Safe control design for uncertain linear systems under input saturation using Lyapunov barrier functions

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Abstract— **This paper proposes conditions for the design of safe, robust state-feedback controllers and barrier certificates for uncertain polytopic continuous-time linear systems subject to input saturation. The generalized sector condition is employed to assess the presence of saturation and the level set of the Lyapunov function is utilized to define the barrier function. Numerical experiments are used to illustrate the features of the proposed method.**

I. INTRODUCTION

Safety is a fundamental concern in control engineering, especially in automation. The safety principles may find applications in several domains including robotics, autonomous vehicles, and industrial automation, among others. The safe operation and reliability of these systems is a significant challenge, and disregarding their importance can lead to economic and societal impacts. Effective safety measures are essential to prevent accidents, and failures in autonomous driving [1], UAV operations [2], robotics [3], and cruise control [4], for instance.

The closed-loop states must be confined to a certain region of the state space to guarantee that the safety constraints are satisfied. This requirement becomes even more complex when considering the presence of uncertainties and input saturation, which introduces nonlinearities in the control systems. Uncertainties may arise from unmodeled dynamics, component imperfections, or measurement inaccuracies, while saturation can result from hardware limitations, energy constraints, or the need to prevent system degradation. If not properly accounted for during the design phase, these factors can lead to undesirable behavior, pushing the system into unsafe regions or even causing instability [5]. Actuator saturation has been extensively studied in various contexts, including state-feedback control for continuous uncertain systems [6] and discrete-time linear parameter-varying (LPV) systems [7].

The barrier function has been employed to guarantee that the system does not leave the safe region of the state space [8]. In [9], sum-of-squares (SOS) conditions are proposed to deal with the safety of continuous and discrete-time polynomial systems. Conditions for linear continuous-time systems considering the unit peak input were also introduced using the SOS method and an iterative approach. In [10], the focus is concerned with designing a feedback controller for linear systems through SOS programming using a co-convex approach where the $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_\infty$ norm constrained limitations are all addressed using convex constraints. In [11], the authors introduced the input constraints to the problem of safety certificates and solves this using the SOS programming and the S-procedure confronting the CBC (Control barrier certificates) with CBF (Control barrier function) on numerical conservatism. In [12], a model predictive control-based approach is employed to design a safe controller avoiding point-wise safety guarantees without adding multiple constraints. In [13] a logarithmic barrier Lyapunov function was employed to deal with the robust stabilization of input saturated uncertain linear systems with perturbed measurements. In [14] the authors propose the use of control Lyapunov-barrier function-based model predictive control to deal with the stabilization problem of saturated nonlinear systems, the conditions are presented in form of an optimization problem.

Although there has been an increasing interest in studies related to barrier functions, formulations based on linear matrix inequalities (LMIs) are still relatively scarce. Moreover, the study of safety regarding the presence of uncertainties is not fully addressed in the literature. LMI-based approaches are particularly valuable in addressing the presence of uncertainties in the model, as they enable a convex formulation of the problem, allowing for efficient and reliable computational solutions using convex optimization techniques. Furthermore, existing specialized computational packages make it straightforward to model and solve LMIs, facilitating their implementation. It is important to highlight that in [6] the control design for LPV systems with saturating actuators was addressed. However, no safety considerations have been included in the formulation.

This paper introduces new conditions in the form of LMIs to provide barrier Lyapunov functions and safe controllers that will ensure the safe operation of the dynamical system. Differently from existing conditions, a non-iterative method is proposed. We consider continuous-time uncertain linear systems under input saturation. The goal is to design a safe robust state-feedback controller to stabilize the system. The safe set, initial conditions, and barrier certificate are all represented in ellipsoidal form. To handle input saturation, we employ the generalized sector condition, while the barrier set is defined using a level set of the Lyapunov function. Numerical examples are used to illustrate the efficacy of the

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proposed conditions.

Notation: $X \in \mathbb{R}^{m\times n}$ represents a matrix with real entries of m rows and n columns. The transpose of a matrix X is denoted by $X^\top$. $P > 0$ ($P < 0$) represents a positive definite (negative definite) symmetric matrix. The identity and zero matrices of appropriate dimensions are denoted by I and 0, respectively.

II. PRELIMINARIES

A. Linear uncertain system

Consider the following continuous-time uncertain linear system

$$\dot{x}(t) = A(\alpha)x(t) + B(\alpha)u(t), \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state vector, and $u \in \mathbb{R}^{n_u}$ is the control input. The parameter-dependent matrices $A(\alpha) \in \mathbb{R}^{n_x\times n_x}$ and $B(\alpha) \in \mathbb{R}^{n_x\times n_u}$ can be written in terms of their vertices as follows

$$A(\alpha) = \sum_{i=1}^{N} \alpha_i A_i,\ B(\alpha) = \sum_{i=1}^{N} \alpha_i B_i.$$

The time-invariant parameters α_i belong to the unit simplex satisfying the following conditions

$$\Lambda_N = \left\{ \alpha \in \mathbb{R}^N : \sum_{i=1}^{N} \alpha_i = 1,\ \alpha_i \geq 0,\ i = 1, \ldots, N \right\}.$$

By using a robust state-feedback controller

$$u(t) = Kx(t), \tag{2}$$

with $K \in \mathbb{R}^{n_u\times n_x}$, yields the following closed-loop system:

$$\dot{x}(t) = A_{cl}(\alpha)x(t). \tag{3}$$

with $A_{cl}(\alpha) = A(\alpha) + B(\alpha)K$.

B. Linear uncertain system under input saturation

To deal with the presence of input saturation, let us consider the following uncertain continuous-time system

$$\dot{x}(t) = A(\alpha)x(t) + B(\alpha)\text{sat}(u(t)). \tag{4}$$

The control input is subject to the component-wise saturation map $\text{sat}(\cdot) : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ defined as

$$\text{sat}\left(u_{(j)}\right) = \text{sign}\left(u_{(j)}\right) \min\left(\left|u_{(j)}\right|, \mu_{0(j)}\right), \forall j \in 1, \ldots, n_u,$$

where the positive scalar $\mu_{0(j)} \in \mathbb{R}$, bounds the (j)-th control input component due to the presence of input saturation.

Consider the robust state-feedback controller given by

$$u(t) = Kx(t). \tag{5}$$

Let us define the following nonlinearity:

$$\Psi(u(t)) = \text{sat}(u(t)) - u(t). \tag{6}$$

The use of (5), and (6), allows us to write (4) as:

$$\dot{x}(t) = (A(\alpha) + B(\alpha)K)x(t) + B(\alpha)\Psi(u(t)). \tag{7}$$

The generalized sector condition will be employed to address the dead zone nonlinearity introduced by the input saturation.

Lemma 1 ([15]) *Let the set*

$$\mathcal{D}_u = \left\{ x(t) \in \mathbb{R}^{n_x} : \left| (K - G)_{(j)}\, x(t) \right| \leq \mu_{0(j)} \right\}, \tag{8}$$

$j = 1, \ldots, n_u$. If $x(t) \in \mathcal{D}_u$, then for any $u(t) \in \mathbb{R}^{n_u}$, one has

$$\Psi(u(t))^\top T\left[\Psi(u(t)) + Gx(t)\right] \leq 0, \tag{9}$$

for any positive definite diagonal matrix $T \in \mathbb{R}^{n_u\times n_u}$ and matrix $G \in \mathbb{R}^{n_u\times n_x}$.

C. Safety Definitions

Before presenting the main results, let us define the following sets that are of interest in the next Section[1].

- The set of initial conditions:

$$\mathcal{I} = \mathcal{E}(H, h) = \left\{ x \in \mathbb{R}^n, \quad x^\top Hx \leq h \right\}. \tag{10}$$

- The safe set:

$$\mathcal{S} = \mathcal{E}(S, s) = \left\{ x \in \mathbb{R}^n, \quad x^\top Sx \leq s \right\}. \tag{11}$$

- The barrier set:

$$\mathcal{B} = \mathcal{E}(P, p) = \left\{ x \in \mathbb{R}^n, \quad x^\top Px \leq p \right\}. \tag{12}$$

Definition 1 (Safety) *A dynamic system is said to be* safe *if, for any $x_0 \in \mathcal{I}$ and $t \in \mathcal{T} := [0, T_f]$, there exists a control input $u \in \mathcal{U}$ such that the resulting trajectory satisfies $x(t, u, x_0) \in \mathcal{S}$. Any such control input is referred to as a* safe control input*.*

Definition 2 (Forward Invariance for Linear Systems) *Consider the continuous-time uncertain linear system* (1)*. A set $\mathcal{B} := \{x \in \mathbb{R}^{n_x} : b(x) \geq 0\}$, where $b : \mathbb{R}^{n_x} \to \mathbb{R}$ is a continuously differentiable function is said to be forward invariant for the system if*

$$\frac{\partial b(x)}{\partial x}(A(\alpha)x + B(\alpha)u) \geq 0, \quad \forall x \in \partial\mathcal{B}. \tag{13}$$

If such a control input u exists that ensures forward invariance, then $\mathcal{B}$ is referred to as a control invariant set*, the function $b(x)$ as a* control barrier function (CBF)*, and u as a* safe controller*.*

This paper aims to establish new LMI-based conditions for certifying the safety of linear uncertain systems, specifically for the models described by (3) and (7) with a given gain K. Moreover, we aim to develop conditions to co-design the barrier set and the state-feedback controller that will be able to keep the system in the safety set. The following problems are addressed in this paper:

$\mathcal{P}_1$ - Given a safe set $\mathcal{S}$, a set of initial conditions $\mathcal{I}$, and a robust stabilizing gain matrix $K \in \mathbb{R}^{n_x\times n_u}$, the goal is to determine a barrier set $\mathcal{B}$ such that $\mathcal{I} \subseteq \mathcal{B} \subseteq \mathcal{S}$.

$\mathcal{P}_2$ - Given a safe set $\mathcal{S}$, a set of initial conditions $\mathcal{I}$, design a robust stabilizing gain matrix $K \in \mathbb{R}^{n_x\times n_u}$, and a barrier set $\mathcal{B}$ such that $\mathcal{I} \subseteq \mathcal{B} \subseteq \mathcal{S}$. Additionally, to account for input saturation, it is required that $\mathcal{B} \subset \mathcal{D}_u$, ensuring that the control input remains within admissible limits throughout the barrier set.

[1]Hereafter the dependence on t is omitted to simplify the proofs and formulations.

III. MAIN RESULTS

Theorem 3 (Safety certificates for linear systems) *Given a safe set $\mathcal{S}$, a set of initial conditions $\mathcal{I}$, and a stabilizing gain matrix $K \in \mathbb{R}^{n_x \times n_u}$, if there exist symmetric positive definite matrices $P_i \in \mathbb{R}^{n_x \times n_x}$, matrices $X_1 \in \mathbb{R}^{n_x \times n_x}$, $X_2 \in \mathbb{R}^{n_x \times n_x}$, and positive scalars σ_1, σ_2 and $p \in \mathbb{R}$ such that the following conditions hold*

$$-P_i + \sigma_2 H > 0, \ i = 1, \dots N, \tag{14}$$

$$p - \sigma_2 h > 0, \tag{15}$$

$$P_i - \sigma_1 S > 0, \ i = 1, \dots N, \tag{16}$$

$$-p + \sigma_1 s > 0, \tag{17}$$

$$\begin{bmatrix} X_1 + X_1^\top & -P_i - X_1 A_{cl_i} + X_2^\top \\ \star & -X_2 A_{cl_i} - A_{cl_i}^\top X_2^\top \end{bmatrix} > 0, \ i = 1, \dots N, \tag{18}$$

then, for initial conditions starting in the set $\mathcal{I}$ as in (10)*, the trajectories of the system* (3) *will not leave the barrier set $\mathcal{B} = \mathcal{E}(P(\alpha), p) = \{x \in \mathbb{R}^n, \quad x^\top P(\alpha) x \le p\}$. Moreover, the invariant set $\mathcal{B}$ is such that $\mathcal{I} \subseteq \mathcal{B} \subseteq \mathcal{S}$.*

Proof: Multiplying (14) by α_i, $i = 1, \dots, N$ and summing up one has

$$-\sum_{i=1}^{N} \alpha_i P_i + \left(\sum_{i=1}^{N} \alpha_i \right) \sigma_2 H > 0.$$

Note that $\sum_{i=1}^{N} \alpha_i = 1$ so we can write

$$-P(\alpha) + \sigma_2 H > 0. \tag{19}$$

Pre-and post multiplying it by $x^\top$ and x, respectively and summing with (15) one has

$$p - x^\top P x - \sigma_2 (h - x^\top H x) > 0,$$

that ensures that $\mathcal{I} \subseteq \mathcal{B}$.

The same procedure is applied to (16) and (17) leading to

$$x^\top P x - p + \sigma_1 (s - x^\top S x) > 0,$$

that guarantees that $\mathcal{B} \subseteq \mathcal{S}$. Multiplying (18) by α_i, $i = 1, \dots, N$, and summing up results in

$$\begin{bmatrix} X_1 + X_1^\top & -P(\alpha) - X_1 A_{cl}(\alpha) + X_2^\top \\ \star & -X_2 A_{cl}(\alpha) - A_{cl}(\alpha)^\top X_2^\top \end{bmatrix} > 0. \tag{20}$$

Pre- and post multiplying it by $x^\top \begin{bmatrix} A_{cl}^\top & I \end{bmatrix}$, and its transpose, respectively gives

$$-x^\top (A_{cl}(\alpha)^\top P + P A_{cl}(\alpha)) x > 0,$$

which implies that $\dot{b}(x) > 0$ with $b(x) = p - x^\top P(\alpha) x$. As discussed in [16], the boundary condition can be omitted since it is defined by a convex and compact set. ■

Remark 1 *The level set $\mathcal{E}(P(\alpha), p) = \{x \in \mathbb{R}^n, \quad x^\top P(\alpha) x \le p\}$, can be characterized by the intersection of all ellipses in the uncertain domain: $\mathcal{E}(P(\alpha), p) = \bigcap_{\forall \alpha \in \Lambda_N} \mathcal{E}(P(\alpha), p)$. It has been shown in [17] that for linear parameter-dependent matrices in the form of $P(\alpha) = \sum_{i=1}^{N} \alpha_i P_i$ this problem can be equivalently addressed using a set of finite conditions $\mathcal{E}(P(\alpha), p) = \bigcap_{i=1,\dots,N} \mathcal{E}(P_i, p)$. This offline procedure can be tackled numerically or by a simple optimization problem. Given the matrices P_i from the solution of Theorem 3, find an ellipsoid $\mathcal{E}(V) \subseteq \mathcal{E}(P(\alpha), p)$ that satisfies:*

$$\mathcal{OP}_1 \equiv \begin{cases} \min & \mathtt{trace}(V) \\ s.t. & V - P_i \ge 0, \ i = 1, \dots, N. \end{cases} \tag{21}$$

Theorem 4 *Given a safe set $\mathcal{S}$, a set of initial conditions $\mathcal{I}$, if there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n_x \times n_x}$, a positive diagonal matrix $Y \in \mathbb{R}^{n_u \times n_u}$, matrices $Z \in \mathbb{R}^{n_u \times n_x}$, $R \in \mathbb{R}^{n_u \times n_x}$, positive scalars σ_1, σ_2 and $p \in \mathbb{R}$ such that $p < 1$, and the following conditions hold*

$$p - \sigma_2 h > 0, \tag{22}$$

$$\sigma_1 s - p > 0, \tag{23}$$

$$\begin{bmatrix} W & I \\ I & \sigma_2 H \end{bmatrix} > 0, \tag{24}$$

$$\begin{bmatrix} W & \sigma_1 W S \\ \sigma_1 S W & \sigma_1 S \end{bmatrix} > 0, \tag{25}$$

$$\begin{bmatrix} W A_i^\top + R^\top B_i^\top + A_i W + B_i R & B_i Y - Z^\top \\ Y B_i^\top - Z & -2Y \end{bmatrix} < 0, \quad i = 1, \dots, N, \tag{26}$$

$$\begin{bmatrix} W & R_{(j)}^\top - Z_{(i)}^\top \\ R_{(j)} - Z_{(j)} & \mu_{0(j)}^2 \end{bmatrix} > 0, \ j = 1, \dots, n_u, \tag{27}$$

then, the state-feedback control gain $K = R W^{-1}$ will ensure that for initial conditions starting in the set $\mathcal{I}$ as in (10)*, the trajectories of the system* (7) *will not leave the barrier set $\mathcal{B}$ given in* (12) *. Moreover, the invariant set is such that $\mathcal{I} \subseteq \mathcal{B} \subseteq \mathcal{S}$, and $\mathcal{E}(P, p) \subset \mathcal{E}(P, 1) \subset \mathcal{D}_u$ with $G = Z W^{-1}$, and $P = W^{-1}$.*

Proof: Using the Schur complement in (24), we obtain

$$\sigma_2 H - W^{-1} > 0.$$

Pre- and post-multiplying it by $x^\top$ and x, respectively, and summing with (22), yields

$$\sigma_2 x^\top H x - x^\top W^{-1} x + p - \sigma_2 h > 0.$$

Substituting $W^{-1} = P$ gives

$$b(x) - \sigma_2 (h - x^\top H x) > 0,$$

where $b(x) = p - x^\top P x$, ensuring that $\mathcal{I} \subseteq \mathcal{B}$.

Once again using the Schur complement to (25), we have

$$W - \sigma_1 W S W > 0.$$

Pre- and post-multiplying by $x^\top W^{-1}$ and its transpose, respectively, and replacing $P = W^{-1}$, we obtain

$$x^\top P x - \sigma_1 x^\top S x > 0.$$

Combining this with (23), we get

$$x^\top Px - \sigma_1 x^\top Sx + \sigma_1 s - p > 0,$$

which is equivalent to

$$-b(x) + \sigma_1(s - x^\top Sx) > 0.$$

guaranteeing that $\mathcal{B} \subseteq \mathcal{S}$.

Pre- and post-multiplying (26) by $\begin{bmatrix} W^{-1} & 0 \\ 0 & Y^{-1} \end{bmatrix}$, and using $R = KW$, $Y^{-1} = T$, $W^{-1} = P$, and $Z = GW$, we obtain

$$\begin{bmatrix} (A_i + B_iK)^\top P + P(A_i + B_iK) & PB_i - G^\top T \\ B_i^\top P - TG & -2T \end{bmatrix} < 0,$$

$i = 1, \ldots, N$. Multiplying it by α_i, $i = 1, \ldots, N$, and summing up yields

$$\begin{bmatrix} A_{cl}(\alpha)^\top P + PA_{cl}(\alpha) & PB(\alpha) - G^\top T \\ B(\alpha)^\top P - TG & -2T \end{bmatrix} < 0,$$

Pre- and post-multiplying this inequality by $\begin{bmatrix} x^\top & \Psi(u)^\top \end{bmatrix}$ and its transpose, and simplifying, yields

$$-\dot{b}(x) - 2\Psi(u)^\top T(\Psi(u) + Gx) < 0.$$

Since the sector condition in Lemma 1 is nonpositive, it follows that $-\dot{b}(x) < 0$.

Applying the Schur complement to (26), we have

$$W - \frac{1}{\mu_{0(j)}^2}\left(R_{(j)}^\top - Z_{(j)}^\top\right)\left(R_{(j)} - Z_{(j)}\right) > 0.$$

Substituting $Z = GW$, $R = KW$ and pre- and post-multiplying by W^{-1}, gives

$$W^{-1} - \frac{1}{\mu_{0(j)}^2}\left(K_{(j)}^\top - G_{(j)}^\top\right)\left(K_{(j)} - G_{(j)}\right) > 0.$$

Pre-multiplying it by $x^\top$ and post-multiplying by x, and using $P = W^{-1}$, we can write

$$x^\top\left(K_{(j)}^\top - G_{(j)}^\top\right)\left(K_{(j)} - G_{(j)}\right)x < \mu_{0(j)}^2 x^\top Px.$$

Since $b(x) = p - x^\top Px > 0$, it follows that $x^\top Px < p$, and because $p < 1$, we can write

$$x^\top\left(K_{(j)}^\top - G_{(j)}^\top\right)\left(K_{(j)} - G_{(j)}\right)x < \mu_{0(j)}^2,$$

or

$$\left|(K - G)_{(j)}\, x\right| \leq \mu_{0(j)},\ j = 1, \ldots, n_u,$$

and conclude that $\mathcal{E}(P,p) \subset \mathcal{E}(P,1) \subset \mathcal{D}_u$. ■

Remark 2 *Condition* (25) *in Theorem 4, involves a product between the scalar decision variable $\sigma_1 > 0$ and the positive definite decision matrix W. This coupling poses challenges for convex optimization, as it introduces bilinearity between decision variables. To circumvent this issue in the numerical experiments, we fix the scalar parameter $\sigma_1 = 1$, preserving convexity while enabling tractable computation.*

Remark 3 *The inverse of the matrix W is used to recover the robust state-feedback control gain, and for this reason it cannot depend on the uncertain parameter α. However, the following optimization problem can be used to increase the size of the barrier certificate*

$$\mathcal{OP}_2 \equiv \begin{cases} \max & \texttt{trace}(W) \\ \textit{s.t.} & \textit{Theorem } 4. \end{cases} \tag{28}$$

Theorem 4 can be adapted to consider the case when no actuator saturation is present. The following Corollary gives the conditions for this case.

Corollary 5 *Given a safe set $\mathcal{S}$, a set of initial conditions $\mathcal{I}$, if there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n_x \times n_x}$, matrix $R \in \mathbb{R}^{n_u \times n_x}$, positive scalars σ_1, σ_2 and $p \in \mathbb{R}$ such that* (22), (23), (24), (25), *and*

$$WA_i^\top + R^\top B_i^\top + A_iW + B_iR < 0,\ i = 1, \ldots, N, \tag{29}$$

then, the state-feedback control gain $K = RW^{-1}$ will ensure that for initial conditions starting in the set $\mathcal{I}$ as in (10), *the trajectories of the system* (3) *will not leave the barrier set $\mathcal{B}$ given in* (12). *Moreover, the invariant set is such that $\mathcal{I} \subseteq \mathcal{B} \subseteq \mathcal{S}$, with $P = W^{-1}$.*

Proof: The proof follows the same steps of the proof of Theorem 4. ■

Remark 4 *All the conditions presented in this paper can be easily adapted for the case without uncertainties by simple removing the dependence i related to the uncertain parameter α.*

IV. NUMERICAL EXPERIMENTS

The numerical experiments were performed using MATLAB (R2020a) 64 bits for Windows 11, in a machine with Intel Core i5-8265U (1.8 GHz) processor and 8 GB RAM. The codes were implemented by using the packages YALMIP [18], and the solver Mosek [19].

A. Example 1

Consider the continuous-time linear system, borrowed from [16], with matrices:

$$A = \begin{bmatrix} 0.8 & 0.7 \\ -0.4 & -0.6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{30}$$

We consider the presence of actuator saturation with $\mu_{0(1)} = \mu_{0(2)} = 0.3$. The safe set is the interior of a disc:

$$\mathcal{S} = \mathcal{E}(S,s) = \left\{x \in \mathbb{R}^n,\ \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 1\right\},$$

and the initial conditions set is given by

$$\mathcal{I} = \mathcal{E}(H,h) = \left\{x \in \mathbb{R}^n,\ \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 0.16\right\}.$$

The system is open-loop unstable. We used Theorem 4 to co-design a controller and a barrier set that will guarantee the safety of the system. Figure 1 illustrates the set $\mathcal{I}$ of initial conditions, the safe set $\mathcal{S}$, the barrier set $\mathcal{B}$ derived from Theorem 4, and the barrier set $\mathcal{B}_{Opt}$ obtained with

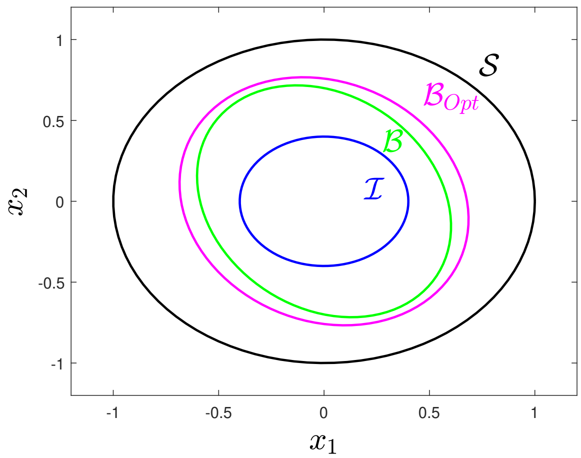

Fig. 1: Set of initial conditions ($\mathcal{I}$), barrier set obtained with Theorem 4 ($\mathcal{B}$), barrier set obtained with Optimization Problem 28 ($\mathcal{B}_{Opt}$), and safe set ($\mathcal{S}$).

the Optimization Problem (28). In both scenarios, we have used $\sigma_1 = 1$. Optimization Problem has expanded the barrier region, yielding a less conservative result.

Figure 2 displays the trajectories originating from randomly selected initial conditions on the boundary of the barrier set. The motivation is to show that even if the trajectories reach the barrier set, they are guided towards the origin. This behavior illustrates that the barrier function and the safe control input effectively maintain the system's trajectories within the safe set. It is possible to notice that the barrier set is contained within the set $\mathcal{D}_u$ ensuring that the control input remains within admissible limits. This containment allows the system to be stabilized even in the presence of actuator saturation.

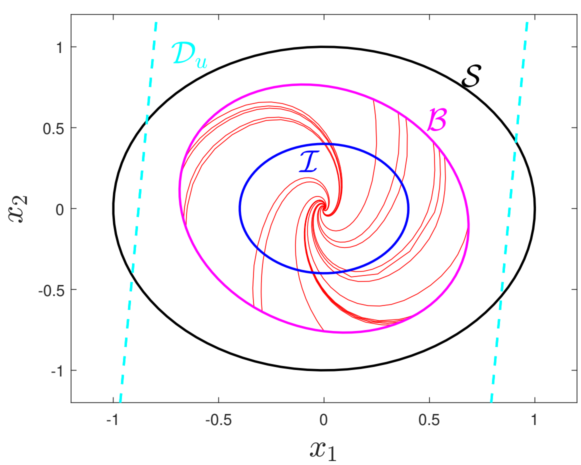

Fig. 2: Set of initial conditions ($\mathcal{I}$), barrier set obtained with Optimization Problem 28 ($\mathcal{B}$), safe set ($\mathcal{S}$), system trajectories in red color — and the set $\mathcal{D}_u$ $--$.

By increasing the value of h in the initial condition set and applying the Optimization Problem (28) we could find feasible solutions up to $h = 0.68$. Figure 3 shows the results for this case. We can see that the barrier set increased considerably. However, it does not leave the safe set or the limits imposed by the actuator saturation, region $\mathcal{D}_u$.

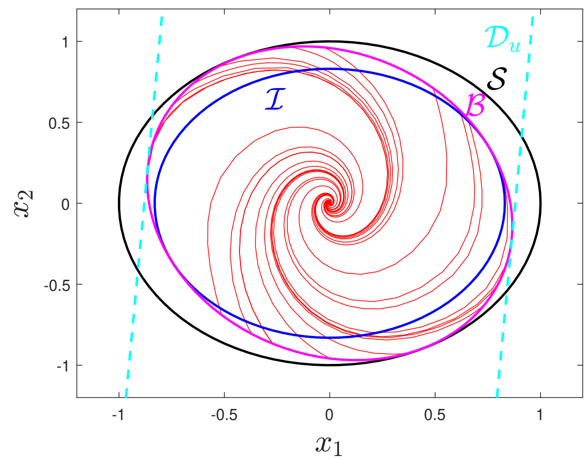

Fig. 3: Set of initial conditions ($\mathcal{I}$), barrier set obtained with Optimization Problem 28 ($\mathcal{B}$), safe set ($\mathcal{S}$), system trajectories in red color — and the set $\mathcal{D}_u$ $--$.

B. Example 2

Consider the spring-mass system on a horizontal surface borrowed from [6], where the system matrices are as follows

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{K_e}{m} & -\frac{K_v}{m} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \tag{31}$$

where m is the mass, K_e is the stiffness of the spring and K_v is the viscous friction coefficient. The nominal parameter values are: $m = 1$, $K_e = 1$, $K_v = 1$, and it is assumed that each parameter belong to an uncertain interval within $\pm 50\%$ from its nominal value. This system can be represented as a polytope with 8 vertices. The safe set and the set of initial conditions are the same as those used in Experiment 1.

By setting $\mu_0 = 0.5$ and applying Theorem 4, we co-designed a controller and a barrier set to ensure the system's safety. The results are shown in Figure 4. The barrier set $\mathcal{B}$ obtained with Theorem 4 ($\sigma_1 = 1$), is contained within the barrier set $\mathcal{B}_{Opt}$ obtained with the Optimization Problem (28) ($\sigma_1 = 1$). It can be seen that the optimization process has expanded the barrier region even in the presence of uncertainty. The matrices obtained from the solution of the Optimization Problem (28) are

$$K = \begin{bmatrix} -0.1231 & -1.0461 \end{bmatrix}, \; G = \begin{bmatrix} 0.0122 & -0.4599 \end{bmatrix},$$

$$P = W^{-1} = \begin{bmatrix} 1.5953 & 0.4833 \\ 0.4833 & 1.3924 \end{bmatrix}.$$

Figure 5 depicts the trajectories for 20 randomly generated initial conditions starting at the boundary of $\mathcal{B}$ to illustrate that the trajectories will converge to the origin. We can see that the region $\mathcal{B}$ is conservative in this case. It can be related to the fact that a single Lyapunov function is used to design the controller for a polytopic system with 8 vertices.

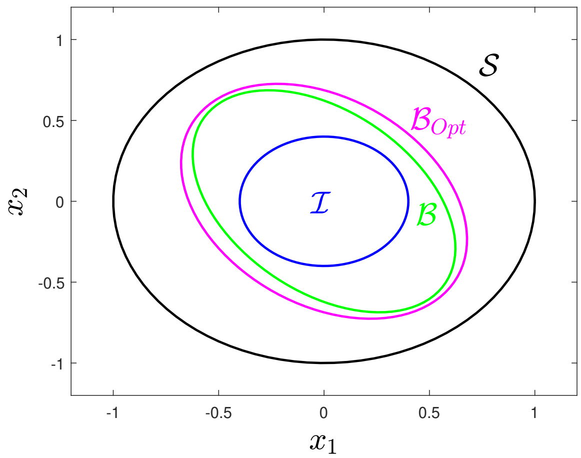

Fig. 4: Set of initial conditions ($\mathcal{I}$), barrier set obtained with Theorem 4 ($\mathcal{B}$), barrier set obtained with Optimization Problem 28 ($\mathcal{B}_{Opt}$), and safe set ($\mathcal{S}$) for uncertain system.

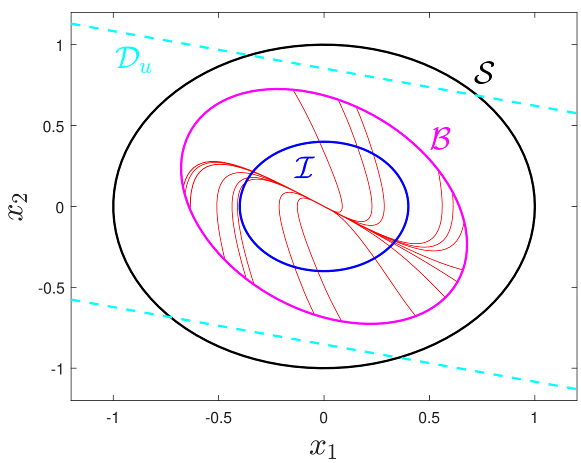

Fig. 5: Set of initial conditions ($\mathcal{I}$), barrier set obtained with Optimization Problem 28 ($\mathcal{B}$), safe set ($\mathcal{S}$), system trajectories in red color – and the set $\mathcal{D}_u$ −−.

V. CONCLUSIONS

This paper introduced conditions in the form of LMIs for the design of safe, robust state-based feedback controllers for continuous-time uncertain linear systems, considering actuator saturation. Moreover, a barrier Lyapunov function based on a defined ellipsoid was employed to derive the conditions. The proposed optimization problems helped reduce the conservatism of the solutions. Numerical experiments illustrated the effectiveness and potential of this technique. In future work, the authors are investigating the analysis/design of barrier functions for discrete-time systems and the use of output-feedback control laws.

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