - H^*_{\psi} \,].q_{1}^2 \qquad [7.16b.b]$$

Droping the '*' and subscripts from equations [7.15b], [7.16b], [7.17b], [7.18b] and [7.19b] yields the set

$$[B.r_{\psi}]_{\psi} + [B^{-1}.(\ln r)_{\Phi}]_{\Phi} = Ø \qquad [7.15c]$$

$$[\ln(B)]_{\psi} = (H_{\psi} - G_{\psi}/r).q^2 \qquad [7.16c]$$

$$q^2 = r_{\psi}^2 + r_{\Phi}^2/(B^2.r) \qquad [7.17c]$$

$$x_{\Phi} = B.r_{\psi} \qquad [7.18c]$$

$$x_{\psi} = (-1/B).(\ln r)_{\Phi} \qquad [7.19c]$$

Boundary Conditions

(1) Upstream/Inlet: The values of the radius at inlet depend upon the inlet axial velocity profile which is chosen as the parabolic form

$$q = a.y^2 + b.y + c \qquad \text{where a,b,c are constant.}$$

These constants are determined by three pairs of radii and velocities chosen at will and are defined by the relations

$$a = [q_1 (r_2 - r_3) + q_2 (r_3 - r_1) + q_3 (r_1 - r_2)]/e$$

$$b = - [q_1 (r_2 - r_3) + q_2 (r_3 - r_1) + q_3 (r_1 - r_2)]/e$$

$$c = [q_1 r_2 r_3 (r_2 - r_3) + q_2 r_3 r_1 (r_3 - r_1) + q_3 r_1 r_2 (r_1 - r_2)]/e$$

$$e = (r_1 - r_3)(r_3 - r_2)(r_2 - r_1)$$

The maximium/minimum values of the flow occur at

$$r_m = - b/(2.a) \ ; \ q_m = (4.a.c - b^2)/(4.a)$$

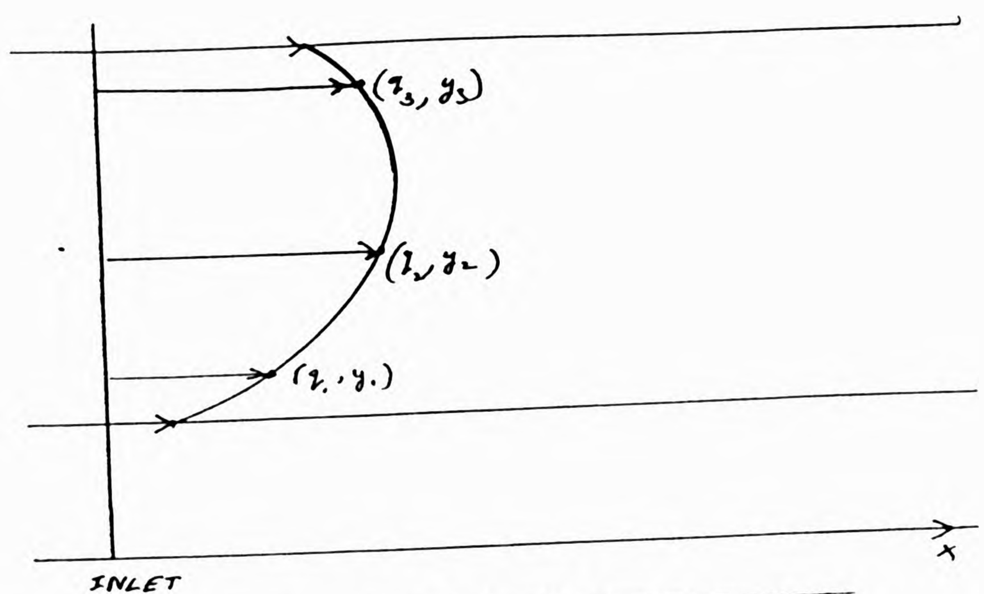


Fig.7.1

If $q_1 = q_3$ then

$$a = (r_1 - r_3)(q_1 - q_2)/e$$

$$b = -(r_1 - r_3)(r_1 + r_3)(q_1 - q_2)/e$$

$$c = (r_1 - r_3).[q_1 r_2 (r_1 - r_2 + r_3) - q_2 r_1 r_3]/e \quad \text{with } r_m = (r_1 + r_3)/2$$

If $q_1 = q_2 = q_3$ then the parabola collapses to a straight line corresponding to uniform inlet flow. Any randomly selected profile could be chosen but the 'natural' choice would be to take $q_1 = q_3$ and to choose r_1 and r_3 to correspond to the inner and outer walls. The value of q to be chosen will then correspond to the maximum/minimum speed q_m of the inlet profile which will occur at the mid-point.

<u>(3)Inlet Swirl Velocity Profile.</u>

The distribution of the swirl velocity at inlet (and upstream) is chosen as $w = e.y + f/y$ (where e, f are constant).

If $e = 0$ then the swirl velocity corresponds to that consistant with irrotational flow although the flow will only be irrotational if the inlet axial velocity is constant across the duct. If $e \neq 0$ and $f = 0$ then the inlet swirl corresponds to solid body rotation with angular velocity 'e'. An arbirary relation between 'e' and 'f' was chosen in order to limit the multiplicity of independent parameters that can now be varied to define the upstream flow conditions.

In this case $e = n.f$ where $n \neq 0$ {n = .25.(arbitrary)}.

The inlet swirl velocity is of the form

$w^* = f.(n.y + 1/y)$; $w^* = n - y^{-2}$; Min/Max = $\pm$ [$n^{-1/2}, 2.n^{1/2}$]

and a plot of some examples of possible inlet swirl profiles is shown in Fig 7.2 in which $w^* = w/f$ is plotted against y.

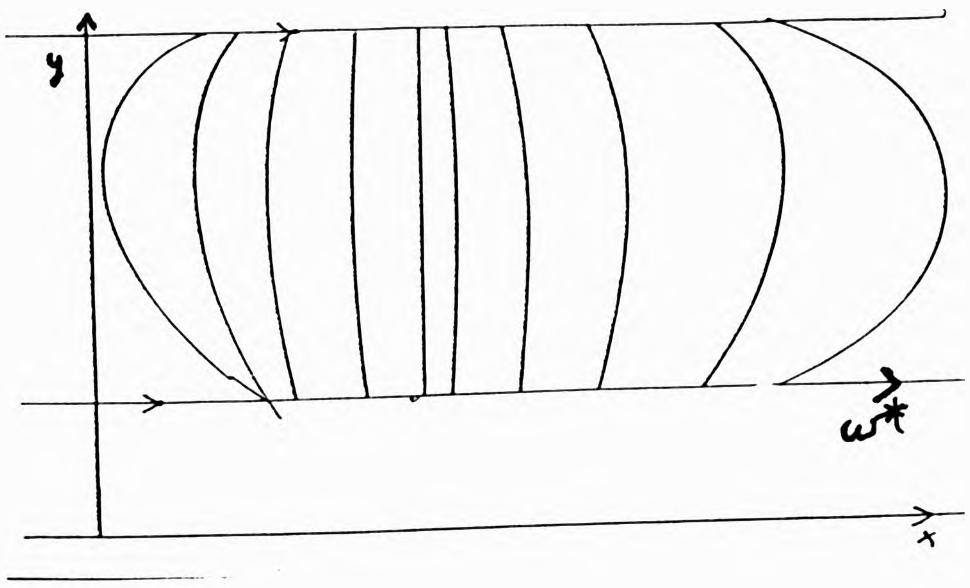


FIG 7.2

It can be seen from Fig. 7.2.1 that for the range of values of radii 'not close' to the axis, the 'solid body' part of the swirl velocity function dominates the value of the swirl velocity near $n = 2$. Because the calculation is made dimensionless on division by the reference length (y = inner duct radius), the singularity of the swirl velocity profile at $y = 0$ is removed and therefore the inlet speed will not become infinite, however it may be allowed to increase without limit by increasing the value of 'f'. It should be noted that the swirl profiles given in Fig7.2 cannot be compared quantitavely with each other since each profile has been scaled to its own inlet speed on the inner wall. The profiles are indicative only of shape but can be scaled up to any value by the factor f.

Case 1. $n >= 0$

Since the radius of the inner wall at inlet = 1 by definition

and if we define $w^* = w/f = n.y + 1/y$

(a) if $n > 1$, w^* has a minimum for $y<1$, at $y = n^{-1/2}$; $w = 2n^{1/2}$

(b) if $n = 1$, w^* has a minimum at $y = 1$ (i.e hub)

(c) if $n < 1$, w^* can have a minimum above the inner wall.

Case 2. $n < 0$. If $n<0$ then w^* is monotonic decreasing without limit. Thus there will be a stream line at some point of the flow at which the swirl speed will be zero while being non zero on the inner and outer casing but of opposite sign giving a contrarotating flow. Since the swirl velocity must be represented by a function of the form $w = C(Y)/y$ both at inlet , outlet and within the transition region, it follows that for some 'y' there exists a $C(Y) = 0$.

Inlet Velocity Profiles & Shapes for Inlet Swirl Speed Distributions of the Form

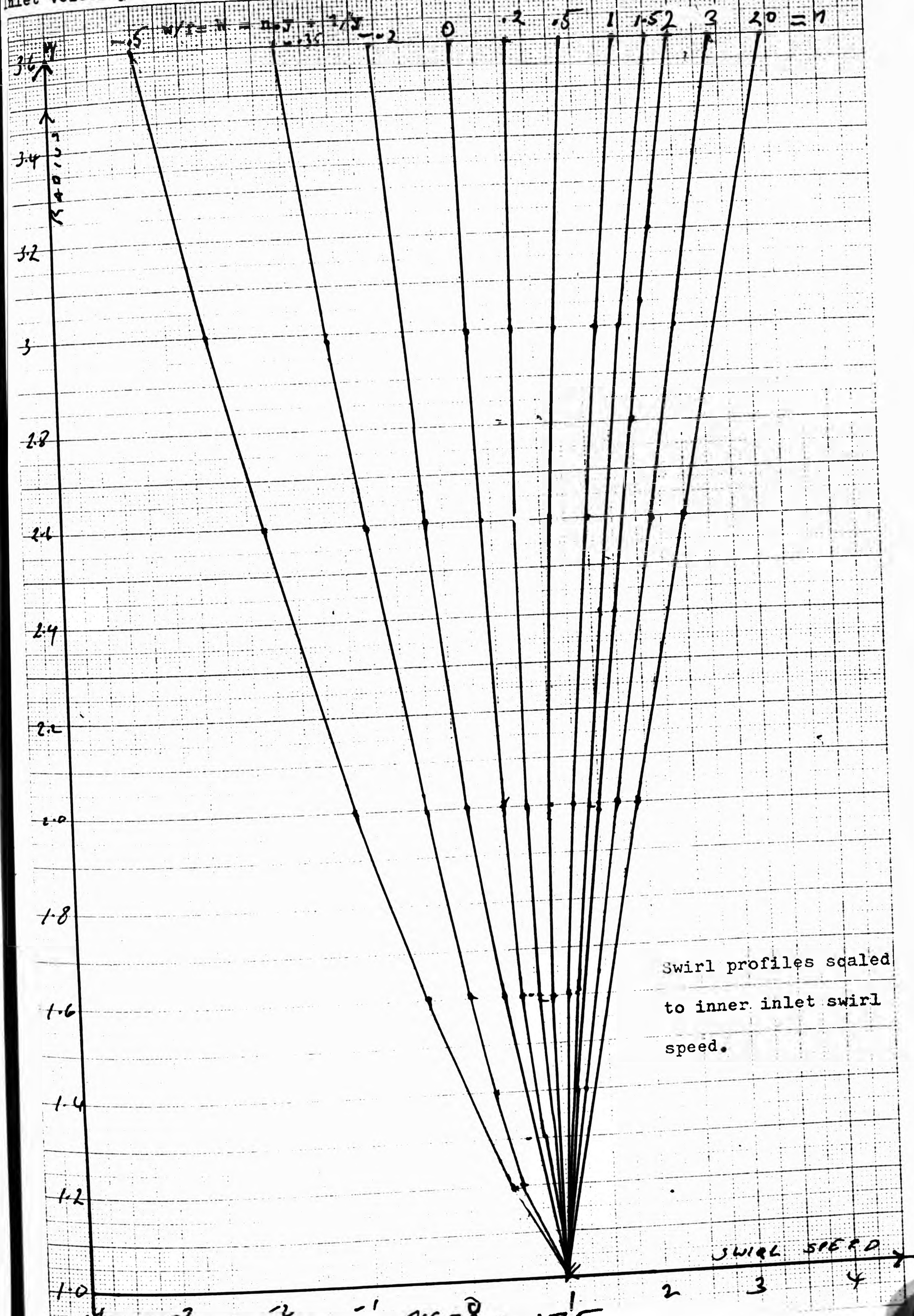


FIG 7.2.1

Hence C = Ø for some y at inlet. It follows that there is a surface of revolution throughout the flow on which the swirl speed is zero and with swirl velocities of different sign on opposite sides of this stream surface.

(3) Inlet Distribution of y with Ψ.

The prescription of inlet velocity profile together with a choice of inner inlet radius allow the derivation of the inlet y distribution at equal delta Ψ.

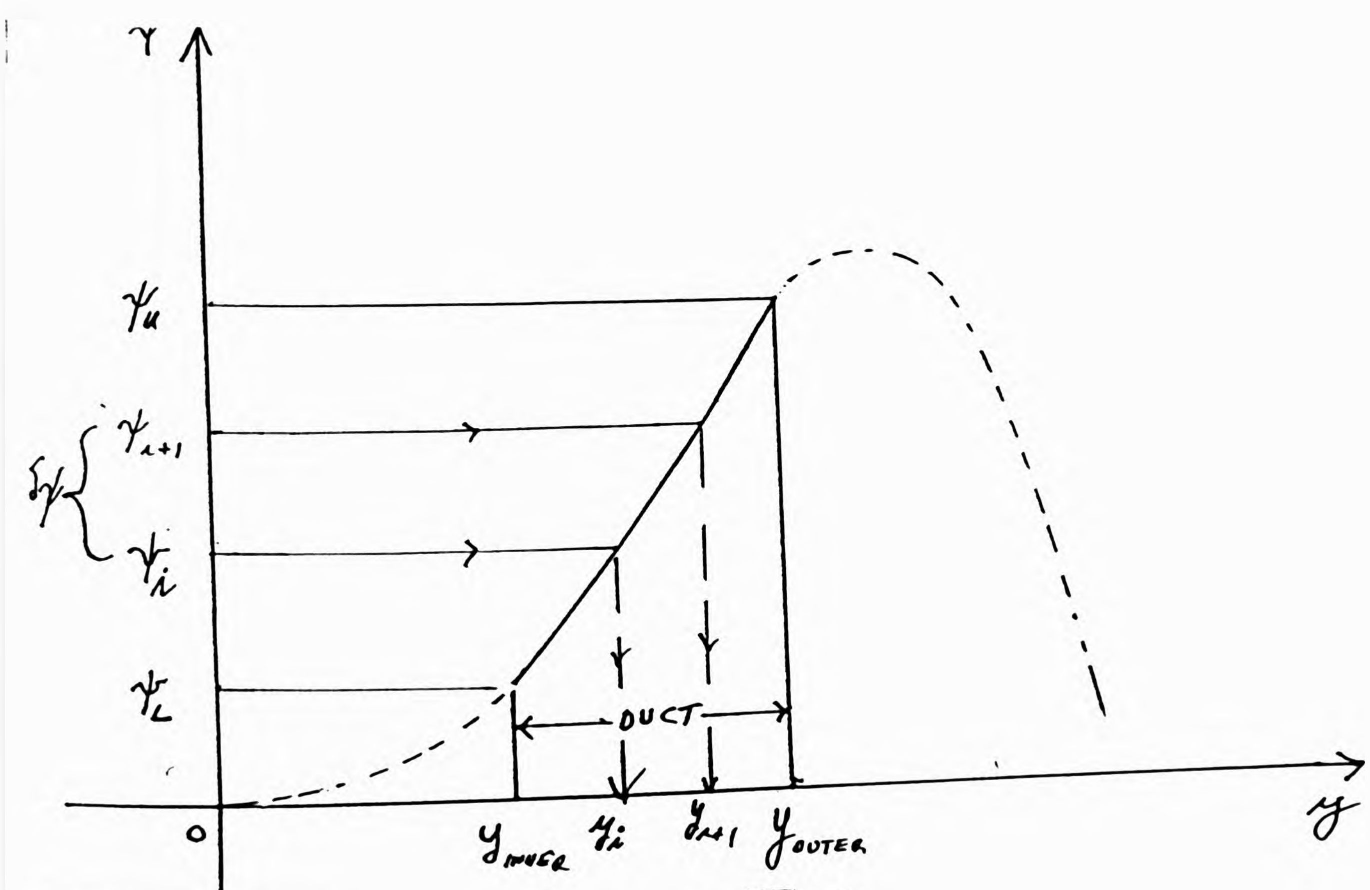


Fig 7.3

From [1.11.6(i)] we have $u = q^2.y_\Psi/A$

With A = 1/y and q = u (since at inlet v = Ø)

$q = q^2.y.y_\Psi \Rightarrow d\Psi = q.y.dy.$

But $q = ay^2 + b.y + c \Rightarrow d\Psi = (a.y^3 + b.y^2 + c).dy$ [7.32]

Integrating w.r.t Υ along the Φ inlet characteristic from inner wall to some point y we have

$$\Upsilon - \Upsilon^* = [ay^4/4 + by^3/3 + cy^2/2\ ]_{y=y^*}^{y=y}$$ where Υ^* and y^* are known.

Rearranging

$$ay^4/4 + by^3/3 + cy^2/2 = \Upsilon - \Upsilon^* - [ay^{*4}/4 + by^{*3}/3 + cy^{*2}/2] \quad [7.33]$$

This expression is a quartic in the unknown inlet y for a chosen value of Υ. Choosing values of Υ at equal intervals between Υ_L and Υ_U (Fig. 7.3) we use [7.33] to calculate the corresponding y's. Equation [7.33] has in general four distinct roots and an iterative algorithm was used to determine the appropriate one. The method chosen was that of 'bisection' where it was assumed that the required root lies between $y=0$ and $y=y_m$ where y_m is that value of y corresponding to the maximum value of Υ there being no guarantee that other iterative routines would converge to the required root. An approximation for the inlet y's could be obtained from [7.32] but the values of 'y' would become increasingly more inaccurate with increasing grid size.

(4) Wall Boundary Conditions.

The wall boundary conditions are similar to those applied in the previous section for irrotational flow ; Velocity prescription on the hub and casing of a 'Stratford' diffusion type together with regions of constant velocity and/or radii at inlet and outlet sections on either of the walls if required. Again, accelerating velocity distributions can be used instead.

(5) Outlet Conditions.

A parallel flow condition is imposed at outlet. However as in the previous chapter, this is not mandatory and a variety of velocity based outlet conditions might be considered depending on particular circumstances, an example being a velocity distribution 'joining' the hub and casing along the outlet Φ characteristic or possibly some 'mixed' condition of a similar type to the wall B.Cs.

<u>(6) Definition of The 'C' Functions.</u>

At inlet the swirl velocity w is given by $w = e.y + f/y$

and throughout the flow $w = C(\Upsilon)/y$

Hence denoting an inlet value of y by y*

$$w = C(\Upsilon)/y^* = e.y^* + f/y^* \Rightarrow C(\Upsilon) = e.y^{*2} + f \qquad [7.34]$$

Thus the values of $C(\Upsilon)$ calculated from [7.34] for a given stream line are constant along that stream throughout the flow.

<u>Finite Difference Forms.</u>

$$[\, B.r_{\Psi} \,]_{\Psi} + [\, B^{-1}.(\ln r)_{\Phi} \,]_{\Phi} = \emptyset \qquad [7.15c]$$

$$[\, \ln(B) \,]_{\Psi} = (\, H_{\Psi} - G_{\Psi}/r \,).q^2 \qquad [7.16c]$$

$$q^2 = r^2_{\Psi} + r^2_{\Phi}/(B^2.r) \qquad [7.17c]$$

Define $dt^k Z$ as the k^{th} finite difference of Z in the 't' direction then

$$d\Phi r = r_{i,j+1} - r_{i,j-1} \quad ; \quad d\Psi r = r_{i+1,j} - r_{i-1,j}$$

$$d\Phi^2 r = r_{i,j+1} - 2.r_{i,j} + r_{i,j-1} \quad ; \quad d\Psi^2 r = r_{i+1,j} - 2.r_{i,j} + r_{i-1,j}$$

$$d\Psi B = B_{i+1,j} - B_{i,j}$$

and similarly for C and R, B however being replaced by a forward difference.

Let $R = \ln(r)$; $C = B^{-1}$ then from [7.15c]

$B_\Psi . r_\Psi + B_\Psi . r_\Psi + C_\Phi . R_\Phi + C_\Phi . R_\Phi = 0$

$$\Rightarrow \frac{d_\Psi B . d_\Psi r}{d\Upsilon . d\Upsilon} + \frac{B . d_\Psi^2 r}{(d\Upsilon)^2} + \frac{d_\Phi C . d_\Phi R}{d\Phi \quad d\Phi} + \frac{C . d_\Phi^2 R}{(d\Phi)^2} = 0$$

$$- B . d_\Psi^2 r = d_\Psi B . d_\Psi r + (d\Upsilon/d\Phi)^2 . (d_\Phi C . d_\Phi R + C . d_\Phi^2 R) \qquad [7.15c']$$

Similarly for [7.16c] and [7.17c]

$$(1/B).(d_\Psi B/d\Upsilon) = (H_\Psi - G_\Psi /r).q^2 \qquad [7.16c']$$

$$q^2 = (d_\Psi r/d\Upsilon)^2 + (d_\Phi r/d\Phi)^2/(B^2.r) \qquad [7.17c']$$

Now since $C = B^{-1}$ then $d_\Phi C = - B^{-2} . d_\Phi B$

Substituting finite difference forms for these expressions for B, $d_\Psi B$, C, $d_\Phi C$, R, $d_\Phi R$, $d_\Phi^2 R$, $d_\Psi r$ and $d_\Psi^2 r$ and rearranging yields

$$r_{i,j} = (r_{i+1,j} + r_{i-1,j})/2 + [d_\Psi . d_\Psi r + D_1 . (d_\Phi^2 R - d_\Phi B . d_\Phi R/B)/B]/(2.B) \qquad [7.15c'']$$

Hence the explicit finite difference forms of [7.15.c] to [7.19c] are

$$r_{i,j} = (r_{i+1,j} + r_{i-1,j})/2 + \{ (B_{i+1,j} - B_{i,j}).(r_{i+1,j} - r_{i,j}) + D_1 . [\ln(r_{i,j+1} . r_{i,j-1} / r^2_{i,j}) - (B_{i,j+1} - B_{i,j})(r_{i,j+1} - r_{i,j})/B_{i,j}]/B_{i,j} \} /(2.B_{i,j}) \qquad [7.15d]$$

$$B_{i,j} = B_{i-1,j} . [1 - G_{\Psi\, i-1} / r_{i-1,j} - H_{\Psi\, i-1}).q^2_{i-1,j} . d\Upsilon] \qquad [7.16d]$$

$$q^2_{i,j} = [(r_{i+1,j} - r_{i-1,j})/(2d\Upsilon)]^2 + [(r_{i,j+1} - r_{i,j-1})/(2d\Phi)]^2/B^2_{i,j} . r_{i,j}) \qquad [7.17d]$$

$$x_{i,j+1} = x_{i,j} + B_{i,j} . D_4 . (r_{i+1,j} - r_{i,j}) \qquad [7.18.d]$$

$$x_{i+1,j} = x_{i,j} - (1/B_{i,j}) . D_3 . \ln(r_{i,j+1} / r_{i,j}) \qquad [7.19d]$$

Suitable programs using these finite difference forms give solutions to the design problem for rotational incompressible flows with swirl the results of which are discussed below.

Numerical Checks for Consistancy.

(1) Global Error Check.

As in the case of irrotational flow it was possible to obtain an estimate of the 'global' error in the solution by integrating, numerically, the fundamental equations over the whole flow field.

Similarly manner we have from $x_\Phi = B.r_\Psi$; $x_\Psi = (-1/B).[\ln(r)]_\Phi$

in finite difference form

$$(x_{i,j+1} - x_{i,j})/d\Phi = B_{i,j}.(r_{i+1,j} - r_{i,j})/d\Upsilon \; ; \; i=1..I; \; j = 1..J-1 \quad [7.35]$$

$$(x_{i+1,j} - x_{i,j})/d\Upsilon = (-1/B_{i,j})[\ln(r_{i,j+1}/r_{i,j})]/d\Phi \; ; \; i=1..I-1; \; j=1..J \quad [7.36]$$

Summing for i,j over the whole flow field for both [7.35] and [7.36]

$$\sum_{i=1}^{i=I}(1/d\Phi).(x_{i,out} - x_{i,in}) = \sum_{Lower}^{i=I} (1/d\Upsilon) . \sum_{Inlet}^{j=J-1} B_{i,j}.(r_{i+1,j} - r_{i,j})$$

$$(1/d\Upsilon).\sum_{j=1}^{j=J}\{ (x_{upper,j} - x_{lower,j}) \}= \sum_{inlet}^{j=J}(-1/d\Phi) \sum_{lower}^{i=I+1} (1/B_{i,j}).[\ln(r_{i,j+1}/r_{i,j})]$$

Summations in the x variable can be performed over 'j' and 'i' repectively for [7.35] and [7.36] since the left hand side does not involve the $B_{i,j}$. Comparisons for the values obtained for each side of the equations gives an estimate of the consistancy for the distribution for both 'x' and 'r' over the flow field.

Let

$$Z_1 = (1/d\Phi).\sum_{i=1}^{i=I} (x_{i,J} - x_{i,1})$$

$$Z_2 = (1/d\Upsilon).\sum_{i=1}^{i=I} \sum_{j=1}^{j=J-1} [B_{i,j}.(r_{i+1,j} - r_{i,j})]$$

$$Z_3 = (1/d\Upsilon).\sum_{j=1}^{j=J} (x_{I,j} - x_{1,j})$$

$$Z_4 = -(1/d\Phi).\sum_{i=1}^{i=I-1} \sum_{j=1}^{j=J} (1/B_{i,j}).\ln(r_{i,j+1}/r_{i,j})$$

Then the quantities (Z_1/Z_2) and (Z_3/Z_4) may be taken as error estimates for x and r.

As in the case for non swirling flow, a mass flow calculation in the axial direction is used to estimate flow errors in the distributions of x, r and q in the flow field. A similar calculation in the azimuthal plane is used as an estimate for the swirl speed. The flow through a Φ line joining two lines of constant Υ in the (Φ,Υ) plane is approximately that around the surface of the frustum of a cone in the (x,y) plane (See Fig 7.4).

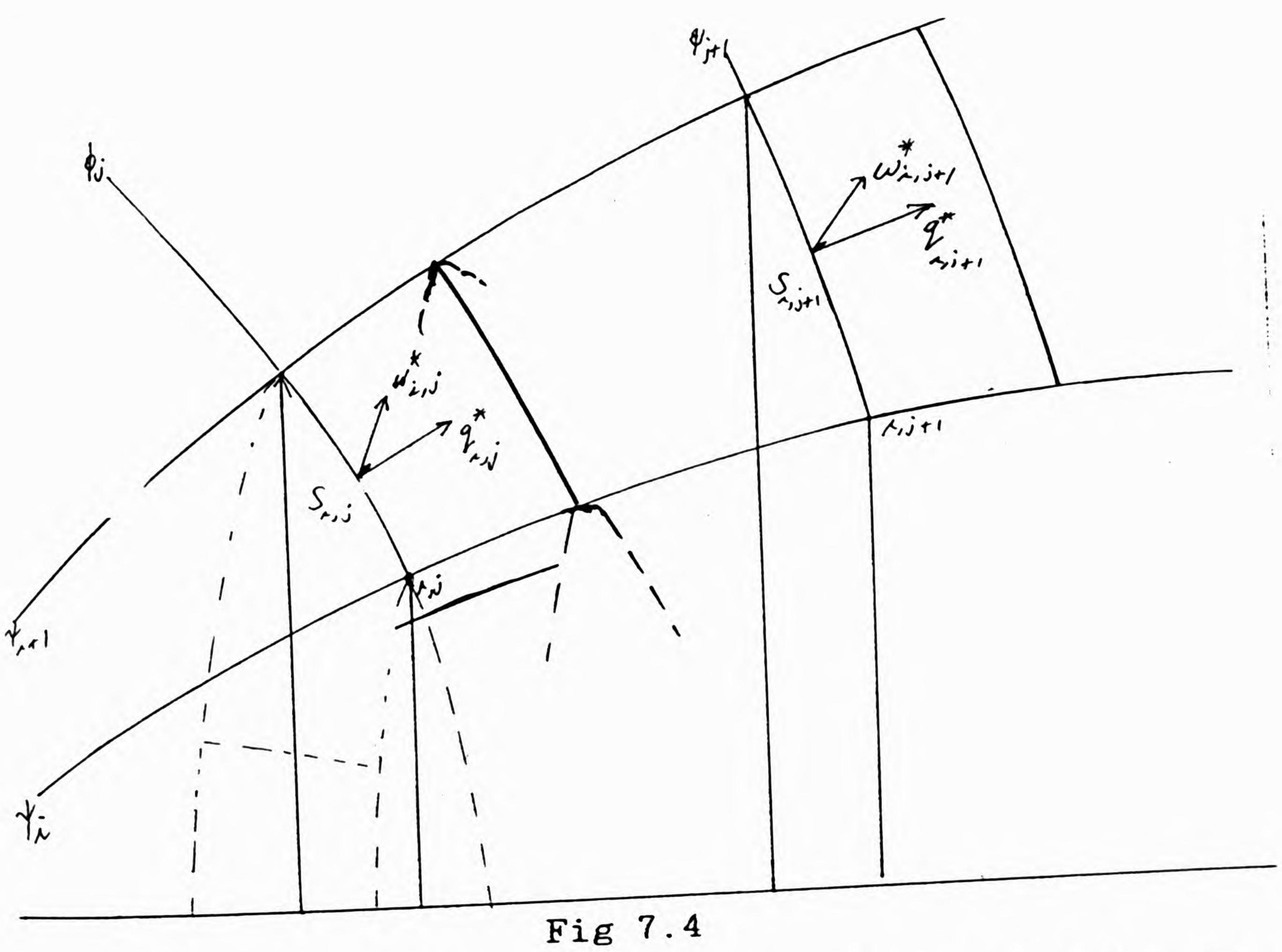


Fig 7.4

The cylindrical element with the curved surface of the frustum of a cone as its base will have a height proportional to the speed normal to this surface (i.e the mass flow through the surface is given by surface area times normal speed. Angular momentum of this thin ring is given by mass times speed in the azimuthal direction.)

Hence the volume of this element is given by

$$V^*_{i,j} = S_{i,j} . k . q^*_{i,j}$$

where k is a constant of proportionality throughout the flow and $q^*_{i,j}$ $w^*_{i,j}$ are some quantities representing the speeds, q and w, across and over the base of the element. Hence the momentum of the fluid is given by $A_{i,j} = k.S_{i,j} . q^*_{i,j} . w^*_{i,j}$

The momenta of successive elements may be calculated from either of the the ratios

(a) $A_{i,j+1}/A_{i,j}$ for $j = 1$ to $J - 1$

or (b) $A_{i,j+1}/A_{i,1}$ for $j = \emptyset$ to $J - 1$

where in (a) successive values are compared for all i,j and in (b) all momenta are compared with the 'exact' inlet value, $A_{i,1}$, which is calculated from the algebraic expressions for the axial and swirl velocities.

Calculation of Angular and Axial Momentum at Inlet.

Consider the thin ring inner radius y with thickness dy.

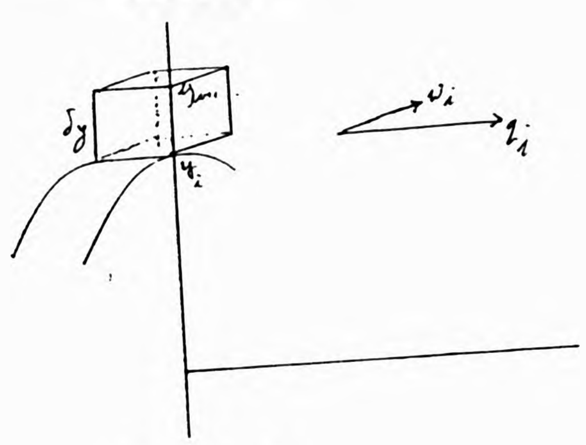


Fig.7.5

The mass flow through the surface of this disk is dm where

$$dm = 2.\pi.y.dy.q = 2.\pi.y.(a + b.y + c.y^2).dy$$

If the swirl speed is w then the angular momentum is given by

$$dA = 2.\pi.y.q.w.dy$$

$$\Rightarrow \quad A_{a,b} = 2.\pi. \int_{ya}^{yb} w.q.y.dy$$

In the present case the inlet axial and swirl speeds are of the form

$$q = a + b.y + c.y^2 \ ; \ w = e.y + f/y.$$

Hence

$$A_{a,b} = 2.\pi. \int (a + b.y + c.y^2).(e.y + f/y).y.dy$$

$$= [a_1.y + a_2.y^2 + a_3.y^3 + a_4.y^4 + a_5.y^5]_{y=ya}^{y=yb}$$

where a_1 =a.f; a_2 =b.f/2; a_3 =(a.e + f.c)/3; a_4 = a.b/4; a_5 = e.c/5:

If y_a and y_b are taken as successive, inlet radii in the discrete form of the p.d.e then the expression gives the momentum of the annular ring and this value can be compared with the values of the A.M. calculated from the converged solution downstream of inlet. The total angular momentum can be calculated by taking $y_a = y_{inner}$ and $y_b = y_{outer}$ radii respectively. Similarly the exact mass flow for the whole annulus at inlet is given by

$$M_{a,b} = 2.\pi.[b_1.y^2 + b_2.y^3 + b_3.y^4]_{y=ya}^{y=yb} \ ; \ b_1 = a/2 \ ; \ b_2 = b/3 \ ; \ b_3 = c/4$$

Thus $M_{a,b}$ and $A_{a,b}$ can be used to evaluate the accuracy of the linear and angular momenta at any station of the flow.

Hence the flow around the surface at some station, j, between the two stream surfaces at i and i+1 is constant for all j.

$$\text{Thus} \int_{i}^{i+1} w.ds = S_i.W^*_j = A_i \text{ for all j.} \qquad [7.37]$$

$$\text{and} \quad A.M = \int_{i,j}^{i+1,j} w.dV = w^*_{i,j}.V^*_{i,j} \quad \text{where } V^*_{i,j} = S_{i,j}.q^*_{i,j}.$$

Choosing $w^*_{i,j} = (w_{i,j} + w_{i,j})/2$ then [7.37] provides an estimate of the accuracy of the swirling flow. Since the swirl velocity is prescribed algebraically at inlet the quantities A_i may be calculated exactly. In particular, the total angular momentum at inlet may be compared with that at outlet. (the flow at the inlet station should ideally be parallel although in a numerical calculation this will only be the case if a sufficiently long inlet section exists to maintain a parallel flow regime).

Calculation of A_i At inlet $w = e.y + f.y$

The mass flow around an annulus of inner radius y width dy is

$$dm = 2.\pi.y.dy.w = 2.\pi.y.(e.y + f/y).dy = 2.\pi.(e.y^2 + f).dy$$

Thus $m = 2.\pi.(e.y^3/3 + f.y) + K_1$

Let $m = 0$ when $y = y_1$ then $K_1 = -2.\pi.(e.y_1{}^2/3 + f.y_1) = -2.\pi.K_2$

Hence $m/(2.\pi) = e.y^3/3 + f.y - K_2$

Then the A_i representing the inlet mass flow between successive annuli are given by

$$A_i = (m_{i+1} - m_i)/(2.\pi) = (e.y^3_{i+1}/3 + f.y_{i+1}) - (e.y^3_i/3 - f.y_i)$$

these A_i are to be compared with the angular momentum calculated for each cell throughout the flow.

Numerical Solution and Results.

In order to test the consistancy and validity of the programme code, the numerical solutions derived from the programme catering for vorticity and swirl [with vorticity set equal to zero throughout the flow] were compared with those produced by earlier programmes in which vorticity was absent. Agreement between the two programmes was almost exact the basis for comparison being a plot [15x15 grid] for

a flow with zero vorticy and the current programme 11x11 grid].
Similar comparisons made for the turbulent B.L give the same level of agreement. The initial values throughout the flow field for the various parameters were

(1) 'r' ; linear interpolation throughout the Φ,Ψ domain;

(2) 'B' ; the value of the 'B' function was set equal to 1 throughout the flow corresponding to an initial irrotational state.

(3) 'q' ; initial values of q along a streamline set equal to its inlet value ; i.e $q_{i,j} = q_{i,1}$ $j > 1$, throughout the duct.

(4) 'Ω'; Initial values of the function $\Omega\theta/r^{1/2}$ are calculated from [7.27] using the interpolated values of r and the prescribed inlet values of G_ψ and H_ψ giving the vorticity distribution across the inlet Φ characteristic which varies between streamlines.

Comparisons can then made between duct shapes by

(i) Increasing the vorticity parameter by altering the inlet parabolic speed profile for zero swirl;

(ii) Keeping the vorticy parameter constant and 'winding up' the swirl;

(iii)Examining the effect of an inlet flow in which the swirl velocity rotates in opposite senses on the duct walls; i.e there is a stream surface, other than the wall boundaries, for which the swirl velocity is zero;

(iv) Effect of a concave inlet flow profile in the axial direction;

(v) Examples of accelerating flows;

(vi) Patches of constant radii and/or speed on the walls at inlet or outlet;

The programme in fact produces three converged distributions for each solution; i.e the 'r', 'q' and 'B' distributions over the flow field. The order in which these quantities are calculated within the numerical routines can sometimes be significant when considering the number of iterations required for convergence.

Various combinations were tried including using the most recently caculated values in the iteration. However it was found that this sometimes produced instability and that the 'steadiest' convergence was obtained by

(i) calculating the 'B' distribution for all i,j,

(ii) " the $r_{i,j}$ and $q_{i,j}$ 'simultanaeously' ($q_{i,j}$ not appearing explicitly in the expressions for $r_{i,j}$ except on the boundaries and therefore its influence on the value of r only being applied via the 'B' function).

The parameters which define the wall geometry for these ducts are

(1) Laminar or Turbulent B.L ; (2) Swirl 'strength'

(3) Vorticity ; (4) Upper Wall PVD.; (5) Lower Wall PVD.

By varying the parameters controlling swirl, vorticity for combinations of fixed and variable upper and lower walls their effect on the duct shape may be judged and an indication of the type of numerical experimentation and investigation that might be undertaken is given below by a selection of computer runs and results.

Case (A) Boundary conditions

(i) Upper Wall: 'Stratford' type (S-.4)velocity distribution at 40% of the critical separation value.

(ii) Lower Wall: Fixed radius equal to inlet radius.

(iii) Outlet : Parallel flow.

(iv) Inlet : Increasing vorticity of parabolic axial profile.

The increase in the deviation from uniform flow by increasing the vorticity is shown in Fig 7.6.

Effects: Increased vorticity causes

(1) Upper wall; lowered by increasing the vorticity.

(2) Lower wall; fixed.

(3) Upper Vel.; Little change for increasing vorticity.

(4) Lower Vel.; The velocity profile is raised however but remains monotonic decreasing to outlet.

(5) Duct Length; The axial length of the duct is shortened.

(6) Cross-Stream vel profile; There appears to be little change in this profile.

These comments apply to a velocity distriution calculated on the basis of an S-.4 'Stratford' distribution. It seems that by increasing the vorticity of the inlet flow (having the effect of lowering the outer wall) we can continue to 'wind up' the Stratford number to its full value on the outer wall. [See Fig 7.6]

Case (B) Boundary Conditions.

(1) Upper Wall Fixed; (2) Lower Wall S-.3;(3) Outlet Parallel flow;

(4) Increasing non-uniform parabolic profile;

Effects: Increase in vorticity causes

(1) Upper Wall: Fixed N/A;

(2) Lower wall: position of wall raised;

(3) Upper Vel Dist; Slow diffusion on fixed wall.

(4) Lower Vel Dist; Velocity profile lowered very slightly.
[See Fig. 7.7]

Note For these runs constant wall radii sections imposed automatically because of the occurance of negative radii.

Case(C) Boundary Conditions.

(1) Upper Wall S + .5; (2) Lower Wall Fixed; (3) Inlet non-uniform flow. (4) Outlet parallel flow.

Effects: Increasing vorticity causes

(1) Upper wall; Little change;

(2) Lower wall; Fixed N/A;

(3) Upper Vel. Dist.; Little or no change;

(4) Lower Vel. Dist.; Smooth slow decelleration

(5) Cross-stream Vel profile; Keeps shape with little change.

(6) Duct Length; Shortened. [See Fig 7.8]

Case (D) Boundary Conditions.

(1) Upper wall Fixed; (2) Lower wall S +.5; (3) Inlet non-uniform flow.(4) outlet parallel flow.

Effects: Increasing vorticity causes

(1) Upper wall: Fixed N/A

(2) Lower wall Lowered.

(3) Upper vel Dist.

(4) Lower Vel. Dist; Unchanged.

(5) Duct Length ; Shortened.

A conclusion that can be drawn from the above examples is that for the ranges considered the increase in vorticity caused by accentuating the inlet parabolic velocity profile shortens the duct for accelerating flows and lengthens it for deccelerating ones.

For deccelerating flows increased vorticity narrows the duct

For accelerating flows " " widens " "

Fig. 7.9 shows the effect on duct shape of increasing the inlet vorticity distribution.

TURB. B.L. EFFECT OF INCREASING INLET VORTICITY BY INCREASING NON-UNIFORMITY OF INLET FLOW MA.

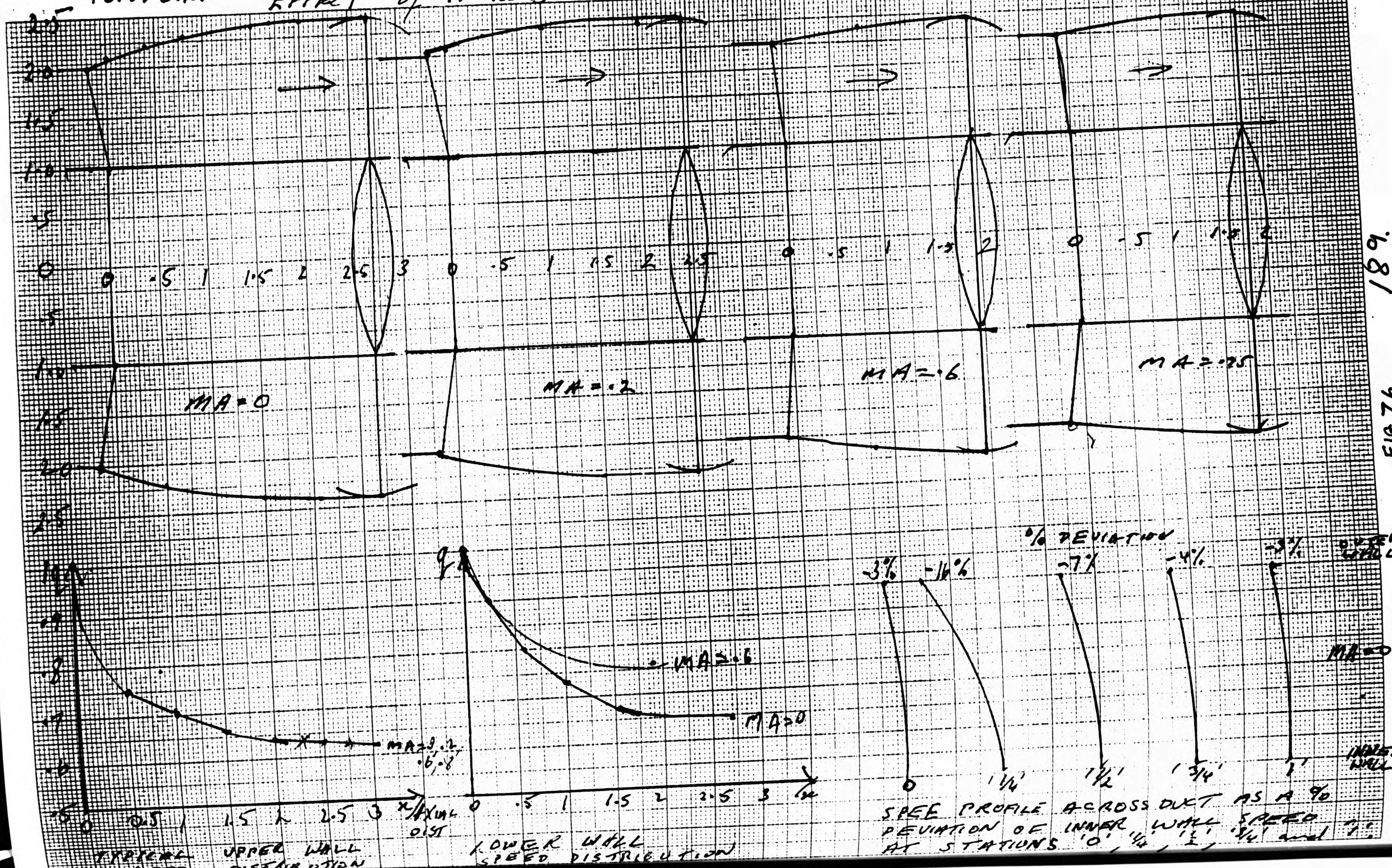


FIG. 7.6

TURB. B.L: SWIRL = 0 INLET = NON-UNI : OUTLET = PARA : UPP = FIXED : LOW = S = -.3 GRID = 11x11

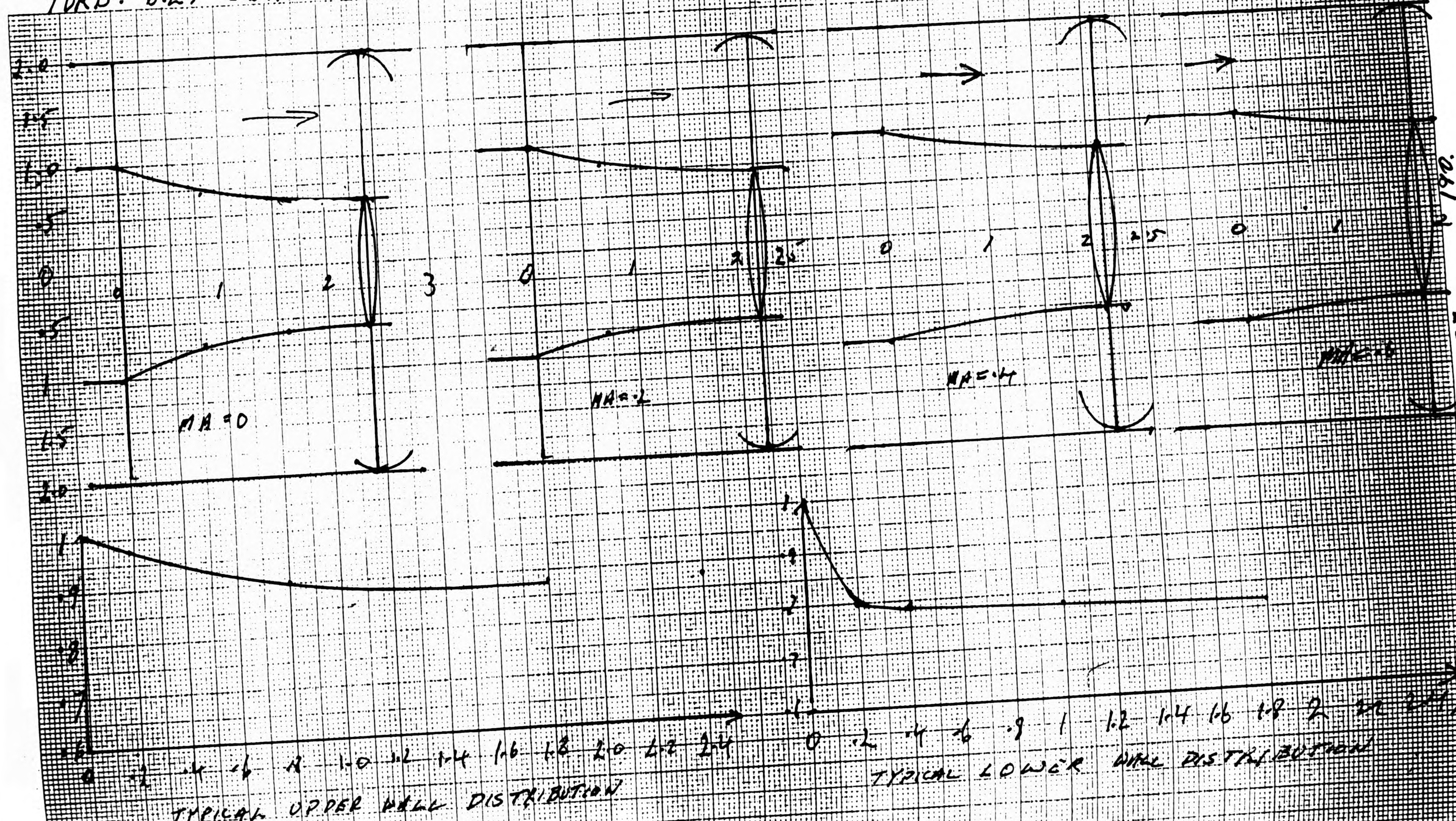
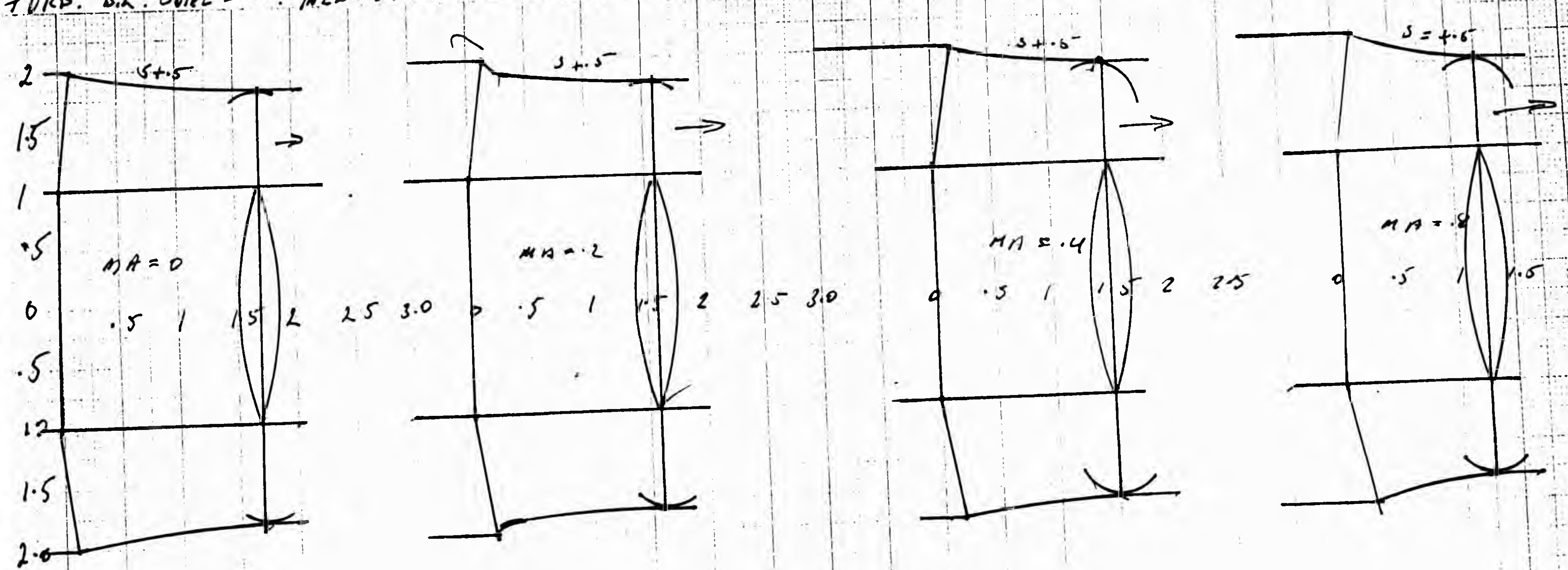
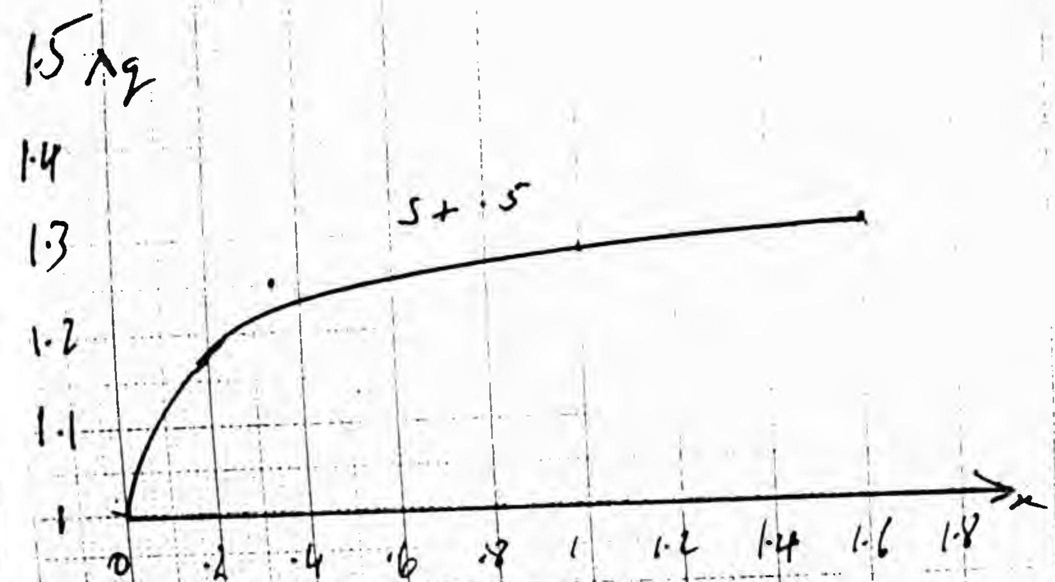


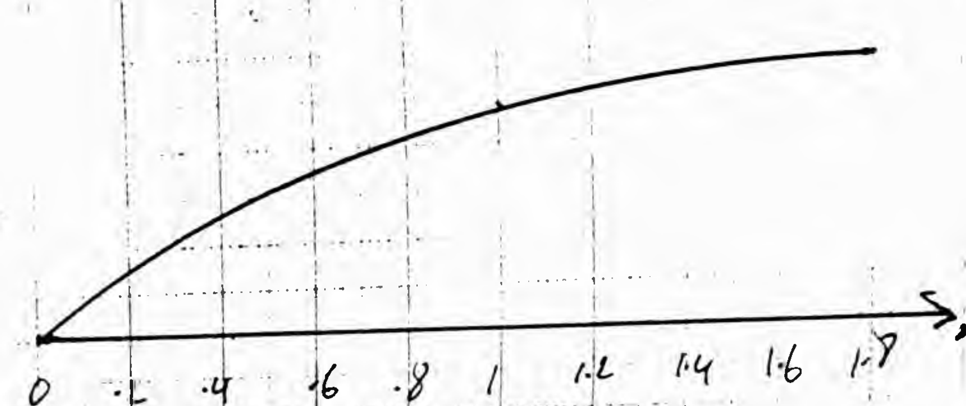
FIG 77

TURB. B.L : SWIRL = 0 : INLET = NON. UNI : OUTLET = PARA : UPP = S+.5 : LOW = FIXED : GRID 11x11



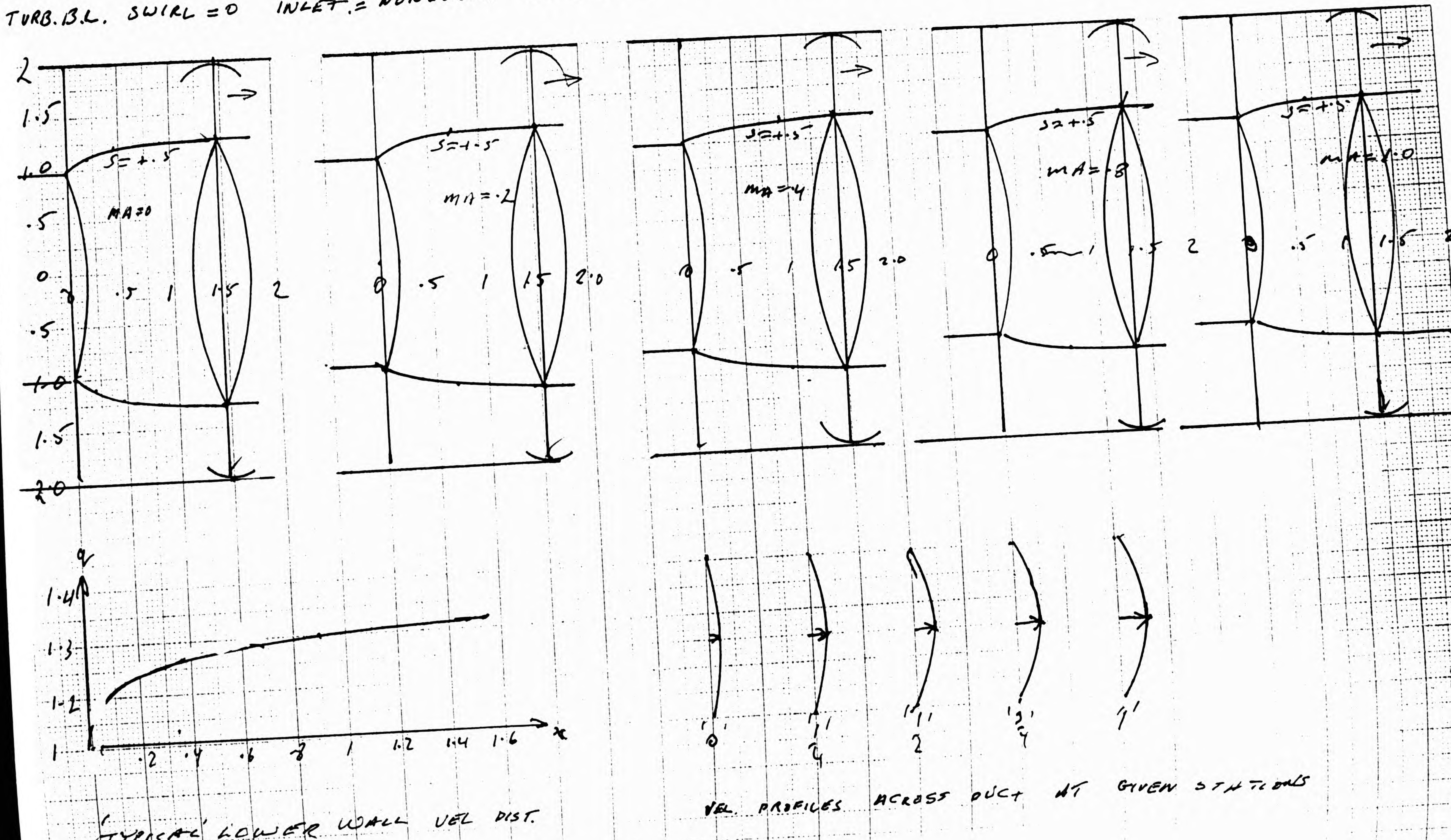


TYPICAL UPPER WALL DISTRIBUTION



FIXED LOWER WALL DISTRIBUTION.

TURB. B.L. SWIRL = 0 INLET = NON-UNI. OUTLET = PARA. UPPER = FIXED LOWER = S+.5 GRID 11x11



192. FIG. 7.9

LAM. B.L INLET = NON-UNI : OUTLET = PARA : UPPER = S-1 : LOWER = S-1 : SWIRL = 0 GRID 11x11

MA ≡ INLET NON-UNI. PARAMETER

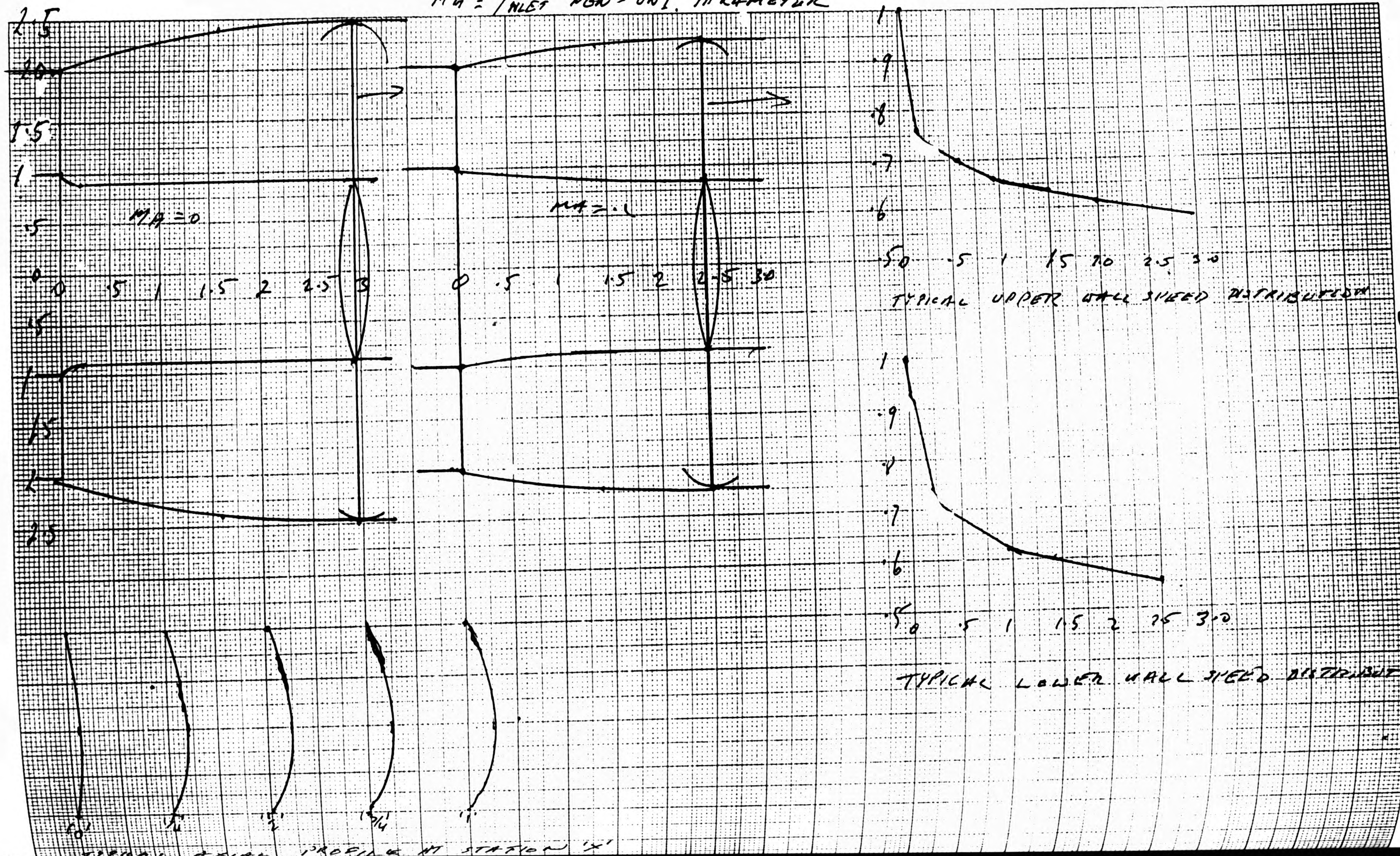


FIG. 7.10

The results obtained for flow with vorticity may be compared with the shapes of those ducts in Chapter 6.

Since the PVD control the values of the wall radii only indirectly a recurring numerical problem is the occurance of negative values of the wall radii on the lower wall. In this event the B.Cs were relaxed and replaced by either

(i) a constant wall radius condition or

(ii) a constant wall speed condition or

(iii) a 'winding down' of the parameter controling the PVD. This alteration will generally speaking be accompanied by a reduction in the ammount of diffusion occuring at this point but may be 'wound up' to an optimum value by the numerical routine if conditions allow. Of the three possibilities (iii) would be the most 'natural' variation to apply given that the B.Cs are essentially velocity based whilst application of (ii) will relax the control over the behaviour of the B.L, in (i) all control is lost.

Chapter 8

In this chapter the effect of compressibility is allowed for in the design scheme to investigate its influence on the flow behaviour and any consequential change in the duct geometry. The mathematical treatment allows the prescription of arbitrary stagnation conditions for isentropic flow of a gas. Numerical results are obtained and compared with the incompressible case. The equations of motion for an axisymmetric compressible flow with a non zero vorticity vector are

$$u.u_x + v.u_y = (-1/\rho).p_x \qquad [8.1]$$

$$u.v_x + v.v_y - w^2/y = (-1/\rho).p_y \qquad [8.2]$$

$$u.w_x + v.w_y + v.w/y = 0 \qquad [8.3]$$

$$(\rho.y.u)_x + (\rho.y.v)_y = 0 \qquad [8.4]$$

$$\underline{\Omega}^* = [\ (1/y).(y.w)_y\ ].\underline{x} + [\ -w_x\ ].\underline{y} + [\ v_x - u_y\ ].\hat{\underline{\theta}} \qquad [8.5]$$

As in the incompressible case the set of design plane equations is

(a)	(b)	
$[\ \ln(A)\ ]_\Phi = \epsilon^*.B/q^2$	$\epsilon^* = u_x + v_y$	[8.6]
$[\ \ln(B)\ ]_\Psi = -\Omega_\theta.A/q^2$	$\Omega_\theta = v_x - u_y$	[8.7]
$x_\Phi = [\ B/A]y_\Psi$	$\Omega_x = -w_x$	[8.8]
$x_\Psi = [-A/B].y_\Phi$	$\Omega_y = (1/y).(y.w)_y$	[8.9]
$[\ (A/B).y_\Phi\ ]_\Phi + [\ (B/A).y_\Psi\ ]_\Psi$	$= 0$	[8.10]
$[\ y^2_\Psi\ /A^2\ ] + [\ y^2_\Phi\ /B^2\ ] = 1/q^2$		[8.11]

where $q^2 = u^2 + v^2$ and $Q^2 = q^2 + w^2 = u^2 + v^2 + w^2$ [8.12]

<u>Evaluation of the 'A' Function</u> From the continuity equation [8.4], we can derive an expression for ϵ^* to be substituted into [8.6(a)] and hence evaluate the function A.

Thus $u.(\rho y)_x + v.(\rho y)_y + (\rho.y).(u_x + v_y) = 0$

$$\epsilon^* = u_x + v_y = -[u.(\rho.y)_x + v.(\rho.y)_y]/(\rho.y)$$

$$= -[u.(\ln(\rho.y))_x + v.(\ln(\rho.y))_y]$$

$$= -[q.\mathrm{Cos}\theta.(\ln(\rho.y))_x + q.\mathrm{Sin}\theta.(\ln(\rho.y))_y]$$

$$= -q.[(\ln(\rho.y))_x.x_s + (\ln(\rho.y))_y.y_s]$$

$$\epsilon^* = -q.[\ln(\rho y)]_s$$

But by definition of Φ we have for any F, $(F)_s = (q/B).(F)_\Phi$

Hence $\epsilon^* = -q.[\ln(\rho.y)]_\Phi.q/B = -[\ln(\rho.y)]_\Phi.q^2/B$

Substituting into [8.6(a)] gives

$$[\ln(A)]_\Phi = -[\ln(\rho.y)]_\Phi.(q^2/B).(B/q^2) = -[\ln(\rho y)]_\Phi$$

$\Rightarrow [\ln(A.y.\rho)]_\Phi = 0 \Rightarrow A.y. = g_1(\Upsilon) \Rightarrow A = g_1(\Upsilon)/(\rho.y)$

Since $g_1(\Upsilon)$ is arbitrary let $g_1(\Upsilon) = 1 \Rightarrow A = 1/(\rho.y)$.

From equation [1.11.7]

$\Upsilon_x = -v/A$; $\Upsilon_y = u/A \Rightarrow \Upsilon_x = -\rho.y.v$; $\Upsilon_y = \rho.y.u$

Substituting for A into [8.7(a)], [8.10], [8.11] gives

$$[\ln(B)]_\Psi = -\Omega_0/(\rho.y.q^2) \qquad [8.7(a).1]$$

$$[\rho.B.y.y_\Psi]_\Psi + [(1/\rho B).(1/y).y_\Phi]_\Phi = 0 \qquad [8.10.1]$$

$$[\rho.y.y_\Psi)^2 + (y_\Phi)^2/B^2] = 1/q^2 \qquad [8.11.1]$$

From [8.3] we may deduce that $(y.w)_s = 0 \Rightarrow (B/q).(y.w)_\Phi = 0$

Hence (y.w) is a function of Υ only and (as in the incompressible case) we can write $y.w = C(\Upsilon) \Rightarrow w = C(\Upsilon)/y$ [8.13]

In a manner similar to the former, expressions for the x and y components of vorticity may be derived in terms of the arbitrary function $C(\Upsilon)$.

Thus

$$\Omega_x = (1/y).(y.w)_y = (1/y).(C(\Upsilon))_y = (1/y).[C_\Psi.\Upsilon_y + C_\Phi.\Phi_y] =$$

$$= (1/y).C_\Psi.\Upsilon_y = \rho\,u.C_\Psi \qquad [8.14]$$

and

$$\Omega_y = -w_x = -[(1/y).C(\Upsilon)]_x = -(1/y)[C(\Upsilon)]_x = -(1/y).[C_\Psi.\Upsilon_x + C_\Phi.\Phi_x] =$$

$$= -(1/y).\Upsilon_x.C_\Psi = -(-\rho.v).C_\Psi = \rho.v.C_\Psi \qquad [8.15]$$

In order to proceed further with the solution set, it is necessary to obtain an expression for the θ component of vorticity in [8.7(a).1]. This is done by considering equation [8.2]

$$u.v_x + v.v_y - w^2/y = -(1/\rho).p_y \Rightarrow$$

$$u.u_y + v.v_y + w.w_y + u.v_x - u.u_y - w.w_y - w^2/y = -(1/\rho).p_y \Rightarrow$$

$$(1/2)[u^2 + v^2 + w^2]_y + u.(v_x - u_y) = -(1/\rho).p_y + w.w_y + w^2/y$$

$$= -(1/\rho).p_y + (C/y).(C/y)_y + C^2/y^3$$

$$= -(1/\rho).p_y + (C/y).(-C/y^2 + C_y/y) + C^2/y^3$$

$$= -(1/\rho).p_y + C.C_y/y^2 \Rightarrow$$

$$(1/2).[Q^2]_y + u.\Omega_\theta = -(1/\rho).p_y + C.C_y/y^2 \Rightarrow$$

$$u.\Omega_\theta = -[(1/2).(Q^2)_y + (1/\rho)p_y] + C.C_y/y^2 \qquad [8.16(a)]$$

By considering the equations of isentropic flow of a gas, expressions may be derived for the θ component of vorticity in terms of the radial coordinate y and quantities defined in the upstream region (where the flow is known).

Pressure, Density, Temperature and Speed Relations.

Consider a particle of the fluid, with the speed, pressure, density and temperature at the point X_i denoted by Q_i, p_i, ρ_i and T_i. If K_i is a quantity depending on the value of the entropy, S, at X_i, then

$$p_i . \rho_i^{-\gamma} = K_i (S) \qquad [8.16.1]$$

Let the total energy (enthalpy) of the gas at X_i be denoted by H_i, then

$$H_i = cp . T_i + \tfrac{1}{2} . Q_i^2 \qquad [8.16.2]$$

Also $\quad p/\rho = R.T \qquad [8.16.3]$

where $\quad R = cp - cv \quad ; \gamma = cp/cv \qquad [8.16.4,5]$

and the speed of sound is given by

$$c^2 = dp/d\rho \qquad [8.16.6]$$

Suppose that a fluid element, isolated from the surrounding medium, is brought adiabatically to rest.

Then $Q_i = 0$, and by definition the stagnation values of p, ρ, T at the point X_i (denoted by the subscripts 'i,o')

satisfy $\quad p_{i,o} . \rho_{i,o}^{-\gamma} = K_i \qquad [8.16.7]$

and

$$H_{i,o} = cp . T_{i,o} \qquad [8.16.8]$$

Since, by definition the enthalpy is unchanged by this process then

$$H_i = H_{i,o} \quad ; \quad Q_{i,o} = 0 \qquad [8.16.8a]$$

Hence from [8.16.2] and [8.16.7]

$$H_{i,o} = cp . T_i + \tfrac{1}{2} . Q_i^2 = cp . T_{i,o} \qquad [8.16.9]$$

and $\quad p_i . \rho_i^{-\gamma} = p_{i,o} . \rho_{i,o}^{-\gamma} \qquad [8.16.10]$

From [8.16.3]

$$T_i = p_i / R.\rho_i \qquad T_{i,o} = p_{i,o} / R.\rho_{i,o} \qquad [8.16.11]$$

substituting from [8.16.11] into [8.16.9] gives

$$cp.p_i / R.\rho_i + ½.Q^2_i = cp.p_{i,o} / R.\rho_{i,o}$$

substituting from [8.16.10] $\quad p_i = p_{i,o}\rho^{\gamma}_i.\rho^{-\gamma}_{i,o} \quad$ gives

$$cp.p_{i,o}.\rho_i / (R.\rho_{i,o}.\rho_i) + ½.Q^2_i = cp.p_{i,o} / (R.\rho_{i,o})$$

which gives (after some rearrangement)

$$\rho_i = \rho_{i,o}[\,1 - \frac{(\gamma-1)}{2\gamma}.(\rho_{i,o}/p_{i,o}).Q^2_i\,]^{1/(\gamma-1)} \qquad [8.16.12]$$

From [8.16.6] $\quad c^2_i = \left(\frac{dp}{d\rho}\right)_i$

Hence the stagnation speed of sound is $c^2_{i,o} = \gamma.p_{i,o}/\rho_{i,o}$

Then [8.16.12] becomes

$$(\rho_i/\rho_{i,o}) = [\,1 - ½.(\gamma-1).Q^2_i/c^2_{i,o}\,]^{1/(\gamma-1)} \qquad [8.16.13]$$

$$(p_i/p_{i,o}) = [\,1 - ½.(\gamma-1).Q^2_i/c^2_{i,o}\,]^{\gamma/(\gamma-1)} \qquad [8.16.14]$$

$$(T_i/T_{i,o}) = [\,1 - ½.(\gamma-1).Q^2_i/c^2_{i,o}\,] \qquad [8.16.15]$$

since $(p_i/p_{i,o}) = (\rho_i/\rho_{i,o})^{\gamma}$ and $(T_i/T_{i,o}) = (\rho_i/\rho_{i,o})^{(\gamma-1)}$.

Also $H_i = H_{i,o} = cp.T_{i,o} = cp.p_{i,o}/(R.\rho_{i,o}) = cp.c^2_{i,o}/(R\gamma) =$

$$= (cp/(cp-cv)\gamma).c^2_{i,o} = (\gamma/(\gamma-1).\gamma).c^2_{i,o} = c^2_{i,o}/(\gamma-1)$$

From [8.16.2] and [8.16.8a]

$$H_i = H_{i,o} = cp.T_i + ½.Q^2_i =$$

$$= (cp/R).(p_i/\rho_i) + ½.Q^2_i$$

$$= (c_p/(c_p - c_v)).(p_i/\rho_i) + \tfrac{1}{2}.Q_i^2$$

$$H_i = (\gamma/(\gamma-1)).(p_i/\rho_i) + \tfrac{1}{2}.Q_i^2$$

If the flow is isentropic then the quantities K_i are the same at all points along a given stream line and equations [8.16.13/14/15] now define the relationship of p, ρ, T, Q along the i^{th} streamline rather than a point.

$$(\gamma/(\gamma-1)).(p/\rho) + (1/2).Q^2 = c_{i,o}^2/(\gamma-1) \qquad [8.17a]$$

Since the stagnation speed of sound may vary between streamlines we may write

$$c_{i,o}^2/(\gamma-1) = H(\Upsilon) = (\gamma/(\gamma-1)).(p/\rho) + (1/2).Q^2 \qquad [8.17b]$$

The evaluation of the expression on the RHS of [8.16a] is obtained by differentiating [8.17b] with respect to y, thus

$$H_y = (\gamma/(\gamma-1)).(p/\rho)_y + (1/2).(Q^2)_y$$

Now $(p/\rho)_y = (k.\rho^{\gamma-1})_y = k.(\gamma-1).\rho^{\gamma-2}.\rho_y = k(\gamma-1).\rho^{\gamma-2}.p_y/(k.\gamma.\rho^{\gamma-1})$

$$= [(\gamma-1)/\gamma].(1/\rho).p_y \quad ; \text{(since } p_y = k.\gamma.\rho^{\gamma-1}.\rho_y)$$

$$\Rightarrow \quad H_y = (\gamma/(\gamma-1)).((\gamma-1)/\gamma).(1/\rho).p_y + (1/2).(Q^2)_y$$

$$\Rightarrow \quad H_y = (1/\rho).p_y + (1/2).(Q^2)_y$$

Substituting for H_y into [8.16a] gives

$$u.\Omega_o = - H_y + C.C_y/y^2$$

Now $H_y = H_\Psi.\Upsilon_y + H_\Phi.\Phi_y = H_\Psi.\rho.y.u + \emptyset$ and $C_y = C_\Psi.\Upsilon_y = C_\Psi.\rho.y.u$

$$\Rightarrow \quad u.\Omega_o = - \rho.y.u.H_\Psi + \rho.y.u.C.C_\Psi/y^2$$

$$\Rightarrow \quad \frac{\Omega_o}{(\rho.y)} = \frac{1}{2.y^2}.(C^2)_\Psi - H_\Psi \qquad [8.18]$$

This expression combining density and vorticity is that required in equation [8.7(a).1]. Since H and C are functions of Ψ only they may be prescribed upstream of the transition region in the cylindrical flow regime.

The density speed/pressure/temperature relations are given by equations [8.16.13/14/15] although the absolute density and pressure throughout the flow will not be uniquely determined until some base pressure is specified.

This completes the solution sets (1) and (2) listed below

<u>Set (1)</u>

$$[\rho.B.y.y_{\psi}]_{\psi} + [(1/\rho.B).(\ln y)_{\Phi}]_{\Phi} = 0 \qquad [8.10.1]$$

$$[\rho.y.y_{\psi}]^2 + [y_{\Phi}/B]^2 = 1/q^2 \qquad [8.11.1]$$

$$[\ln(B)]_{\psi} = -(\Omega_0/\rho.y).(1/q^2) \qquad [8.7(a).1]$$

$$(\Omega_0/\rho.y) = [C^2/2]_{\psi}/(y^2) - H_{\psi} \qquad [8.18]$$

$$[\ln B]_{\psi} = (H_1 - C_1/y^2)/q^2 \qquad [8.18a]$$

<u>Set (2)</u>

$$\rho/\rho_0 = [1 - \{(\gamma-1)/2\}.(Q/c_0)^2]^{1/(\gamma-1)} \qquad [8.19]$$

$$y.w = C(\Psi) \qquad [8.13]$$

$$[\gamma/(\gamma-1)].(p/\rho) + (1/2).Q^2 = c_0^2/(\gamma-1) = H(\Psi) \qquad [8.17b]$$

where $Q^2 = q^2 + w^2 = u^2 + v^2 + w^2$; $c^2 = \gamma p/\rho$

Letting $C_1 = [C^2/2]_{\psi}$; $H_1 = H_{\psi}$ and eliminating Ω_0 from [8.18] and [8.7(a).1] then

$$\Omega_0/(\rho.y) = C_1/y^2 - H_1 \quad \Rightarrow$$

$$[\ln(B)]_{\psi} = [H_1 - C_1/y^2]/q^2 \qquad [8.18a]$$

Using the transform of Chapter 3 to map onto the unit square gives

$y^2 = r = c_1 . r_1$; $x = c_2 . x_1$; $q = c_3 . q_1$;

$Q = c_3 / Q_1$; $w = c_3 / w_1$; $v = c_3 / v_1$; $u = c_3 / u_1$;

$\Psi_1 = (\Psi - c_4)/c_5$; $\Phi_1 = (\Phi - c_6)/c_7$

$c_1 = (c_5/c_7)^2$; $c_2 = (c_5/c_7)/2$; $c_3 = 2.(c_7{}^2/c_5)$.

Applying this transform to [8.10.1], [8.11.1] and [8.18a] we have

$$[\rho.B.r_{\Psi}]_{\Psi} + [\,(1/\rho.B).(\ln r)_{\Phi}]_{\Phi} = 0 \qquad [8.20]$$

$$\rho^2.[\,(r_{\Psi})^2 + (1/r).(r_{\Phi}/\rho.B)^2\,] = q^2 \qquad [8.21]$$

$$[\,\ln B_{\Psi}\,] = [\,H_1 - G_1/r\,].q^2 \qquad [8.22]$$

where the subscripts have been ommitted and all variables and constants are quantities in the transformed plane. The value of the density, ρ, and all other quantities required for is determination are calculated from the transformed equivalent of equation set (2). Suitable specification of conditions on the physical boundaries together with some choice of the stagnation quantities of the flow will enable us to use the numerical equivalents of [8.20/21/22] to calculate the required duct geometry and flow patterns.

The range of Ψ and Φ may be normalized in the transformed design plane by choosing c_4, c_6 as the minima and c_5, c_7 as the ranges of Φ and Ψ.

Thus $0 =< \Psi =< 1$; $0 =< \Phi =< 1$.

Boundary Conditions

From Crocco's equation $\nabla H = T\nabla S + \underline{V} \wedge \underline{\Omega}$ [8.22.1]

Far upstream, in the region of cylidrical, axisymmetric flow

$$\frac{\partial}{\partial\theta} \equiv 0 \quad ; \quad \frac{\partial}{\partial x} \equiv 0 \quad ; \quad v = 0 \qquad [8.22.2]$$

From equations [8.7/8/9] the prescription of the axial and circumferential swirl velocity profiles will necessarily define the vorticity vector in the (y,θ) plane

$$\underline{\Omega} = (0).\underline{\hat{x}} + ((1/y).(y.w)_y).\underline{\hat{y}} + (- u_y). \underline{\hat{\theta}}$$

Further, taking the 'dot' product of Crooco's equation with $\underline{V}$ gives

$$\underline{V}.\nabla H = T\,\underline{V}.\nabla S + \underline{V}.(\underline{V} \wedge \underline{\Omega}) = T\,\underline{V}.\nabla S \quad \text{[Vector Identity]}$$

Hence $\underline{V}.\nabla H + T\ \underline{V}.\nabla S = 0$

Thus if H is defined such that $\underline{V}.\nabla H = 0$,
i.e the total energy of a particle flowing along a streamline is constant, it necessarily follows that $\underline{V}.\nabla S = 0$
i.e the rate of change of entropy inthe direction of the flow is zero and constant along a streamline giving isentropic flow.
Similarly isentropic flow implies that the total energy H of a particle is constant along a stream line.
The conditions on the physical boundaries in compressible flow may be chosen from the same range available in the incompressible case.

Thus

(1) Inlet: An invariant distribution of the radial coordinate r ($=y^2$) based on a non-uniform inlet speed profile having a parabolic variation across the duct together with a swirl speed distribution of the form $w = a.y + b/y$.

(2) Outlet: Parallel flow condition across the duct $r_{\Phi} = Ø$.

(3) Upper Wall: Prescribed velocity distributions based on 'mixed' B.C or altrernative acccelerating flows.

(4) Inner Wall: As for (3). Condition (3) and (4) may be applied piecewise in conjunction with constant velocity and/or radius distributions if desired.

In the case of compressible flow some choice of the stagnation quantities c_o, p_o, ρ_o must be made in order to specify the density uniquely in [8.19]. If the flow were potential then the stagnation speed of sound would be constant throughout the medium, however in general its value, c_o, may vary for every stream line. Arbitrary stagnation conditions may be prescribed by expressing the stagnation quantities on each stream line as a function of the corresponding values at some station (inlet say). Having established this set of stagnation conditions we can express the variables of state p, ρ, T and c in terms of the local speed of the medium. For an isentropic flow of a gas with constant specific heats we have

$$[\gamma/(\gamma-1)].p/\rho + (1/2).Q^2 = k \qquad [8.23]$$

$$c^2 = \gamma.p/\rho \qquad [8.24]$$

$$p = K.\rho^{\gamma} \qquad [8.25]$$

$$p/ \; = R.T \quad ; \quad T = k^*.c^2 \qquad [8.26]$$

Stagnation Conditions

The stagnation conditions on the i^{th} stream line being denoted by the subscripts 'i,o we have

$$[\gamma/(\gamma-1)].p_{i,o}/\rho_{i,o} = k_i \qquad =>$$

$$[\gamma/(\gamma-1)].p/\rho + (1/2)Q^2 = [\gamma/(\gamma-1)].p_{i,o}/\rho_{i,o} \qquad =>$$

$$c^2/(\gamma-1) + (1/2).Q^2 = c^2_{i,o}/(\gamma-1) \qquad . \qquad [8.23a]$$

Critical Values

Suppose that at some station 'j' on the streamline 'i' the speed of the gas becomes equal to the local speed of sound, then from [8.23a]

$$Q_{i,j} = c_{i,j} = c^*_{i,j}$$

$$c^{*2}_{i,j}/(\gamma-1) + (1/2).c^{*2}_{i,j} = c^2_{i,o}/(\gamma-1) \qquad =>$$

$c^*_{i,j} = (2/\gamma+1)^{½}.c_{i,o} = c^*_i$; (since the critical sound speed is constant for a given streamline.)

For $\gamma = 1.4$, $c^*_i = Ø.9128.c_{i,o}$

It follows that for the flow to remain subsonic on a given streamline

$$Q_{i,j} < c^*_i = (2/\gamma+1)^{1/2}.c_{i,o} = Ø.9128.c_{i,o}$$

The other critical values of the variables of state may be obtained from [8.23] to [8.26]. In order to ensure subsonic flow we may choose a constant k_4 ,say, such that $Q_{i,j} =< k_4.c_{i,Ø}$ for all $Q_{i,j}$.

Density Speed Relation

Eliminating 'p' [8.23a] and [8.25] we have

$$[\gamma/(\gamma-1)].K.\rho^{\gamma-1}_{i,j} = [c^2_{i,o}/(\gamma-1) - (1/2).Q^2_{i,j}] \qquad =>$$

$$\rho_{i,j} = (1/\gamma K)^{[1/(\gamma-1)]}.[\, c^2_{i,o} - \{(\gamma-1)/2\}.Q^2_{i,j}\,]^{[1/(\gamma-1)]}$$

Choosing some particular density $\rho_{a,b}$ (say) we have

$$\rho_{a,b} = \left\{(1/\gamma K).[c^2_{a,o} - \{(\gamma-1)/2\}.Q^2_{a,b}\right\}^{1/(\gamma-1)}$$

Refering all densities to this arbitrary density $\rho_{a,b}$ we have

$$\rho_{i,j}/\rho_{a,b} = \{[c^2_{i,o} - (\gamma-1)/2.Q^2_{i,j}\,)]/[c^2_{a,o} - (\gamma-1)/2.Q^2_{a,b}\,]\}^{1/(\gamma-1)} \qquad [8.29a]$$

If $\rho_{a,b}$ is chosen as the stagnation density on stream line 'a' then

$Q_{a,b} = \emptyset$ and $\rho_{a,b} = \rho_{a,o}$ =>

$$\rho_{i,j}/\rho_{a,o} = [(c_{i,o}/c_{a,o})^2 - \{(\gamma-1)/(2.c^2_{a,o})\}.Q^2_{i,j}\,]^{1/(\gamma-1)} \qquad [8.29b]$$

If, in addition, the stagnation speed of sound is the same for each stream line then $\rho_{a,o} = \rho_o$ and $c_{a,o} = c_o$ (say) => $(c_{i,o}/c_{a,o})^2 = 1$ =>

$$\rho_{i,j} = \rho_o.[\, 1 - \{(\gamma-1)/2\}.Q^2_{i,j}/c^2_o\,]^{1/(\gamma-1)} \qquad [8.29c]$$

If [8.29c] applies then we have adiabatic isentropic flow and the density/speed relationship is identical for each stream line throughout the flow. If, on the other hand, stagnation conditions vary across the flow then the more general relationship [8.29a] holds. In [8.29a], let i=1, j=1; i.e the reference density is that at the inner inlet point of the transition region then

$$\rho_{1,1} = (1/\gamma K)^{[1/(\gamma-1)]}.[c^2_{1,o} - \{(\gamma-1)/2\}.Q^2_{1,1}\,]^{[1/(\gamma-1)]}$$

But $c^2_{1,1} = c^2_{1,o} - [(\gamma-1)/2].Q^2_{1,1}$ => $\rho_{1,1} = (c^2_{1,1}/\gamma K)^{1/(\gamma-1)}$

Refering all densities to $\rho_{1,1}$ and setting $\rho_{1,1} = 1$ we have

$\rho_{1,1}$ => $c^2_{1,1}/\gamma K = 1$; $K = c^2_{1,1}/\gamma$ =>

$$\rho_{i,j}/\rho_{1,1} = \rho_{i,j} = [\,(c_{i,o}/c_{1,1})^2 - [(\gamma-1)/(2c^2_{1,1}\,).Q^2_{i,j}\,]^{1/(\gamma-1)} \qquad [8.29]$$

Thus [8.29] is the numerical form of the speed-density [8.19a]

$$\rho_{i,j} = [\, P_i - P_o . Q^2_{i,j} \,]^{1/(\gamma-1)}$$

where $P_i = (c_{i,o} / c_{1,1})^2$ and $P_o = (\gamma-1)/(2.c^2_{1,1})$

where the P_i may in general be different for each streamline.

The choice of the P_i is arbitrary but some rationale is necessary in order to produce a feasible flow regime in the transition region. The choice of P_i across the duct will implicit ly define the density variation in the transition region and also the inlet conditions (e.g Mach number) across the duct. These in turn will define the upstream values of density, pressure and temperature. Similarly a choice of distribution of ρ, p or T in some region of the flow will imply the distribution for P_i.

<u>Choice of P_i.</u>

Suppose that on the i[th] streamline we wish to impose a density variation of the order of D_i across the duct at inlet. Then

$$\rho_{i,j} = \rho_{1,1} + D_i = 1 + D_i \quad (\text{since } \rho_{1,1} = 1)$$

$$= [\, P_i - P_o . Q^2_{i,j} \,]^{1/(\gamma-1)}$$

If the desired density variation is now chosen then the P_i are defined since $Q_{i,j}$ is prescribed at inlet. Thus let D_i be set equal to some % variation of the inlet density

$$\therefore \quad P_i = (1 + D_i)^{(\gamma-1)} + P_o . Q_{i,j} \qquad [8.29d]$$

The value of P_o depends upon the value of the inlet Mach number on the hub.

Thus let $M_1 = Q_{1,1} / c_{1,1}$; Hence $P_o = (\gamma-1).M^2_1/(2.Q^2_{1,1})$ is known since $Q_{1,1}$ is prescribed at inlet.

Since all P_i and P_o are now defined the density-speed relation for each of the streamlines is specified ,

i.e $\rho_{i,j} = [\, P_i - P_o . Q^2_{i,j} \,]^{1/(\gamma-1)}$.

In summary (i) The inner inlet Mach number, M_1 on the hub is chosen which defines $P_o = (\gamma-1).M^2_1/(2.Q^2_{1,1})$ since all $Q_{i,1}$ are prescribed.

(ii) Some % variation (D_i) of density, ρ, are chosen which defines the P_i's since $Q_{i,1}$ are prescribed at inlet station.

Since the P_i are now defined, the corresponding value of the Mach numbers at inlet may be calculated from the relation

$P_i = (c_{i,o} / c_{1,1})^2$

Since $c^2_{i,j} + A.Q^2_{i,j} = c^2_{i,o}$ where $A = (\gamma-1)/2$

for j=1 $c^2_{i,1} + A.Q^2_{i,1} = c^2_{i,o}$

=> $P_i = c^2_{i,1}/c^2_{1,1} + A.Q^2_{i,1}/c^2_{1,1}$

But the inlet Mach numbers are defined by

$M_i = Q_{i,1}/c_{i,1}$; $M_1 = Q_{1,1}/c_{1,1}$ and eliminating c's

$P_i = (M_1 Q_{i,1}/Q_{1,1})^2.[\, (1/M_i)^2 + A \,]$

which expresses P_i in terms of quantities at inlet. Solving for the inlet Mach numbers M_i gives

$M_i = M_1.[P_i.(Q_{1,1}/Q_{i,1})^2 - A.M^2_1]^{-1/2}$

Alternative methods of defining the density-speed relation can be based on choosing different distributions of other flow variables at some station of the flow.

This particular choice suffices only to establish a feasible relationship between the quantities ρ and Q at the inlet station. However the the technique does indicate a method by which other appropriate properties of the flow such as temperature, pressure or functions of them might be prescribed. Equation [8.29] shows that the density varies in a sense opposite to that of speed for a given streamline (i.e in the Φ, j direction). However this is not necessarily the case along any other vector particularly the Υ characteristic since the density is a function of the inlet Mach number for any given line and it would be possible to choose inlet conditions to change the 'sense' of the density speed relation.

Thus let

$A = (c_{i,o}/c_{1,1})^2 > 0$; $B = (\gamma-1)(2c^2_{1,1}) > 0$; $N = 1/(\gamma-1)$; $A - B.Q^2 > 0$

where B is constant but A is a function of the stagnation speed of sound on the streamlines and hence a function of the Mach numbers.

Thus $\rho = [\ A - B.Q^2\ ]^N$

$$d\rho = (\rho_A).dA + (\rho_Q).dQ$$

$$= N.[\ A - B.Q^2\ ]^{n-1}.dA + N.[\ A - B.Q^2\ ]^{N-1}(-2.B.Q).dQ$$

$$= N.[\ A - B.Q^2\ ]^{N-1}.(\ dA - 2.B.Q.dQ\ )$$

$$\frac{d\rho}{dQ} = N.[\ A - B.Q^2\ ]^{N-1}.(\ dA/dQ - 2.B.Q\ ) \qquad [8.30]$$

Now if (i) A = constant $\frac{d\rho}{dQ} = (-)(2.B.N.Q).[\ A - B.Q^2\ ]^{N-1} < 0$;

and (ii) A $\neq$ constant $\frac{d\rho}{dQ} = 2.N.Q.[\ dA/d(Q^2) - B].[\ A - B.Q^2\ ]^{N-1}$

Thus if the choice of inlet Mach number is such that $dA/d(Q^2) - B > 0$ for some region of the inlet then ρ will vary in the same sense as Q.

Finite Difference Forms

The complete set of equations giving the solution for compressible flow comprises

$$[(\rho.B).r_{\psi}]_{\psi} + [\{1/(\rho.B)\}.(\ln r)_{\Phi}]_{\Phi} = 0 \qquad [8.20]$$

$$\rho^2.[(r_{\psi})^2 + (1/r).(r_{\Phi}/\{\rho.B\})^2] = q^2 \qquad [8.21]$$

$$[\ln B]_{\psi} = [H_1 - C_1/r].q^2 \qquad [8.22]$$

$$\rho = [P_i - P_o.Q^2]^{\gamma^*} \qquad [8.23]$$

where $\gamma^* = 1/(\gamma-1)$; $q^2 = u^2 + v^2$; $Q^2 = u^2 + v^2 + w^2$ and H, G, P_i and P_o are known functions of Υ.

Comparison with the incompressible case shows that the array [B] has, effectively, been replaced by the matrix [ρ.B] (see below).

Thus from equations [7.15c], [7.16c], [7.17c]

$$[B.r_{\psi}]_{\psi} + [(1/r).(\ln r)_{\Phi}]_{\Phi} = 0 \qquad [7.15c]$$

$$(r_{\psi})^2 + (1/r).(r_{\Phi}/B)^2 = q^2 \qquad [7.17c]$$

$$[\ln B]_{\psi} = [H_1 - C_1/r].q^2 \qquad [7.16c]$$

Thus in the numerical iteration, the density is calculated at each point of the grid from [8.23] and used to calculate the new B matrix.

The structure of the finite difference equations for compressible flow is substantially the same as for the incompressible case, the modifications being given in the appendix.

The 'x' coordinate is obtained from the equivalent finite difference form of [8.8a] and [8.9a]. i.e

$$x_{\Phi} = [B/A].y_{\psi} \quad [8.8a] \qquad ; \qquad x_{\psi} = [-A/B].y_{\Phi} \quad [8.9a]$$

With the transform of Chapter 3 these become

$$x_{i,j+1} = x_{i,j} + B_{i,j}\,\rho_{i,j}\,.(r_{i,j+1} - r_{i,j}).d\Phi/d\Upsilon \qquad [8.29]$$

$$x_{i+1,j} = x_{i,j} + [\{-1/(B_{i,j}\,.\,\rho_{i,j})\}.\ln(r_{i,j+1}/r_{i,j})].d\Upsilon/d\Phi \qquad [8.30]$$

Equations [8.29] and [8.30] are used to evaluate the 'x' coordinate at each grid point.

Summary of Results.

OUTLET VALUES ((HUB AL, CASING AP,)

R(AL,AI)	R(AP,AI)	Q(AL,AI)	Q(AP,AI)
0.916996039	2.47786903	.584282606	.542575350
0.898233270	2.44923131	.5861876253	.543979439
0.833708864	2.36262394	.592780901	.548270577
0.633074333	2.08096679	.614223702	.563302140

The above results are for compressible flow with zero swirl and uniform inlet flow. The preliminary result seems to be that the whole duct shape is depressed downwards for increasing Mach number . The exit speeds across the duct are approximately uniform.

Numerical results and observations. The program thus far developed for the numerical solutions has the following variable set of input parameters which define the flow

(1) Upper wall Distributions of functions of (r,q) of the following types (a) constant radii patches, (b) constant velocity patches, (c) decellerating flows , (d) accellerating flows.

(2) Lower Wall As for (1) above.

(3) Inlet Axial Profile Parabolic axial velocity profile with variable maximum and inlet wall speeds contributing to a non zero vorticity vector.

(4) Inlet Swirl Swirl velocity profile of the form a/y + b.y.

(5) Outlet speed Distribution Parallel flow condition.

(6) Boundary Layer Choice of boundary layers (a) laminar,
(b) turbulent.

(7) Variable Density Distribution of Mach numbers across the flow at inlet implying density variation throughout the flow.

In addition there are other subsidiary parameters such as ratio of inlet duct radii and duct 'length' which may be varied.

The multiplicity of combinations of variable parameters makes it impossible to investigate the widest drange of possiblities but some general conclusions are given below.

(1) Fig 8.1 & 8.1.a. : For a fixed lower boundary and a laminar boundary layer on the point of separation on the upper wall, an increase in Mach number at inlet causes the upper wall to 'move' inwards. The wall speed distribution is proportionally little changed by a change in the contour.

(2) Fig 8.2 & 8.2.a.: For a fixed Mach number at inlet and fixed lower wall an increase in the swirl parameter raises the outer wall. Except for geometrical displacement the speed distribution is little changed.

Alternative flow constraints and prescriptions may be applicable depending upon circumstance and suitable numerical formulations will allow their inclusion in the design scheme to produce duct contours satisfying these requirements.

INL=U : OUTL=P: INN=F; UPP.=S-1: SW=0: DL= ; GR= 11x11 : ACC= .9 : LAM.B.L

DUCT PROFILE EFFECT of INCREASING MACH NO.

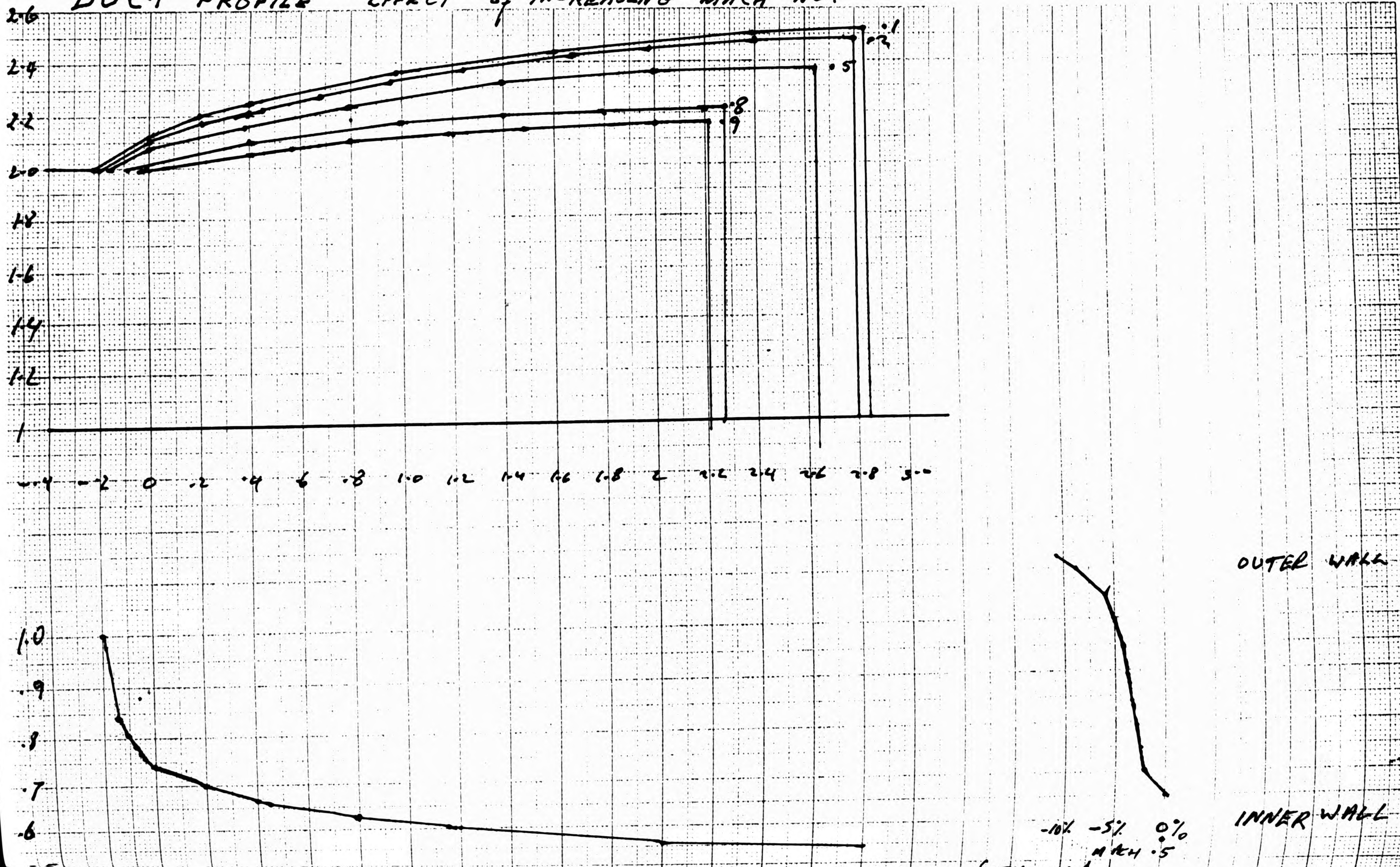


FIG. 8.1

EFFECT OF INCREASING INLET MACH NO.

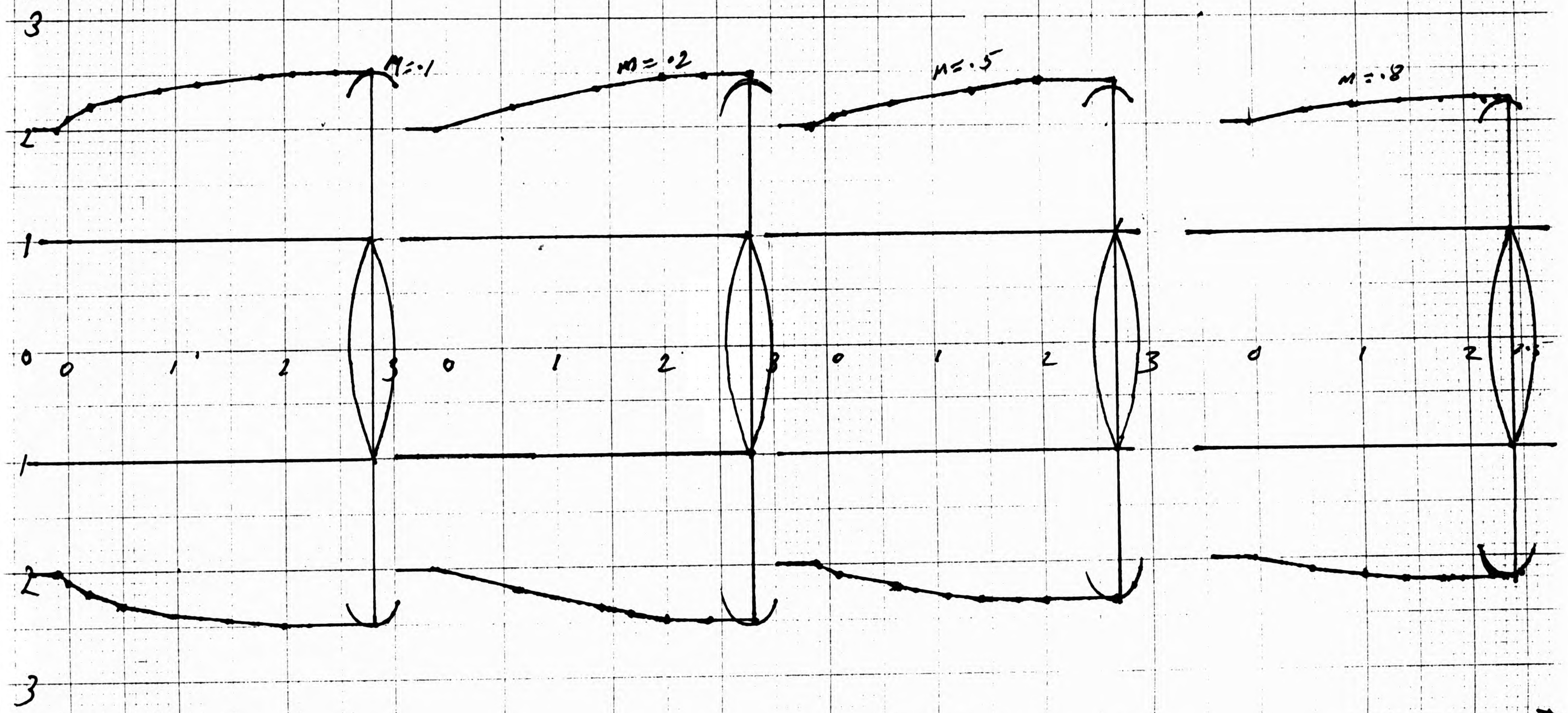


FIG. 8.1...

INL = V : OUTL = P : INN = F ; UPP. = S-1 : SW = x : DL = 3 ; GR = 11.11 : ACC = .9

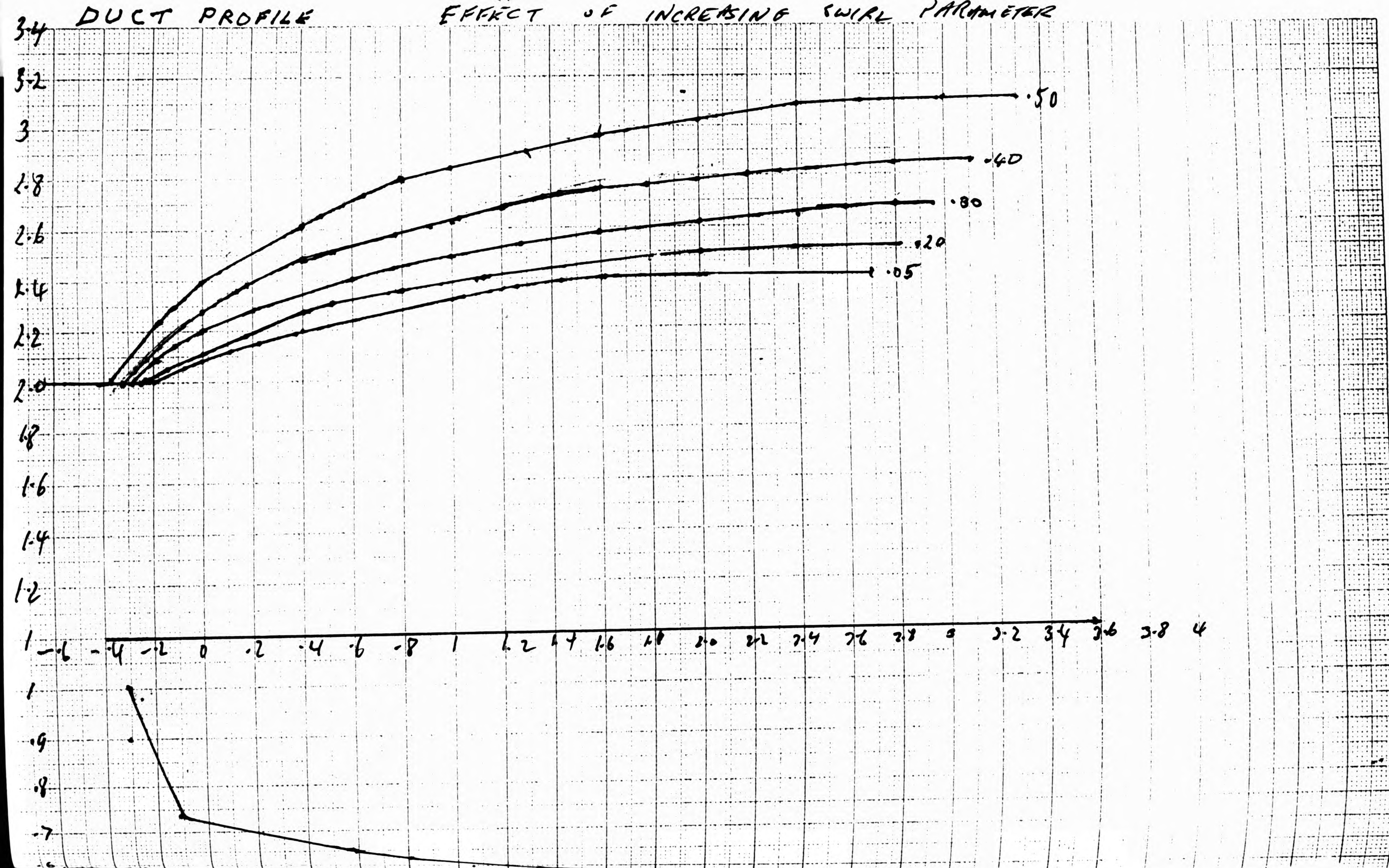
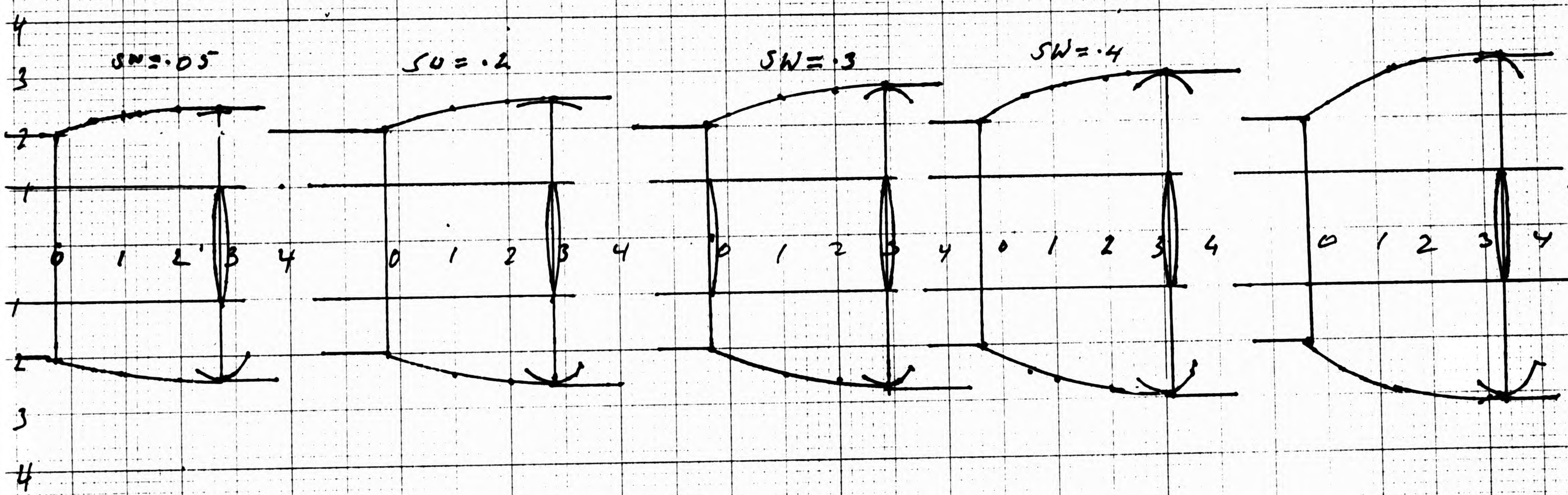


FIG. 8.2.

EFFECT OF INCREASING SWIRL



Conclusion

The formulation of the design problem in this thesis allows the incorporation of the following flow parameters in the numerical techniques to influence the flow pattern and hence the geometry of the annular duct.

(1) Inlet Vorticity Distributions and Velocity Profiles.

(2) Swirl Component parameter.

(3) Density/pressure/ Mach Number distributions at Inlet.

(4) Inner and outer wall prescribed velocity distributions and/or radius.

(5) Effect of Laminar or Turbulent Boundary Layers and Separation Criteria.

The investigation in detail of the effect that variation in these parameters might have on the flow geometry, either individually or in concert, would be made by a substantial amount of numerical experimentation and optimum configurations deduced. The separation criteria applied in calculating the wall velocity distributions are applicable to situations where the wall curvature is not large. This is usually a reasonable assumption in the axial direction but in the case of swirling flows, if the inner wall collapses towards the axis, the swirl velocity increases substantially implying a large pressure gradient across the boundary layer to support the inward acceleration. This contravenes the usual B.L assumption (for flows where there are no large changes in curvature) that the presssure gradient of the free stream is 'impressed' upon the B.L. In the case of the outer wall

this is not such a serious drawback since (in the examples considered) the curvature of the wall in the θ direction is of the same order of magnitude as the 'small' axial curvature.
Current boundary layer theory does not provide us with detailed knowledge of the behaviour of skewed boundary layers that would be expected in the case in swirling flows, however there is no reason, in principle, why alternative boundary conditions based on further analysis of boundary layer behaviour together with general fluid flow considerations could not be incorporated into the general numerical design approach presented in this thesis and extend our ability to generate duct shapes supporting fluid regimes with arbitrary but consistant flow properties.

APP.1

Appendices

(1)
$z = x + i.y$; $z_x = 1$; $z_y = i$; $z^* = x - i.y$; $z^*_x = 1$; $z^*_y = -i$

For any function, F, $F_x = F_z.z_x + F_{z*}.z^*_x = F_z + F_{z*}$

$F_y = F_z.z_y + F_{z*}.z^*_y = i [F_z + F_{z*}]$

$x = (z + z^*)/2$; $x_z = 1/2$; $x_{z*} = 1/2$

$y = (z - z^*)/(2i)$; $y_z = -i/2$; $y_{z*} = i/2$

$F_z = F_x.x_z + F_y.y_z = [F_x - i F_y]/2$

$F_{z*} = F_x.x_{z*} + F_y.y_{z*} = [F_x + i.F_y]/2$

$F_{zz*} = [F_{xx} + F_{yy}]/4 = \nabla^2[F/4]$

----

(2)
If $Q = u - i.v$; $\epsilon = u_x + v_y$; $\Omega = v_x - u_y$; $Q = q.e^{-i\theta}$; $q^2 = u^2 + v^2$

Then $Q = [(u_x + v_y) - i.(v_x - u_y)]/2 = [\epsilon - i.\Omega]/2$

----

(3)
For any function F

$F_s = F_x.x_s + F_y.y_s = \text{Cos}\theta.F_x + \text{Sin}\theta.F_y$: $x_s = \text{Cos}\theta$: $y_s = \text{Sin}\theta$:

$F_n = F_x.x_n + F_y.y_n = -\text{Sin}\theta.F_x + \text{Cos}\theta.F_y$: $x_n = \text{Sin}\theta$: $y_n = \text{Cos}\theta$:

$ds = dx.\text{Cos}\theta + dy.\text{Sin}\theta$: $dn = -dx.\text{Sin}\theta + dy.\text{Cos}\theta$

APP 2.

(4)

$$F_s + i.F_n = [\ \cos\theta.F_x + \sin\theta.F_y\ ] + i.[\ -i.\sin\theta.F_x + \cos\theta.F_y\ ]$$
$$= [\ \cos\theta - i.\sin\theta\ ].F_x + i.[\ \cos\theta - i.\sin\theta\ ].F_y$$
$$= e^{-i\theta}.[\ F_x + F_y\ ] = 2e^{-i\theta}.F_{z*}$$

(5)

$$(\ln Q)_s + i.(\ln Q)_n = 2.e^{-i\theta}.(\ln Q)_{z*} \qquad \text{[from (4)]}$$
$$= 2.e^{-i\theta}.(1/Q).Q_{z*}$$
$$= 2.e^{-i\theta}/(q.e^{-i\theta}).Q_{z*} \qquad \text{[from (2)]}$$
$$= [\ \epsilon - i.\Omega\ ]/q \qquad \text{[from (2)]}$$

(6)

$$(\ \ln Q\ )_s + i.(\ \ln Q\ )_n = [\ \ln(q.e^{-i\theta})_s\ ] + i.[\ \ln(q.e^{-i\theta})_n\ ]$$
$$= [\ \ln q - i\theta\ ]_s + i.[\ \ln q - i\theta\ ]_n$$
$$= [\ (\ln q)_s + \theta_n\ ] + i.[\ (\ln q)_n - \theta_s\ ]$$
$$= [\ \epsilon - i.\Omega\ ]/q \qquad \text{[from (5)]}$$

(7)

$$F_s = F_\Phi.\Phi_s + F_\Upsilon.\Upsilon_s \quad ; \quad F_n = F_\Phi.\Phi_n + F_\Upsilon.\Upsilon_n$$

But from definitions of Φ and Υ ; $\Phi_n = \emptyset$ and $\Upsilon_s = \emptyset$

$\Phi_s = q_2$ and $\Upsilon_n = q_1$

Hence $F_s = q_2.F_\Phi$; $F_n = q_1.F_\Upsilon$

(8)

$$d\Phi = \Phi_s.ds + \Phi_n.dn = q_2.ds + \emptyset = q_2.ds \quad ; \quad ds = d\Phi/q_2$$
$$d\Upsilon = \Upsilon_s.ds + \Upsilon_n.dn = \emptyset + q_1.dn = q_1.dn \quad ; \quad dn = d\Upsilon/q_1$$

(9)

From (3) $ds = dx.\cos\theta + dy.\sin\theta$; $dn = -dx.\sin\theta + dy.\cos\theta$

$dx = ds.\cos\theta - dn.\sin\theta$; $dy = ds.\sin\theta + dn.\cos\theta$

$$dz = dx + i.dy = (ds.\cos\theta - dn.\sin\theta\ ) + i\ (ds.\sin\theta + dn.\cos\theta)$$
$$= ds.(\cos\theta + i.\sin\theta) + i.dn.(\cos\theta + i.\sin\theta) = e^{i\theta}.(ds + i.dn)$$

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