Integral Equations and Operator Theory

Representations of nilpotent groups on spaces with indefinite metric.

--Manuscript Draft--

Manuscript Number:	IEOT-D-16-00183R1
Full Title:	Representations of nilpotent groups on spaces with indefinite metric.
Article Type:	Original Research
Keywords:	Nilpotent group, representation, indefinite metric, cocycle, decomposable
Corresponding Author:	Edward Kissin, Dr. London Metropolitan University London, UNITED KINGDOM
Corresponding Author Secondary Information:	
Corresponding Author's Institution:	London Metropolitan University
Corresponding Author's Secondary Institution:	
First Author:	Edward Kissin, Dr.
First Author Secondary Information:	
Order of Authors:	Edward Kissin, Dr.
	Victor Shulman
Order of Authors Secondary Information:	
Funding Information:	
Abstract:	The paper studies the structure of J-unitary representations of connected nilpotent groups on Pk-spaces,that is, the representations on a Hilbert space preserving a quadratic form "with a finite number of negative squares". Apart from some comparatively simple cases, such representations can be realized as double extensions of finite-dimensional representations by unitary ones. So their study is based on some special cohomological technique. We concentrate mostly on the problems of the decomposition of these representations and the classification of "non- decomposable" ones.
Response to Reviewers:	Thank you very much for refereeing our paper. And for your valuable comments. We accept all of them. 1. On pages 3 and 19 the definitions of J-decomposable representation are given; these Definitions are not the same. We removed our “definition” – statement on page 3. 2. The Theorem 3.3 can be stated as follows. Let _ be a representation in M⊕N⊕K, where M and M⊕N are invariant. Let we have orthogonal sums M = M1⊕M2 and N = N1⊕N2, and let M1⊕N1 and M2⊕N2 be invariant. Then there exist decompositions K = K1⊕K2 with M1⊕N1⊕K1 and M2⊕N2⊕K2 invariant. This version of the Theorem 3.3 is very transparent and there is no need to use "cohomological mashinery"in stating (and in proving) this Theorem. We renamed this Theorem into Proposition. Before it, according to your comment, we mention that if ξ_{12} = 0 and ξ_{21} = 0, then the result follows immediately. We also write there that we need to consider a more complicated case when these cocycles are non-zero coboundaries, for using further in the proofs of Corollary 4.3 and Theorem 6.4. Therefore we leave “cohomological” language in the proposition.

	We also simplified its proof. 3. In Theorem 6.1. we deal actually with commutative groups. So it is reasonable to restrict (in stating and in the Proof of Theorem) with commutative groups. According to your comments, we changed the statement of the theorem and prove it for commutative groups. In a Remark after the theorem we write that the result also holds for a wider class of groups. We also simplified the proof of the theorem. Edward Kissin and Victor Shulman

Representations of nilpotent groups on spaces with indefinite metric.

Edward Kissin and Victor S. Shulman

Abstract. The paper studies the structure of J-unitary representations of connected nilpotent groups on Π_k-spaces, that is, the representations on a Hilbert space preserving a quadratic form "with a finite number of negative squares". Apart from some comparatively simple cases, such representations can be realized as *double extensions* of finite-dimensional representations by unitary ones. So their study is based on some special cohomological technique. We concentrate mostly on the problems of the decomposition of these representations and the classification of "non-decomposable" ones.

1. Introduction

Irreducible unitary representations of connected nilpotent groups were studied in works of Dixmier, Lenglends, Guichardet, Pukanski, Kirillov and other mathematicians. For Lie groups Kirillov [Kir] developed the famous method of orbits relating structure of irreducible representations with symplectic geometry. The study of general unitary representations is simplified by the fact that they uniquely decompose in direct integrals of the unitary ones.

The situation is more complicated for non-unitary representations. Though all irreducible finite-dimensional representations are still one-dimensional and correspond to characters of the group, but the general finite-dimensional representations do not decompose in the sums of irreducible ones. Thus it is natural to take non-decomposable (but not necessarily irreducible) representations as building blocks – by the Krull-Schmidt theorem, the decomposition of an arbitrary finite-dimensional representation in the sum of non-decomposable ones is unique up to isomorphism. Unfortunately the classification of non-decomposable finite-dimensional representations is a "wild" problem even for a simple commutative group $G = \mathbb{R}^2$.

An intermediate, or mixed situation – the combination of finite-dimensional and unitary representations – naturally arises when one considers J-unitary representations on spaces with indefinite scalar products. Let H be a complex Hilbert space with an indefinite sesquilinear form $[\cdot,\cdot]$ and let

$$[x,y] = (Jx,y) \text{ for all } x,y \in H$$

and some *connecting* operator $J^* = J \in B(H)$ with bounded inverse. The initial scalar product plays an auxiliary role and can be changed if necessary by an equivalent one in such a way that J is an involution: $J^2 = \mathbf{1}_H$. Such scalar products are called J-*admissible*; it is convenient to fix one of them and to use it in topological constructions. It should be noted that the symbol J plays two roles in the theory: it denotes a concrete connecting involution and indicates that some term is used in "indefinite" sense (e.g. a J-unitary operator — an operator preserving the form $[\cdot,\cdot]$).

If J is a connecting involution then $(\mathbf{1}_H - J)/2$ is an orthoprojection on a subspace H_-, so that

$$H = H_- \oplus H_+,\ [x,x] < 0 \text{ for } x \in H_-,\ [x,x] > 0 \text{ for } x \in H_+,$$

$$\text{and } J = \begin{pmatrix} -\mathbf{1}_{H_-} & 0 \\ 0 & \mathbf{1}_{H_+} \end{pmatrix}.$$

Set $k_\pm = \dim(H_\pm)$ and $k = \min(k_\pm)$. The value of k is the same for all J-admissible scalar products; if $k < \infty$, H is called a *Pontryagin space* or Π_k-*space*. We assume that $k = k_- = \dim H_- \leq \dim H_+$. A subspace K is *neutral* if $[x,x] = 0$, *positive* if $[x,x] > 0$ and *negative* if $[x,x] < 0$ for $0 \neq x \in K$.

A representation π of a topological group G on H is *irreducible* if it has no closed invariant subspaces, *weakly continuous* if $(\pi(g)x,y)$ is continuous on G for $x,y \in H$. It is J-*unitary* if

$$[\pi(g)x, \pi(g)y] = [x,y] \text{ for all } x,y \in H \text{ and all } g \in G,$$
$$\text{i.e., } J\pi(g)^*J = \pi(g^{-1}). \tag{1.1}$$

J-unitary representations of locally compact groups were investigated by Araki [A], Ismagilov [Is1, Is2, Is3], Kissin and Shulman [KS], Naimark [N1, N2], Naimark and Ismagilov [NI], Sakai [Sa] and others. They were also considered in relation to the study of various problems in the quantum theory ([DT], [MPS], [Sc], [Sc1], [St], [SW]). It is well known that bounded representations of amenable groups are similar to unitary ones. Recently it was shown in [OST] that bounded J-unitary representations of all groups on Π_k-spaces are similar to unitary representations.

J-unitary representations naturally fall into two classes: non-singular and singular representations. A representation is *non-singular* if it has no neutral invariant subspaces; otherwise it is *singular*. Non-singular representations decompose in the J-orthogonal sum of a finite number of irreducible components and a unitary representation (see [Is]); in general, the irreducible components are not similar to unitary representations.

Naimark [N1] studied J-unitary representations of connected solvable groups on Π_k-spaces.

Theorem 1.1. [N1] *Let G be a connected, locally compact solvable group and let π be a weakly continuous J-unitary representations on Π_k-space H. Then*

(i) *π has a k-dimensional non-positive invariant subspace.*
(ii) *If π is non-singular then it is bounded, similar to a unitary representation and*

$$H = N[+]P, \text{ where } N, P \text{ are invariant subspaces,}$$

N is negative, $\dim(N) = k$, and P is positive. The representations $\pi|_N$ and $\pi|_P$ are similar to unitary representations.

Later Sakai [Sa] extended this result to amenable groups. Unlike non-singular representations, singular representations of solvable groups can be unbounded and, therefore, not similar to unitary representations. Thus the "decomposition" they admit is not the decomposition into irreducible components. Rather they "decompose" into non-Π-decomposable representations.

Definition 1.2. *A representation π on a Π_k-space H is* Π*-decomposable if $H = H_1[+]H_2$, where H_1 and H_2 are invariant and not positive. Otherwise, π is called non-*Π*-decomposable.*

The underlying space of a non-Π-decomposable representation may have a decomposition $H = H_1[+]H_2$, where H_1 and H_2 are invariant, but one of them must be positive.

This paper is a continuation of [KS1] that studied cohomology of nilpotent groups, normal cocycles and the extensions of representations generated by cocycles of these groups. In Section 2 we review some of its results.

In Section 3 we provide further information about geometry of Π_k-spaces ([AI], [B], [KS]) which is different from geometry of Hilbert spaces and often counter-intuitive. We consider some general properties of J-unitary representations and show that singular representations can be constructed as *double extensions* $\mathfrak{ee}(\lambda, U, \xi, \gamma)$, where λ is a representation on a finite-dimensional space, U is a non-singular representations and ξ and γ are some cohomological data. We also obtain some useful criteria of Π-decomposability of the representations $\mathfrak{ee}(\lambda, U, \xi, \gamma)$.

In Section 4 the results of Section 3 are refined for the case of nilpotent groups. In Section 5 we partially describe the structure of finite-dimensional J-unitary representations of connected nilpotent groups G. First we consider important classes $\{\pi_{k,m}\}$ and $\{\pi_{\chi,\chi^*}\}$ of these representations, where $k, m \in \mathbb{N}$ and χ are non-unitary characters on G. It is shown that each finite-dimensional J-unitary representation of G decomposes in the J-orthogonal sum of the representations $\pi_{k,m}$, π_{χ,χ^*} and one-dimensional unitary representations. Even for small k and m, the structure of $\pi_{k,m}$-representations can be very complicated. Using some results of [KS1] about neutral cocycles of nilpotent groups, we get a description of representations $\pi_{1,m}$. It allows us in Corollary 5.4 to describe transparently these representations for the groups $\mathcal{T}_n$ of all $n \times n$ real upper triangular matrices with identity on the main diagonal.

Although each $\pi_{1,m}$ representation is non-Π-decomposable, it can be *J-decomposable*, i.e., it can decompose in the J-orthogonal sum of two representations. In Theorem 5.9 we give some necessary and sufficient conditions for them to be non-J-decomposable. Similar results are obtained for the representations π_{χ,χ^*}.

A singular representation π of a nilpotent group G is called *primary* if, for some maximal invariant neutral subspace L of π, $\pi|_L$ has only one eigen-character, i.e., a character χ of G such that $\pi(g)x = \chi(g)x$ for some $0 \neq x \in L$ and all $g \in G$. The representations $\pi_{k,m}$ and π_{χ,χ^*} are examples of primary representations.

In Section 6 we show that all non-Π-decomposable representations of commutative groups are primary. On the other hand, we prove that if characters of G are not separated in the dual space of G (e.g., $G = \mathcal{T}_3$ is the Heisenberg group of all 3×3 real upper triangular matrices $g = (g_{ij})$ with $g_{ii} = 1$), then G has a non-Π-decomposable representation which is not primary.

We say that a maximal neutral invariant subspace L *splits* a singular representation π on H if there is an invariant subspace K, $\dim K < \infty$, such that $L \subset K$ and $H = K[+]K^{[\perp]}$, where $K^{[\perp]}$ is the J-orthogonal complement of K. In Section 7 we show that L always either splits or *approximately splits* π, i.e., there are invariant subspaces $\{H_m\}_{m=1}^{\infty}$ such that $L \subset H_{m+1} \subset H_m$,

$$\dim H_m = \infty,\ H = H_m[+]H_m^{[\perp]} \text{ and } \dim(\cap_m H_m) < \infty.$$

The subspaces $H_m^{[\perp]}$ increase, the representations $\pi|_{H_m^{[\perp]}}$ are similar to unitary ones and the invariant subspace $\mathcal{N} = \cap_m H_m$ (the "nucleus") is degenerate, finite-dimensional and contains L. Thus the representations $\pi|_{H_n}$ are "infinitely close" to $\pi|_{\mathcal{N}}$ and the representations $\pi|_{H_m^{[\perp]}}$ give an *"approximate decomposition"* of π.

We are very grateful to the referee for many helpful suggestions.

2. Cohomology of groups with coefficients in bimodules.

We first recall some cohomological notions in a version convenient for our study. For Banach spaces L and $\mathfrak{H}$, let $B(\mathfrak{H}, L)$ be the space of all bounded operators from $\mathfrak{H}$ to L and $B(\mathfrak{H}) = B(\mathfrak{H}, \mathfrak{H})$. Let λ and U be representations of a topological group G on L and $\mathfrak{H}$ respectively. Let C^n be the space of all continuous functions from G^n to $B(\mathfrak{H}, L)$. Define the map $d^1_{\lambda,U}\colon C^1 \to C^2$ by

$$d^1_{\lambda,U}(\xi)(g,h) = \lambda(g)\xi(h) - \xi(gh) + \xi(g)U(h) \text{ for } \xi \in C^1. \tag{2.1}$$

The space $\mathcal{Z}^1(\lambda, U) = \ker d^1_{\lambda,U}$ of (λ, U)-cocycles consists of all functions $\xi\colon G \to B(\mathfrak{H}, L)$ satisfying

$$\xi(gh) = \lambda(g)\xi(h) + \xi(g)U(h) \text{ for all } g, h \in G. \tag{2.2}$$

The space $\mathcal{B}^1(\lambda, U)$ of (λ, U)-coboundaries consists of all functions ξ: $G \to B(\mathfrak{H}, L)$ satisfying

$$\xi(g) = \lambda(g)X - XU(g), \text{ for all } g \in G \text{ and some } X \in B(\mathfrak{H}, L). \tag{2.3}$$

Then $\mathcal{B}^1(\lambda, U) \subseteq \mathcal{Z}^1(\lambda, U)$ and $\mathcal{H}^1(\lambda, U) = \mathcal{Z}^1(\lambda, U)/\mathcal{B}^1(\lambda, U)$ is the 1st cohomology group of G with coefficients in (λ, U)-bimodule $B(\mathfrak{H}, L)$.

Let H_1, H_2 be Hilbert spaces. For a map u: $G \to B(H_1, H_2)$, define the map $u^\sharp$: $G \to B(H_2, H_1)$ by:

$$u^\sharp(g) = u(g^{-1})^*. \tag{2.4}$$

If u is a (π_1, π_2)-cocycle (coboundary), where π_i are representations on H_i, then $u^\sharp$ is a $(\pi_2^\sharp, \pi_1^\sharp)$-cocycle (coboundary); if $H_1 = H_2$ and u is a representation then $u^\sharp$ is also representation.

A (λ, U)-cocycle ξ is called *neutral* if the function $-\xi(g)\xi^\sharp(h)$ from $G\times G$ to $B(L)$ is the $(\lambda, \lambda^\sharp)$-coboundary of some function γ (called a *prechain* of ξ) from G to $B(L)$:

$$d^1_{\lambda,\lambda^\sharp}(\gamma)(g,h) \overset{(2.1)}{=} \lambda(g)\gamma(h) - \gamma(gh) + \gamma(g)\lambda^\sharp(h) = -\xi(g)\xi^\sharp(h). \tag{2.5}$$

The map γ is determined up to a cocycle. Neutral cocycles were introduced by Ismagilov [Is3] and systematically studied in [KS1]. They and their generalizations play an important role in what follows.

For a subgroup H of G, let $[G, H]$ be the minimal *closed* subgroup of G containing all commutators $[g, h] = ghg^{-1}h^{-1}$, $g \in G$, $h \in H$. Set $G^{[1]} = [G, G]$, $G^{[2]} = [G, G^{[1]}]$,..., $G^{[n]} = [G, G^{[n-1]}]$; G is nilpotent if $G^{[n]} = \{e\}$ for some n.

Consider the following example. If $L = \mathbb{C}$, $\lambda(g) \equiv 1$ and $U(g) \equiv \mathbf{1}_\mathfrak{H}$ are trivial representations of G, then a (λ, U)-cocycle can be identified with a continuous map α: $G \to \mathfrak{H}$, satisfying

$$\alpha(gh) = \alpha(g) + \alpha(h) \text{ for } g, h \in G. \tag{2.6}$$

Proposition 2.1. ([KS1]) *Let G be a connected locally compact group and let a continuous map $\alpha : G \to \mathfrak{H}$ satisfy* (2.6). *Then there are $n := n_G \in \mathbb{N}$, a normal subgroup G_0 of G, $G^{[1]} \subseteq G_0 \subseteq \ker(\alpha)$, an isomorphism $\theta : G/G_0 \to \mathbb{R}^n$ and a linear map $\beta : \mathbb{R}^n \to \mathfrak{H}$ such that*

$$\alpha(g) = \beta(\omega(g)) = \beta(x_1, ..., x_n) = x_1u_1 + ... + x_nu_n$$

for some $u_1, ..., u_n \in \mathfrak{H}$, where $\omega : G \to \mathbb{R}^n$ is the composition of the canonical homomorphism $G \to G/G_0$ with θ, so that $\omega(g) = (x_1, ..., x_n) \in \mathbb{R}^n$.

The following result obtained in [KS1] is important for the rest of the paper.

Theorem 2.2. *Let λ and U be representations of a nilpotent group G. If* $\mathrm{Sp}(\lambda(h)) \cap \mathrm{Sp}(U(h)) = \varnothing$ *for some $h \in G$, then $\mathcal{H}^1(\lambda, U) = \mathcal{H}^1(U, \lambda) = 0$.*

A complex-valued function χ on G is *a character* if $\chi(gh) = \chi(g)\chi(h)$ for all $g, h \in G$. Then

$$\chi^*(g) = \overline{\chi(g^{-1})} = \overline{\chi(g)^{-1}} \text{ for } g \in G, \text{ is a character.} \tag{2.7}$$

If $\chi = \chi^*$, i.e., $|\chi(g)| = 1$ for $g \in G$, χ is called *unitary.*

If $\dim L = n$ and G is nilpotent and connected then, by Lie-Kolchin Theorem, λ has upper triangular form in some basis in L with characters $\{\chi_i\}_{i=1}^n$ on the diagonal (they may repeat). The *set* of these characters (each taken only once) is denoted by $\mathrm{sign}(\lambda)$. It coincides with the set of all eigenfunctionals so does not depend on the choice of a basis.

If $\mathrm{sign}(\lambda)$ consists of one character χ, we say that λ is *monothetic*, or a *χ-representation.*

Corollary 2.3. (Corollary 2.18 [KS1].) *Each finite-dimensional representation λ on L of a connected nilpotent group uniquely decomposes in the direct sum of monothetic representations*:

$$\lambda = \sum_{\chi \in \mathrm{sign}(\lambda)} \dot{+} \lambda_\chi \textit{ and } L = \sum_{\chi \in \mathrm{sign}(\lambda)} \dot{+} L_\chi, \tag{2.8}$$

where each $\lambda_\chi = \lambda|_{L_\chi}$ is an χ-representation.

Simple examples show that Corollary 2.3 does not extend to solvable groups.

We say that a representation U of G on $\mathfrak{H}$ and a character χ of G are

1) *eigen-disjoint* if

$$\mathfrak{H}^\chi = \{x \in \mathfrak{H} : U(g)x = \chi(g)x \text{ for all } g \in G\} = \{0\};$$

2) *spectrally disjoint* if $\chi(h) \notin \mathrm{Sp}(U(h))$ for some $h \in G$;

3) *sectionally spectrally disjoint* if, with respect to some decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus ... \oplus \mathfrak{H}_n$, U has an upper triangular form such that χ is spectrally disjoint with each diagonal block U_i.

A set Ω of characters of G and a representation U of G are *eigen-disjoint*, *spectrally disjoint*, *sectionally spectrally disjoint*, if this is true for U and each $\chi \in \Omega$.

Combining Theorem 2.2 and Corollary 2.3 yields

Corollary 2.4. ([KS1]) *Let λ, U be representations of a connected nilpotent group and let λ be finite-dimensional. If* $\mathrm{sign}(\lambda)$ *and U are sectionally spectrally disjoint then $\mathcal{H}^1(\lambda, U) = \mathcal{H}^1(U, \lambda) = 0$.*

We will later need the following result.

Lemma 2.5. ([KS1]) *Let χ and $\{\chi_i\}_{i=1}^r$ be continuous characters on a connected group G.*

(i) *If $\chi(g) \in \{\chi_i(g)\}_{i=1}^r$ for each $g \in G$, then χ coincides with one of the characters $\chi_1, ..., \chi_r$.*

(ii) *Let* $U = \chi \mathbf{1}_{\mathfrak{H}}$ *be a representation of a connected locally compact group* G *on* $\mathfrak{H}$ *and* λ *be a* χ*-representation of* G *on* L, $\dim L < \infty$. *For any* (λ, U)*-cocycle* ξ, *the codimension of the space* $\cap_{g\in G} \ker \xi(g)$ *in* $\mathfrak{H}$ *does not exceed* $n_G \dim L$ (*see Proposition* 2.1).

3. J-unitary representations of groups on Π_k-spaces

First, we provide some additional information about geometry of Π_k-spaces ([AI], [B], [KS]). Let $H = H_- \oplus H_+$ be a Π_k-space with a connecting involution $J = \begin{pmatrix} -\mathbf{1}_{H_-} & 0 \\ 0 & \mathbf{1}_{H_+} \end{pmatrix}$, so that $[x, y] = (Jx, y)$ for all $x, y \in H$, and $k = \min(k_\pm)$, where $k_\pm = \dim(H_\pm)$. We assume that $k = k_- = \dim H_- \leq \dim H_+$. All subspaces of H we consider will be closed.

Let K be a subspace of H. The J-orthogonal complement of K is defined by

$$K^{[\perp]} = \{y \in H\colon [x, y] = 0 \text{ for all } x \in K\}.$$

Subspaces K and M of H are J-*orthogonal* if $[x, y] = 0$ for $x \in K$ and $y \in M$. We write $H = K[+]M$ if H is also the direct sum of K and M. Then there is a J-admissible scalar product on H with respect to which K and M are orthogonal and we write $H = K[\oplus]M$. In particular, $H = H_-[\oplus]H_+$.

Subspaces L and M are *skew-related* if for each $x \in L$, there is $y \in M$ such that $[x, y] \neq 0$ and vice versa. In this case $\dim L = \dim M$. A subspace L is neutral if and only if $L \subseteq L^{[\perp]}$; it is *non-degenerate* if $L \cap L^{[\perp]} = \{0\}$.

For example, if

$$H = \mathbb{C}e_1 \oplus \mathbb{C}e_2, \ J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } [x, y] = (Jx, y) \text{ for } x, y \in H,$$

then H is a Π_1-space. For $\alpha \in \mathbb{C}$, the vector $x_\alpha = \alpha e_1 \oplus e_2$ is negative if $1 < |\alpha|$, positive if $1 > |\alpha|$ and $H = \mathbb{C}x_\alpha[+]\mathbb{C}x_{\overline{\alpha}^{-1}}$. If $|\alpha| = 1$ then $\mathbb{C}x_\alpha$ is neutral and $(\mathbb{C}x_\alpha)^{[\perp]} = \mathbb{C}x_\alpha$.

A projection p in $B(H)$ is J-*orthogonal* if the following equivalent conditions hold

$$Jp^* = pJ \Longleftrightarrow H = pH[+](\mathbf{1} - p)H \Longleftrightarrow [px, y] = [x, py] \tag{3.1}$$

for $x, y \in H$. The following facts are well known (see, for example, [KS]).

Proposition 3.1. *Let* H *be a* Π_k*-space. For any subspace* K *of* H,

$$(K^{[\perp]})^{[\perp]} = K,$$

$$K \text{ is non-degenerate} \Longleftrightarrow K \cap K^{[\perp]} = \{0\} \Longleftrightarrow H = K[+]K^{[\perp]}. \tag{3.2}$$

If K *is a non-positive* (*e.g. neutral or negative*) *subspace of* H *then* $\dim K \leq k$.

If K *is non-degenerate, then it is a* Π_n*-space and* $K^{[\perp]}$ *is a* Π_m*-space,*

$$n_- + m_- = k_- \text{ and } n_+ + m_+ = k_+. \tag{3.3}$$

We consider now J-unitary representations π of topological groups on Π_k-spaces (see (1.1)). As usual, J-unitary representations π and ρ on H and K are *similar* if $\rho = S\pi S^{-1}$ for $S \in B(H,K)$. They are *J-equivalent* if also $[Sx, Sy]_{_K} = [x,y]_{_H}$ and *J-antiequivalent* if $[Sx, Sy]_{_K} = -[x,y]_{_H}$ for $x, y \in H$.

Recall that π is singular if it has a non-zero invariant neutral subspace. We say that π is *completely singular* (or *generic* [KS]) if it has an invariant neutral subspace of dimension k (equivalently all maximal invariant neutral subspaces are k-dimensional).

Remark 3.2. Let π be a J-unitary representation on a Π_k-space $H = H_-[\oplus]H_+$, $k_\pm = \dim(H_\pm)$.

(i) If $S \in B(H)$ has a bounded inverse, H is also a Π_k-space with respect to the indefinite metric $[x,y]_1 = [S^{-1}x, S^{-1}y]$. The representation $\rho = S\pi S^{-1}$ on $(H, [\cdot,\cdot]_1)$ is J-unitary and J-equivalent to π.

(ii) Suppose that $k_+ < k_-$, so that $k = k_+$. Then H is also a Π_k-space with metric $[\cdot,\cdot]_1 = -[\cdot,\cdot]$ and $H = H'_-[\oplus]H'_+$, where $H'_- = H_+$, $H'_+ = H_-$ and $k = k_+ = \dim H'_-$. The representation π on $(H, [\cdot,\cdot]_1)$ is J-unitary and J-antiequivalent to π on $(H, [\cdot,\cdot])$. ■

We focus our attention on the study of singular representations π. Let L be a maximal neutral π-invariant subspace of H. Then $\dim L \le k$, $L^{[\perp]}$ is invariant and contains L. Set $\mathfrak{H} = L^{[\perp]} \ominus L$ and $M = JL$. Then $H = L \oplus \mathfrak{H} \oplus M$, $\mathfrak{H}$ is non-degenerate and invariant for J; M is neutral and skew-related to L.

By Corollary 3.4 [KS], $(\mathfrak{H}, [\cdot,\cdot])$ is a Π_n-space, $n = k - \dim(L)$, with a connecting operator $I = J|_{\mathfrak{H}}$. As L and M are skew-related, identifying M with L via the map τ: $M \to L$, $(x, \tau(y)) = [x,y]$, we can write that $H = L \oplus \mathfrak{H} \oplus L$,

$$\pi(g) = \begin{pmatrix} \lambda(g) & \xi(g) & \gamma(g) \\ 0 & U(g) & \eta(g) \\ 0 & 0 & \mu(g) \end{pmatrix} \text{ and } J = \begin{pmatrix} 0 & 0 & \mathbf{1}_L \\ 0 & I & 0 \\ \mathbf{1}_L & 0 & 0 \end{pmatrix} \tag{3.4}$$

where $\lambda = \pi|_{_L}$ and U, μ are representations of G on $\mathfrak{H}$ and L, respectively.

As π is J-unitary, we have from (1.1) that $\pi(g^{-1}) = J\pi(g)^*J$. Hence (see (2.4))

$$\mu = \lambda^\sharp, \ \eta = I\xi^\sharp, \ \gamma^\sharp = \gamma \text{ and } U(g^{-1}) = IU(g)^*I, \tag{3.5}$$

where $u^\sharp(g) = u^*(g^{-1})$ for $g \in G$. Thus U is J-unitary with connecting operator I. It is *non-singular*, as L is a maximal neutral invariant subspaces in H. As π is a representation, the maps ξ and γ satisfy

$$\begin{aligned} \xi(gh) &= \lambda(g)\xi(h) + \xi(g)U(h), \\ \gamma(gh) &= \lambda(g)\gamma(h) + \xi(g)I\xi^\sharp(h) + \gamma(g)\lambda^\sharp(h). \end{aligned} \tag{3.6}$$

In other words, ξ is a cocycle and

$$d^1_{\lambda,\lambda^\sharp}(\gamma)(g,h) = -\xi(g)I\xi^\sharp(h). \tag{3.7}$$

We often write L for $L \oplus \{0\} \oplus \{0\}$, M for $\{0\} \oplus \{0\} \oplus L$, η for $I\xi^\sharp$ and μ for $\lambda^\sharp$.

If π is completely singular then $\mathfrak{H}$ is a Hilbert space with scalar product $[x, y]$ and U is a unitary representation. In this case $I = \mathbf{1}_{\mathfrak{H}}$, so that (3.7) implies that cocycle ξ is neutral (see (2.5)) and γ is its prechain. Conversely, starting with a unitary representation U on a Hilbert space $\mathfrak{H}$, a representation λ on an n-dimensional Hilbert space L, a neutral cocycle $\xi \in \mathcal{B}^1(\lambda, U)$ and a prechain γ of ξ, one can define a completely singular representation π on a Π_n-space $H = L \oplus \mathfrak{H} \oplus L$ via the construction in (3.4). All completely singular representations can be obtained in this way.

To catch the general case, we will slightly extend our approach. Now U must be a non-singular representation on a Π_m-space $\mathfrak{H}$ with connecting operator I. We say that a cocycle $\xi \in \mathcal{B}^1(\lambda, U)$ is *I-neutral*, if there is a map γ: $G \to B(L, L)$ such that (3.7) holds. Starting with λ, U, ξ and γ, we define a representation π of G on the Π_{m+n}-space $H = L \oplus \mathfrak{H} \oplus L$ with connecting operator J as in (3.4). We denote π by $\mathfrak{ee}(\lambda, U, \xi, \gamma)$ and call it a *double extension* of a non-singular representation U by λ defined by ξ. It follows from the previous considerations that any singular J-unitary representation on a Π_k-space is J-unitary equivalent to a representation of this form.

Now we will find some conditions for the double extension $\pi = \mathfrak{ee}(\lambda, U, \xi, \gamma)$ to be Π-decomposable.

Let $L = L_1 \dotplus L_2$ and $\mathfrak{H} = \mathfrak{H}_1[+]\mathfrak{H}_2$, where L_i are λ-invariant and $\mathfrak{H}_i$ are U-invariant subspaces. Let p be a projection on L_1 along L_2. Then p commutes with λ. Set $M_1 = p^*M$ and $M_2 = (\mathbf{1}_M - p^*)M$. As p^* commutes with $\lambda^\sharp$, M_i are $\lambda^\sharp$-invariant subspaces and $M = M_1 \dotplus M_2$. If $x \in L_2$ and $y \in M_1$ then $y = p^*y$ and, by (3.4), $[x, y] = (x, p^*y) = (px, y) = 0$. Thus M_1 is J-orthogonal to L_2. Similarly, M_2 is J-orthogonal to L_1, M_1 is skew-related to L_1 and M_2 is skew-related to L_2.

Thus $H = (L_1 \dotplus L_2)[\oplus](\mathfrak{H}_1[+]\mathfrak{H}_2) \oplus (M_1 \dotplus M_2)$ and with respect to this decomposition

$$\pi = \begin{pmatrix} \lambda_1 & 0 & \xi_{11} & \xi_{12} & \gamma_{11} & \gamma_{12} \\ 0 & \lambda_2 & \xi_{21} & \xi_{22} & \gamma_{21} & \gamma_{22} \\ 0 & 0 & U_1 & 0 & \eta_{11} & \eta_{12} \\ 0 & 0 & 0 & U_2 & \eta_{21} & \eta_{22} \\ 0 & 0 & 0 & 0 & \lambda_1^\sharp & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2^\sharp \end{pmatrix}, \tag{3.8}$$

where $\lambda_i = \pi|_{L_i}$, $U_i = U|_{\mathfrak{H}_i}$, $\lambda_i^\sharp = \lambda^\sharp|_{M_i}$.

If $\xi_{12} = \xi_{21} = 0$ then $\eta_{12} = \eta_{21} = 0$ as $\eta = I(\xi)^\sharp$. If also $\gamma_{21} = 0$ then $H_1 = (L_1[+]\mathfrak{H}_1) \dotplus M_1$ is π-invariant and non-degenerate. Thus $H = H_1[+]H_1^{[\perp]}$, $H_1^{[\perp]}$ is π-invariant and $L_2 \subset H_1^{[\perp]}$. We extend this now to the case when $\xi_{12} + \xi_{21}$ is a coboundary to use it in the proof of Corollary 4.3. Its inverse (Theorem 3.4) gives some sufficient condition for $\xi_{12} + \xi_{21}$ to be a coboundary and will be used to prove Theorem 6.5.

Note that the I-orthogonal projection q on $\mathfrak{H}_1$ along $\mathfrak{H}_2$ commutes with U.

Proposition 3.3. *Let* $\pi = \mathfrak{ee}(\lambda, U, \xi, \gamma)$ *have form* (3.8). *Let*

$$\xi_{12} + \xi_{21} \text{ be a } (\lambda, U)\text{-coboundary and } \mathcal{H}^1(\lambda_2, \lambda_1^\sharp) = 0. \tag{3.9}$$

Then $H = H_1[+]H_2$ *is the* J*-orthogonal sum of invariant subspaces,* $H_1 = (L_1[+]\mathfrak{H}_1') \dotplus M_1'$, *where* $\mathfrak{H}_1' = \{-T_1x \dotplus x\colon x \in \mathfrak{H}_1\}$ *for some* $T_1 \in B(\mathfrak{H}_1, L_2)$ *and* M_1' *is skew-related to* L_1, *and* $L_2 \subset H_i$.

Proof. As $\xi_{12} + \xi_{21}$ is a (λ, U)-coboundary, $\xi_{12} + \xi_{21} = \lambda X - XU$ for some $X \in B(\mathfrak{H}, L)$. Then $\xi_{21} = (\mathbf{1}_L - p)(\lambda X - XU)q = \lambda_2 T_1 - T_1 U_1$ for $T_1 = (\mathbf{1}_L - p)Xq \in B(\mathfrak{H}_1, L_2)$. Similarly, $\xi_{12} = \lambda_1 T_2 - T_2 U_2$ for $T_2 = pX(\mathbf{1}_{\mathfrak{H}} - q)$. Thus ξ_{21} and ξ_{12} are coboundaries.

Set $\mathfrak{H}_i' = \{-T_i x \dotplus x\colon x \in \mathfrak{H}_i\}$, $i = 1, 2$. Then $L_i \dotplus \mathfrak{H}_i'$ are π-invariant. For example, for $i = 1$,

$$\begin{aligned}\pi(g)(-T_1x \dotplus x) &= \xi_{11}(g)x \dotplus (-\lambda_2(g)T_1x \dotplus \xi_{21}(g)x) \dotplus U_1(g)x \\ &= \xi_{11}(g)x \dotplus (-T_1U_1(g)x \dotplus U_1(g)x) \in L_1 \dotplus \mathfrak{H}_1'.\end{aligned}$$

Consider a new J-admissible scalar product on H such that L is orthogonal to $\mathfrak{H}' = \mathfrak{H}_1'[+]\mathfrak{H}_2'$. Then

$$H = (L_1 \dotplus L_2)[\oplus](\mathfrak{H}_1'[+]\mathfrak{H}_2') \oplus (M_1 \dotplus M_2).$$

With respect to this decomposition

$$\pi = \begin{pmatrix} \lambda_1 & 0 & \xi_{11}' & 0 & \gamma_{11}' & \gamma_{12}' \\ 0 & \lambda_2 & 0 & \xi_{22}' & \gamma_{21}' & \gamma_{22}' \\ 0 & 0 & U_1' & 0 & \eta_{11}' & \eta_{12}' \\ 0 & 0 & 0 & U_2' & \eta_{21}' & \eta_{22}' \\ 0 & 0 & 0 & 0 & \lambda_1^\sharp & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2^\sharp \end{pmatrix} \text{ where } \eta' = \begin{pmatrix} \eta_{11}' & \eta_{12}' \\ \eta_{21}' & \eta_{22}' \end{pmatrix} = I'(\xi')^\sharp$$

and $I' = J|_{\mathfrak{H}'}$. By (3.1), the I'-orthogonal projection q' on $\mathfrak{H}_1'$ along $\mathfrak{H}_2'$ satisfies $I'(q')^* = q'I'$. Hence

$$\begin{aligned}\eta_{21}'(g) &= (\mathbf{1}_{\mathfrak{H}} - q')\eta(g)p^* = (\mathbf{1}_{\mathfrak{H}} - q')I'(\xi')^\sharp(g)p^* \\ &= I'(\mathbf{1}_{\mathfrak{H}} - (q')^*)\xi'(g^{-1})^*p^* \\ &= I'(p\xi'(g^{-1})(\mathbf{1}_{\mathfrak{H}} - q'))^* = I'\xi_{12}'(g^{-1}) = 0.\end{aligned}$$

Thus γ_{21}' is a $(\lambda_2, \lambda_1^\sharp)$-cocycle. By (3.9), it is a coboundary: $\gamma_{21}' = \lambda_2 S - S\lambda_1^\sharp$ for some $S \in B(M_1, L_2)$.

The space $M_1' = \{-Sz \dotplus z\colon z \in M_1\}$ is skew-related to L_1 and $\pi(g)M_1' \subseteq L_1 \dotplus \mathfrak{H}_1' \dotplus M_1'$, as

$$\begin{aligned}\pi(g)(-Sz \dotplus z) &= \gamma_{11}'(g)z \dotplus (-\lambda_2(g)Sz + \gamma_{21}'(g)z) \dotplus \eta_{11}'(g)z \dotplus \lambda_1^\sharp(g)z \\ &= \gamma_{11}'(g)z \dotplus \eta_{11}'(g)z \dotplus (-S\lambda_1^\sharp(g)z \dotplus \lambda_1^\sharp(g)z)\end{aligned}$$

belongs to $L_1 \dotplus \mathfrak{H}_1' \dotplus M_1'$. Hence the subspace $H_1 = (L_1[+]\mathfrak{H}_1') \dotplus M_1'$ is π-invariant. As the subspace $(L_1[+]\mathfrak{H}_1') \dotplus M_1$ is non-degenerate (see (3.2)) and J-orthogonal to L_2, the subspace H_1 is also non-degenerate and J-orthogonal to L_2. Hence, by (3.2), $H = H_1[+]H_2$ and $L_2 \subset H_2$. □

Our next result is a partial inverse of Proposition 3.3. We will use it later to prove that some special representations are non-Π-decomposable. Recall that a finite-dimensional representation is *semisimple* if it is a direct sum of irreducible representations.

Theorem 3.4. *Let* λ *be semisimple and not irreducible. If* $\pi = \mathfrak{ee}(\lambda, U, \xi, \gamma)$ *is* Π-*decomposable then*

$$\sigma := p\xi(\mathbf{1}_{\mathfrak{H}} - q) + (\mathbf{1}_L - p)\xi q \text{ is a } (\lambda, U)\text{-coboundary,} \tag{3.10}$$

for some projections p *and* $q = q^\sharp$ *commuting with* λ *and* U*, respectively. If* $p = 0$ *then* q *maps* $\mathfrak{H}$ *on a subspace which is not positive*; *if* $p = \mathbf{1}_L$ *then* $\mathbf{1}_{\mathfrak{H}} - q$ *maps* $\mathfrak{H}$ *on a subspace which is not positive.*

In particular, if π *is completely singular then* $p \neq 0, \mathbf{1}_L$ *in* (3.10).

Proof. Let P be a J-orthogonal projection such that the decomposition $H \overset{(3.1)}{=} PH[+](\mathbf{1}_H - P)H$ is a Π-decomposition of H. It commutes with π and has form $P = (p_{ij})_{i,j=1}^3$ with respect to the decomposition $H = L \oplus \mathfrak{H} \oplus L$ (we may assume that $p_{11} \neq 0$; otherwise replace P by $1_H - P$). Hence

$$p_{31}\lambda(g) = \mu(g)p_{31} \text{ and } p_{31}\xi(g) = \mu(g)p_{32} - p_{32}U(g) \text{ for } g \in G.$$

Assume firstly that $p_{31} \neq 0$. As λ is semisimple and not irreducible, $\mathbf{1}_L = \sum_{i=1}^n r_i$, $n > 1$, where r_i are projections commuting with λ and $\lambda|_{L_i}$ are irreducible, $L_i = r_iL$. As $\mu(g) = \lambda(g^{-1})^*$, the projections r_i^* commute with μ and $\mu|_{M_i}$ are irreducible, $M_i = r_i^*L$. Clearly, there are i, j such that $0 \neq r_i^* p_{31} r_j \in B(L, M)$. Set $t = r_i^* p_{31}$. As r_i^* commutes with μ,

$$t\lambda(g) = \mu(g)t \text{ and } t\xi(g) = \mu(g)r_i^*p_{32} - r_i^*p_{32}U(g) \text{ for } g \in G. \tag{3.11}$$

We claim that there is an operator $s\colon M \to L$ such that $s\mu(g) = \lambda(g)s$ and $st \neq 0$. Indeed, the restriction $t' = t|_{L_j}$ considered as operator from L_j to M_i is non-zero and satisfies, by (3.11), the condition $t'\lambda(g)z = \mu(g)t'z$ for $z \in L_j$. As $\lambda|_{L_j}$ and $\mu|_{M_i}$ are irreducible, t' is invertible by the Shur Lemma. Denote by $s' : M_i \to L_j$ the inverse of t' and extend s' to $s\colon M \to L$ by setting $s = s'r_i^*$. Then $sty = st'y = s'r_i^*t'y = s't'y = y$ for $y \in L_j$. In particular, $st \neq 0$.

Let us show that $\lambda(g)s = s\mu(g)$. For $y \in L$, we have $x := r_i^*y \in M_i$ and $z := s'x \in L_j$, so that

$$\begin{aligned}\lambda(g)sy &= \lambda(g)s'x = s't'\lambda(g)z = s'\mu(g)t'z = s'\mu(g)t's'x \\ &= s'\mu(g)r_i^*y = s'r_i^*\mu(g)y = s\mu(g)y.\end{aligned}$$

Thus $st\lambda(g) = s\mu(g)t = \lambda(g)st$, so that st belongs the commutant $\lambda(G)'$ of $\lambda(G)$ and, by (3.11),

$$st\xi(g) = s\mu(g)r_i^*p_{32} - sr_i^*p_{32}U(g) = \lambda(g)T - TU(g), \text{ where } T = sr_i^*p_{32}.$$

We have proved that the set S of all operators $w \in \lambda(G)'$, for which the map $g \mapsto w\xi(g)$ is a coboundary, is non-zero. The algebra $\lambda(G)'$ is semisimple, since it is isomorphic to the direct sum of full matrix algebras by the Schur

Lemma. As S is a left ideal in $\lambda(G)'$, it contains a non-zero projection r (see [H, Lemma 1.3.1]). If $r \neq \mathbf{1}_L$, take $p = r$ and $q = 0$ in (3.10); if $r = \mathbf{1}_L$ then all r_i belong to S and we may set $p = r_1$ and $q = 0$.

Now let $p_{31} = 0$. Then the condition $P^2 = P$ implies $p_{32}p_{21} = 0$. As P is J-orthogonal ($P = JP^*J$), it follows that $p_{32} = p_{21}^* I$. So $p_{21}^* I p_{21} = p_{32}p_{21} = 0$ and the subspace $F = p_{21}L$ of $\mathfrak{H}$ is neutral, since $[p_{21}x, p_{21}y] = (Jp_{21}x, p_{21}y) = (p_{21}^* I p_{21}x, y) = 0$ for $x, y \in L$. Moreover, F is invariant under U. Indeed, as $p_{31} = 0$ and P commutes with π, we have from (3.4) that $U(g)p_{21}x = p_{21}\lambda(g)x \in p_{21}L$ for $x \in L$. As U is non-singular, $p_{21}L = \{0\}$. Thus $p_{21} = 0$, so that $p_{32} = p_{21}^* I = 0$.

Set $p = p_{11}$ and $q = p_{22}$. Since P is a projection, p and q are projections and $q = Iq^*I = q^\sharp$, as $P = JP^*J$. As $P\pi(g) = \pi(g)P$, the projections p and q commute with λ and U, respectively, and $p\xi(g) - \xi(g)q = \lambda(g)p_{12} - p_{12}U(g)$. Hence $p\xi(g)(\mathbf{1}_\mathfrak{H} - q)$ is a (λ, U)-coboundary, since

$$\begin{aligned} p\xi(g)(\mathbf{1}_\mathfrak{H} - q) &= (p\xi(g) - \xi(g)q)(\mathbf{1}_\mathfrak{H} - q) \\ &= \lambda(g)p_{12}(\mathbf{1}_\mathfrak{H} - q) - p_{12}(\mathbf{1}_\mathfrak{H} - q)U(g). \end{aligned}$$

Similarly, $(\mathbf{1}_L - p)\xi(g)q$ is a (λ, U)-coboundary. Thus σ is a (λ, U)-coboundary.

In particular, if $p = p_{11} = 0$ then $p_{33} = p_{11}^* = 0$ and the projection P maps H into $L \oplus \mathfrak{H}$. Since $[x + y, x + y] = [y, y]$ for all $x \in L$, $y \in \mathfrak{H}$, and PH cannot be a positive subspace of H, we have that the subspace $q\mathfrak{H} = p_{22}\mathfrak{H}$ is not positive.

If $p = \mathbf{1}_L$ then $p_{33} = \mathbf{1}_L$ and the projection $\mathbf{1}_H - P$ maps H into $L \oplus \mathfrak{H}$. Repeating the above argument, we obtain that $\mathbf{1}_\mathfrak{H} - q$ maps $\mathfrak{H}$ on a subspace which is not positive.

If π is completely singular then $\mathfrak{H}$ is positive. Hence the cases $p = 0, \mathbf{1}_L$ are not possible. □

4. Decomposition of J-unitary representations of nilpotent groups

From now on G is a connected, locally compact nilpotent group. As the structure of non-singular representations of nilpotent groups is described in Theorem 1.1, we restrict our study to singular representations on Π_k-spaces, that is, double extensions $\pi = \mathfrak{ee}(\lambda, U, \xi, \gamma)$ on $L \oplus \mathfrak{H} \oplus M$, $\lambda = \pi|_L$ and $\dim L < \infty$. Since $\lambda^\sharp$ is a representations on L and we identify L and M, we have from (2.8) that

$$L = \sum_{\chi \in \operatorname{sign}(\lambda)} \dot{+} L_\chi \text{ and } M = \sum_{\omega \in \operatorname{sign}(\lambda^\sharp)} \dot{+} M_\omega, \tag{4.1}$$

where L_χ are λ-invariant and M_ω are $\lambda^\sharp$-invariant. For $\chi \in \operatorname{sign}(\lambda)$, let p_χ be the projection on L_χ along the sum of all other $L_{\chi'}$.

Let Ω_1, Ω_2 be sets of characters on G. We write

$$\Omega_1 \mathrel{\mathring{=}} \Omega_2 \text{ if } \Omega_1 \cup \Omega_1^* = \Omega_2 \cup \Omega_2^*, \text{ where } \Omega^* = \{\chi^* \colon \chi \in \Omega\}. \tag{4.2}$$

The representation π may have several maximal neutral invariant subspaces L. The following lemma describes the dependance of $\operatorname{sign}(\lambda)$ and $\operatorname{sign}(\lambda^\sharp)$ on the choice of L.

Lemma 4.1. (i) $\operatorname{sign}(\lambda^\sharp) = \operatorname{sign}(\lambda)^*$ *and* $M_{\chi^*} = p_\chi^* M$ *for* $\chi \in \operatorname{sign}(\lambda)$.

(ii) M_{χ^*} *and* L_χ *are skew-related and* M_{χ^*} *is* J*-orthogonal to all* $L_{\chi'}$, $\chi' \neq \chi$.

(iii) *Let* L, L' *be maximal neutral invariant subspaces of* π, $\lambda = \pi|_L$ *and* $\lambda' = \pi|_{L'}$. *Then* $\dim L = \dim L'$ *and* $\operatorname{sign}(\lambda) \mathrel{\mathring{=}} \operatorname{sign}(\lambda')$, *so that they have the same unitary characters.*

Proof. (i) As p_χ commute with λ, p_χ^* commute with $\lambda^\sharp$. Thus the subspace $p_\chi^* M \approx p_\chi^* L$ is invariant for $\lambda^\sharp$. Set $n_\chi = \dim L_\chi$. Then $(\lambda(g) - \chi(g)\mathbf{1}_L)^{n_\chi} p_\chi = 0$ for all $g \in G$. Hence

$$(\lambda^\sharp(g) - \chi^\sharp(g)\mathbf{1}_L)^{n_\chi} p_\chi^* = p_\chi^* (\lambda^\sharp(g) - \chi^\sharp(g)\mathbf{1}_L)^{n_\chi}$$
$$= ((\lambda(g^{-1}) - \chi(g^{-1})\mathbf{1}_L)^{n_\chi} p_\chi)^* = 0.$$

Thus $\operatorname{sign}(\lambda^\sharp) = \{\chi^* \colon \chi \in \operatorname{sign}(\lambda)\} = \operatorname{sign}(\lambda)^*$ and $M_{\chi^*} = p_\chi^* M$ for each $\chi \in \operatorname{sign}(\lambda)$.

(ii) Let $0 \neq x \in L_\chi$. Then $x = p_\chi x$. Set $y = p_\chi^* x$ and consider it as an element of M. Then $y \in M_{\chi^*}$ and $[x, y] = (x, Jy) = (x, y)$, where y is considered as an element of L. Hence $[x, y] = (x, y) = (x, p_\chi^* x) = (p_\chi x, x) = (x, x) \neq 0$. In the same way we show that, for each $0 \neq z \in M_{\chi^*}$, there is $u \in L_\chi$ such that $[u, z] \neq 0$. Thus M_{χ^*} and L_χ are skew-related. Similarly, M_{χ^*} is J-orthogonal to L_ω, $\omega \neq \chi$, as $p_\omega p_\chi = 0$.

(iii) It follows from Corollary 1.12(ii) [KS] that $\dim L = \dim L'$. If $L \cap L' = \{0\}$ then L and L' are skew-related. As in (i) and (ii), we have $\operatorname{sign}(\lambda') = \operatorname{sign}(\lambda)^*$ and each subspace L_χ is skew-related to L'_{χ^*} and J-orthogonal to all $L'_{\chi'}$, $\chi' \neq \chi^*$.

If $K = L \cap L' \neq \{0\}$, then π generates a quotient J-unitary representation ρ on the Π_n-space $K^{[\perp]}/K$, $n < k$ (see [KS]). The subspaces $\widehat{L} = L/K$ and $\widehat{L'} = L'/K$ of $K^{[\perp]}/K$ are maximal neutral subspaces invariant for ρ and $\widehat{L} \cap \widehat{L'} = \{0\}$. As above, $\operatorname{sign}(\rho_{\widehat{L'}}) = \{\chi^* \colon \chi \in \operatorname{sign}(\rho_{\widehat{L}})\}$. As $\operatorname{sign}(\lambda) = \operatorname{sign}(\pi_K) \cup \operatorname{sign}(\rho_{\widehat{L}})$ and $\operatorname{sign}(\lambda') = \operatorname{sign}(\pi_K) \cup \operatorname{sign}(\rho_{\widehat{L'}})$, we conclude the proof. □

If $\chi \in \operatorname{sign}(\lambda)$ is non-unitary, χ^* may belong to $\operatorname{sign}(\lambda')$ while χ does not. Indeed, let $H = \mathbb{C}e_1 \oplus \mathbb{C}e_2$,

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad [x, y] = (Jx, y) \text{ and } \pi(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

for $t \in \mathbb{R}$. Then π is a J-unitary representation of $\mathbb{R}$ on a Π_1-space H, $L = \mathbb{C}e_1$ and $M = \mathbb{C}e_2$ are skew-related maximal neutral invariant subspaces, $\chi(t) = e^t$ is a non-unitary character on $\mathbb{R}$, $\operatorname{sign}(\pi|_L) = \chi$ and $\operatorname{sign}(\pi|_M) = \chi^*$.

When G is nilpotent the non-singular part U of a singular representation π can be described more precisely. If π is completely singular then $\mathfrak{H}$ is a positive subspace and U is unitary. In the following proposition we consider the case that π is not completely singular.

Proposition 4.2. (i) *The representation* U *on* $\mathfrak{H}$ *in* (3.4) *is similar to a unitary representation, and* $\mathfrak{H}$ *uniquely decomposes in the* J*-orthogonal sum* $\mathfrak{H} = N[+]P$, *where* N *is a negative and* P *is a positive* U*-invariant subspaces.*

(ii) *The projection* q *on* N *along* P *is* J*-orthogonal,* $I = \mathbf{1}_{\mathfrak{H}} - 2q$ *is an isometry in the scalar product* $\langle u, v\rangle = [Iu, v]$, *for* $u, v \in \mathfrak{H}$, *and* $\mathfrak{H} = N \langle + \rangle P$.

(iii) U *is unitary in* $\langle \cdot, \cdot \rangle$. *If* p *is the orthoprojection in* $\langle \cdot, \cdot \rangle$ *on a* U*-invariant space* K *in* $\mathfrak{H}$, *then* p *is* J*-orthogonal* ($Ip^*I = p$), *commutes with* q *and* $K = (K \cap N) \langle + \rangle (K \cap P)$.

Proof. As U is non-singular, (i) follows from Theorem 1.1 and (ii) from Proposition 3.1.

(iii) As U is J-unitary and commutes with q, it is unitary in $\langle \cdot, \cdot \rangle$, since

$$\langle U(g)u, v\rangle = [IU(g)u, v] = [U(g)Iu, v] = [Iu, U(g)v] = \langle u, U(g)v\rangle$$

for $u, v \in \mathfrak{H}$. As $\dim N < \infty$ and U_N is unitary, $N = N_{\chi_1} \oplus ... \oplus N_{\chi_n}$, each N_{χ_k} is U-invariant and $\dim N_{\chi_k} = 1$. As U is non-singular, $\text{sign}(U_N)$ and U_P are eigen-disjoint. Hence U_N and U_P have no non-zero intertwining operators. Indeed, if $WU_N = U_P W$ for $W \in B(N, P)$, then $(U_P(g) - \chi_k(g)\mathbf{1})Wx = W(U_N(g) - \chi_k(g)\mathbf{1})x = 0$ for $x \in N_{\chi_k}$. Hence $Wx = 0$, as U_P has no χ_k-eigenvectors. Thus $W = 0$.

Let p have form $p = (p_{ij})$ with respect to the decomposition $\mathfrak{H} = N \langle + \rangle P$. As p and U commute, $U|_N p_{12} = p_{12} U|_P$. By the above, $p_{12} = 0$. As $p^* = p$, we have $p_{21} = 0$. Thus p commutes with q and with $I = \mathbf{1}_{\mathfrak{H}} - 2q$. Hence $Ip^*I = p$ and $K = (K \cap N) \langle + \rangle (K \cap P)$. □

We now obtain an important corollary of Proposition 3.3. For $\Omega \subseteq \text{sign}(\lambda)$, set $L_\Omega = \sum_{\chi \in \Omega} \dot{+} L_\chi$.

Corollary 4.3. *Let* $\pi = \mathfrak{ee}(\lambda, U, \xi, \gamma)$ *be a representation on* $H = L[\oplus]\mathfrak{H} \oplus M$. *Suppose that*

1) $\mathfrak{H} = \mathfrak{H}_1 \dot{+} \mathfrak{H}_2$ *where* $\mathfrak{H}_1$, $\mathfrak{H}_2$ *are* U*-invariant*;

2) $\text{sign}(\lambda) = \Omega_1 \cup \Omega_2$ *and* $\Omega_1 \cap \Omega_2 = \Omega_1^* \cap \Omega_2 = \varnothing$;

3) Ω_1 *is sectionally spectrally disjoint with* $U|_{\mathfrak{H}_2}$, Ω_2 *is sectionally spectrally disjoint with* $U|_{\mathfrak{H}_1}$.

Then $H = H_1[+]H_2$, *where* $H_i = (L_{\Omega_i}[+]\mathfrak{H}_i') \dot{+} M_i$ *are invariant subspaces,* L_{Ω_i} *are maximal neutral invariant subspaces of* H_i *and* $\dim M_i = \dim L_{\Omega_i}$ *for* $i = 1, 2$. *Moreover,* $\mathfrak{H}_i' = \{-T_i x \dot{+} x\colon x \in \mathfrak{H}_i\}$ *for some bounded operators* $T_i \in B(\mathfrak{H}_i, L_{\Omega_j})$, $i \neq j$, *so that* $\dim \mathfrak{H}_i' = \dim \mathfrak{H}_i$.

If $\Omega_1 = \text{sign}(\lambda)$ *then* $H_1 = L \dot{+} \mathfrak{H}_1 \dot{+} M_1$, $H_2 = \{-Tx \dot{+} x : x \in \mathfrak{H}_2\}$ *for some* $T \in B(\mathfrak{H}_2, L)$, *and the representation* $\pi|_{H_2}$ *is non-singular.*

If $\mathfrak{H}_2 = \mathfrak{H}$ *then* $H_1 = L_{\Omega_1} \dot{+} M_1$, *where* M_1 *is skew-related to* L_{Ω_1}, *and* L_{Ω_2} *is a maximal neutral invariant subspace of* H_2.

Proof. As U is non-singular, $\mathfrak{H} = \mathfrak{H}_1[+]\mathfrak{H}_1^{[\perp]}$ and $\mathfrak{H}_1^{[\perp]}$ is U-invariant. As $\mathfrak{H} = \mathfrak{H}_1 \dot{+} \mathfrak{H}_2$, $\mathfrak{H}_1^{[\perp]} = \{Tx + x\colon x \in \mathfrak{H}_2\}$ for a $T \in B(\mathfrak{H}_2, \mathfrak{H}_1)$, and $TU_{\mathfrak{H}_2} =$

$U_{\mathfrak{H}_1}T$. The operator $S = T + \mathbf{1}_{\mathfrak{H}_2}$ from $\mathfrak{H}_2$ to $\mathfrak{H}_1^{[\perp]}$ has bounded inverse and $SU_{\mathfrak{H}_2} = U_{\mathfrak{H}_1^{[\perp]}}S$. Hence $\mathrm{Sp}(U_{\mathfrak{H}_1^{[\perp]}}(g)) = \mathrm{Sp}(U_{\mathfrak{H}_2}(g))$ for $g \in G$. As Ω_1 is sectionally spectrally disjoint with $U_{\mathfrak{H}_2}$, it is sectionally spectrally disjoint with $U_{\mathfrak{H}_1^{[\perp]}}$.

The projection p on L_{Ω_1} along L_{Ω_2} commutes with λ. The projection q on $\mathfrak{H}_1$ along $\mathfrak{H}_1^{[\perp]}$ is J-orthogonal and commutes with U. Set $\lambda_i = \lambda|_{L_{\Omega_i}}$. Since Ω_1 and $U_2 = U|_{\mathfrak{H}_1^{[\perp]}}$ are sectionally spectrally disjoint, and since Ω_2 and $U_1 = U|_{\mathfrak{H}_1}$ are sectionally spectrally disjoint, $\mathcal{H}^1(\lambda_1, U_2) = \mathcal{H}^1(\lambda_2, U_1) = 0$ by Corollary 2.4. Hence the (λ_2, U_1)-cocycle $\xi_{21} = (\mathbf{1}_L - p)\xi q$ and the (λ_1, U_2)-cocycle $\xi_{12} = p\xi(\mathbf{1}_{\mathfrak{H}} - q)$ are coboundaries. As $\mathrm{sign}(\lambda_i) = \Omega_i$, we have $\mathrm{sign}(\lambda_1^\sharp) = \Omega_1^*$ by Lemma 4.1. As $\Omega_1^* \cap \Omega_2 = \varnothing$, it follows from Corollary 2.4 that $\mathcal{H}^1(\lambda_2, \lambda_1^\sharp) = 0$. The rest follows from Proposition 3.3. $\square$

As above, let L be a maximal neutral invariant subspace of a representation $\pi = \mathfrak{ee}(\lambda, U, \xi, \gamma)$ in (3.4) on $H = L[\oplus]\mathfrak{H} \oplus M$, where $\mathfrak{H}$ is a Π_n-space, $n < k$, and M ($M \approx L$) is skew-related to L. For $\chi \in \mathrm{sign}(\lambda)$, consider the χ-eigenspace $\mathfrak{H}^\chi$ of U

$$\mathfrak{H}^\chi = \{x \in \mathfrak{H}\colon U(g)x = \chi(g)x \text{ for all } g \in G\}. \tag{4.3}$$

By Proposition 4.2, U is non-degenerate and similar to a unitary representation. Hence if $\mathfrak{H}^\chi \neq \{0\}$ then χ is unitary and $\mathfrak{H}^\chi$ is positive or negative; otherwise it has a neutral U-invariant subspace. Set

$$\mathrm{usign}(\lambda) = \{\chi = \chi^* \in \ \mathrm{sign}(\lambda) : \mathfrak{H}^\chi \neq \{0\}\}. \tag{4.4}$$

All subspaces $\mathfrak{H}^\chi$, $\chi \in \mathrm{usign}(\lambda)$, are mutually J-orthogonal. Hence there is a U-invariant subspace $\mathfrak{H}^0 \subseteq \mathfrak{H}$ such that each $\chi \in \mathrm{sign}(\lambda)$ is eigen-disjoint with $U|_{\mathfrak{H}^0}$. Thus $H = L[\oplus]\mathfrak{H} \oplus M$,

$$\mathfrak{H} = \mathfrak{H}^\Omega[+]\mathfrak{H}^0,\ \mathfrak{H}^\Omega = \sum_{\chi \in \mathrm{usign}(\lambda)} [+]\mathfrak{H}^\chi \text{ and } L = \sum_{\chi \in \mathrm{sign}(\lambda)} \dot{+} L_\chi. \tag{4.5}$$

Lemma 4.4. *The representation π on H has χ-eigenspaces E^χ, $\chi \in$ usign(λ), such that*

1) $E = \sum_{\chi \in \mathrm{usign}(\lambda)} [+]E^\chi$ *is a non-degenerate subspace and $\pi|_E$ is non-singular,*
2) $H = K[+]E$ *where K is π-invariant and decomposition* (4.5) *has form $K = L[\oplus]\mathfrak{K} \oplus M'$, where $\mathfrak{K} = \mathfrak{K}^\Omega[+]\mathfrak{H}^0$,*

$$\mathfrak{K}^\Omega = \sum_{\chi \in \mathrm{usign}(\lambda)} [+]\mathfrak{K}^\chi \textit{ and } \dim \mathfrak{K}^\chi \le n_G \dim L_\chi \textit{ for all } \chi. \tag{4.6}$$

Proof. Set $V = U|_{\mathfrak{H}^\Omega}$. Then $L[\oplus]\mathfrak{H}^\Omega$ is invariant and $\pi|_{L[\oplus]\mathfrak{H}^\Omega} = \begin{pmatrix} \lambda & \xi \\ 0 & V \end{pmatrix}$, where $\xi = (\xi_{\omega\chi})_{\omega \in \mathrm{sign}(\lambda), \chi \in \mathrm{usign}(\lambda)}$ is a (λ, V)-cocycle and $\xi_{\omega\chi} \in B(\mathfrak{H}^\chi, L_\omega)$. As λ and V are block-diagonal, each $\xi_{\omega\chi}$ is a $(\lambda_\omega, V|_{\mathfrak{H}^\chi})$-cocycle. By Corollary

2.4, $\mathcal{H}^1(\lambda_\omega, V|_{\mathfrak{H}^\chi}) = 0$ if $\omega \neq \chi$. Hence, by (2.3), $\xi_{\omega\chi}(g) = \lambda_\omega(g)T_{\omega\chi} - T_{\omega\chi}U_{\mathfrak{H}^\chi}(g)$ for some operators $T_{\omega\chi} \in B(\mathfrak{H}^\chi, L_\omega)$. For $\chi \in \mathrm{usign}(\lambda)$, set

$$T_\chi = \sum_{\omega\in\mathrm{sign}(\lambda),\chi\neq\omega} T_{\omega\chi} \text{ and } \mathfrak{L}^\chi = \{-T_\chi y \dotplus y\colon y \in \mathfrak{H}^\chi\} \subset L \dotplus \mathfrak{H}^\chi.$$

Since each $\mathfrak{H}^\chi$ is positive or negative, each $\mathfrak{L}^\chi$ is positive or negative. Set also $\mathfrak{L}^\chi = \{0\}$ for $\chi \in \mathrm{sign}(\lambda)\backslash\mathrm{usign}(\lambda)$. Then the spaces $L_\chi \dotplus \mathfrak{L}^\chi$ are invariant, $\dim \mathfrak{L}^\chi = \dim \mathfrak{H}^\chi$,

$$L[+]\sum_{\chi\in\mathrm{usign}(\lambda)}[+]\mathfrak{H}^\chi = \sum_{\chi\in\mathrm{sign}(\lambda)}[+](L_\chi \dotplus \mathfrak{L}^\chi),$$

$$\pi|_{L_\chi\dotplus\mathfrak{L}^\chi} = \begin{pmatrix} \lambda_\chi & \xi_\chi \\ 0 & \chi\mathbf{1}_{\mathfrak{L}^\chi}\end{pmatrix} \text{ if } \chi \in \mathrm{usign}(\lambda).$$

Set $E^\chi = \cap_{g\in G}\ker\xi_\chi(g) \subseteq \mathfrak{L}^\chi$ for $\chi \in \mathrm{usign}(\lambda)$. Then E^χ is a χ-eigenspace of π and $\mathfrak{L}^\chi = \mathfrak{K}^\chi[+]E^\chi$ for some $\mathfrak{K}^\chi \subseteq \mathfrak{L}^\chi$. By Lemma 2.5, $\dim \mathfrak{K}^\chi \le n_G \dim L_\chi$. Thus we have

$$H = \left(L[+]\mathfrak{K}^\Omega[+]\mathfrak{H}^0[+]E\right) \oplus M,$$
$$\text{where } \mathfrak{K}^\Omega = \sum_{\chi\in\mathrm{usign}(\lambda)}[+]\mathfrak{K}^\chi \text{ and } E = \sum_{\chi\in\mathrm{usign}(\lambda)}[+]E^\chi.$$

As $\mathfrak{L}^\chi$ are positive or negative, E^χ are positive or negative. Thus E is non-degenerate. By Proposition 3.1, $H = K[+]E$, $K = E^{[\perp]}$ and $L[+]\mathfrak{K}^\Omega[+]\mathfrak{H}^0 \subseteq K$. As K is non-degenerate, there is a scalar product on K and a subspace M' skew-related to L such that

$$K = \left(L[\oplus]\mathfrak{K}^\Omega[+]\mathfrak{H}^0\right) \oplus M' \text{ and } \dim M' = \dim M = \dim L$$

which completes the proof. □

Corollary 4.5. *Let* $\mathrm{sign}(\lambda) = \Omega_1 \cup \Omega_2$, $(\Omega_1 \cup \Omega_1^*) \cap \Omega_2 = \varnothing$. *If* Ω_1 *is sectionally spectrally disjoint with* $U|_{\mathfrak{H}^0}$ *in* (4.5), *then*

$$H = H_1[+]H_2, \text{ where } H_1, H_2 \text{ are invariant subspaces,}$$
$$L_{\Omega_i} = \sum_{\omega\in\Omega_i}\dotplus L_\omega \text{ are maximal neutral invariant subspaces in } H_i \tag{4.7}$$

and $\dim H_1 < \infty$.

Proof. By Lemma 4.4, $H = K[+]E$, $L \subset K$, E is the J-orthogonal sum of eigenspaces of π and K has decomposition (4.6). Set $\Phi = \mathrm{usign}(\lambda)$ (see (4.4)), $R_1 = \oplus_{\chi\in\Omega_1\cap\Phi}\mathfrak{K}^\chi$ and $R_2 = \mathfrak{H}^0 \oplus (\oplus_{\chi\in\Omega_2\cap\Phi}\mathfrak{K}^\chi)$. Each $\chi \in \Omega_1$ is sectionally spectrally disjoint with all $U|_{\mathfrak{K}^\omega} = \omega\mathbf{1}_{\mathfrak{K}^\omega}$, $\omega \in \Omega_2 \cap \Phi$, and with $U|_{\mathfrak{H}^0}$. Thus Ω_1 and $U|_{R_2}$ are sectionally spectrally disjoint. Similarly, Ω_2 and $U|_{R_1}$ are sectionally spectrally disjoint.

By Corollary 4.3, $K = K_1[+]K_2$, where K_i, $i = 1, 2$, are invariant subspaces, L_{Ω_i} are maximal neutral invariant subspaces in K_i and

$$K_1 = (L_{\Omega_1}[+]\mathcal{H}) \oplus M_1,$$
$$L_{\Omega_1}^{[\perp]} \cap K_1 = L_{\Omega_1}[+]\mathcal{H},\ M_1 \text{ is skew-related to } L_{\Omega_1},$$
$$\mathcal{H} = \{-Tx + x\colon x \in R_1\},\ \text{for some operator } T \in B(R_1, L_{\Omega_2}).$$

Then $\dim M_1 = \dim L_{\Omega_1}$. By (4.6),

$$\dim \mathcal{H} = \dim R_1 = \sum_{\chi\in\Omega_1} \dim \mathfrak{K}^{\chi} \le n_{_G} \sum_{\chi\in\Omega_1} \dim L_{\chi} < \infty.$$

Thus $\dim K_1 < \infty$. Set $H_1 = K_1$, $H_2 = K_2[+]E$. □

Let N be a maximal negative invariant subspace. Then $\dim N \le k$ and $H = N[+]N^{[\perp]}$. If $\pi_{N^{[\perp]}}$ is Π-decomposable then $N^{[\perp]} = H_1[+]H_2$, where H_1, H_2 are invariant subspaces. By Proposition 3.1, they are Π_{n_1}- and Π_{n_2}-spaces, $0 < \max(n_1, n_2) < k$. Continuing this and using (3.3), we get

Lemma 4.6. *Let π be a J-unitary representation on a Π_k-space H and N be a maximal negative invariant subspace. Then either*

$$\textit{either } H = N[+]P,\ \textit{where } \dim N = k \textit{ and } P \textit{ is positive},$$
$$\textit{or } H = N[+]H_1[+]...[+]H_n, \tag{4.8}$$

where all H_i are invariant Π_{k^i}-spaces, $k^i > 0$, $\pi|_{H_i}$ are non-Π-decomposable.

Note that for some summands in (4.8) the inequality $k_-^i \le k_+^i$ can fail. As $\pi|_N$ is similar to unitary, it decomposes into a finite sum of one-dimensional unitary representations.

Consider now some particular cases of Corollary 4.5. Let π be a non-Π-decomposable representation of G on H and $\chi \in \text{sign}(\lambda)$. Set $\Omega_1 = \{\chi, \chi^*\} \cap \text{sign}(\lambda)$ and $\Omega_2 = \text{sign}(\lambda) \diagdown \Omega_1$.

Let χ be non-unitary. As $U|_{\mathfrak{H}^0}$ is similar to a unitary representation, it is spectrally disjoint with Ω_1. Since usign(λ) in (4.4) consists of unitary characters, $\Omega_1 \cap$ usign$(\lambda) = \varnothing$. As π is non-Π-decomposable, it follows from Corollary 4.5 and its proof that $\text{sign}(\lambda) = \Omega_1 \subseteq \{\chi, \chi^*\}$, that $L = L_{\Omega_1}$, $H_1 = L \oplus M$, where L and M are skew-related, and H_2 is positive.

Let χ be unitary and $\dim H < \infty$. Since G is nilpotent and U is similar to a unitary representation, $\mathfrak{H}^0 = \sum_{i=1}^n \oplus \mathfrak{H}^{\omega_i}$ is a finite sum of ω_i-eigenspaces of U. As χ is eigen-disjoint with $U|_{\mathfrak{H}^0}$, they are spectrally disjoint. If $\Omega_2 \ne \varnothing$, then π is Π-decomposable by Corollary 4.5. Thus $\text{sign}(\lambda) = \{\chi\}$.

Combining all this, we have the following summary of the results of this subsection.

Theorem 4.7. *Each J-unitary representation of a connected nilpotent group G on a Π_k-space decomposes in a finite sum of summands of the following types*:

1) *a representation on a positive subspace similar to a unitary one*;
2) *a unitary representation on a one-dimensional negative space*;

3) *a finite-dimensional non-*Π*-decomposable representation with* $\operatorname{sign}(\lambda) = \{\chi\}$ *for a unitary* χ;
4) *a finite-dimensional non-*Π*-decomposable representation on* $L \oplus M$, *where* L *is neutral, invariant and skew-related to* M, *and* $\operatorname{sign}(\lambda) \subseteq \{\chi, \chi^*\}$ *for a non-unitary* χ;
5) *a non-*Π*-decomposable representation* $\mathfrak{ee}(\lambda, U, \xi, \gamma)$ *such that* $\operatorname{sign}(\lambda)$ *consists of unitary characters and* $U = U^\Omega \oplus U^0$, *where* U^Ω *acts on a space* $\mathfrak{H}^\Omega$ *with* $\dim \mathfrak{H}^\Omega \le n_G \dim L$ *and* $\operatorname{sign}(U^\Omega) \subseteq \operatorname{sign}(\lambda)$, *and where* U^0 *acts on a space* $\mathfrak{H}^0$ *with* $\dim \mathfrak{H}^0 = \infty$ *and is eigen-disjoint but not spectrally disjoint with each* $\chi \in \operatorname{sign}(\lambda)$.

More information about cases 3) and 4) will be obtained in the further sections.

5. Finite-dimensional representations on Π_k-spaces

In this section we consider some important classes of finite-dimensional J-unitary representations of connected, locally compact nilpotent groups and prove that each finite-dimensional J-unitary representation of such a group is the direct sum of these representations.

5.1. Representations $\pi_{k,m}$.

Let $\dim L = k \in \mathbb{N}$ and $\dim \mathfrak{H} = m \in \mathbb{N} \cup \{0\}$. Let λ be a χ_e-representation of G on L, where χ_e is the identity character on G, and let $U(g) = \mathbf{1}_{\mathfrak{H}}$ be a trivial representation of G on $\mathfrak{H}$. We say that $\pi = \mathfrak{ee}(\lambda, U, \xi, \gamma)$ in (3.4) is $\pi_{k,m}$ *representation.*

The following lemma allows us to consider $\pi_{k,m}$-representations only for $m \le kn_G$.

Lemma 5.1. *Let* $\dim \mathfrak{H} > kn_G$. *Then* $H = K[\oplus]P$, *where* K *and* P *are* π*-invariant subspaces,* P *is positive,* $K = L \oplus \mathcal{K} \oplus L$, $\mathcal{K} \subseteq \mathfrak{H}$ *and* $\dim \mathcal{K} \le kn_G$.

Proof. Set $P = \cap_{g \in G} \ker \xi(g)$ and $\mathcal{K} = \mathfrak{H} \ominus P$. By (3.4), P is π-invariant, positive, J-orthogonal to $K = L \oplus \mathcal{K} \oplus L$. Then $H = K[\oplus]P$ and K is π-invariant. By Lemma 2.5, $\dim \mathcal{K} \le kn_G$. □

The structure of $\pi_{k,m}$-representations depends on the structure of λ, ξ and γ. Since non-unitary finite-dimensional representations do not admit reasonable classification even for commutative groups, one cannot hope for a constructive description of the class $\pi_{k,m}$ in general. However, such a description is possible, though quite complicated in a very special and important case of $\pi_{1,m}$-representations on Π_1-spaces.

Representations $\pi_{1,m}$**.** Let $L = \mathbb{C}e$ and $\dim \mathfrak{H} = m \le n_G$. Then $H = L \oplus \mathfrak{H} \oplus L$ is a Π_1-space. Let $\lambda = \lambda^\sharp = \iota$ be the trivial representation of G on L: $\iota(g) \equiv \mathbf{1}_L$. Then

$$\pi_{1,m}(g) = \begin{pmatrix} \mathbf{1}_L & \xi(g) & \gamma(g) \\ 0 & \mathbf{1}_{\mathfrak{H}} & \xi^\sharp(g) \\ 0 & 0 & \mathbf{1}_L \end{pmatrix} \text{ for } g \in G, \qquad (5.1)$$

where ξ is a neutral (ι, U)-cocycle, $\xi(g) \in M_{1,m}(\mathbb{C})$ and γ is a prechain of ξ:

$$\xi(gh) = \xi(h) + \xi(g),\ \gamma(gh) = \gamma(h) + \xi(g)\xi^{\sharp}(h) + \gamma(g),\ \gamma(g)^* = \gamma(g^{-1})$$

for $g, h \in G$. The description of neutral (ι, U)-cocycles and their prechains was obtained in [KS1]. Here we will summarize the results obtained there.

Let $n = n_G$ and $\omega\colon G \to \mathbb{R}^n$ be the composition of the canonical homomorphism $G \to G/G_0$ with an isomorphism $G/G_0 \to \mathbb{R}^n$ (see Proposition 2.1), so that $\omega(g) \in \mathbb{R}^n$ is a column.

For $x \in \mathfrak{H}$, $y \in L$, a rank one operator $x \otimes y$ acts from $\mathfrak{H}$ to L by the formula

$$(x \otimes y)z = (z, x)_{\mathfrak{H}}\, y \text{ for } z \in \mathfrak{H}, \tag{5.2}$$

where $(\cdot, \cdot)_{\mathfrak{H}}$ is the scalar product on $\mathfrak{H}$. Then, for $x, v \in \mathfrak{H}$, $y, u \in L$, $A \in B(L)$ and $B \in B(\mathfrak{H})$

$$\begin{aligned} &(x \otimes y)^* = y \otimes x,\ (x \otimes y)(u \otimes v) = (v, x)(u \otimes y),\\ &A(x \otimes y) = x \otimes Ay,\ (x \otimes y)B = B^* x \otimes y. \end{aligned} \tag{5.3}$$

Recall that $G^{[1]}$ is the closed subgroup of G generated by all commutators $[g, h] = ghg^{-1}h^{-1}$ where $g, h \in G$, and $G^{[2]}$ is the closed subgroup of G generated by all $[g, h]$, where $g \in G$, $h \in G^{[1]}$.

Theorem 5.2. ([KS1]) (i) *Each (ι, U)-cocycle has form $\xi(g) = A\omega(g) \otimes e$, where A is an $m \times n$ matrix. It is neutral if and only if there exists a continuous real-valued function ε on G satisfying*

$$\varepsilon(gh) = \varepsilon(g) + \varepsilon(h) - \operatorname{Im}(A^* A\omega(g), \omega(h))_{\mathfrak{H}} \text{ for } g, h \in G. \tag{5.4}$$

Let $(\cdot, \cdot)_{\mathbb{R}^n}$ be the scalar product in $\mathbb{R}^n$. For each $\zeta \in \mathbb{R}^n$, the corresponding prechain γ_ζ has form

$$\begin{aligned} &\gamma_\zeta(g) = \phi_\zeta(g)\mathbf{1}_L, \text{ where}\\ &\phi_\zeta(g) = -\|A\omega(g)\|^2/2 + i(\zeta, \omega(g))_{\mathbb{R}^n} + i\varepsilon(g). \end{aligned} \tag{5.5}$$

The representation $\pi_{1,m} = \mathfrak{ee}(\iota, U, \xi, \gamma)$ on H has form (5.1) *with $\xi^{\sharp}(g) = \xi(g^{-1})^* = -e \otimes A\omega(g)$.*

(ii) *If the $n \times n$ matrix A^*A has real entries then the (ι, U)-cocycle $\xi = A\omega \otimes e$ is neutral and the functions $\phi_\zeta(g)$ have form* (5.5) *with $\varepsilon = 0$. If $G^{[2]} = G^{[1]}$ (for example, G is commutative) then a cocycle $\xi = A\omega \otimes e$ is neutral if and only if the matrix A^*A has real entries.*

To formulate conditions of neutrality of the (ι, U)-cocycle $\xi = A\omega \otimes e$ in general, that is, when $G^{[2]} \neq G^{[1]}$, we need some additional notation.

Let $E = G/G^{[2]}$ and $Z = E^{[1]}$. Then $H := E/Z \neq \{0\}$, as $G^{[2]} \neq G^{[1]}$. Let $p\colon G \to E$ and $q\colon E \to H$ be the quotient maps. By Proposition 2.1, there are continuous epimorphisms $\omega_H\colon H \to \mathbb{R}^l$ and $\omega_Z\colon Z \to \mathbb{R}^k$ for $l := n_H, k := n_Z \in \mathbb{N}$. It was proved in Corollary 4.5 [KS1] that there exist

1) a Borel locally bounded right inverse $\rho\colon H \to E$ of the map q: $q(\rho(h)) = h$ for $h \in H$;

2) real-valued $n \times n$ ($n = n_G$) matrices $T_1, ..., T_k$ such that, for all $h, h' \in H$,

$$\omega_Z(h \diamond h') = (u_1, ..., u_k) \in \mathbb{R}^k, \text{ where } h \diamond h' = \rho(hh')^{-1}\rho(h)\rho(h') \in Z,$$
$$u_i = (T_i\omega_H(h), \omega_H(h'))_{\mathbb{R}^l}$$

and $(\cdot,\cdot)_{\mathbb{R}^l}$ is the scalar product on $\mathbb{R}^l$. Let $n = n_G$. For an $m \times n$ matrix A, consider an $n \times n$ matrix $S = A^*A = (s_{ij})$.

Theorem 5.3. ([KS1, Theorem 4.7]) *A (ι, U)-cocycle $\xi(g) = (A\omega(g)) \otimes e$ is neutral if and only if*

$$\mathrm{Im}(S) = (\mathrm{Im}\, s_{ij}) = \frac{1}{2}\sum_{j=1}^{k} \sigma_j(T_j - T_j^*) \textit{ for some } \sigma = (\sigma_1, ..., \sigma_k) \in \mathbb{R}^k.$$

Let $(\cdot,\cdot)_{\mathbb{R}^k}$ be the scalar product on $\mathbb{R}^k$. The function ε on G satisfying (5.4) *has form*

$$\varepsilon(g) = (\sigma, \omega_Z(\rho(h_g)^{-1}p(g)))_{\mathbb{R}^k} - \frac{1}{2}(\sigma, \omega_Z(h_g \diamond h_g))_{\mathbb{R}^k},$$

where $h_g = q(p(g)) \in H$ and $g \in G$.

The above construction is more transparent for the nilpotent group $\mathcal{T}_k$ of all $k \times k$ real upper triangular matrices $g = (g_{ij})$ with identity on the main diagonal. Then $g = (\widehat{g}_1, ..., \widehat{g}_{k-1})$, where $\widehat{g}_i = (g_{1,1+i}, ..., g_{k-i,k}) \in \mathbb{R}^{k-i}$ are the diagonals of g. We have (see Proposition 2.1) $n_G = k-1$, $G^{[1]} = \{g \in G$: $\widehat{g}_1 = 0\}$,

$$\begin{aligned} G^{[2]} &= \{g \in G\colon \widehat{g}_1 = \widehat{g}_2 = 0\}, \\ E &= G/G^{[2]} \cong \{g \in G : \widehat{g}_i = 0 \text{ for } i \geq 3\}, \\ H &\cong G/G^{[1]} \cong \mathbb{R}^{k-1}, \quad Z \cong G^{[1]}/G^{[2]} \cong \mathbb{R}^{k-2}, \\ \omega(g) &= \widehat{g}_1 \text{ and } n_Z = k-2. \end{aligned}$$

For $\widehat{g}_1 = (g_{12}, .., g_{(k-1),k}) \in \mathbb{R}^{k-1}$, set

$$\widehat{g}_1 \boxtimes \widehat{g}_1 = (g_{12}g_{23}, g_{23}g_{34}, ..., g_{(k-2),(k-1)}g_{(k-1),k}) \in \mathbb{R}^{k-2} \cong Z.$$

If $h = (h_1, ..., h_{k-1}) \in H \cong \mathbb{R}^{k-1}$, we have $\rho(h) = (\widehat{g}_1, 0, ..., 0) \in E$ with $\widehat{g}_1 = h$. Hence

$$\begin{aligned} h \diamond h &= \rho(h+h)^{-1}\rho(h)^2 = (h+h, 0, ..., 0)^{-1}(h, 0, ..., 0)^2 \\ &= (0, h \boxtimes h, 0, ..., 0) \bmod G^{[2]}. \end{aligned}$$

We have $p(g) = (\widehat{g}_1, \widehat{g}_2, 0, ..., 0) \in E$, so that $h_g = q(p(g)) = \widehat{g}_1 \in H$ and $\omega_Z(h_g \diamond h_g) = \widehat{g}_1 \boxtimes \widehat{g}_1 \in \mathbb{R}^{k-2}$. Continuing these calculations and applying Theorems 5.2 and 5.3, we obtain

Corollary 5.4. *Let $\iota(g) = \mathbf{1}_L$ and $U(g) = \mathbf{1}_{\mathfrak{H}}$ for all $g \in \mathcal{T}_k$, where $L = \mathbb{C}e$ and $\mathfrak{H} = \mathbb{C}^m$, $m \leq k-1$. For a matrix $A \in M_{m\times(k-1)}(\mathbb{C})$, let $S := A^*A = (s_{ij})$*

and $\sigma = (\sigma_1, ..., \sigma_{k-2}) \in \mathbb{R}^{k-2}$ *with* $\sigma_i = 2s_{i,i+1}$. *Then each* (ι, U)*-cocycle has form* $\xi(g) = A\widehat{g}_1 \otimes e$. *It is neutral if and only if*

$$\operatorname{Im} s_{ij} = 0, \ \textit{when } |i-j| > 1. \tag{5.6}$$

If (5.6) *holds then, for each* $\zeta \in \mathbb{R}^{k-1}$, *the prechain* γ_ζ *has form*

$$\gamma_\zeta(g) = \phi_\zeta(g)\mathbf{1}_L, \ \textit{where}$$
$$\phi_\zeta(g) = -\left\|A\widehat{g}_1\right\|^2/2 + i(\zeta, \widehat{g}_1)_{\mathbb{R}^{k-1}} + i(\sigma, \widehat{g}_2 - \frac{1}{2}\widehat{g}_1 \boxtimes \widehat{g}_1)_{\mathbb{R}^{k-2}}. \tag{5.7}$$

The corresponding representation $\pi_{1,m} = \mathfrak{ee}(\iota, I, \xi, \gamma_\zeta)$ *of* $\mathcal{T}_k$ *on* $H = L \oplus \mathfrak{H} \oplus L$ *has form*

$$\pi_{1,m}(g) = \begin{pmatrix} 1 & A\widehat{g}_1 \otimes e & \phi_\zeta(g)\mathbf{1}_L \\ 0 & \mathbf{1}_{\mathfrak{H}} & -e \otimes A\widehat{g}_1 \\ 0 & 0 & 1 \end{pmatrix} \ (\textit{see } (5.2)).$$

We shall now consider two particular cases: $k = 3$ and $k = 4$.

Example 5.5. (i) For $k = 3$,

$$\mathcal{T}_3 = \left\{ g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \tag{5.8}$$

is the real Heisenberg group. Then $m = 0, 1, 2$, $\sigma = (\sigma_1)$ and $\zeta \in \mathbb{R}^2$. If $m \neq 0$ then A^*A is a 2×2 matrix and condition (5.6) holds automatically. Thus (ι, I)-cocycles $\xi(g) = A \begin{pmatrix} x \\ y \end{pmatrix} \otimes e$ are neutral for all $m \times 2$ matrices A.

If $m = 0$ then $A = 0$ and $\sigma_1 = 0$.

If $m = 1$ then $A = (a_{11}, a_{12})$ and $\sigma_1 = 2s_{12} = 2\operatorname{Im}(a_{12}\overline{a_{11}})$.

If $m = 2$ then $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\sigma_1 = 2s_{12} = 2\operatorname{Im}(a_{12}\overline{a_{11}} + a_{22}\overline{a_{21}})$.

Thus Corollary 5.4 gives a complete description of all $\pi_{1,m}$ representations of the group $\mathcal{T}_3$.

(ii) For the group $\mathcal{T}_4$, $0 \le m \le 3$, A is an $m \times 3$ matrix, $S := A^*A = (s_{ij})$, $\sigma = (2s_{12}, 2s_{23})$, $\widehat{g}_1 = (g_{12}, g_{23}, g_{34})$ and $\zeta \in \mathbb{R}^3$. By (5.6), the cocycle $\xi(g) = A\widehat{g}_1 \otimes e$ is neutral if $s_{13} \in \mathbb{R}$.

If $m = 0$ then $A = 0$ and $\sigma = (0, 0)$.

If $m = 1$ then $A = (a_{11}, a_{12}, a_{13})$, $s_{13} = \overline{a_{11}}a_{13} \in \mathbb{R}$, $\sigma_1 = 2\operatorname{Im} a_{12}\overline{a_{11}}$ and $\sigma_2 = 2\operatorname{Im} a_{13}\overline{a_{12}}$.

Similarly, we can consider cases $m = 2, 3$ and obtain a full list of representations $\pi_{1,m}$ of $\mathcal{T}_4$.

Representations $\pi_{k,0}$. Let $m=0$. Then $H=L\oplus L$ with $\dim L=k$,

$$J=\begin{pmatrix} 0 & \mathbf{1}_L \\ \mathbf{1}_L & 0 \end{pmatrix},\ \pi_{k,0}(g)=\begin{pmatrix} \lambda(g) & \gamma(g) \\ 0 & \lambda^\sharp(g) \end{pmatrix},$$
$$\gamma(gh)=\lambda(g)\gamma(h)+\gamma(g)\lambda^\sharp(h) \tag{5.9}$$

and $\gamma(g)^*=\gamma(g^{-1})$. Then γ is a $(\lambda,\lambda^\sharp)$-cocycle and H is a $2k$-dimensional Π_k-space.

If $\gamma\equiv 0$, (5.9) trivially holds. For $\gamma\neq 0$, consider a particular case when $\lambda(g)\equiv \mathbf{1}_L$. Then

$$\gamma(g)=\gamma(g^{-1})^*,\ \gamma(e)=0 \text{ and } \gamma(gh)=\gamma(g)+\gamma(h) \text{ for } g,h\in G.$$

It follows from Proposition 2.1 that there is a linear map δ from $\mathbb{R}^{n_G}$ into the space of $k\times k$ symmetric matrices such that $\gamma(g)=i\delta(\omega(g))$, where ω is the canonical homomorphism from G onto $G/G_0\approx\mathbb{R}^{n_G}$. Thus $\pi_{k,0}(g)=\begin{pmatrix} \mathbf{1}_L & i\delta(\omega(g)) \\ 0 & \mathbf{1}_L \end{pmatrix}$ is a J-unitary representation of G.

For $\lambda\neq\mathbf{1}_L$, we consider the following example. Let $G=\mathbb{R}$ and $\dim L=2$. For $t\in\mathbb{R}$, let

$$\lambda(t)=\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\ \lambda^\sharp(t)=\lambda(-t)^*=\begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix},$$
$$\gamma(t)=it\begin{pmatrix} t^2/3 & t/2 \\ t/2 & 1 \end{pmatrix}\lambda^\sharp(t).$$

Then γ satisfies (5.9) and $\pi_{k,0}(t)=\begin{pmatrix} \lambda(t) & \gamma(t) \\ 0 & \lambda^\sharp(t) \end{pmatrix}$ is a J-unitary representation of G.

5.2. Representations π_{χ,χ^*}.

For a non-unitary character χ, let λ be a χ-representation of G on L, $\dim L=k$, and $\lambda^\sharp(g)=\lambda(g^{-1})^*$. Then $H=L\oplus M$ $(M\sim L)$ is a Π_k-space with $[x,y]=(Jx,y)$, where

$$J=\begin{pmatrix} 0 & \mathbf{1}_L \\ \mathbf{1}_L & 0 \end{pmatrix} \text{ and } \pi_{\chi,\chi^*}(g)=\begin{pmatrix} \lambda(g) & 0 \\ 0 & \lambda^\sharp(g) \end{pmatrix} \tag{5.10}$$

is a J-unitary representation.

For example, the character $\chi(t)=e^t$ on $\mathbb{R}$ is non-unitary and $\pi_{\chi,\chi^*}(t)=\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ is a J-unitary representation of $\mathbb{R}$ on a 2-dimensional Π_1-space.

For the real Heisenberg group $\mathcal{T}_3$ (see (5.8)), let $\lambda(g)\equiv g$ be its identity representation on L, $\dim L=3$. For $\alpha,\beta\in R$, $\chi(g)=e^{\alpha x+\beta y}$ is a non-unitary character on $\mathcal{T}_3$. Its representation π_{χ,χ^*} on a Π_3-space $L\oplus L$ has form $\pi_{\chi,\chi^*}(g)=\begin{pmatrix} \chi(g)g & 0 \\ 0 & \chi(g^{-1})(g^{-1})^* \end{pmatrix}$ for $g\in\mathcal{T}_3$.

5.3. Decomposition of finite-dimensional representations

We will now show the universality of constructions introduced above.

Theorem 5.6. *Let G be a connected locally compact nilpotent group. Each finite-dimensional J-unitary representation of G on a Π_k-space H is the J-orthogonal sum of unitary representations on one-dimensional positive and negative subspaces and*

1) *representations $\chi\pi_{k,m}$, or J-antiequivalent to $\chi\pi_{k,m}$ for unitary characters χ of G;*
2) *representations π_{χ,χ^*} for non-unitary characters χ of G.*

Proof. By Lemma 4.6 and Theorem 4.7, we only need to consider two types of finite-dimensional non-Π-decomposable representations π on a Π_k-space H:

a) $\mathrm{sign}(\lambda) = \{\chi\}$ for a unitary χ;
b) $H = L \oplus M$ and $\mathrm{sign}(\pi|_L) \subseteq \{\chi, \chi^*\}$ for a non-unitary χ.

Case a). In decomposition (4.8) of a representation into non-Π-decomposable components it may happen that $k^i = k^i_+ < k^i_-$ for some invariant Π_{k^i}-spaces H_i. As in Remark 3.2, H_i is a Π_{p^i}-space in the new metric $[\cdot,\cdot]_1 = -[\cdot,\cdot]$ with $p^i = k^i$, $p^i = p^i_- < p^i_+$ and the representation $\pi|_{H_i}$ on $(H_i, [\cdot,\cdot]_1)$ is J-antiequivalent to $\pi|_{H_i}$ on $(H_i, [\cdot,\cdot])$. Thus we can only consider the case $k_- \leq k_+$

As $\dim H < \infty$, we can also assume that π has no invariant subspaces K such that $\pi|_K$ is non-singular, since then, by Theorem 1.1, $\pi|_K$ is a sum of representations on a negative and positive invariant subspaces. As $\mathrm{sign}(\lambda) = \{\chi\}$, we have from (4.5) that

$$H = L[\oplus]\mathfrak{H} \oplus M, \;\; L = L_\chi, \;\; \mathfrak{H} = \mathfrak{H}^\chi[+]\mathfrak{H}^0$$

and the character χ is eigen-disjoint with $U|_{\mathfrak{H}^0}$. As G is nilpotent, $U|_{\mathfrak{H}^0}$ is a finite sum of one-dimensional representations. Since χ is eigen-disjoint with $U|_{\mathfrak{H}^0}$, they are spectrally disjoint. If $\mathfrak{H}^0 \neq \{0\}$, we have from Corollary 4.3 that $H = H_1[+]H_2$ and the representation $\pi|_{H_2}$ is non-degenerate. This contradiction shows that $\mathfrak{H}^0 = \{0\}$.

If $\mathfrak{H}^\chi = \{0\}$ then $H = L \oplus M$, L is skew-related to M, so π is a $\pi_{k,0}$ representation (see (5.9)).

Let $\mathfrak{H}^\chi \neq \{0\}$. Then $H = L[\oplus]\mathfrak{H}^\chi \oplus M$. Note that $\mathfrak{H}^\chi$ can be either negative or positive. If $\mathfrak{H}^\chi$ is a negative subspace then $k_+ = \dim L < k_- = \dim L + \dim \mathfrak{H}^\chi$ which contradicts our assumption. Hence $\mathfrak{H}^\chi$ is positive and $\dim L = k_+ = k$. As π has no positive invariant subspaces, $m = \dim \mathfrak{H}^\chi \leq kn_G$ by Lemma 5.1. Thus $\pi = \chi\pi'$, where π' is a $\pi_{k,m}$ representation.

Case b). By (3.4), (3.5) and Lemma 4.1, $\mathrm{sign}(\pi) = \mathrm{sign}(\pi|_L) \cup \mathrm{sign}(\pi|^\sharp_L) = \{\chi, \chi^*\}$. By Corollary 2.3, $H = L_\chi \dotplus L_{\chi^*}$ where L_χ, L_{χ^*} are π-invariant and $\lambda := \pi|_{L_\chi}$ is a χ-representation.

Let us show that the subspaces L_χ, L_{χ^*} are neutral. Indeed, as $L_\chi^{[\perp]}$ is π-invariant, $K = L_\chi \cap L_\chi^{[\perp]}$ is neutral and π-invariant. If $K \neq L_\chi$ then (see [KS]) $R_\chi = L_\chi/K$ is a Π_n-space and the quotient representation $\widehat{\lambda}$ on R_χ is

J-unitary. By (3.4), $R_\chi = l \oplus \mathfrak{h} \oplus \mathfrak{m}$, where l is a maximal neutral invariant subspace of R_χ. Since λ is a χ-representation, $\widehat{\lambda}$ is also a χ-representation. Hence $\widehat{\lambda}|_l$ and the representation ρ that $\widehat{\lambda}$ generates on $\mathfrak{m}$ are χ-representations. However, as $\rho = (\widehat{\lambda}|_l)^\sharp$ by (3.5), ρ is a χ^*-representation. Thus $\chi = \chi^*$, so χ is unitary. This contradiction shows that $K = L_\chi$ is neutral. Similarly L_{χ^*} is neutral.

As H is non-degenerate, L_χ, L_{χ^*} are skew-related subspaces. Hence they are maximal neutral and $\dim L_\chi = \dim L_{\chi^*}$. Identifying L_{χ^*} with L_χ, we have that, with respect to the decomposition $H = L_\chi \dotplus L_\chi$, π has the same form as π_{χ,χ^*} in (5.10). □

Theorem 5.6 implies that all non-Π-decomposable finite-dimensional representations are either one-dimensional, or of type $\pi_{k,m}$, or of type π_{χ,χ^*}. However, it does not mean that all representations of type $\pi_{k,m}$ or π_{χ,χ^*} are non-Π-decomposable.

It is also interesting to study the following stronger notion of non-decomposability.

Definition 5.7. *A J-unitary representation on H is J-decomposable if there exists a decomposition $H = H_1[+]H_2$, where H_1 and H_2 are invariant subspaces. Otherwise, it is non-J-decomposable.*

We will see later that representations on infinite-dimensional spaces cannot be non-J-decomposable. For finite-dimensional representations non-Π- and non-J-decomposability are closely related: if π is non-Π-decomposable then, choosing the maximal positive invariant subspace P, we have that $H = K[+]P$ where $\pi|_K$ is non-J-decomposable. For π_{χ,χ^*}-representations they are equivalent.

Proposition 5.8. *Set $\pi = \pi_{\chi,\chi^*}$. The following conditions are equivalent.*

(i) *L does not decompose into a direct sum of invariant subspaces.*
(ii) *The representation π is non-Π-decomposable.*
(iii) *The representation π is non-J-decomposable.*

Proof. (ii) $\Longrightarrow$ (i). Assume that $L = L_1 \dotplus L_2$ and L_1, L_2 be π-invariant. Denote by p the projection on L_1 along $L_2 \oplus M$. As L_1 and $L_2 \oplus M$ are π-invariant, $\pi p = p\pi$. Then $p^\sharp = Jp^*J$ is also a projection, as $J^2 = \mathbf{1}_H$, $[p^\sharp x, y] = [x, py]$ for $x, y \in H$, and $p^\sharp$ commutes with π, since

$$\pi(g)p^\sharp \overset{(1.1)}{=} J\pi(g^{-1})^* J p^\sharp = J(p\pi(g^{-1}))^* J$$
$$= J(\pi(g^{-1})p)^* J \overset{(1.1)}{=} p^\sharp \pi(g).$$

Thus the subspace $M_1 := p^\sharp H$ is π-invariant. For $x \in H$ and $y \in M$, $[p^\sharp x, y] = [x, py] = 0$, as $pM = \{0\}$. As M is a maximal neutral subspace, $p^\sharp x \in M$. Hence $M_1 \subseteq M$. If $u \in L_1$ then $[x, u] \neq 0$ for some $x \in H$. Thus $p^\sharp x \in M_1$ and $[p^\sharp x, u] = [x, pu] = [x, u] \neq 0$. Similarly, if $v \in M_1$ then $[z, y] \neq 0$ for some $z \in L_1$. Thus L_1 and M_1 are skew-related, and $K = L_1 \oplus M_1$ is a non-degenerate π-invariant subspace. By (3.2), $H = K[+]K^{[\perp]}$ and $K^{[\perp]}$ is

π-invariant. For $x \in L_2$, $y \in M_1$, we have $[x,y] = [x, p^\sharp y] = [px, y] = 0$. Thus $L_2 \subset K^{[\perp]}$ and π is Π-decomposable.

(iii) $\Longrightarrow$ (ii) is evident.

(i) $\Longrightarrow$ (iii) Assume that $H = K[+]K^{[\perp]}$ and both subspaces are π-invariant. As $\operatorname{sign}(\pi) = \{\chi, \chi^*\}$, we have from Corollary 2.3 that $K = K_\chi \dot{+} K_{\chi^*}$ and $K^{[\perp]} = T_\chi \dot{+} T_{\chi^*}$, where $K_\chi, K_{\chi^*}, T_\chi, T_{\chi^*}$ are π-invariant subspaces. It is easy to see that $K_\chi, T_\chi \in L$ and $K_{\chi^*}, T_{\chi^*} \in M$. As $H = K[+]K^{[\perp]}$, we have $K_\chi \dot{+} T_\chi = L$. If $K_\chi = 0$ then $K \subset M$ and the decomposition $H = K[+]K^{[\perp]}$ does not hold (Proposition 3.1). Thus $K_\chi \neq \{0\}$. Similarly, $T_\chi \neq \{0\}$ which contradicts (i). $\square$

For $\pi_{k,m}$-representations, the problem is more difficult as it needs an analysis of general finite-dimensional representations on L. Below we get a criteria of non-J-decomposability for the case $k = 1$, where the non-Π-decomposability is evident.

We saw earlier that each representation $\pi_{1,m}$ has form (5.1) with $\xi(g) = A\omega(g) \otimes e$ (see Theorem 5.2), where $\omega \colon G \to \mathbb{R}^{n_G}$ is the standard homomorphism, A is a $m \times n_G$ matrix satisfying the conditions of Theorem 5.3 and $\gamma(g) = \phi(g)\mathbf{1}_L$, where the function ϕ is given in (5.5).

Theorem 5.9. *A representation $\pi := \pi_{1,m}$ is non-J-decomposable if and only if* $\ker A^* = \{0\}$.

Proof. Note first that $\ker A^* = \{0\}$ if and only if the cocycle $\xi(g) = A\omega(g) \otimes e$ satisfies the condition $\cap_{g\in G} \ker \xi(g) = \{0\}$. Indeed, $\ker \xi(g)$ is the orthogonal complement of $A\omega(g)$ by (5.2). As ω is surjective, we conclude that $\cap_{g\in G} \ker \xi(g)$ is the orthogonal complement of the image of A.

Now if $\ker A^* \neq \{0\}$ then $K = \cap_{g\in G} \ker \xi(g) \neq \{0\}$ is a non-degenerate invariant subspace. Thus π is J-decomposable. Conversely, let $\cap_{g\in G} \ker \xi(g) = \{0\}$. Then

$$\xi(g)x = 0, \text{ for all } g \in G \text{ and some } x \in \mathfrak{H}, \text{ implies } x = 0. \tag{5.11}$$

If π is J-decomposable then, as $H = L \oplus \mathfrak{H} \oplus M$, there is a J-orthogonal projection $p = (p_{i,j})_{i,j=1}^3 \neq 0, \mathbf{1}_H$ commuting with π. Then $p\pi(g) \equiv \pi(g)p$ implies $p_{31}\xi(g) \equiv 0$ and $\xi(g)p_{21} + \phi(g)p_{31} \equiv 0$. Since $\dim L = 1$, p_{11}, p_{13}, p_{31}, p_{33} are numbers. As $\xi(g) \not\equiv 0$, we have $p_{31} = 0$. Thus $\xi(g)p_{21} \equiv 0$. By (5.11), $\xi(g)p_{21}e \equiv 0 \Longrightarrow p_{21} = 0$. As p is J-orthogonal, $p = Jp^*J$ by (3.1). Then $p_{11} = \overline{p_{33}}$ and $p_{32} = 0$. Since $p^2 = p$, either $p_{33} = 0$, or $p_{33} = 1$. If $p_{33} = 0$ then $p_{11} = \overline{p_{33}} = 0$. Hence $p\pi(g) = \pi(g)p$ for all $g \in G$, so that $\xi(g)p_{22} = 0$. By (5.11), $p_{22} = 0$. Thus $p^3 = 0$, a contradiction.

If $p_{33} = 1$ then $p_{11} = 1$. As $p\pi = \pi p$, we have $\xi(g)(\mathbf{1}_{\mathfrak{H}} - p_{22}) = 0$ for $g \in G$. By (5.11), $p_{22} = \mathbf{1}_{\mathfrak{H}}$. Thus, as $p^2 = p$, we have $2p_{12} = p_{12}$ and $2p_{23} = p_{23}$. Hence $p_{12} = p_{23} = 0$. Then $p^2 = p$ implies $2p_{13} = p_{13}$. Hence $p_{13} = 0$, so that $p = \mathbf{1}_H$, a contradiction. Thus π is non-J-decomposable. $\square$

Remark 5.10. The condition $\ker A^* = 0$ can be rewritten in the following way: $\cap_{g\in G} \ker \xi(g) = \{0\}$. This condition is necessary for a representation

$\pi_{k,m}$ with arbitrary $k \geq 1$ to be non-J-decomposable. What conditions guarantee that a $\pi_{k,m}$-representation is non-Π- or non-J-decomposable?

6. Primary and completely singular representations.

It follows from Theorem 5.6 that singular finite-dimensional non-Π-decomposable representations of connected nilpotent groups on Π_k-spaces possess two features that deserve consideration in the general context. They are 1) *completely singular*, i.e., $\dim L = k$; 2) *primary*, i.e., the restriction $\lambda = \pi|_L$ to a maximal neutral invariant subspace is monothetic: $\text{sign}(\lambda)$ consists of one character.

Our aim here is to understand to which groups and to which representations these properties extend. Firstly, it should be noted that each bounded singular non-Π-decomposable representation π of any group G on a Π_k-space H is completely singular. Indeed, π is similar to a unitary representation ([OST]), so that $H = N[+]P$, where N, P are invariant, N is negative, $\dim N = k$, and P is positive. As π is non-Π-decomposable, $\pi|_N$ is irreducible. Hence, if L is a neutral invariant subspace then $L = \{x + Tx\colon x \in N\}$, where $T \in B(N,P)$, $\pi T|_N = T\pi|_N$ and $[x,x] + [Tx,Tx] = 0$. Thus $\dim L = \dim N = k$.

In particular, continuous representations of compact groups are bounded. So they are completely singular by above. If π is bounded and G is nilpotent then $\dim L = \dim N = 1$, as $\pi|_N$ is irreducible. Thus π is also primary.

On the other hand, if G is not nilpotent, it may have an unbounded finite-dimensional singular non-Π-decomposable representation which is not completely singular. Consider the group

$$G = QU(2) = \left\{ g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C},\ \ |a|^2 - |b|^2 = 1 \right\}$$

and $I = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

The space $K = \mathbb{C}e \oplus \mathbb{C}e$ with indefinite metric $[x,y] = (Ix,y)$ is a Π_1-space. The representation ρ of G on K given by $\rho(g)x = gx$ for $g \in G$, $x \in K$, is irreducible and J-unitary, as $Ig^*I = g^{-1}$.

Consider the group $\widetilde{G} = G \times K \times \mathbb{R}$ with operation $(g,x,t)(h,y,s) = (gh, y + h^*x, t + s + \operatorname{Im}(h^*x, Iy))$ for $g, h \in G$, $x, y \in K$, $t, s \in \mathbb{R}$. Let $L = \mathbb{C}u$. Then $H = L \oplus K \oplus L$ is a Π_2-space with $[\xi,\eta] = (J\xi,\eta)$, where

$$J = \begin{pmatrix} 0 & 0 & \mathbf{1}_L \\ 0 & I & 0 \\ \mathbf{1}_L & 0 & 0 \end{pmatrix} \quad \text{and}$$

$$\pi(g,x,t) = \begin{pmatrix} \mathbf{1}_L & x \otimes e & \left(-\frac{1}{2}(Ix,x) + it\right)(e \otimes e) \\ 0 & g & -e \otimes gIx \\ 0 & 0 & \mathbf{1}_L \end{pmatrix}$$

is a J-unitary representation of $\widetilde{G}$ on H. The subspaces L, $L \oplus K$ are the only non-trivial invariant subspaces. As L is a maximal neutral invariant subspace and $\dim L = 1$, π is not completely singular.

Theorem 6.1. *Each connected locally compact, commutative group G such that $G/G^{[1]}$ is not compact has a singular non-Π-decomposable representation which is not completely singular.*

Proof. As $G/G^{[1]}$ is not compact, it follows from Theorem 26 [M] that G has a normal subgroup G_0 containing $G^{[1]}$ such that $G/G_0 \approx \mathbb{R}^n$ for some $n \neq 0$. Setting $\ker \pi = G_0$, we only have to show that the commutative groups $G = \mathbb{R}^n$ have representations which are not completely singular.

Let $\mathcal{H} = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$, $\mathfrak{H} = L_2(G, dm)$ and $H = \mathcal{H} \oplus \mathfrak{H}$. Let (see (5.2),(5.3))

$$J = I \oplus \mathbf{1}_{\mathfrak{H}}, \text{ where } I = e_1 \otimes e_3 - e_2 \otimes e_2 + e_3 \otimes e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Consider the indefinite form $[x, y] = (Jx, y)$ on H. Then $H_- = \mathbb{C}(e_1 - e_3) \oplus \mathbb{C}e_2$ and $H_+ = \mathbb{C}(e_1 + e_3) \oplus \mathfrak{H}$ are negative and positive subspaces of H and $H = H_- [\oplus] H_+$. Thus H is a Π_2-space.

Let φ be a non-zero additive map from $G = \mathbb{R}^n$ onto $\mathbb{R}$: $\varphi(g + h) = \varphi(g) + \varphi(h)$ for $g, h \in \mathbb{R}^n$ (for example, $\varphi(g) = (g, u)_{\mathbb{R}^n}$ for some $u \in \mathbb{R}^n$). Then the map

$$g \in G \to \sigma(g) = \begin{pmatrix} 1 & \varphi(g) & \frac{\varphi(g)^2}{2} \\ 0 & 1 & \varphi(g) \\ 0 & 0 & 1 \end{pmatrix}$$

is a representation of G on $\mathcal{H}$. Let U be the regular representation of G on $\mathfrak{H}$. Then

$$g \in G \to \pi(g) = \sigma(g) \oplus U(g)$$

is a representation of G on H. Moreover, π is J-unitary, as (see (1.1)),

$$J\pi(g)^* J = I\sigma(g)^* I \oplus U(g)^* = \sigma(-g) \oplus U(-g) = \pi(-g).$$

Let $\mathbb{C}(x + y)$ be an eigenspace of π, where $x \in \mathcal{H}$ and $y \in \mathfrak{H}$. Then $\mathbb{C}y$ is an eigenspace of U. As U has no eigenspaces, $y = 0$ and $\mathbb{C}x$ is an eigenspace of σ. It is easy to see that only $\mathbb{C}e_1$ is an eigenspace of σ and, therefore, of π.

If π is Π-decomposable then $H = H_1[+]H_2$, where H_i are invariant Π_1-subspaces. By Theorem 1.1, both summands have eigenspaces, a contradiction. Thus π is non-Π-decomposable.

It remains to show that π is not completely singular, i.e., it does not have a two-dimensional neutral invariant subspace. Suppose, to the contrary, that N is such a subspace. As G is connected and commutative, N has a basis (f_1, f_2) such that

$$\pi(g)f_1 = \lambda(g)f_1 \text{ and } \pi(g)f_2 = \nu(g)f_1 + \mu(g)f_2 \text{ for } g \in \mathbb{R}^n.$$

As only $\mathbb{C}e_1$ is an eigenspace of π, we can assume that $f_1 = e_1$ and (changing if necessary f_2 by $f_2 - \lambda f_1$ with an appropiate λ) $f_2 = \beta e_2 \oplus \gamma e_3 \oplus y \in N$.

As N is neutral, $0 = [e_1, f_2] = (Je_1, f_2) = (e_3, f_2) = \overline{\gamma}$. Thus $f_2 = \beta e_2 \oplus y$ and, for $g \in G$,

$$\pi(g)f_2 = \sigma(g)\beta e_2 \oplus U(g)y = \beta\varphi(g)e_1 \oplus \beta e_2 \oplus U(g)y.$$

Since $\pi(g)f_2 = \nu(g)e_1 + \mu(g)(\beta e_2 \oplus y)$, we get that $U(g)y = \mu(g)y$. As U has no eigenspaces, $y = 0$. Thus $f_2 = \beta e_2 \in N$ and $\beta \neq 0$. Since f_2 is not neutral but negative, we have a contradiction. □

Remark 6.2. *Theorem* 6.1 *extends to all connected locally compact groups G with non-compact $G/G^{[1]}$ (for example, to the Heisenberg group), since, by Theorem* 26 [M], *G has a normal subgroup G_0 such that $G/G_0 \approx \mathbb{R}^n$.*

We turn now to the question, for which nilpotent groups all non-decomposable singular representations are primary. Our first aim is to show that this is true for commutative groups.

Theorem 6.3. *Let G be a commutative connected, locally compact group. Each singular non-Π-decomposable representation π of G on a Π_k-space H is primary.*

Proof. Denote by G^* the group of all unitary characters of G. As in (4.5), let $H = L[\oplus]\mathfrak{H} \oplus L$, where $L = \sum_{\chi \in \mathrm{sign}(\lambda)} L_\chi$ is a maximal neutral invariant subspace and $\lambda = \pi|_L$. Since π is non-Π-decomposable, it follows from Corollary 4.5 that it suffices to prove our result in the case when $\mathrm{sign}(\lambda)$ has no non-unitary characters, i.e., $\mathrm{sign}(\lambda) \subset G^*$.

Let $\chi \in \mathrm{sign}(\lambda)$. As the representation U on $\mathfrak{H}$ is similar to a unitary representation,

$$\mathfrak{H} = \int_{G^*}^{\oplus} \mathfrak{H}_\omega dP(\omega) \text{ and } U(g) = \int_{G^*}^{\oplus} \omega(g) dP(\omega) \text{ for } g \in G,$$

where P is a spectral measure on G^*. Set $\Omega_1 = \{\chi\}$ and $\Omega_2 = \mathrm{sign}(\lambda) \setminus \{\chi\}$. By Lemma 2.5, there is $h \in G$ such that $\chi(h) \notin \{\phi(h)\}_{\phi \in \Omega_2}$.

Set $\varepsilon = \frac{1}{3}\min\{|\chi(h) - \phi(h)|\colon \phi \in \Omega_2\}$ and consider the sets

$$\begin{aligned} V &= \{\omega \in G^*\colon\ |\chi(h) - \omega(h)| < \varepsilon\} \text{ and} \\ G^* \setminus V &= \{\omega \in G^*\colon\ |\chi(h) - \omega(h)| \geq \varepsilon\} \end{aligned} \tag{6.1}$$

in G^*. Then $\Omega_2 \subset G^* \setminus V$. The subspaces

$$\mathfrak{H}_1 = \int_V^{\oplus} \mathfrak{H}_\omega dP(\omega) \text{ and } \mathfrak{H}_2 = \int_{G^* \setminus V}^{\oplus} \mathfrak{H}_\omega dP(\omega)$$

are invariant for U, $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and

$$\begin{aligned} \chi(h) &\overset{(6.1)}{\notin} \mathrm{Sp}(U(h)|_{\mathfrak{H}_2}) = \overline{\{\omega(h)\}_{\omega \in G^* \setminus V}} \text{ and} \\ \phi(h) &\overset{(6.1)}{\notin} \mathrm{Sp}(U(h)|_{\mathfrak{H}_1}) = \overline{\{\omega(h)\}_{\omega \in V}}, \end{aligned}$$

for each $\phi \in \Omega_2$. Thus Ω_1, $U|_{\mathfrak{H}_2}$ are spectrally disjoint, and Ω_2, $U|_{\mathfrak{H}_1}$ are spectrally disjoint.

Applying Corollary 4.3, we have that $H = H_1[+]H_2$ is the sum of invariant subspaces, L_χ is a maximal neutral invariant subspace of H_1 and $L_{\Omega_2} = \sum_{\omega\in\Omega_2} \dot{+} L_\omega$ is a maximal neutral invariant subspace of H_2. As π is non-Π-decomposable, $\Omega_2 = \varnothing$. Thus $\mathrm{sign}(\lambda) = \{\chi\}$. □

For a unitary representation π of a group G, denote by $E(\pi)$ the set of all matrix elements of π: the functions $g \mapsto (\pi(g)x, x)$, where $x \in \mathfrak{H}$. For unitary equivalent representations π and ρ, $E(\pi) = E(\rho)$. The dual object of G is the set $\widehat{G}$ of all unitary equivalence classes $\widehat{\pi}$ of irreducible unitary representations of G, supplied with the topology of uniform convergence of matrix elements: $\widehat{\pi}$ belongs to the closure of $M \subset \widehat{G}$ if each element of $E(\widehat{\pi})$ can be uniformly approximated on compacts by matrix elements of representations in M. This topology is usually non-Hausdorff, but there is a large class of groups for which $\widehat{G}$ is a T_0-space, i.e., the intersection of all neighborhoods of each point contains only this point. This class contains all groups of type I [D1, 4.4.1] and, in particular, all connected nilpotent locally compact groups (see [Kir]).

Each unitary character χ of G, identified with the equivalence class of one-dimensional representations $\widehat{\chi\iota}$, is contained in $\widehat{G}$. The open sets

$$W_{K,\varepsilon}(\chi) = \{\widehat{\pi} \in \widehat{G}\colon |\varphi(g) - \chi(g)| < \varepsilon \text{ for all } g \in K \text{ and some } \varphi \in E(\pi)\}, \tag{6.2}$$

where $K \subset G$ are compacts and $\varepsilon > 0$, form a base of neighbourhoods for χ. Characters χ and ω are *separated* in $\widehat{G}$ if they have non-intersecting neighbourhoods in $\widehat{G}$. Note that they are separated if and only if the trivial character χ_e and the character $\overline{\chi}\omega$ are separated in $\widehat{G}$.

As an example, we consider the dual space of the real Heisenberg group $G = \mathcal{T}_3$ (see (5.8)). It is known (see [ShZ]) that the unitary characters χ of G and the corresponding one-dimensional unitary representations ι_χ on $\mathbb{C}u$ have form

$$\chi_{\alpha,\beta}(g(x,y,z)) = e^{i(\alpha x+\beta y)}, \text{ for } \alpha,\beta \in \mathbb{R}, \text{ and } \iota_{\chi_{\alpha,\beta}}(g)u = \chi_{\alpha,\beta}(g)u.$$

In particular $\chi_{0,0} = \chi_e$, and $\iota_{\chi_{0,0}} = \iota$, the trivial representation.

Infinite-dimensional unitary irreducible representations of G act on $L^2(\mathbb{R})$ by the formula

$$U_\sigma(g(x,y,z))f(t) = e^{i\sigma(z+ty)}f(t+x), \text{ for } f \in L^2(\mathbb{R}), \tag{6.3}$$

where $0 \neq \sigma \in \mathbb{R}$.

Proposition 6.4. *Every two characters of* $G = \mathcal{T}_3$ *(see* (5.8)*) cannot be separated in* $\widehat{G}$.

Proof. It suffices to prove the proposition for $\chi_{0,0}$ and each character χ. Consider the increasing sequence of compacts $K_m = \{g = g(x,y,z)\colon |x| + |y| + |z| \leq m\}$. As $G = \cup_m K_m$, the sets $W_{K_m,\varepsilon}(\chi_{0,0})$ (see (6.2)) form a base of neighbourhoods of $\chi_{0,0}$.

Consider the set of representations $\{U_{\sigma_n}\colon \sigma_n = n^{-6},\ n \in \mathbb{N}\}$. Define f_n in $L^2(\mathbb{R})$ by $f_n(t) = n^{-2}$ for $t \in [0, n^4]$, and $f_n(t) = 0$ for $t \notin [0, n^4]$. Then $\|f_n\| = 1$ and, for $g = g(x, y, z)$,

$$\begin{aligned}
|(U_{\sigma_n}(g)f_n, f_n) - 1| &\le \|U_{\sigma_n}(g)f_n - f_n\| \\
&\le \left\| \left(e^{i(z+ty)/n^6} - 1\right) f_n(t+x) \right\| + \|f_n(t+x) - f_n(t)\| \\
&= n^{-2} \left(\int_{-x}^{n^4 - x} \left| e^{i(z+ty)/n^6} - 1 \right|^2 dt \right)^{1/2} \\
&+ n^{-2} \left| \int_{-x}^{0} dt + \int_{n^4 - x}^{n^4} dt \right|^{1/2} \\
&\le \max_{-x \le t \le n^4 - x} \left| e^{i(z+ty)/n^6} - 1 \right| + n^{-2}(2\,|x|)^{1/2} \\
&\le (|y| + (2\,|x|)^{1/2})n^{-2}.
\end{aligned}$$

Hence the matrix elements $(U_{\sigma_n}(g)f_n, f_n)$ uniformly tend to 1 on each K_m. This means that each neighbourhood $W_{K_m,\varepsilon}(\chi_{0,0})$ of $\chi_{0,0}$ contains representation U_{σ_n} for all n starting for some N.

On the other hand, it should be noted that if $U_\sigma \in W_{K,\varepsilon}(\chi_{0,0})$ then $U_\sigma \in W_{K,\varepsilon}(\chi)$ for each character $\chi = \chi_{\alpha,\beta}$. To see this, note that the unitary operator $V = V_{\alpha,\beta,\sigma}$ on $L^2(\mathbb{R})$ that acts by

$$(Vf)(t) = e^{i\alpha(t - \frac{\beta}{\sigma})} f\left(t - \frac{\beta}{\sigma}\right) \text{ for } f \in L^2(\mathbb{R}),$$

satisfies $V\chi(g)U_\sigma(g) = U_\sigma(g)V$ for all $g \in G$. Hence

$$|(U_\sigma(g)Vf, Vf) - \chi(g)| = |(U_\sigma(g)f, f) - 1|$$

for $0 \ne \sigma \in \mathbb{R}$, $f \in L^2(\mathbb{R})$ and $g \in G$. Thus if $U_{\sigma_n} \in W_{K_m,\varepsilon}(\chi_{0,0})$ then $U_{\sigma_n} \in W_{K_m,\varepsilon}(\chi)$, so that $\chi_{0,0}$ and χ cannot be separated. □

We shall show now that if G has unitary characters not separated in $\widehat{G}$, then it has a non-Π-decomposable Π_k-representation which is not primary.

Theorem 6.5. *Let G be a connected locally compact nilpotent group. Suppose that G has unitary characters not separated in $\widehat{G}$. Then there is a finite-dimensional representation λ of G on L, a unitary representation U on $\mathfrak{H}$ and a neutral (λ, U)-cocycle ξ such that the double extension $\pi = \mathfrak{ee}(\lambda, U, \xi, \gamma)$ is a non-Π-decomposable representation on $H = L \oplus \mathfrak{H} \oplus L$ and not primary.*

Proof. We mentioned above that if two unitary characters are not separated in $\widehat{G}$, there is a unitary character χ which is not separated in $\widehat{G}$ from the trivial character χ_e. Define a unitary representation λ on the 2-dimensional Hilbert space $L = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ by

$$\lambda(g) = e_1 \otimes e_1 + \chi(g)(e_2 \otimes e_2) \text{ (see (5.2)).} \tag{6.4}$$

As χ is unitary, $\lambda^\sharp(g) \overset{(2.4)}{=} \lambda(g^{-1})^* \overset{(2.7)}{=} \lambda(g)$ for $g \in G$.

Since connected locally compact groups are σ-compact, choose compacts $\{e\} \in K_1 \subset K_2 \subset ...$ such that $G = \cup_{n=1}^{\infty} K_n$. As χ_e, χ are not separated in $\widehat{G}$, $W_{K_n,2^{-n}}(\chi_e) \cap W_{K_n,2^{-n}}(\chi) \neq \varnothing$ (see (6.2)). This means that there are irreducible unitary representations π_n of G on $\mathfrak{H}_n$ and $u_n, v_n \in \mathfrak{H}_n$ such that, for $g \in K_n$,

$$|(\pi_n(g)u_n, u_n) - 1| < 2^{-n} \text{ and } |(\pi_n(g)v_n, v_n) - \chi(g)| < 2^{-n}. \qquad (6.5)$$

As $e \in K_n$, $|\|u_n\|^2 - 1| < 2^{-n}$ and $|\|v_n\|^2 - 1| < 2^{-n}$. Changing u_n,v_n if necessary, we may assume that $\|u_n\| = \|v_n\| = 1$. Since $\widehat{G}$ is a T_0-space, the representations π_n can be chosen pairwise non-equivalent. As each π_n is irreducible, it either coincides with ι, or $\chi\iota$, or it has no χ_e- and χ-eigenspaces. It follows from (6.5) that, starting from some n, π_n can not coincide with ι, or $\chi\iota$. Thus, without any loss of generality, we assume that χ_e and χ are eigen-disjoint with all π_n.

Set $\mathfrak{H} = \oplus \mathfrak{H}_n$, $U = \oplus_{n=1}^{\infty} \pi_n$,

$$u_n(g) = u_n - \pi_n(g)^* u_n \text{ and } v_n(g) = \overline{\chi(g)} v_n - \pi_n(g)^* v_n. \qquad (6.6)$$

Set also $u(g) = \oplus_{n=1}^{\infty} u_n(g)$ and $v(g) = \oplus_{n=1}^{\infty} v_n(g)$ for $g \in G$. Then

$$\|u_n(g)\|^2 = 2\,\mathrm{Re}(1 - (u_n, \pi_n(g)u_n)) \overset{(6.5)}{\leq} 2^{-(n-1)} \qquad (6.7)$$

$$\text{and } \|u(g)\|^2 \overset{(6.7)}{\leq} \sum_{k=1}^{\infty} 2^{-(k-1)} < \infty, \text{ so that } u(g) \in \mathfrak{H} \text{ for } g \in K.$$

Similarly, $v(g) \in \mathfrak{H}$. As

$$u_n(gh) = u_n(h) + \pi_n(h)^* u_n(g) \text{ and } v_n(gh) = \overline{\chi(g)} v_n(h) + \pi_n(h)^* v_n(g),$$

we have, for $g, h \in G$,

$$u(gh) = u(h) + U(h)^* u(g), v(gh) = \overline{\chi(g)} v(h) + U(h)^* v(g). \qquad (6.8)$$

Let us define maps ξ: $G \to B(\mathfrak{H}, L)$ and γ: $G \to B(L)$ by

$$\begin{aligned} \xi(g) &= u(g) \otimes e_1 + v(g) \otimes e_2 \in B(\mathfrak{H}, L), \\ \gamma(g) &= -\sum_{n=1}^{\infty} ((u_n, u_n(g))(e_1 \otimes e_1) + (u_n(g^{-1}), v_n)(e_1 \otimes e_2) \\ &\quad + (v_n, u_n(g))(e_2 \otimes e_1) + (v_n, v_n(g))(e_2 \otimes e_2)). \end{aligned} \qquad (6.9)$$

The series converges uniformly on compacts because of condition (6.7). Then ξ is a (λ, U)-cocycle by (6.8), $\xi^\sharp(g) = \xi(g^{-1})^* = e_1 \otimes u(g^{-1}) + e_2 \otimes v(g^{-1})$ and

$$-\xi(g)\xi^\sharp(h) = a^{11}(e_1 \otimes e_1) + a^{12}(e_1 \otimes e_2) + a^{21}(e_2 \otimes e_1) + a^{22}(e_2 \otimes e_2),$$

where

$$a^{11}=\sum_{n=1}^{\infty}(u_n,u_n(gh)-u_n(h)-u_n(g)),$$
$$a^{12}=\sum_{n=1}^{\infty}(u_n(h^{-1}g^{-1})-\chi(g)u_n(h^{-1})-u_n(g^{-1}),v_n),$$
$$a^{21}=\sum_{n=1}^{\infty}(v_n,u_n(gh)-u_n(h)-\overline{\chi(h)}u_n(g)),$$
$$a^{22}=\sum_{n=1}^{\infty}(v_n,v_n(gh)-\overline{\chi(g)}v_n(h)-\overline{\chi(h)}v_n(g)).$$

By direct calculations we obtain, using (5.3), that

$$(d_{\lambda,\lambda}\gamma)(g,h)=\lambda(g)\gamma(h)-\gamma(gh)+\gamma(g)\lambda(h)=-\xi(g)\xi^{\sharp}(h).$$

In other words, ξ is a neutral (λ,U)-cocycle and γ is its prechain (see (2.5)).

Set $H=L\oplus\mathfrak{H}\oplus L$ and $\pi=\mathfrak{ee}(\lambda,U,\xi,\gamma)$. Then H is a Π_2-space and $\mathrm{sign}(\lambda)=\{\chi_e,\chi\}$, so that π is not primary. We have to show that π is non-Π-decomposable. Suppose that it is Π-decomposable. By Theorem 3.4, there is a projection $p\neq\mathbf{0},\mathbf{1}_L$ commuting with λ and a projection $q=q^*$ commuting with U such that $\eta=\xi-p\xi q-(\mathbf{1}_L-p)\xi(\mathbf{1}_{\mathfrak{H}}-q)$ is a (λ,U)-coboundary: $\eta(g)=\lambda(g)S-SU(g)$ for some $S\in B(\mathfrak{H},L)$ and all $g\in G$. Then

$$p\eta(g)(\mathbf{1}_{\mathfrak{H}}-q)=\lambda(g)T-TU(g),\ \text{where } T=pS(\mathbf{1}_{\mathfrak{H}}-q). \tag{6.10}$$

As p commutes with λ, either $p=e_1\otimes e_1$, or $p=e_2\otimes e_2$. Let $p=e_1\otimes e_1$. Then, by (5.3), $T=(e_1\otimes e_1)S(\mathbf{1}_{\mathfrak{H}}-q)=x\otimes e_1$ for some $x\in\mathfrak{H}$. Hence, by (6.9) and (6.10),

$$\begin{aligned}p\eta(g)(\mathbf{1}_{\mathfrak{H}}-q)&=p(\xi-p\xi q-(\mathbf{1}_L-p)\xi(\mathbf{1}_{\mathfrak{H}}-q))(\mathbf{1}_{\mathfrak{H}}-q)\\&=p\xi(g)(\mathbf{1}_{\mathfrak{H}}-q)=(\mathbf{1}_{\mathfrak{H}}-q)u(g)\otimes e_1\\&\overset{(6.10)}{=}\lambda(g)T-TU(g)\overset{(5.3)}{=}(\mathbf{1}_{\mathfrak{H}}-U(g)^*)x\otimes e_1.\end{aligned}$$

Thus $(\mathbf{1}_{\mathfrak{H}}-q)u(g)=(\mathbf{1}_{\mathfrak{H}}-U(g)^*)x$. As q commutes with U and all π_n are pairwise non-equivalent, q is the projection on a subspace $\oplus_{n\in E}\mathfrak{H}_n$ for some $E\subseteq\mathbb{N}$. Let $x=\oplus_{n=1}^{\infty}x_n$, $x_n\in\mathfrak{H}_n$. Then

$$(\mathbf{1}_{\mathfrak{H}_n}-\pi_n(g)^*)u_n\overset{(6.6)}{=}u_n(g)=(\mathbf{1}_{\mathfrak{H}_n}-\pi_n(g)^*)x_n$$

for $n\notin E$ and all $g\in G$. As χ_e is eigen-disjoint with all π_n, we have $u_n=x_n$ for $n\notin E$. Taking into account that $\|u_n\|=1$ and $\|x\|^2=\sum\|x_n\|^2<\infty$, we conclude that the set $\mathbb{N}\setminus E$ is finite.

Similarly,

$$(\mathbf{1}_L-p)\eta(g)q=(\mathbf{1}_L-p)\xi(g)q=qv(g)\otimes e_2=(\overline{\chi_2(g)}-U(g)^*)z\otimes e_2,$$

for some $z=\oplus_{n=1}^{\infty}z_n\in\mathfrak{H}$, $z_n\in\mathfrak{H}_n$, so that $qv(g)=(\overline{\chi_2(g)}-U(g)^*)z$. Repeating the above argument, we get that $v_n=z_n$ for $n\in E$. As $\|v_n\|=1$ and

$\|z\|^2 = \sum \|z_n\|^2 < \infty$, we conclude that the set E is finite, a contradiction. Thus π is non-Π-decomposable. $\square$

7. Splitting and approximate splitting of singular representations

While non-singular J-unitary representations of nilpotent groups are similar to unitary representations (Theorem 1.1), singular representations have much more complicated structure. Although some of them decompose into sums of finite-dimensional representations (their structure was described in Corollary 5.6) and representations similar to unitary, this situation is comparatively rare.

In this section we will show that all singular representations admit an "approximate" decomposition of this kind. We will start by introducing some terminology.

Definition 7.1. *We say that a maximal neutral invariant subspace L of a representation π on H*

(i) *splits π if $H = K[+]K^{[\perp]}$, where K is invariant,* $\dim K < \infty$ *and $L \subset K$;*

(ii) *approximately splits π, if it does not split π, but there are non-degenerate invariant subspaces $\{H_m\}_{m=1}^{\infty}$ of H such that $L \subset H_{m+1} \subsetneqq H_m$,* $\dim H_m = \infty$ *and* $\dim(\cap_m H_m) < \infty$.

We will show that, for arbitrary J-unitary representation π of a connected nilpotent group G, each maximal neutral invariant subspace L either splits or approximately splits π. Moreover, this does not depend on the choice of L.

Note that, in representations π considered in Theorem 6.1, maximal neutral subspaces split π, while in Theorem 6.5 maximal neutral subspaces approximately split π.

Let π be a J-unitary representation on a Π_k-space H and (see (4.5))

$$H = L \oplus \mathfrak{H} \oplus M, \ \mathfrak{H} = \mathfrak{H}^{\Omega}[+]\mathfrak{H}^0, \ \lambda = \pi|_L, \tag{7.1}$$

and $\mathrm{sign}(\lambda)$ is eigen-disjoint with $U|_{\mathfrak{H}^0}$. It was proved in Proposition 3.3 [KS1] that

$$\mathfrak{H}^0 = \oplus_{n=1}^{N} \mathfrak{H}_n, \text{ where } N \le \infty, \tag{7.2}$$

$\mathfrak{H}_n$ are U-invariant subspaces such that each $U|_{\mathfrak{H}_n}$ and $\mathrm{sign}(\lambda)$ are spectrally disjoint.

Proposition 7.2. *If $U|_{\mathfrak{H}^0}$ is not sectionally spectrally disjoint with some $\chi \in$* $\mathrm{sign}(\lambda)$, *then $\mathfrak{H}^0 = \oplus_{n=1}^{\infty} \mathfrak{H}_n$ and there are non-degenerate invariant subspaces $\{H_m\}_{m=1}^{\infty}$ such that* $\dim H_m = \infty$,

$$L \subset H_{m+1} \subsetneqq H_m, \quad L \oplus \mathfrak{H}^{\Omega} = L^{[\perp]} \cap (\cap_m H_m)$$

and $\pi|_{H_m^{[\perp]}}$ are non-singular.

Proof. If $\mathfrak{N} < \infty$ in (7.2) then each $\omega \in \operatorname{sign}(\lambda)$ is sectionally spectrally disjoint with $U|_{\mathfrak{H}^0}$. As χ is not sectionally spectrally disjoint with $U|_{\mathfrak{H}^0}$, we have $\mathfrak{N} = \infty$. Thus $\mathfrak{H}^0 = \oplus_{n=1}^{\infty}\mathfrak{H}_n$.

We prove the rest of the theorem by induction. In (7.1) set

$$\mathcal{H}^k = \mathfrak{H}^\Omega \oplus (\oplus_{n=k}^{\infty}\mathfrak{H}_n) \text{ for all } k \geq 1.$$

For $k = 1$, $\mathcal{H}^1 = \mathfrak{H}$ and $H_1 := H = L \oplus \mathcal{H}^1 \oplus M_1$ for $M_1 = M$. Assume that there are subspaces M_k, $k = 2, ...m$, skew-related to L such that $H_k = L \oplus \mathcal{H}^k \dotplus M_k$ are π-invariant, non-degenerate and $H_m \subsetneqq H_{m-1} \subsetneqq ... \subsetneqq H_1$. Then $H_m = L \oplus (\mathcal{H}^{m+1} \oplus \mathfrak{H}_m) \dotplus M_k$ and $\mathcal{H}^{m+1}$, $\mathfrak{H}_m$ are U-invariant subspaces. Set $\pi_m = \pi|_{H_m}$. As $\operatorname{sign}(\lambda)$ and $U|_{\mathfrak{H}_m}$ are spectrally disjoint, Corollary 4.3 implies that there is $M_{m+1} \subset H_m$ skew-related to L such that $H_{m+1} = (L \oplus \mathcal{H}^{m+1}) \dotplus M_{m+1}$ is a non-degenerate, π_m-invariant subspace. Hence H_{m+1} is π-invariant, $\dim H_{m+1} = \infty$ and $H_{m+1} \subsetneqq H_m$. By induction, there is a decreasing chain $\{H_m\}_{m=1}^{\infty}$ of invariant non-degenerate subspaces containing L.

As $L^{[\perp]} = L \oplus \mathfrak{H}$, we have $L^{[\perp]} \cap H_m = L \oplus \mathcal{H}^m$. As $\cap_m \mathcal{H}^m = \mathfrak{H}^\Omega$, we have

$$L^{[\perp]} \cap (\cap_m H_m) = \cap_m (L^{[\perp]} \cap H_m) = \cap_m (L \oplus \mathcal{H}^m) = L \oplus \mathfrak{H}^\Omega.$$

As all spaces $H_m^{[\perp]}$ above have no neutral invariant subspaces, $\pi|_{H_m^{[\perp]}}$ are non-singular. □

Theorem 7.3. (i) *If* $\operatorname{sign}(\lambda)$ *and* $U|_{\mathfrak{H}^0}$ *are sectionally spectrally disjoint then* L *splits* π.

(ii) *The following conditions are equivalent*:

1) L *splits* π;

2) $H = M[+]P$, *where* M *and* P *are invariant subspaces*, $\dim M < \infty$ *and* P *is positive*;

3) *each non-degenerate invariant subspace* R *of* H *has a decomposition* $R = M_R[+]P_R$, *where* M_R *and* P_R *are invariant subspaces*, $\dim M_R < \infty$ *and* P_R *is positive*;

4) π *has a minimal non-degenerate invariant subspace containing* L.

(iii) *If* L *splits* π *then each maximal neutral invariant subspace splits* π.

Proof. (i) Setting $\Omega_1 = \operatorname{sign}(\lambda)$ and $K = H_1$ in Corollary 4.5, we get that L splits π.

(ii) 1) $\Longrightarrow$ 4) is evident.

4) $\Longrightarrow$ 1). Let K be a minimal non-degenerate invariant subspace containing L. As in (7.1), $K = L \oplus (\mathfrak{K}^\Omega[+]\mathfrak{K}^0) \oplus M$. Assume that $\dim K = \infty$. If $\operatorname{sign}(\lambda)$ is sectionally spectrally disjoint with $U|_{\mathfrak{K}^0}$ then, by (i), L splits $\pi|_K$: there is a non-degenerate invariant subspace K_1 of K such that $\dim K_1 < \infty$ and $L \subset K_1$ – a contradiction. Thus $U|_{\mathfrak{K}^0}$ is not sectionally spectrally disjoint with some $\chi \in \operatorname{sign}(\lambda)$. By Proposition 7.2, there are non-degenerate invariant subspaces $\{K_m\}_{m=1}^{\infty}$ of K containing L. This contradicts the assumtion that K is minimal. Thus $\dim K < \infty$.

1) $\Longrightarrow$ 2). If L splits π, $H = K[+]K^{[\perp]}$, $\dim K < \infty$ and $L \subset K$. Thus $K^{[\perp]}$ has no neutral invariant subspaces, so $\pi|_{K^{[\perp]}}$ is non-singular. By Theorem 1.1, $K^{[\perp]} = N[+]P$, P is a positive and N is a negative invariant spaces, $\dim N < \infty$. Set $M = K[+]N$. Then $\dim M < \infty$ and $H = M[+]P$.

2) $\Longrightarrow$ 1). Let $H = M[+]P$. As M and P are π-invariant, the projection q on P along M commutes with π. As $\dim L < \infty$, the subspace $R = qL$ of P is invariant, $\dim R \leq \dim L$ and $L \subseteq M[+]R$. Then $K := M[+]R$ is invariant, non-degenerate, $\dim(K) < \infty$ and $L \subset K$. Thus $H = K[+]K^{[\perp]}$, so L splits π.

2) $\Longrightarrow$ 3) Let $H = M[+]P$. If R is a non-degenerate invariant subspace then $P_R = R \cap P$ is invariant, positive and has finite codimension in R, as $\operatorname{codim}(P) < \infty$. By (3.2), $R = M_R[+]P_R$, M_R is invariant and $\dim M_R < \infty$. 3) $\Longrightarrow$ 2) is evident.

(iii) Let L split π. By (ii), $H = M[+]P$, M, P are invariant, $\dim M < \infty$ and P is positive. Let L' be a maximal neutral invariant subspace. As in 2) $\Longrightarrow$ 1), we get that $H = K[+]Q$, where K, Q are invariant, $\dim K < \infty$, $L' \subset K$ and Q is positive. Thus L' splits π. $\square$

If L splits π then $H = K[+]K^{[\perp]}$, $\dim K < \infty$ and $\pi|_K$ decomposes in a finite J-orthogonal sum of one-dimensional unitary representations and of representations $\chi\pi_{k,m}$, π_{χ,χ^*} (Theorem 5.6). The representation $\pi|_{K^{[\perp]}}$ is non-singular and similar to a unitary representation.

Theorem 7.4. *Let L be a maximal neutral invariant subspace of a singular representation π on H.*

(i) *L either splits π or approximately splits π.*
(ii) *L approximately splits π if and only if H does not have a decomposition $H = M[+]P$, where M, P are invariant subspaces, $\dim M < \infty$ and P is positive.*
(iii) *If L approximately splits π, all maximal neutral invariant subspaces approximately split π.*

Proof. (i) By Lemma 4.4, $H = K[+]E$, where E is a sum of eigenspaces of π and

$$K = L \oplus \mathfrak{K} \oplus M, \text{ where } \mathfrak{K} = \mathfrak{K}^{\Omega} \oplus \mathfrak{K}^0 \text{ and } \dim \mathfrak{K}^{\Omega} < \infty,$$

is a non-degenerate invariant space. If L does not split π, it does not split $\pi|_K$. Hence we have from Theorem 7.3 that $\operatorname{sign}(\lambda)$ has a character which is not sectionally spectrally disjoint with $U|_{\mathfrak{K}^0}$. Then, by Proposition 7.2, there are non-degenerate invariant subspaces $\{K_m\}_{m=1}^{\infty}$ in K such that $L \subset K_{m+1} \subsetneq K_m$, for all m,

$$L \oplus \mathfrak{K}^{\Omega} = L^{[\perp]} \cap (\cap_m K_m) \text{ and } L^{[\perp]} = L \oplus \mathfrak{K}$$

is the J-orthogonal complement of L in K. As $\dim(L \oplus \mathfrak{K}^{\Omega}) < \infty$ and $\operatorname{codim}(L^{[\perp]}) = \dim M < \infty$, we have that $\dim(\cap_m K_m) < \infty$. Setting $H_m = K_m$, we get that L approximately splits π.

(ii) follows from (i) and from part (ii) 2) of Theorem 7.3.

(iii) follows from (i) and from part (iii) of Theorem 7.3. $\square$

If L approximately splits π then $H = H_m[+]H_m^{[\perp]}$ for $m \in \mathbb{N}$, and the representations $\pi|_{H_m^{[\perp]}}$ are similar to unitary representations. The spaces H_m decrease and the invariant subspace $\mathcal{N} = \cap_m H_m$ (the "nucleus") is degenerate, finite-dimensional and contains L. Thus the representations $\pi|_{H_n}$ are "infinitely close" to $\pi|_{\mathcal{N}}$ and we have an "approximately decomposition" of π: the representations $\pi|_{H_m^{[\perp]}}$ "almost approximate" π.

Since singular representations π, as a rule, do not decompose into irreducible components, this is the closest we can get to the decomposition of π.

If π is non-Π-decomposable then all $H_m^{[\perp]}$ are positive subspaces.

References

[A] Araki H., Indecomposable representations with invariant inner product, Commun. Math. Phys., 97(1985), 149-159.

[AI] Azizov T.Ya and Iokhvidov I.S., Linear operators in spaces with an Indefinite Metric, J. Wiley and Sons, 1989.

[B] Bognar J., Indefinite Inner-Product Spaces, Springer-Verlag, Berlin 1974.

[D1] Dixmier J., Les C*-algebres et leurs representations, Gauthier-Villars, Paris, 1969.

[DT] Dubin D. A. and Tarski J., J. Math. Phys. 7, 574 (1966).

[H] Herstein I.N., Noncommutative rings, John Wiley and sons, Inc., 1971.

[Is] Ismagilov R.S., Rings of operators in a space with an indefinite metric, Dokl. Akad. Nauk SSSR, No 2, 171(1966), 269-271 = Soviet Math. Dokl. 7, No 6(1966), 1460-1462.

[Is1] Ismagilov R.S., Unitary representations of the Lorentz group in spaces with indefinite metric, Izv. Akad. Nauk SSSR, 30, No 3(1966), 497-522.

[Is2] Ismagilov R.S., On irreducible representations of the discrete group SL$(2, P)$ which are unitary with respect to an indefinite metric, Izv. Akad. Nauk SSSR, 30, No 4(1966), 923-950.

[Is3] Ismagilov R.S., On the problem of extension of representations, Matem. Zametki, 35, No 1(1984), 99-105.

[Kir] Kirillov A.A., Unitary representations of nilpotent Lie groups, Russian mathematical surveys, 17(1962), 53-104.

[KS] Kissin E. and Shulman V. S., "Representations on Krein Spaces and Derivations of C*-algebras", Addison Wesley Longman, London, 1997.

[KS1] Kissin E. and Shulman V. S., Non-unitary representations of nilpotent groups, I: cohomologies, extensions and neutral cocycles, 269(2015), 2564-2610. DOI: 10.1016/j.jfa.2015.07.003.

[MPS] Morchio G., Pierotti D. and Strocchi F. , Infrared and Vacuum Structure in Two-Dimensional Local Quantum Field Theory, J. Math. Phys. 31 (1990), no. 6, 1467-1477.

[M] Morris S. A., Pontryagin duality and the structure of locally compact Abelian groups, London Math. Soc., Lecture Notes Series 29, Cambridge University Press, 1977.

[N1] Naimark M.A., On unitary representations of solvable groups in spaces with an indefinite metric, Izv. Akad. Nauk SSSR, 27(1963), 1181-1185 (AMS Translations 49(1966), 86-91).

[N2] Naimark M.A., Structure of unitary representations of locally bicompact groups and symmetric representations of algebras in Pontryagin Π_k-spaces, Izv. Akad. Nauk SSSR, 30, No 5(1966), 1111-32 = Math. USSR Izvestija, 36-58.

[NI] Naimark M.A. and Ismagilov R.S., Representations of groups and algebras in spaces with indefinite metric, Mat. Anal., 1968, Moscow 1969, 73-105.

[OST] Ostrovskii M. I., Shulman V.S., Turowska L, Fixed points of holomorphic transformations of operator balls, Quartely J. of Math., (Oxford) 62 (2011), 173–87.

[Sa] Sakai K., On J-unitary representations of amenable groups, Sci. Rep. Kagoshima Univ., 26(1977), 33-41.

[Sc] Schmidt A. U., Mathematical Problems of Gauge Quantum Field Theory: A Survey of the Schwinger Model, Univ. Iagellonicae Acta Mathematica Fasciculus XXXIV(1997), 113-134.

[Sc1] Schmidt A. U., Infinite infrared regularization and a state space for the Heisenberg algebra, J. Math. Phys. 43, 243 (2002).

[ShZ] Shtern A.I., Zhelobenko D.P., Representations of Lie groups, Moscow, 1983 (in Russian).

[St] Strocchi F., Selected Topics on the General Properties of Quantum Field Theory, Lecture Notes in Physics, vol. 51, World Scientific, Singapore, London, Hong Kong, 1993.

[SW] Strocchi F. and Wightmam A.S., J. Math. Phys., 15, 2198(1974).

e-mail: e.kissin@londonmet.ac.uk
STORM, London Metropolitan University
166-220 Holloway Road,
London N7 8DB, Great Britain.

e-mail: shulman.victor80@gmail.com
Department of Mathematics,
Vologda State University,
Vologda 160000, Russia.

Edward Kissin and Victor S. Shulman