Minimax in the theory of operators on Hilbert spaces and Clarkson-McCarthy estimates for $l_q(S^p)$ spaces of operators in Schatten ideals

Teresa Formisano

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Abstract

The main results in this thesis are the minimax theorems for operators in Schatten ideals of compact operators acting on separable Hilbert spaces, generalized Clarkson-McCarthy inequalities for vector l_q-spaces $l_q(S^p)$ of operators from Schatten ideals S^p, inequalities for partitioned operators and for Cartesian decomposition of operators. All Clarkson-McCarthy type inequalities are in fact some estimates on the norms of operators acting on the spaces $l_q(S^p)$ or from one such space into another.

Contents

Chapter 1 Introduction

The study of linear operators and functionals on Banach and Hilbert spaces aims at producing results and techniques that help us to understand the structure and properties of these spaces. This study was developed in twentieth-century and attracted some of the greatest mathematicians such as D. Hilbert, F. Riesz, J. von Neumann and S. Banach. It grew and became a branch of mathematics called functional analysis. It includes the study of vector spaces, spaces of functions and various classes of operators defined on them. Some of the most important theorems in functional analysis are: Hahn-Banach theorem, uniform boundedness theorem, open mapping theorem and the Riesz representation theorem. There are numerous applications of this theory in algebra, real and complex analysis, numerical analysis, calculus of variations, theory of approximation, differential equations, representation theory, physics (for example boundary value problems and quantum mechanics), engineering and statistics.

Functional analysis uses language, concepts and methods of logic, real and complex analysis, algebra, topology and geometry in the study of functions on linear spaces and function spaces.

The first minimax theorem was proved by von Neumann in 1928 - it was a result related to his work on games of strategy. No new development occurred for the next ten years but, as time went on, minimax theorems became an object of study not only in the game theory but also in other branches of mathematics. Minimax theory consists of establishing sufficient and necessary conditions for the following equality to hold:

$$\inf_{x\in X}\sup_{y\in Y} f(x,y)=\sup_{y\in Y}\inf_{x\in X} f(x,y), \tag{1.1}$$

where $f(x,y)$ is a function defined on the product of spaces X and Y. Minimax theory is applied in decision theory, game theory, optimization, computational theories, philosophy and statistics, for example to maximize potential gain. For overview on minimax theory and its applications see [34] and [15].

This thesis has two aims and, consequently, is divided into two parts that correspond to them. The first part consists of Chapters 2, 3 and 4. In these chapters we verify whether the general minimax conditions hold in various settings of the operator theory. We also identify necessary and sufficient conditions for which minimax theorems can be proved for certain classes of functionals and operators on Hilbert spaces.

The second part consists of Chapters 5 and 6. Its aim is to obtain generalized Clarkson-McCarthy inequalities for l_q-spaces of operators from Schatten ideals S^p. We apply these generalized inequalities to prove various estimates for partitions and Cartesian decomposition of operators from $S^p\left(H,H^\infty\right)$ and $l_q\left(S^p\right)$ spaces.

Borenshtein and Shulman proved in [10] that if Y is a compact metric space, X is a real interval and f is continuous, then (1.1) holds provided that, for each $y\in Y$, the function $f\left(\cdot,y\right)$ is convex and, for each $x\in X$, every local maximum of the function $f\left(x,\cdot\right)$ is a global maximum. Some weaker conditions on f that ensure the validity of (1.1) were established by Saint Raymond in [36] and Ricceri in [33]. Minimax theory has various applications in the operator theory; see, for example, Asplund-Ptak equality

$$\inf_{\lambda \in \mathbb{C}} \sup_{\|x\|=1} \|Ax - \lambda Bx\| = \sup_{\|x\|=1} \inf_{\lambda \in \mathbb{C}} \|Ax - \lambda Bx\|,$$

where H is a Hilbert space, $x \in H$, $\mathbb{C}$ is the the set of complex numbers and A and B are bounded linear operators on H [2].

In our work we wanted to identify new minimax theorems that hold for seminorms and linear operators that act on separable Hilbert spaces. In Chapter 3 we obtain some minimax results that hold for sequences of bounded seminorms. We illustrate these results with examples of seminorms on the Hilbert space l_2. Next we consider and prove some simple minimax formula for operators. The formula works also for bilinear functionals on a Hilbert space. The main results on minimax conditions obtained in this thesis are the minimax conditions for operators in Schatten ideals of compact operators. The details of this theory are explained in Chapter 4, and the results, namely Proposition 4.8 and Theorems 4.9, 4.11 and 4.15, have been published in our joint paper in [19, pp.29-40] under the joint authorship of T. Formisano and E. Kissin, where the second author contributed to various stages and to its final version.

Clarkson proved in [12] famous inequalities for Banach spaces of sequences l_p, $p > 1$. He used these inequalities to show that the l_p spaces, for $p > 1$, are uniformly convex. McCarthy obtained in [28] non-commutative analogues of Clarkson estimates for pairs of operators in Schatten ideals S^p. Using them, he proved that the spaces S^p are uniformly convex, for $1 < p < \infty$, and therefore they are reflexive Banach spaces [39, p.23]. The Clarkson-McCarthy estimates play an important role in analysis and operator theory. They were generalized to a wider class of norms that include the p-norms by Bhatia and Holbrook [6] and Hirzallah and Kittaneh [24]. In [9] Bhatia and Kittaneh proved analogues of Clarkson-McCarthy inequalities for n-tuples of operators of special type. Kissin [25] extended these estimates to all n-tuples of operators. He also extended the results of Bhatia and Kittaneh in [7] and [8] on estimates for partitioned operators and for Cartesian decomposition of operators.

In Chapters 5 and 6 we develop a theory that allows us to extend the result of Kissin [25] and to obtain an analogue of generalized Clarkson-McCarthy inequality for $l_q(S^p)$ spaces. We also establish various relevant results for operators that belong to $l_q(S^p)$ and $S^p(H, H^\infty)$ spaces. Making use of this, we prove that the spaces $l_p(S^p)$ are p-uniformly convex, for $p \geq 2$. We also analyze partition of operators from S^p spaces and Cartesian decomposition of operators from $l_q(S^p)$ spaces. In fact, we extend the results obtained in [25, Theorems 1 and 4-5] to infinite families of projections and operators. This extension requires new techniques and a new approach to the theory of $l_q(S^p)$ spaces and their relation to $S^p(H, H^\infty)$ spaces.

Finally, we draw conclusions in Chapter 7. We provide elementary background of the theory of Hilbert spaces in the next chapter. In most cases, the reader can find proofs of known results in the referenced literature. In some instances, we give the proofs of some well known results for the readers convenience.

Part I

Minimax

Chapter 2 Preliminaries and background

A linear space X over $\mathbb{R}$ (real numbers) or $\mathbb{C}$ (complex numbers) is called a normed linear space if it is equipped with a norm $\|\cdot\|$, that is, each $x \in X$ is associated with a non-negative number $\|x\|$ – the norm of x, with the properties:

$$(i)\ \ \|x\| \geq 0 \text{ and } \|x\| = 0 \text{ if and only if } x = 0; \tag{2.1}$$

$$(ii)\ \ \|x + y\| \leq \|x\| + \|y\| \text{ for all } x, y \in X; \tag{2.2}$$

$$(iii)\ \ \|\alpha x\| = |\alpha| \, \|x\| \text{ where } \alpha \text{ is a scalar.} \tag{2.3}$$

The distance between $x, y \in X$ is defined by $\|x - y\|$. The concept of norm generalizes the notion of absolute value and, more generally, the notion of the length of a vector. For example if $\mathbb{R}$ is the real line with usual arithmetic and $x \in \mathbb{R}$ then the usual absolute value, $|x|$, is a norm. Having the distance function given by a norm, we can extend familiar concepts from calculus to this more general setting.

Definition 2.1 *Let (x_n) be a sequence in a normed space $(X, \|\cdot\|)$.*

(i) *It is a Cauchy sequence if for every $\varepsilon > 0$ there is an integer N such that $m, n \geq N$ implies $\|x_n - x_m\| < \varepsilon$.*

(ii) *It has a limit $x \in X$ (in other words, (x_n) converges to x) provided that, for every $\varepsilon > 0$, there exists an integer N such that $n \geq N$ implies $\|x_n - x\| < \varepsilon$. We write $\lim_{n\to\infty} \|x_n - x\| = 0$, $\|x_n - x\| \to_{n\to\infty} 0$, $\lim_{n\to\infty} x_n = x$ or $x_n \to_{n\to\infty} x$.*

A function from X into another normed space Y is continuous at $x \in X$ provided *that for every sequence* (x_n) *in* X *converging to* x, *the sequence* $(f(x_n))$ *converges to* $f(x)$.

If every Cauchy sequence in a normed linear space X has a limit in X then X is said to be *complete*. A complete normed linear space is called *a Banach space* in honour of the Polish mathematician Stefan Banach.

Let X be a Banach space with norm $\|\cdot\|_X = \|\cdot\|$. For $1 \leq p < \infty$, the $l_p(X)$ space consists of all infinite sequences $x = (x_1, ..., x_n, ...)$ of elements $x_n \in X$ such that

$$\|x\|_p = \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p} < \infty.$$

For $p = \infty$, the $l_\infty(X)$ space consists of all infinite sequences $x = (x_1, ..., x_n, ...)$ of elements $x_n \in X$ such that

$$\|x\|_\infty = \sup_n \|x_n\| < \infty.$$

The proof below (see Lemma 2.2 - Theorem 2.4) that all $l_p(X)$, $1 \leq p < \infty$, are Banach spaces is based on proofs developed in [30, pp.45-46] and [32, pp.78-81].

Recall that a real-valued function f defined on an interval I of $\mathbb{R}$ is convex if

$$f(\alpha a + (1-\alpha) b) \leq \alpha f(a) + (1-\alpha) f(b), \text{ for all } 0 \leq \alpha \leq 1 \text{ and all } a, b \in I.$$

In other words, if $a, b \in I$ then the graph of the function f restricted to the interval $[a, b]$ lies beneath the line segment joining the points $(a, f(a))$ and $(b, f(b))$. Positivity of the second derivative is a sufficient condition for convexity, showing that, in particular, the function $f(t) = e^t$ is convex.

Consider numbers $p \geq 1$, $q \geq 1$ satisfying

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{2.4}$$

If one of the numbers is 1, we assume that the other is ∞.

Lemma 2.2 [30, Lemma 2.36], [32, Lemma IX.1] *If* $s \geq 0, t \geq 0$ *then, for* $p, q > 1$ *satisfying* (2.4),

$$st \leq \frac{s^p}{p} + \frac{t^q}{q}.$$

Proof. If $st = 0$, the lemma is evident. Let $s > 0$ and $t > 0$. Set $a = p \ln s$ and $b = q \ln t$. Then $s = e^{a/p}$ and $t = e^{b/q}$. Thus $s^p = e^a$ and $t^q = e^b$. By convexity of $f(t) = e^t$, we obtain

$$st = e^{a/p} e^{b/q} = e^{\left(\frac{1}{p}a + \left(1 - \frac{1}{p}\right)b\right)} \leq \frac{1}{p} e^a + \left(1 - \frac{1}{p}\right) e^b = \frac{1}{p} s^p + \frac{1}{q} t^q$$

which completes the proof. ■

The above lemma allows us to prove easily the following important Holder's and Minkowski's inequalities.

Proposition 2.3 (i) [32, Theorem IX.2] (Holder's inequality) *Let* $p > 1, q > 1$ *satisfy* (2.4). *Then, for any* $n \in \mathbb{N}$ *and* $a_i, b_i \in \mathbb{C}$, $i = 1, ..., n$,

$$\sum_{i=1}^{n} |a_i b_i| \leq \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q \right)^{1/q}. \tag{2.5}$$

(ii) [32, Theorem IX.3] (Minkowski's inequality) *If* $p \geq 1$ *then, for any* $n \in \mathbb{N}$ *and* $a_i, b_i \in \mathbb{C}$, $i = 1, ..., n$,

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p}. \tag{2.6}$$

Proof. (i) Let $A = (\sum_{i=1}^{n} |a_i|^p)^{1/p}$ and $B = (\sum_{i=1}^{n} |b_i|^q)^{1/q}$. If $A = 0$ or $B = 0$, the proof is evident. Otherwise, by Lemma 2.2, we obtain

$$\frac{|a_i|}{A}\frac{|b_i|}{B} \leq \frac{\left(\frac{|a_i|}{A}\right)^p}{p} + \frac{\left(\frac{|b_i|}{B}\right)^q}{q} = \frac{|a_i|^p}{pA^p} + \frac{|b_i|^q}{qB^q}.$$

Thus

$$|a_i|\,|b_i| \leq \frac{AB}{pA^p}\,|a_i|^p + \frac{AB}{qB^q}\,|b_i|^q\,.$$

Hence, summing up, we obtain

$$\sum_{i=1}^{n} |a_i b_i| \leq \frac{AB}{pA^p}\sum_{i=1}^{n} |a_i|^p + \frac{AB}{qB^q}\sum_{i=1}^{n} |b_i|^q = \frac{AB}{pA^p}A^p + \frac{AB}{qB^q}B^q =$$

$$= AB\left(\frac{1}{p} + \frac{1}{q}\right) = AB.$$

(ii) For $p = 1$, the inequality is evident. Let $p > 1$ and let p, q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then $1 + \frac{p}{q} = p$ and applying the triangle inequality, we have

$$|a_i + b_i|^p = |a_i + b_i|\,|a_i + b_i|^{p/q} \leq (|a_i| + |b_i|)\,|a_i + b_i|^{p/q} =$$

$$= |a_i|\,|a_i + b_i|^{p/q} + |b_i|\,|a_i + b_i|^{p/q}\,.$$

Thus, summing up and applying (2.5), we get

$$\sum_{i=1}^{n} |a_i + b_i|^p \leq \sum_{i=1}^{n} |a_i|\,|a_i + b_i|^{p/q} + \sum_{i=1}^{n} |b_i|\,|a_i + b_i|^{p/q} \leq$$

$$\stackrel{(2.5)}{\leq} \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{1/q} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{1/q}.$$

If $\sum_{i=1}^{n} |a_i + b_i|^p = 0$, the proof is evident. Otherwise, dividing the above inequality by $(\sum_{i=1}^{n} |a_i + b_i|^p)^{1/q}$ and using $\frac{1}{p} = 1 - \frac{1}{q}$, we obtain

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p}.$$

The proof is complete. ■

Using norm triangle inequality (2.2), we obtain that, for a Banach space X with norm $\|\cdot\|$ and $a_i, b_i \in X$, $i = 1, ..., n$, Minkowski's estimate gives

$$\begin{aligned} \left(\sum_{i=1}^{n} \|a_i + b_i\|^p\right)^{1/p} &\leq \left(\sum_{i=1}^{n} (\|a_i\| + \|b_i\|)^p\right)^{1/p} \\ &\stackrel{(2.6)}{\leq} \left(\sum_{i=1}^{n} \|a_i\|^p\right)^{1/p} + \left(\sum_{i=1}^{n} \|b_i\|^p\right)^{1/p}. \end{aligned} \tag{2.7}$$

We shall now prove that all $l_p(X)$, $1 \leq p < \infty$, are Banach spaces.

Theorem 2.4 *Let X be a Banach space. The space $l_p(X)$, for $1 \leq p < \infty$, is a Banach space – a normed linear space complete with respect to $\|\cdot\|_p$.*

Proof. Let $x = (x_1, ..., x_n, ...) \in l_p(X)$. Then $\|x\|_p = 0$ if and only if $x = 0$, i.e., all $x_n = 0$.

Clearly, $\alpha x \in l_p(X)$ for each $\alpha \in \mathbb{C}$, and $\|\alpha x\|_p = |\alpha| \|x\|_p$.

Let also $y = (y_1, ..., y_n, ...) \in l_p$. It follows from Minkowski's inequality (2.7) that, for all n,

$$\left(\sum_{i=1}^{n}\|x_i+y_i\|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n}\|x_i\|^p\right)^{1/p} + \left(\sum_{i=1}^{n}\|y_i\|^p\right)^{1/p}$$
$$\leq \left(\sum_{i=1}^{\infty}\|x_i\|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty}\|y_i\|^p\right)^{1/p} = \|x\|_p + \|y\|_p .$$

Hence the sequence of partial sums $S_n = \sum_{i=1}^{n}\|x_i+y_i\|^p$ is bounded by $\left(\|x\|_p + \|y\|_p\right)^p$ and monotone increasing. Therefore $\lim S_n$ exists and

$$\lim_{n\to\infty} S_n = \sum_{i=1}^{\infty}\|x_i+y_i\|^p \leq \left(\|x\|_p + \|y\|_p\right)^p .$$

Thus

$$\|x+y\|_p = \left(\sum_{i=1}^{\infty}\|x_i+y_i\|^p\right)^{1/p} = \left(\lim_{n\to\infty} S_n\right)^{1/p} \leq \|x\|_p + \|y\|_p ,$$

so that $\|x+y\|_p \leq \|x\|_p + \|y\|_p$. Thus $\|\cdot\|_p$ is a norm.

The triangle inequality $\|x+y\|_p \leq \|x\|_p + \|y\|_p$ implies that $l_p(X)$ is closed under addition, i.e., if $x, y \in l_p(X)$ then $x+y \in l_p(X)$. Thus $l_p(X)$ is a normed linear space and we only need to show that it is complete.

Let $\{x^k = (x_1^k, ..., x_n^k, ...)\}_{k=1}^{\infty}$ be a Cauchy sequence in $l_p(X)$. Then, for each $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that, if $r, s > N$ then

$$\|x^r - x^s\|_p = \left(\sum_{n=1}^{\infty}\|x_n^r - x_n^s\|^p\right)^{1/p} < \varepsilon.$$

Consequently, for each $n = 1, 2, ...,$ we have $\|x_n^r - x_n^s\| \leq \|x^r - x^s\|_p < \varepsilon$. Thus for each n, the "vertical sequence" $\{x_n^k\}_{k=1}^{\infty}$ is a Cauchy sequence in X. As X is a Banach space and, therefore, is complete, there are $x_n \in X$ such that

$$\lim_{k\to\infty}\left\|x_n^k - x_n\right\| = 0 \text{ for all } n. \tag{2.8}$$

Set $x = (x_1, ..., x_n, ...)$. We shall show that $x \in l_p(X)$, i.e., $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$ and that $\|x - x^k\|_p \to 0$.

As $\{x^k\}_{k=1}^{\infty}$ is a Cauchy sequence in $l_p(X)$, for $\varepsilon = 1$, choose N such that $\|x^k - x^N\|_p \leq 1$ for all $k \geq N$. Setting $s = \|x^N\|_p$, we have

$$\|x^k\|_p = \|x^k - x^N + x^N\|_p \leq \|x^k - x^N\|_p + \|x^N\|_p \leq s + 1 \text{ for all } k \geq N.$$

Suppose that $x \notin l_p(X)$. Then there is q such that $\left(\sum_{n=1}^{q} \|x_n\|^p\right)^{1/p} > s + 3$. Hence, for all $k \geq N$,

$$\begin{aligned} s + 3 < \left(\sum_{n=1}^{q} \|x_n\|^p\right)^{1/p} &= \left(\sum_{n=1}^{q} \|(x_n - x_n^k) + x_n^k\|^p\right)^{1/p} \\ &\overset{(2.7)}{\leq} \left(\sum_{n=1}^{q} \|x_n - x_n^k\|^p\right)^{1/p} + \left(\sum_{n=1}^{q} \|x_n^k\|^p\right)^{1/p} \\ &\leq \left(\sum_{n=1}^{q} \|x_n - x_n^k\|^p\right)^{1/p} + \|x^k\|_p \leq \left(\sum_{n=1}^{q} \|x_n - x_n^k\|^p\right)^{1/p} + s + 1. \end{aligned}$$

By (2.8), we can choose $M \in \mathbb{N}$ such that $\|x_n - x_n^k\| \leq \frac{1}{q^{1/p}}$, for each $n = 1, ..., q$ and all $k \geq M$. Then

$$\sum_{n=1}^{q} \|x_n - x_n^k\|^p \leq q \times \frac{1}{q} = 1.$$

Combining this with the above inequality, we have $s + 3 < s + 2$. This contradiction shows that $x \in l_p(X)$.

Let us show now that $\|x - x^k\|_p \to 0$. As $\{x^k\}_{k=1}^{\infty}$ is a Cauchy sequence, choose N such that

$$\|x^k - x^N\|_p \leq \frac{\varepsilon}{9} \text{ for all } k \geq N. \tag{2.9}$$

For $v \geq 1$, let Q_v be the projection on $l_p(X)$ such that $Q_v y = (y_1, ..., y_v, 0, 0, ..)$ for all $y = (y_n) \in l_p(X)$. Then

$$\|Q_v y\|_p \leq \|y\|_p \text{ and } Q_v y \to y \text{ as } v \to \infty. \tag{2.10}$$

For $\varepsilon > 0$, we can choose m such that

$$\|x - Q_m x\|_p < \frac{\varepsilon}{3} \text{ and } \left\|x^N - Q_m x^N\right\|_p < \frac{\varepsilon}{9}. \tag{2.11}$$

Then, by (2.9)-(2.11), we have for all $k \geq N$,

$$\begin{aligned} \left\|x^k - Q_m x^k\right\|_p &\leq \left\|x^k - x^N\right\|_p + \left\|x^N - Q_m x^N\right\|_p + \left\|Q_m(x^N - x^k)\right\|_p \\ &\leq 2\left\|x^k - x^N\right\|_p + \left\|x^N - Q_m x^N\right\|_p < \frac{2\varepsilon}{9} + \frac{\varepsilon}{9} = \frac{\varepsilon}{3}. \end{aligned} \tag{2.12}$$

By (2.8), we can choose k_0 such that $\left\|x_n - x_n^k\right\| \leq \frac{\varepsilon}{3m^{1/p}}$, for all $n = 1, ..., m$ and all $k \geq k_0$. Then

$$\left\|Q_m(x - x^k)\right\|_p = \left(\sum_{n=1}^{m} \left\|x_n - x_n^k\right\|^p\right)^{1/p} \leq \left(\sum_{n=1}^{m} \left(\frac{\varepsilon}{3m^{1/p}}\right)^p\right)^{1/p} = \frac{\varepsilon}{3}. \tag{2.13}$$

Hence, for $k \geq \max(N, k_0)$, it follows from (2.11) - (2.13) that

$$\left\|x - x^k\right\|_p \leq \|x - Q_m x\|_p + \left\|Q_m(x - x^k)\right\|_p + \left\|Q_m x^k - x^k\right\|_p \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus $\left\|x - x^k\right\|_p \to 0$ as $k \to \infty$. ■

Example 2.5 [32, p.78] Consider the space $l_p = l_p(\mathbb{C})$, for $1 \leq p < \infty$. The elements of l_p are sequences of complex numbers $x = \{x_n\}_1^\infty$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$. If we define the p-norm on l_p by the formula

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p},$$

then we derive from Theorem 2.4 that each l_p is a Banach space.

Remark 2.6 Let X be a Banach space. The space $l_\infty(X)$ is a Banach space. We omit the proof as it is similar to the proof of Theorem 2.4.

It is also known (see for example [16, Lemma 9, XI.9.]) that

$$l_p \subseteq l_q \text{ and } \|x\|_p \geq \|x\|_q\,, \text{ for } x \in l_p, \text{ if } 1 \leq p \leq q.$$

Thus l_1 is the smallest and l_∞ is the largest of the spaces.

Definition 2.7 [1, p.2] *Let X be a linear complex space. An inner-product (or scalar-product) $(\cdot, \cdot)$ is a complex-valued function defined on $X \times X$ which satisfies the conditions:*

1. $(x, y) = \overline{(y, x)}$;
2. $(\alpha x + \beta y, z) = \alpha (x, z) + \beta (y, z)$ *for* $\alpha, \beta \in \mathbb{C}$
3. $(x, x) \geq 0$, *with equality if and only if* $x = 0$.

We can derive from the above conditions that

$$\begin{aligned} (x, \alpha y + \beta z) &= \overline{(\alpha y + \beta z, x)} = \overline{\alpha (y, x) + \beta (z, x)} \\ &= \overline{\alpha (y, x)} + \overline{\beta (z, x)} = \overline{\alpha} (x, y) + \overline{\beta} (x, z)\,. \end{aligned}$$

The Cauchy-Schwarz-Bunyakovsky inequality is one of the most important inequalities in mathematics:

Theorem 2.8 (Cauchy-Schwarz-Bunyakovsky inequality) [1, p.2] *If* $(\cdot,\cdot)$ *is an inner-product on a linear space* X *then*

$$|(x,y)| \leq (x,x)^{\frac{1}{2}} (y,y)^{\frac{1}{2}}, \text{ for all } x,y \in X, \tag{2.14}$$

with equality if and only if x *and* y *are linearly dependent.*

Proof. If $(x,y) = 0$ the theorem is proved. We can assume that $(x,y) \neq 0$. Letting $\theta = \frac{(x,y)}{|(x,y)|}$, we find from Definition 2.7 that, for any real λ,

$$\begin{aligned} 0 &\leq \left(\overline{\theta}x + \lambda y, \overline{\theta}x + \lambda y\right) = |\theta|^2 (x,x) + \lambda\overline{\theta}(x,y) + \lambda\theta\overline{(x,y)} + \lambda^2 (y,y) \\ &= \left|\frac{(x,y)}{|(x,y)|}\right|^2 (x,x) + \lambda\overline{\frac{(x,y)}{|(x,y)|}}(x,y) + \lambda\frac{(x,y)}{|(x,y)|}\overline{(x,y)} + \lambda^2 (y,y) \\ &= (x,x) + \lambda\frac{|(x,y)|^2}{|(x,y)|} + \lambda\frac{|(x,y)|^2}{|(x,y)|} + \lambda^2 (y,y) \\ &= \lambda^2 (y,y) + 2\lambda |(x,y)| + (x,x). \end{aligned} \tag{2.15}$$

We arrived at a non-negative (no roots or one repeated root) quadratic in λ. Thus the discriminant of this quadratic is non-positive:

$$4|(x,y)|^2 - 4(y,y)(x,x) \leq 0$$

Hence $|(x,y)|^2 \leq (x,x)(y,y)$ and we have the inequality (2.14).

The equality in (2.14) holds if and only if the quadratic has a repeated root, in other words if and only if

$$\lambda^2 (y,y) + 2\lambda |(x,y)| + (x,x) = 0, \text{ for some } \lambda \in \mathbb{R}.$$

This implies (see (2.15)) that $\left(\overline{\theta}x + \lambda y, \overline{\theta}x + \lambda y\right) = 0$. Thus $\overline{\theta}x + \lambda y = 0$ for some real λ, so that the vectors x and y are linearly dependent. ■

Let X be a linear space with scalar product $(\cdot, \cdot)$. Set

$$\|x\| = (x, x)^{1/2}.$$

Let us check that $\|\cdot\|$ is a norm on X. From the Definition 2.7 we have $\|x\| \geq 0$ with equality if and only if $x = 0$. Additionally, it follows that $\|\alpha x\|^2 = (\alpha x, \alpha x) = \alpha\overline{\alpha}\,(x, x) = |\alpha|^2 \|x\|^2$ for all scalars α. Thus $\|\alpha x\| = |\alpha|\,\|x\|$ for all scalars α. To prove the triangle inequality, we apply the Cauchy-Schwarz-Bunyakovsky inequality to obtain

$$\begin{aligned} \|x + y\|^2 &= (x + y, x + y) = (x, x) + (x, y) + (y, x) + (y, y) \\ &\overset{(2.14)}{\leq} \|x\|^2 + 2\,\|x\|\,\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2 \text{ for all } x, y \in X. \end{aligned}$$

This implies the triangle inequality $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

A Banach space whose norm comes from a scalar-product as $\|x\| = (x, x)^{\frac{1}{2}}$ is called a Hilbert space in honour of the German mathematician David Hilbert [32].

A normed linear space (not complete) is called a pre-Hilbert space if its norm comes from an inner-product. Hilbert and pre-Hilbert spaces are called inner-product spaces [32].

Example 2.9 [1, p.5-7] Consider the Hilbert space l_2 that consists of sequences $x = \{x_n\}_1^\infty$ of complex numbers such that

$$\sum_{n=1}^{\infty} |x_n|^2 < \infty.$$

As in Example 2.5, it is a Banach space with norm

$$\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2}.$$

The scalar product in the space l_2 has the form

$$(x, y) = \sum_{n=1}^{\infty} x_n \overline{y_n}.$$

The series on the right converges absolutely because

$$|t\overline{s}| = |ts| \leq \frac{1}{2} |t|^2 + \frac{1}{2} |s|^2 \textit{ for all } t, s \in \mathbb{C}.$$

We omit the simple prove that the number (x, y) satisfies all the conditions of a scalar product and the norm $\|x\|$ of each vector $x \in l_2$ satisfies

$$\|x\| = (x, x)^{1/2} = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{\frac{1}{2}}.$$

Definition 2.10 [32] [27, Definition 1.21] *Let X be an inner-product space. Elements $x, y \in X$ are orthogonal (we write $x \perp y$) if their inner-product $(x, y) = 0$. For sets A and B in X, we write $A \perp B$ if $(x, y) = 0$ for all $x \in A$ and $y \in B$. Finally, $A^\perp$ is the set of all vectors $x \in X$ such that $x \perp y$ for all $y \in A$; for any set A this is always a subspace of X, moreover since $A^\perp = \cap_{a \in A} \{a\}^\perp$, $A^\perp$ is a closed subspace by continuity of the inner product.*

A subset S of X is an orthogonal set, if $x, y \in S$ and $x \neq y$ imply $(x, y) = 0$. If each element of an orthogonal set S has norm 1, then S is an orthonormal set.

An orthonormal set S in X is complete if $S \subset T$ and T is another orthonormal set in X imply $S = T$.

One of the most used results in all mathematics and especially in functional analysis is a result taken from logic and it's called Zorn's lemma. It was stated without proof by the man whose name it carries [32]. In fact it is not possible to prove Zorn's lemma in the usual sense of the world. However,it can be shown that Zorn's lemma is logically equivalent to the axiom of choice, which states the following: given any class of non-empty sets, a set can be formed which contains precisely one element taken from each set in the given class. The axiom of choice is intuitively obvious. We therefore treat Zorn's lemma as an axiom of logic [38]. Other, equivalent forms of Zorn's lemma include: Principle of choice, Principle of transfinite induction, Zermelo theorem (Every set can be well ordered), the Tukey-Teichmuller theorem and Hausdorff's theorem. Zorn's lemma is frequently used in place of transfinite induction, since it does not require the sets considered to be well ordered. Usually sets are naturally equipped with a partially ordered relation but not necessary a well ordered relation [31].

Definition 2.11 [32] [31] *Let P be a set and R a relation on P satisfying for $x, y, z \in P$ the following three conditions:*

1. (*reflexive*) xRx

2. (*antisymmetric*) xRy, yRx *implies* $x = y$

3. (*transitive*) xRy, yRz *implies* xRz.

Then (P, R) *is a partially ordered set. If additionally, every two elements of* P *are comparable i.e. for* $x, y \in P$ *either* xRy *or* yRx*, then the set* P *is totally ordered (or linearly ordered). If* $S \subset P$ *then* $m \in P$ *is an upper bound for* S *if* sRm *for all* $s \in S$ *and a lower bound if* mRs *for all* $s \in S$ *(A smallest (largest) element in* S *is an element* $s \in S$ *which serves as a lower bound (upper bound) for* S*). A well-ordered set is a partially ordered set every non-empty subset of which possesses a smallest element. An element* $m \in P$ *is maximal provided* $a \in P$ *and* mRa *implies* $m = a$.

Lemma 2.12 (Zorn's lemma)[32] [38] [31]*Let* P *be a partially ordered set and suppose every totally ordered subset* S *has an upper bound in* P*. Then* P *has at least one maximal element.*

Theorem 2.13 [32] *Let* $X \neq \{0\}$ *be an inner-product space. Then* X *contains a complete orthonormal set.*

Proof. Proof of this theorem uses Zorn's lemma. Let $x \neq 0$ be in X. Then $s = \left\{ \frac{x}{\|x\|} \right\}$ is an orthonormal set. Let P be the collection of all orthonormal sets containing s and ordered by inclusion. Let P_0 be any linearly ordered (totally ordered) subset of P. Consider

$$S_0 = \bigcup_{U \varepsilon P_0} U$$

Let $x, y \in S_0$. Then $x \in U_1$ and $y \in U_2$. Since P_0 is linearly ordered we can assume that $U_1 \subseteq U_2$. Thus $x, y \in U_2$. Since all elements of P_0 are orthonormal sets we have that $x \perp y$ and so S_0 is an orthonormal set. Thus $S_0 \in P$. S_0 is clearly an upper bound for P_0 since for every $U \in P_0$, we have $U \subseteq S_{0.}$ By Zorn's lemma, P has a maximal element T.

Suppose that T is not a complete orthonormal set in X. Then there exists an element $z \in X$ such that $z \notin T$ and $T \cup \{z\}$ is an orthonormal set. This implies that T is not a maximal element in P and we have a contradiction. Thus T is a complete orthonormal set in X and the theorem is proved. ■

Theorem 2.14 [32, p.20] *Let H be a Hilbert space and S a complete orthonormal set in H. Then*

$$x = \sum_{u \in S} (x, u)\, u, \text{ for every } x \in H,$$

where the convergence is unconditional (the series converges to the same element if we rearrange the elements of the series), the number of $u \in S$, for which $(x, u) \neq 0$, is at most countable and

$$\|x\|^2 = \sum_{u \in S} |(x, u)|^2 \,(\text{the Parseval equality}). \tag{2.16}$$

If H is separable (i.e. it contains a countable dense subset), then any complete orthonormal set S is countable, say $S = \{u_n\}_{n=1}^{\infty}$, and

$$x = \sum_{n=1}^{\infty} (x, u_n)\, u_n \text{ and } \|x\|^2 = \sum_{n=1}^{\infty} |(x, u_n)|^2 .$$

A Hilbert space H is the direct sum of its closed subspaces M and N, i.e. $M \oplus N = H$ if $M \cap N = \{0\}$ and each $z \in H$ can be written in the form $z = x + y$, where $x \in M$ and $y \in N$. As $M \cap N = \{0\}$, this representation of z is unique.

Theorem 2.15 [17, Theorem 2.2.4] *For every closed subspace L of a Hilbert space H,*

$$L \oplus L^{\perp} = H.$$

In this thesis we study separable Hilbert spaces.

Definition 2.16 [27, p.31] *Let X and Y be normed linear spaces. A map $T : X \to Y$ is a linear transformation, linear operator or operator (in this thesis all operators are linear) if*

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty, \text{ for all } x, y \in X \text{ and } \alpha, \beta \in \mathbb{C}.$$

It is bounded if there exists $M \geq 0$ such that

$$\|Tx\| \leq M \|x\| \text{ for all } x \in X.$$

The norm $\|T\|$ of a bounded operator T can be defined as

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|, \text{ or equivalently } \|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{x \in X} \frac{\|Tx\|}{\|x\|}.$$

If T is bounded, one-to-one, onto and its inverse T^{-1} is bounded, then T is an isomorphism and we say that the spaces X and Y are isomorphic.

Theorem 2.17 [27, p.32] *The collection $B(X,Y)$ of all bounded operators from a normed linear space X to a normed linear space Y is a normed linear space in the operator norm, where the vector operations are defined pointwise. If, in addition, Y is a Banach space, then $B(X,Y)$ is a Banach space.*

When $X = Y$ we denote $B(X,Y)$ as $B(X)$.

Theorem 2.18 [38, pp.219-220] *Let X, Y be normed spaces and let $T : X \to Y$ be an operator. The following are equivalent:*

1. *T is bounded;*

2. *T is continuous at 0;*

3. *T is continuous on all of X.*

Example 2.19 [27, Example 2.8] Let H be a Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ a bounded sequence of complex numbers. Set $Ae_n = \alpha_n e_n$. Extend A by linearity and continuity to all of H. Then, given $x \in H$, we have $x = \sum_{n=1}^{\infty} (x, e_n) e_n$ and

$$\begin{aligned} \|Ax\|^2 &= \sum_{n=1}^{\infty} |(Ax, e_n)|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2 |\alpha_n|^2 \\ &\leq \left(\sup_n |\alpha_n|^2\right) \sum_{n=1}^{\infty} |(x, e_n)|^2 = \left(\sup_n |\alpha_n|\right)^2 \|x\|^2 . \end{aligned}$$

We see that A is bounded and $\|A\| \leq \sup_n |\alpha_n|$. Consideration of Ae_n shows that

$$\|A\| = \sup_n |\alpha_n| \, .$$

Such an operator A is called a *diagonal operator*, with diagonal sequence $\{\alpha_n\}_{n=1}^{\infty}$.

Definition 2.20 [30, p.86] *Let X be a Banach space and let X^* denote the linear space of all bounded linear operators from X into $\mathbb{C}$. Every $f \in X^*$ is called a linear functional and*

$$\|f\| = \sup \{|f(x)| : \|x\| \leq 1\}$$

is its norm. The space X^* is the dual (or conjugate) space of X.

Theorem 2.21 [27, p.36] (*adjoint of an operator*). *Given Hilbert spaces H and K and $T \in B(H, K)$, there is a unique $T^* \in B(K, H)$ such that*

$$(Tx, y)_K = (x, T^* y)_H \text{ for all } x \in H \text{ and } y \in K.$$

The operator T^* is called the adjoint of T and (see [42, page 78])

$$\|T^*\| = \|T\| \, .$$

Definition 2.22 [30, p.93] *An operator $T \in B(H)$ is self-adjoint if $T = T^*$.*

Theorem 2.23 [30, p.93] *A bounded operator T is self-adjoint if and only if (Tx, x) is real for all $x \in H$.*

With every operator $T : X \to Y$ we associate two important subspaces: The *null space* or the *kernel* denoted by $\ker(T)$ and the *range* or the *image* of T denoted by $R(T)$. The null space consists of all $x \in X$ such that $Tx = 0$ and the the range consists of all $y \in Y$ such that $Tx = y$ for some $x \in X$. The subspace $R(T)$ is not necessarily closed in Y, while $\ker T$ is always a closed subspace of X. [26, page 52].

Theorem 2.24 [30, Proposition 4.27] For all $T \in B(H)$:

(a) $\ker(T^*) = R(T)^{\perp}$;

(b) $\ker(T)^{\perp} = \overline{R(T^*)}$.

Let A be a bounded linear operator on a Hilbert space H. The norm of A (see Definition 2.16) is

$$\|A\| = \sup\{\|Ax\| : \|x\| = 1\} = \sup\left\{(Ax, Ax)^{\frac{1}{2}} : \|x\| = 1\right\}.$$

From Cauchy-Schwarz-Bunyakovsky inequality we obtain for all $x \in H$

$$\sup_{\|y\|=1} |(Ax, y)| \leq \sup_{\|y\|=1} (\|Ax\| \, \|y\|) = \sup_{\|y\|=1} \|Ax\| = \|Ax\|$$

On the other hand, let $\|x\| = 1$ and $Ax \neq 0$. Set $y_0 = \frac{Ax}{\|Ax\|}$. Then

$$\|y_0\| = 1 \text{ and } \|Ax\| = \frac{(Ax, Ax)}{\|Ax\|} = (Ax, y_0) \leq \sup_{\|y\|=1} |(Ax, y)|.$$

Hence

$$\sup_{\|y\|=1} |(Ax, y)| = \|Ax\|. \tag{2.17}$$

Thus

$$\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=\|y\|=1} |(Ax, y)| \, .$$

Definition 2.25 [30, p.93] *An operator $T \in B(H)$ is positive if $(Tx, x) \geq 0$ for all $x \in H$.*

It is clear that 0 and $\mathbf{1}$ are positive, as are T^*T and TT^* for any operator $T \in B(H)$, since for all $x \in H$, we have

$$(T^*Tx, x) = (Tx, Tx) \geq 0 \text{ and } (TT^*x, x) = (T^*x, T^*x) \geq 0.$$

For operators A and B, $A \geq B$ is defined to mean that $A - B \geq 0$; equivalently $A \geq B \iff (Ax, x) \geq (Bx, x)$ for all x.

Theorem 2.26 [30, Theorem 4.32] *Given any positive operator T, there is a unique positive operator A such that $A^2 = T$. The operator A is denoted by $T^{1/2}$. Moreover, $T^{1/2}$ commutes with any operator that commutes with T.*

Definition 2.27 [30, p.95] *If H and K are Hilbert spaces and an operator $U \in B(H, K)$, then U is unitary if $U^*U = \mathbf{1}_H$ and $UU^* = \mathbf{1}_K$.*

Definition 2.28 [38, p.237] *A projection P on a Banach space B is an idempotent in the algebra of all linear bounded operators on B, that is, P is a linear bounded transformation of B into itself such that $P^2 = P$.*

Projections can be described geometrically as follows [38, p.237] (here the symbol $\oplus$ represents direct sum of subspaces):

1. If P is a projection on a Banach space B, then the range $R(P)$ is a closed subspace of B and $B = R(P) \oplus \ker(P)$;

2. a pair of closed linear subspaces M and N of a Banach space B, such that $B = M \oplus N$ determines a projection P whose range and null space are M and N, respectively. (If $z = x + y$ is the unique representation of a vector in B as a sum of vectors in M and N, then P is defined by $Pz = x$).

In the theory of Hilbert spaces we consider projections, sometimes called orthogonal projections, whose range and null space are perpendicular, i.e., $\ker P = (R(P))^{\perp}$.

Definition 2.29 [30, p.94] *An operator $P \in B(H)$ on a Hilbert space H is an orthogonal projection, or ortho-projection, if $P = P^*$ and $P^2 = P$. We will call such operator P just projection.*

By the projection theorem (see [27, p.13]), every non-zero orthogonal projection is of norm 1.

We say that two projections P and Q are orthogonal if $PQ = 0$. It can be proved [38, p.275] that

$$PQ = 0 \iff QP = 0 \iff R(P) \perp R(Q).$$

The following definition holds for Banach spaces but we shall only consider Hilbert spaces.

Definition 2.30 [17, p.59] (i) *A set K in a Banach space X is called a precompact set if, for every sequence $\{x_n\}$ in K, there exists an element $x \in X$ (a limit point) and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x$. It is compact, if all limit points also belong to K.*

(ii) *A linear operator $A : X \to Y$, where X and Y are Banach spaces, is called a compact operator if and only if, for every bounded sequence $\{x_n\}$ in X, the sequence $\{Ax_n\}$ is a precompact set.*

Clearly, a compact operator must be bounded, since the image of the unit ball of X must be a bounded set in Y (otherwise, we can easily find a sequence $\{x_n\}$ inside the unit ball of X such that $\|Ax_n\| \to \infty$ and, therefore the set $\{Ax_n\}$ has no converging subsequence) [17, p.59].

Theorem 2.31 [43, p.10] *If T is a compact operator on a Hilbert space H, then for any bounded linear operator S on H, the operators TS and ST are both compact. If S is also compact, then $T + S$ is compact.*

Note that if T is compact, then $\alpha T = (\alpha \mathbf{1})\, T$ is also compact for all complex numbers α.

Theorem 2.32 [43, p.11]. *A bounded linear operator T on H is compact if and only if T^* is compact, if and only if T^*T is compact, if and only if $|T| = (T^*T)^{1/2}$ is compact.*

Theorem 2.33 [43, p.11]. *If* $\{T_n\}_{n=1}^{\infty}$ *is a sequence of compact operators on* H *and*

$$\|T_n - T\| \rightarrow_{n\rightarrow\infty} 0$$

for some operator T *on* H, *then* T *is also compact.*

Definition 2.34 [30, p.168]. *The spectrum of an operator* $T \in B(H)$, *denoted by* $\sigma(T)$, *is the set of all scalars* λ *such that* $T - \lambda\mathbf{1}$ *is not invertible in* $B(H)$.

Theorem 2.35 (Spectral Theorem for Compact Operators) [30, Theorem 9.16 and 9.18]. *Let* T *be a compact operator in* $B(H)$.

(i) *The set* $\sigma(T) = \{\lambda_n\}$ *is finite or countable. All* $\lambda_n \neq 0$ *are eigenvalues and the corresponding eigenspaces* M_n *are finite-dimensional. If* $\{\lambda_n\}$ *is countable infinite then* $\lambda_n \rightarrow 0$, *as* $n \rightarrow \infty$.

(ii) *If* T *is self-adjoint, then all* λ_n *are real, all eigenspaces* M_n *are mutually orthogonal and their closed linear span is all of* H. *Moreover,* $T = \sum_n \lambda_n P_n$, *where* P_n *are projections on* M_n.

We will need the following version of the spectral theorem also called the Schmidt representation (see [32, pp.64, 75]).

Corollary 2.36 [27, Corollary 4.25]. *Let* T *be a compact self-adjoint operator on a separable Hilbert space* H. *Then there is an orthonormal basis* $\{e_n\}$ *of* H *consisting of eigenvectors for* T *such that*

$$Tx = \sum_n \lambda_n (x, e_n) e_n, \text{ for each } x \in H,$$

where λ_n is the eigenvalue of T corresponding to the eigenvector e_n.

In [21] the authors analyze completely continuous operators that map weakly convergent sequences to norm convergent sequences. In our case of operators on separable Hilbert spaces, completely continuous operators coincide with compact operators, since, for reflexive spaces, the two definitions are equivalent (we know that all Hilbert spaces are reflexive, i.e., if H is a Hilbert space then it is isomorphic to its second dual H^{**}) [11] [43] [38].

Definition 2.37 [38, p.208] *An algebra (real or complex) is a linear space A equipped with a multiplication operation that assigns to each $x, y \in A$ an element $xy \in A$ such that, for all $x, y, z \in A$ and scalars α, the following axioms must be satisfied:*

(1) (Associative law) $x\,(yz) = (xy)\,z$;

(2) (Distributive laws) $x\,(y+z) = xy + xz$ and $(x+y)\,z = xz + yz$;

(3) (Law connecting multiplication and scalar multiplication) $\alpha\,(xy) = (\alpha x)\,y = x\,(\alpha y)$.

An algebra is commutative if $xy = yx$ for all elements of the space.

Definition 2.38 [38, p.302] *A Banach algebra is a real or complex Banach space B, which is also an algebra in which the multiplicative structure is related to the norm by the following requirement*

$$\|xy\| \leq \|x\|\,\|y\| \text{ for all } x, y \in B.$$

For example, the linear space $B(H)$ of all bounded operators on a Hilbert space H endowed with the operator norm is a Banach algebra, where multiplication of operators is their composition.

Definition 2.39 [38, p.324] *A Banach algebra A is called a Banach *-algebra if it has an involution *, that is, if there exists a mapping $x \to x^*$ of A into itself with the following properties*:

(1) $(x+y)^* = x^* + y^*$ *for* $x, y \in A$;

(2) $(\alpha x)^* = \overline{\alpha} x^*$ *for* $x \in A$ *and* $\alpha \in \mathbb{C}$;

(3) $(xy)^* = y^* x^*$ *for* $x, y \in A$;

(4) $x^{**} = x$ *for* $x \in A$;

(5) $\|x^*\| = \|x\|$ *for* $x \in A$.

If H is a Hilbert space, then the algebra $B(H)$ of all bounded linear operators on H is a Banach *-algebra with the adjoint operation $T \to T^*$ as the involution. A subalgebra of the algebra $B(H)$ is said to be self-adjoint, or a *-subalgebra, if it contains the adjoint of each of its operators. All closed self-adjoint subalgebras of $B(H)$ are Banach *-algebras. Moreover, the closed self-adjoint subalgebras of $B(H)$ that satisfy the following condition:

$$\|xx^*\| = \|x\|^2,$$

for all elements x, constitute a special class of Banach *-algebras called C^*-algebras.

Definition 2.40 [38, p.209] *Let A be a complex algebra. Its subset I is a left (respectively, right) ideal of A, if*

(1) $\alpha a + \beta b \in I$ *for all* $a, b \in I$ *and* $\alpha, \beta \in \mathbb{C}$;

(2) $ab \in I$ *(respectively,* $ba \in I$*) for each* $a \in A$ *and* $b \in I$.

It is a two-sided ideal of A, if it is a left and a right ideal of A.

Let $C(H)$ denote the set of all compact operators on H. From the above Theorems 2.31, 2.33 and 2.32 we know that $C(H)$ is a closed, self-adjoint subalgebra and a two-sided ideal of the algebra $B(H)$. Thus $C(H)$ is a C^*-subalgebra of $B(H)$. It is known that $C(H)$ is the only proper closed two-sided ideal of $B(H)$. [43, p.12].

Chapter 3 Minimax and seminorms

3.1 Introduction

Let X and Λ be sets and let f be a real function on $X \times \Lambda = \{(x, \lambda) : x \in X, \lambda \in \Lambda\}$. Recall that the minimax equality is the following equality:

$$\inf_{\lambda \in \Lambda} \left(\sup_{x \in X} f(x, \lambda) \right) = \sup_{x \in X} \left(\inf_{\lambda \in \Lambda} f(x, \lambda) \right).$$

As we shall see in Proposition 3.1, the inequality

$$\inf_{\lambda \in \Lambda} \left(\sup_{x \in X} f(x, \lambda) \right) \geq \sup_{x \in X} \left(\inf_{\lambda \in \Lambda} f(x, \lambda) \right)$$

holds for all functions f. Therefore to prove the minimax equality, one only need to prove the inverse inequality

$$\inf_{\lambda \in \Lambda} \left(\sup_{x \in X} f(x, \lambda) \right) \leq \sup_{x \in X} \left(\inf_{\lambda \in \Lambda} f(x, \lambda) \right).$$

We give below the proof of the following known proposition, as we could not find a reference.

Proposition 3.1 *Let X and Λ be sets and let f be a function from*

$$X \times \Lambda = \{(x, \lambda) : x \in X, \lambda \in \Lambda\}$$

into $\mathbb{R}$. Then

$$\inf_{\lambda \in \Lambda} \left(\sup_{x \in X} f(x, \lambda) \right) \geq \sup_{x \in X} \left(\inf_{\lambda \in \Lambda} f(x, \lambda) \right).$$

Proof. For every $\mu \in \Lambda$, we have

$$\sup_{x\in X} f(x,\mu) \geq \sup_{x\in X} \inf_{\lambda\in\Lambda} f(x,\lambda).$$

Thus

$$\inf_{\mu\in\Lambda} \sup_{x\in X} f(x,\mu) \geq \sup_{x\in X} \inf_{\lambda\in\Lambda} f(x,\lambda)$$

This concludes the proof. ■

To prove some theorems for example Theorem 4.11 we need the following lemma. The lemma is known, but we could not find any reference.

Lemma 3.2 *Let $f : X \times \Lambda \to \mathbb{R}$ be a function on the product of non-empty sets X and Λ. Suppose that there exists $\mu \in \Lambda$ such that*

$$\sup_{\lambda\in\Lambda} f(x,\lambda) = f(x,\mu) \text{ for each } x \in X. \tag{3.1}$$

Alternatively, suppose that there exists $x_0 \in X$ such that

$$\inf_{x\in X} f(x,\lambda) = f(x_0,\lambda) \text{ for each } \lambda \in \Lambda. \tag{3.2}$$

Then

$$\inf_{x\in X}\left(\sup_{\lambda\in\Lambda} f(x,\lambda)\right) = \sup_{\lambda\in\Lambda}\left(\inf_{x\in X} f(x,\lambda)\right) = \inf_{x\in X} f(x,\mu). \tag{3.3}$$

Proof. Applying Proposition 3.1, we always have

$$\inf_{x\in X}\left(\sup_{\lambda\in\Lambda} f(x,\lambda)\right) \geq \sup_{\lambda\in\Lambda}\left(\inf_{x\in X} f(x,\lambda)\right). \tag{3.4}$$

Suppose now that (3.1) holds. Then

$$\inf_{x\in X}\left(\sup_{\lambda\in\Lambda} f(x,\lambda)\right) = \inf_{x\in X} f(x,\mu).$$

Hence

$$\inf_{x\in X}\left(\sup_{\lambda\in\Lambda} f(x,\lambda)\right) = \inf_{x\in X} f(x,\mu) \le \sup_{\lambda\in\Lambda}\left(\inf_{x\in X} f(x,\lambda)\right)$$

Combining this with (3.4), we obtain (3.3). The proof that (3.2) implies (3.3) is similar. ■

In sections 3.2 and 3.3 we consider the validity of the minimax equality for a sequence of seminorms on Banach spaces.

Definition 3.3 [35, p.12] *Let X be a complex vector space. A non-negative, finite, real-valued function g on X is called a seminorm if, for all x, $y \in X$ and scalars λ,*

$$g(\lambda x) = |\lambda|\, g(x) \tag{3.5}$$

$$g(x+y) \le g(x) + g(y). \tag{3.6}$$

In fact, any function on X that satisfies (3.5) and (3.6) is non-negative. Indeed, for each $x \in X$,

$g(0) = g(0x) = |0|\, g(x) = 0,$ so that

$$0 = g(0) = g(x + (-x)) \le g(x) + g(-x) = g(x) + |-1|\, g(x) = 2g(x).$$

Clearly, the set $g^{-1}(0)$ is a linear subspace of X. If $g(x) = 0$ implies $x = 0$, then (see(2.1)-(2.3)) g is a norm, so that (X, g) is a normed space.

Definition 3.4 [1, p.36] *A seminorm g on a normed linear space X is bounded if there exists $M > 0$ such that*

$$g(x) \leq M \|x\| \text{ for all } x \in X.$$

For example, let X be a Banach space with norm $\|\cdot\|$. For each bounded operator T on X, we have that $g_T(x) = \|Tx\|$, for $x \in X$, is a bounded seminorm on X, as

$$g_T(\lambda x) = \|T\lambda x\| = |\lambda| \|Tx\| = |\lambda| g_T(x);$$

$$g_T(x+y) = \|T(x+y)\| \leq \|Tx\| + \|Ty\| = g_T(x) + g_T(y)$$

$$\text{and } g_T(x) = \|Tx\| \leq \|T\| \|x\| \text{ for all } x \in X.$$

A bounded seminorm g on X defines an *equivalent norm* on X and we will write $g \sim \|\cdot\|$, if there exists $0 < k$ such that

$$k \|x\| \leq g(x) \text{ for all } x \in X.$$

In other words, $g \sim \|\cdot\|$ if

$$k = \inf_{x \in X} \frac{g(x)}{\|x\|} = \inf_{\|x\|=1} g(x) > 0. \tag{3.7}$$

For example, if T is a bounded operator on X that has bounded inverse T^{-1} then $g_T \sim \|\cdot\|$, as

$$\|x\| = \left\|T^{-1}Tx\right\| \leq \left\|T^{-1}\right\| \|Tx\| = \left\|T^{-1}\right\| g_T(x) \text{ for all } x \in X,$$

so that $\|T^{-1}\|^{-1} \|x\| \leq g_T(x)$ and $k = \|T^{-1}\|^{-1}$.

It follows from (3.7) that $g \nsim \|\cdot\|$ if and only if there is a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that

$$\|x_n\| = 1 \text{ for all } n \text{ and } g(x_n) \to 0, \text{ as } n \to \infty.$$

The following theorem about seminorms is known. Note that if the seminorms are linear, then the proof of the theorem follows from the uniform boundedness principle and the Banach-Steinhaus theorem (see for example [27, Theorems 3.11 and 3.12]).

Theorem 3.5 [1, p.37] *Let $\{g_k\}_{k=1}^{\infty}$ be a sequence of bounded seminorms on a Hilbert space H. If the sequence is bounded at each point $x \in H$, then the function defined by*

$$g(x) = \sup_n g_n(x) \text{ for } x \in H,$$

is also a bounded seminorm.

3.2 Minimax equality for seminorms

Let $\{g_k\}_{k=1}^{\infty}$ be a sequence of bounded seminorms on a Hilbert space H bounded at each point $x \in H$. Consider the minimax formula:

$$\inf_{\|x\|=1} \sup_n g_n(x) = \sup_n \inf_{\|x\|=1} g_n(x).$$

By Theorem 3.5, $g(x) = \sup_n g_n(x)$ is a bounded seminorm on H. Hence the minimax formula takes the form

$$\inf_{\|x\|=1} g(x) = \inf_{\|x\|=1} \sup_n g_n(x) = \sup_n \inf_{\|x\|=1} g_n(x). \tag{3.8}$$

A particular case could be that there exists $m \in \mathbb{N}$ such that $g_m(x) = \sup_n g_n(x)$.

Proposition 3.6 *Let $\|x\| = (x,x)^{1/2}$ be the norm on a Hilbert space H. Let $\{g_k\}_{k=1}^{\infty}$ be a sequence of bounded seminorms on H bounded at each point $x \in H$ and let $g(x) = \sup_n g_n(x)$. Then*

(i) *If $g \nsim \|\cdot\|$ then (3.8) holds.*

(ii) *If $g \sim \|\cdot\|$ but all $g_n \nsim \|\cdot\|$ then (3.8) doesn't hold.*

(iii) *Let $g \sim \|\cdot\|$. Then (3.8) holds if and only if for each $\varepsilon > 0$ there exists n_ε such that $g_{n_\varepsilon} \sim \|\cdot\|$ and $\inf_{\|x\|=1} g_{n_\varepsilon}(x) \geq \inf_{\|x\|=1} g(x) - \varepsilon$.*

Proof. (i) We know that $g \nsim \|\cdot\|$ if and only if there is a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\|x_n\| = 1$ for all n and $g(x_n) \to 0$, as $n \to \infty$. Then

$$0 \leq \inf_{\|x\|=1} g(x) \leq \inf_{x \in \{x_n\}} g(x) = 0.$$

Thus $\inf_{\|x\|=1} g(x) = 0$. We know from Proposition 3.1 that

$$\inf_{\|x\|=1} g(x) = \inf_{\|x\|=1} \sup_n g_n(x) \geq \sup_n \inf_{\|x\|=1} g_n(x).$$

As all seminorms are non-negative, we have $\sup_n \inf_{\|x\|=1} g_n(x) \geq 0$. Thus

$$\inf_{\|x\|=1} g(x) = \inf_{\|x\|=1} \sup_n g_n(x) = \sup_n \inf_{\|x\|=1} g_n(x) = 0.$$

(ii) Suppose that $g \sim \|\cdot\|$ but all $g_n \nsim \|\cdot\|$. Thus for each n there exists sequence $\{x_j^n\}_{j=1}^{\infty}$ such that $\left\|x_j^n\right\| = 1$, for all n, j, and, for each n, $g(x_j^n) \to 0$, as $j \to \infty$. Hence for all n we have

$$\inf_{\|x\|=1} g_n(x) \leq \inf_{x \in \{x_j^n\}_{j=1}^{\infty}} g_n(x) = 0.$$

Therefore $\sup_n \inf_{\|x\|=1} g_n(x) = 0$. On the other hand, $g \sim \|\cdot\|$ and by (3.7) $\inf\limits_{\|x\|=1} g(x) > 0$. Thus

$$\inf_{\|x\|=1} g(x) = \inf_{\|x\|=1} \sup_n g_n(x) > 0.$$

Therefore LHS > 0 and RHS $= 0$. Hence the minimax (3.8) doesn't hold.

(iii) Let $k_n = \inf_{\|x\|=1} g_n(x)$ and $k = \inf_{\|x\|=1} g(x)$. Then $k_n > 0$ if and only if $g_n \sim \|\cdot\|$. The minimax (3.8) holds if and only if $k = \sup_n k_n$, that is, for each $\varepsilon > 0$ there exists n_ε such that $g_{n_\varepsilon} \sim \|\cdot\|$ and $\inf_{\|x\|=1} g_{n_\varepsilon}(x) \geq \inf_{\|x\|=1} g(x) - \varepsilon$. ■

Case (ii) is a subcase of (iii) but we think that it is worth mentioning it as individual case for clarity. Example 3.7 below illustrates Proposition 3.6 case (ii) when $g = \|\cdot\|_2$.

Example 3.7 Consider the Hilbert space

$$l_2 = \left\{ x = \{x_n\}_1^\infty : \textit{all } x_n \in \mathbb{C}, \|x\|_2 = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2} < \infty \right\}$$

and the following seminorms g_n on l_2 given by $g_n(x) = \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$ where $x \in l_2$. The proof of condition (3.5) is obvious and the proof of the triangle inequality called in this case the Minkowski's inequality

$$g_n(x+y) = \left(\sum_{i=1}^{n} |x_i + y_i|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n} |y_i|^2 \right)^{\frac{1}{2}} = g(x) + g(y)$$

was obtained in Proposition 2.3. Thus g_n are seminorms on l_2.

We have that

$$g(x) = \sup_n g_n(x) = \sup_n \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}} = \|x\|_2 .$$

Thus

$$\text{LHS} = \inf_{\|x\|_2=1} g(x) = \inf_{\|x\|_2=1} \|x\|_2 = 1.$$

On the other hand

$$\text{RHS} = \sup_n \inf_{\|x\|_2=1} g_n(x) = \sup_n 0 = 0.$$

Thus the minimax formula does not hold.

3.3 The minimax in reverse

Let H be a Hilbert space. Let $\{g_k\}_{k=1}^{\infty}$, be a sequence of seminorms in H such that $g_m(x) = \inf_n g_n(x)$ for all $x \in H$ and some $m \in \mathbb{N}$ (for example $\{g_k\}_{k=1}^{\infty}$, could be monotone increasing, i.e. $g_k(x) \leq g_{k+1}(x)$ for all $x \in H$ and we can set $m = 1$). Consider the minimax formula, which is the reverse to minimax (3.8)

$$\inf_n \sup_{\|x\|=1} g_n(x) = \sup_{\|x\|=1} \inf_n g_n(x) \tag{3.9}$$

Theorem 3.8 *The minimax formula* (3.9) *holds and*

$$\inf_n \sup_{\|x\|=1} g_n(x) = \sup_{\|x\|=1} \inf_n g_n(x) = \sup_{\|x\|=1} g_m(x). \tag{3.10}$$

Proof. The inequality in the formula

$$\inf_n \sup_{\|x\|=1} g_n(x) \leq \sup_{\|x\|=1} g_m(x) = \sup_{\|x\|=1} \inf_n g_n(x)$$

is obvious, as the infimum over n of $\sup_{\|x\|=1} g_n(x)$ is not greater than $\sup_{\|x\|=1} g_m(x)$. The reversed inequality holds for all minimax formula (see Proposition 3.1). Hence Eqn. (3.10) holds. ■

Example 3.9 Let us consider the following three examples.

(i) The following seminorms g_n on l_2

$$g_n(x) = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} \quad where\ x \in l_2$$

are monotone increasing and

$$RHS = \sup_{\|x\|=1} \inf_n g_n(x) = \sup_{\|x\|=1} \inf_n \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} = \sup_{\|x\|=1} |x_1| = 1$$

$$LHS = \inf_n \sup_{\|x\|=1} g_n(x) = \inf_n \sup_{\|x\|=1} \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} = \inf_n 1 = 1.$$

Thus the reversed minimax (3.10) holds as equality.

(ii) Consider the following seminorms g_n on l_2

$$g_n(x) = \|P_n x\|_n = \left(\sum_{i=1}^{n} |x_i|^n\right)^{1/n} \quad \text{for } x \in l_2.$$

We have $g_1(x) = |x_1| \leq g_n(x)$ for all $n \in \mathbb{N}$ and all $x \in l_2$. Thus

$$g_1(x) = \inf_n g_n(x) \text{ for all } x \in l_2,$$

$$\left(\sum_{i=1}^{n} |x_i|^n\right)^{1/n} \leq \left(\sum_{i=1}^{\infty} |x_i|^n\right)^{1/n} \leq \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{1/2} = 1,$$
$$\text{for } \|x\|_2 = \|x\| = 1, \text{ and all } n \in \mathbb{N},\ n \geq 2.$$

Hence

$$\sup_{\|x\|=1} g_n(x) \leq 1 \text{ for all } n \in \mathbb{N} \text{ and } \inf_n \sup_{\|x\|=1} g_n(x) \leq \inf_n 1 = 1.$$

We know that $\inf_n g_n(x) = g_1(x) = |x_1|$, for all $x \in l_2$. Thus

$$\sup_{\|x\|=1} \inf_n g_n(x) = \sup_{\|x\|=1} (|x_1|) = 1.$$

Hence, it follows from Proposition 3.1 that

$$1 \geq \inf_n \sup_{\|x\|=1} g_n(x) \geq \sup_{\|x\|=1} \inf_n g_n(x) = \sup_{\|x\|=1} g_1 = 1.$$

Thus the reversed minimax (3.10) holds as equality.

(iii) Let us consider the following seminorms S_n on l_2

$$S_n(x) = \|P_n x\|_{1+\frac{1}{n}} = \left(\sum_{i=1}^{n} |x_i|^{1+\frac{1}{n}}\right)^{1/\left(1+\frac{1}{n}\right)} \quad \text{for } x \in l_2.$$

From Minkowski's inequality we know that S_n's are seminorms.

As the function $f(t) = \left(\sum_{i=1}^{n} s_i^t\right)^{1/t}$, $(0 < t \leq \infty, s_j > 0)$ is nonincreasing [21, p.92], we obtain that the sequence $\{S_n(x)\}_{n=1}^{\infty}$ is monotone increasing, i.e.

$$1 + \frac{1}{n} > 1 + \frac{1}{n+1} \text{ implies } \|x\|_{1+\frac{1}{n+1}} \geq \|x\|_{1+\frac{1}{n}}.$$

We have $S_n(x) \geq \max_{i=1,\ldots,n}(|x_i|) \geq |x_1|$, for all $1 \leq n \in \mathbb{N}$. Thus $\inf_n S_n(x) \geq |x_1|$. On the other hand,

$$\inf_n S_n(x) \leq S_1(x) = \left(\sum_{i=1}^{1} |x_i|^{1+\frac{1}{1}}\right)^{1/\left(1+\frac{1}{1}\right)} = |x_1|.$$

Hence $\inf_n S_n(x) = S_1(x) = |x_1|$. Therefore we have

$$RHS = \sup_{\|x\|=1} \inf_n S_n(x) = \sup_{\|x\|=1} |x_1| = 1.$$

Let us now calculate LHS. We have that, for all n,

$$\sup_{\|x\|=1} S_n(x) \geq S_n((1,0,...,0,...)) = 1.$$

Thus $LHS = \inf_n \sup_{\|x\|=1} S_n(x) \geq \inf_n 1 = 1$. On the other hand

$$LHS = \inf_n \sup_{\|x\|=1} S_n(x) \leq \sup_{\|x\|=1} S_1(x) = \sup_{\|x\|=1} \left(\sum_{i=1}^{1} |x_i|^2 \right)^{1/2} = \sup_{\|x\|=1} |x_1| = 1.$$

Thus the reversed minimax (3.10) holds as equality.

3.4 A minimax theorem for operators

Let H be a Hilbert space and let A be a bounded linear operator on H. The uniform norm of A (see Definition 2.16) is

$$\|A\| = \sup \{\|Ax\| : \|x\| = 1\} = \sup \left\{ (Ax, Ax)^{\frac{1}{2}} : \|x\| = 1 \right\}$$

where (x, y) is the scalar product of elements $x, y \in H$.

Definition 3.10 [30, p.63] *A linear operator $T \in B(H)$ is bounded from below if there is a $k > 0$ such that, $\|Tx\| \geq k$ for all $x \in H, \|x\| = 1$.*

Clearly, being bounded below implies that T is injective as $\ker(T) = \{0\}$. However, the converse is not true in infinite-dimensional spaces.

Theorem 3.11 (The bounded inverse theorem) [30, Theorem 3.6] *For an injective linear operator $T \in B(H)$, the following are equivalent:*

(i) *T^{-1} is bounded;*

(ii) *T is bounded below;*

(iii) $R(T)$ *is closed.*

In this thesis report we will consider various cases when minimax formula holds within the theory of operators on Hilbert spaces. We will start with the following simple case of minimax formula.

Theorem 3.12 *Let H be a Hilbert space and A a bounded operator on H.*

(i) *If A is invertible and*

(a) $\dim H = 1$, *then the minimax formula:*

$$\inf_{\|x\|=1}\left\{\sup_{\|y\|=1}|(Ax,y)|\right\} = \sup_{\|y\|=1}\left\{\inf_{\|x\|=1}|(Ax,y)|\right\} = |a| \tag{3.11}$$

holds, where a is a scalar such that $Ax = ax$ for all $x \in H$.

(b) $\dim H > 1$, *then the minimax condition does not hold:*

$$\inf_{\|x\|=1}\left\{\sup_{\|y\|=1}|(Ax,y)|\right\} = \inf_{\|x\|=1}\|Ax\| = k > 0, \tag{3.12}$$

while

$$\sup_{\|y\|=1}\left\{\inf_{\|x\|=1}|(Ax,y)|\right\} = 0. \tag{3.13}$$

(ii) *If A is not invertible, then*

$$\inf_{\|x\|=1}\left\{\sup_{\|y\|=1}|(Ax,y)|\right\} = \sup_{\|y\|=1}\left\{\inf_{\|x\|=1}|(Ax,y)|\right\} = 0. \tag{3.14}$$

Proof. (i) (a) $\dim H = 1$ implies that one vector spans the space and $Ax = ax$ for all $x \in H$ and some scalar a. Let $e \in H$, $\|e\| = 1$. Then e forms a complete orthonormal set. Thus

$$\inf_{\|x\|=1}\left\{\sup_{\|y\|=1}|(Ax,y)|\right\} = \inf_{\|x\|=1}\left\{\sup_{\|y\|=1}|(ax,y)|\right\} = \inf_{|l_1|=1}\left\{\sup_{|l_2|=1}|(al_1e,l_2e)|\right\} = |a|\,,$$

and

$$\sup_{\|y\|=1}\left\{\inf_{\|x\|=1}|(Ax,y)|\right\} = \sup_{\|y\|=1}\left\{\inf_{\|x\|=1}|(ax,y)|\right\} = \sup_{|l_2|=1}\left\{\inf_{|l_1|=1}|(al_1e,l_2e)|\right\} = |a|\,.$$

(b) Suppose that A is invertible. This implies that A is injective and that A^{-1} is bounded. By theorem 3.11, A is bounded below. Let $\inf_{\|x\|=1}\|Ax\| = k > 0$. Then, for all x such that $\|x\| = 1$

$$\sup_{\|y\|=1}|(Ax,y)| \overset{(2.17)}{=} \|Ax\| \geq k > 0. \tag{3.15}$$

Therefore

$$\inf_{\|x\|=1}\left\{\sup_{\|y\|=1}|(Ax,y)|\right\} = \inf_{\|x\|=1}\|Ax\| = k > 0.$$

Let us now evaluate the right hand side.

$$\sup_{\|y\|=1}\left\{\inf_{\|x\|=1}|(Ax,y)|\right\} = \sup_{\|y\|=1}\left\{\inf_{\|x\|=1}|(x,A^*y)|\right\} = 0$$

as $\dim H > 1$ implies that for each vector A^*y we can find an orthogonal vector x such that $\|x\| = 1$.

(ii) Suppose now that A is not invertible. If A is injective, Theorem 3.11 implies that A is not bounded below i.e. there exists a sequence $\{x_n\}$ such that $\|x_n\| = 1$

for all n and $\lim_{n\to\infty} \|Ax_n\| = 0$. If A is not injective, there is $e \in H$, $\|e\| = 1$, such that $Ae = 0$. Set $x_n = e$ for all n. Then

$$\inf_{\|x\|=1} \left\{ \sup_{\|y\|=1} |(Ax, y)| \right\} \overset{(2.17)}{=} \inf_{\|x\|=1} \|Ax\| \le \inf_{n} \|Ax_n\| = 0.$$

Hence, by Proposition 3.1, if A is not invertible, the minimax (3.14) holds. ■

3.5 Application

Definition 3.13 [1] *We say that a complex function $\Omega : H \times H \to \mathbb{C}$ is a bounded bilinear functional on a Hilbert space H if, for all $x, y, z \in H$, the following conditions are satisfied:*

(a) $\Omega(\alpha_1 x + \alpha_2 y, z) = \alpha_1 \Omega(x, z) + \alpha_2 \Omega(y, z)$;

(b) $\Omega(x, \beta_1 y + \beta_2 z) = \overline{\beta_1} \Omega(x, y) + \overline{\beta_2} \Omega(x, z)$;

(c) $\sup_{\|x\|\le 1, \|y\|\le 1} |\Omega(x, y)| < \infty$.

The scalar product (x, y) on H is an example of a bilinear functional.

The norm of the bilinear functional Ω, is defined by

$$\|\Omega\| = \sup_{\|x\|=1, \|y\|=1} |\Omega(x, y)| = \sup_{x, y \in H} \frac{|\Omega(x, y)|}{\|x\| \|y\|}.$$

Thus $|\Omega(x, y)| \le \|\Omega\| \|x\| \|y\|$ for all $x, y \in H$.

Theorem 3.14 [1] *Each bilinear functional Ω on a Hilbert space H has a representation of the form $\Omega(x, y) = (Ax, y)$ where $A \in B(H)$ and A is uniquely defined by Ω. Furthermore $\|A\| = \|\Omega\|$.*

The minimax theorem can be applied to a bilinear functional Ω on a Hilbert space H as follows. Consider the minimax formula:

$$\inf_{\|x\|=1}\sup_{\|y\|=1}|\Omega(x,y)| = \sup_{\|y\|=1}\inf_{\|x\|=1}|\Omega(x,y)|. \tag{3.16}$$

Corollary 3.15 *Let Ω be a bounded linear functional on a Hilbert space H and let A be the corresponding operator defined in Theorem* 3.14 *such that $\Omega(x,y) = (Ax,y)$.*

(i) *If A is invertible and*

(a) $\dim H = 1$, *then the minimax formula* (3.16) *holds:*

$$\inf_{\|x\|=1}\sup_{\|y\|=1}|\Omega(x,y)| = \sup_{\|y\|=1}\inf_{\|x\|=1}|\Omega(x,y)| = |a|,$$

where a is a scalar such that $Ax = ax$ for all $x \in H$.

(b) $\dim H > 1$, *then the minimax condition* (3.16) *does not hold:*

$$\inf_{\|x\|=1}\sup_{\|y\|=1}|\Omega(x,y)| = \inf_{\|x\|=1}\|Ax\| = k > 0,$$

while

$$\sup_{\|y\|=1}\inf_{\|x\|=1}|\Omega(x,y)| = 0.$$

(ii) *If A is not invertible then the minimax condition* (3.16) *holds:*

$$\inf_{\|x\|=1}\sup_{\|y\|=1}|\Omega(x,y)| = \sup_{\|y\|=1}\inf_{\|x\|=1}|\Omega(x,y)| = 0.$$

3.6 Conclusion

In this chapter we studied minimax condition for sequences of bounded seminorms on a Hilbert space H. We found that its validity depends on comparison of the bounded seminorms with the norm of the Hilbert space H. We illustrated the result with example of bounded seminorms on the space l_2. We also evaluated this minimax in reverse and illustrated it with examples on l_2. We found that, unlike the previous minimax theorem, the minimax in reverse holds in all cases.

Towards the end of this chapter we presented a simple minimax formula for bounded operators. We found that the minimax formula holds if the bounded operator is not invertible and it does not hold if the operator is invertible and $\dim H > 1$. We completed this chapter with application of the minimax condition for operators to bounded bilinear functionals on H.

In the next chapter we study minimax theory for a special class of compact operators - the Schatten class operators on a separable Hilbert space H.

Chapter 4 Minimax and Schatten ideals of compact operators

4.1 Introduction

In this chapter we consider various minimax conditions for norms of compact operators in Schatten ideals. While in majority of cases the restrictions on operators for which these conditions hold are straightforward, in one case considered in Section 4.3 the fulfilment of the minimax condition depends on an interesting geometric property of a family of subspaces $\{L_n = P_n H\}_{n=1}^{\infty}$ of a Hilbert space — approximate intersection of these subspaces.

Before we consider these minimax conditions, let us recall main concepts of theory of Schatten ideals that we will need in this chapter.

Let H be a separable Hilbert space and $B(H)$ be the C^*-algebra of all bounded operators on H with operator norm $\|\cdot\|$. The set $C(H)$ of all compact operators in $B(H)$ is the only closed two-sided ideal of $B(H)$ [21, Corollary 1.1]. However, $B(H)$ has many non-closed two-sided ideals. By Calkin theorem [21, Theorem 1.1], all these ideals of $B(H)$ lie in $C(H)$.

Definition 4.1 [21, pp.68-70] *A two-sided ideal J of $B(H)$ is called symmetrically normed (s. n.), if it is a Banach space in some norm $\|\cdot\|_J$ and*

$$\|AXB\|_J \leq \|A\|\|X\|_J\|B\| \text{ for all } A, B \in B(H) \text{ and } X \in J.$$

The most important class of s. n. ideals - the class of Schatten ideals - is defined in the following way [21, Theorem 7.1]. For $A \in C(H)$, consider the positive operator $|A| = (A^*A)^{1/2}$. The operator $|A|$ is compact [43, Theorem 1.3.7], so that its spectrum $\sigma(|A|)$ contains 0 [30, Remark, p. 196], which is the only limit point of $\sigma(|A|)$. Apart from 0, it consists of countably many positive eigenvalues of finite multiplicity (see Theorem 2.35). Thus $\sigma(|A|)\backslash\{0\}$ can be written as a non-increasing sequence $s(A) = \{s_i(A)\}$ of eigenvalues of $|A|$, taking account of their multiplicities. Hence $s(A)$ belongs to the space c_0 of all sequences of real numbers converging to 0.

For each $0 \leq p < \infty$, consider the following function on c_0 :

$$\phi_p(\xi) = \left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{1/p}, \text{ where } \xi = (\xi_1, ..., \xi_n, ...) \in c_0,$$

and the following subset of compact operators

$$S^p = S^p(H) = \{A \in C(H) : \phi_p(s(A)) = \left(\sum_j s_j^p(A)\right)^{1/p} < \infty\}. \tag{4.1}$$

Then all S^p are two-sided ideals of $B(H)$ [21].

For each $A \in S^p$, consider the norm

$$\|A\|_p = \phi_p(s(A)) = \left(\sum_j s_j^p(A)\right)^{1/p}. \tag{4.2}$$

For all $1 \leq p < \infty$, S^p are Banach*-algebras with respect to the norms $\|\cdot\|_p$ and the adjoint operation as the involution: if $T \in S^p$ then $T^* \in S^p$ [43, Theorem 1.3.6]. Moreover, they are s. n. ideals of $B(H)$ (see [16, Lemma 6 (c)] for the second statement and [21, Theorem 7.1] for the first statement):

$$\|ATB\|_p \leq \|A\|\|T\|_p\|B\| \text{ and } \|T^*\|_p = \|T\|_p, \tag{4.3}$$

for all $T \in S^p$, $A, B \in B(H)$. These ideals are called Schatten ideals.

All Schatten ideals are separable algebras in the $\|\cdot\|_p$ norm topology and the ideal of all finite rank operators in $B(H)$ is dense in each of them [21, p.92]. Moreover [16, Lemma 9 (a)],

$$S^q \subset S^p, \text{ if } q < p \leq \infty, \text{ and } \|A\|_p \leq \|A\|_q \text{ if } A \in S^q. \tag{4.4}$$

We also denote the ideal $C(H)$ of all compact operators by S^∞. Note that [21, p.27]

$$\|A\|_\infty = \|A\| = \sup s_j = s_1. \tag{4.5}$$

Definition 4.2 [27, p.28] [1, p.61] [30, p.164] *A sequence $\{x_n\}$ in a Hilbert space H is said to converge weakly to $x \in H$ if*

$$\lim_{n\to\infty} (x_n, y) = (x, y) \text{ for all } y \in H.$$

Let K be another separable Hilbert space. Let $\{A_n\}_{n=1}^\infty$ be a sequence of operators in $B(H, K)$. It converges to a bounded operator A in the weak operator topology (w.o.t), if

$$(A_n x, y) \to (Ax, y) \text{ for all } x \in H \text{ and } y \in K.$$

It converges to A in the strong operator topology (s.o.t.), if

$$\|Ax - A_n x\|_K \to 0 \text{ for all } x \in H.$$

If $\{x_n\}$ converges to $x \in H$ in norm, then $\{x_n\}$ weakly converges to x. If $\{A_n\}_{n=1}^{\infty}$ uniformly converges to an operator A ($\|A_n - A\| \to 0$) then $\{A_n\}_{n=1}^{\infty} \overset{\text{s.o.t}}{\to} A$; if $\{A_n\}_{n=1}^{\infty} \overset{\text{s.o.t}}{\to} A$ then $\{A_n\}_{n=1}^{\infty} \overset{\text{w.o.t}}{\to} A$.

We can extend the norm $\|\cdot\|_p$ to all operators from $B(H)$, by setting $\|A\|_p = \infty$, if $A \in B(H)$ and $A \notin S^p$. Thus

$$\|A\|_p < \infty \text{ if } A \in S^p, \text{ and } \|A\|_p = \infty \text{ if } A \notin S^p, \text{ for } p \in [1, \infty). \tag{4.6}$$

All Schatten ideals S^p, $p \in [1, \infty)$, share the following important property.

Theorem 4.3 [21, Theorem III.5.1] *Let $p \in [1, \infty)$ and let a sequence $\{A_n\}$ of operators from S^p converge to $A \in B(H)$ in the weak operator topology. If*

$$\sup_n \|A_n\|_p = M < \infty \textit{ then } A \in S^p \textit{ and } \|A\|_p \leq M.$$

Theorem 4.3 implies the following result.

Corollary 4.4 [21, Theorem III.5.2] *Let a sequence $\{T_n\}$ of operators in $B(H)$ converge to $\mathbf{1}_H$ in the strong operator topology. Let $p \in [1, \infty)$ and $A \in B(H)$. The following conditions are equivalent.*

(i) *A belongs to S^p.*

(ii) *For some $M_1 > 0$, A satisfies*

$$\sup_n \|T_n A T_n\|_p = M_1 < \infty. \tag{4.7}$$

(iii) *For some $M_2 > 0$, A satisfies $\sup_n \|T_n A\|_p = M_2 < \infty$.*

Proof. As $\|T_n x - x\| \to 0$, as $n \to \infty$, for all $x \in H$, it follows from the uniform boundedness principle (see, for example, [16, Theorem II.1.17] [30, Theorem 3.10]) that there is $L > 0$ such that $\sup_n \|T_n\| < L$.

(i) $\to$ (iii). If $A \in S^p$ then all $T_n A \in S^p$ and, by (4.3),

$$\|T_n A\|_p \leq \|T_n\| \, \|A\|_p \leq L \, \|A\|_p \, .$$

Hence (iii) holds for $M_2 = L \, \|A\|_p$.

(iii) $\to$ (ii). As $T_n A \in S^p$, the operators $T_n A T_n$ also belong to S^p. By (4.3),

$$\|T_n A T_n\|_p \leq \|T_n A\|_p \, \|T_n\| \leq M_2 L = M_1.$$

(ii) $\to$ (i). Let (4.7) hold. The sequence $\{T_n A T_n\}$ converges to A in s.o.t. Indeed, for each $x \in H$,

$$\begin{aligned} \|Ax - T_n A T_n x\| &\leq \|Ax - T_n Ax\| + \|T_n Ax - T_n A T_n x\| \\ &\leq \|Ax - T_n Ax\| + \|T_n\| \, \|A\| \, \|x - T_n x\| \to 0, \text{ as } n \to \infty, \end{aligned}$$

since $\|z - T_n z\| \to 0$ for all $z \in H$. Hence $\{T_n A T_n\}$ converges to A in w.o.t. As $\|T_n A T_n\|_p \leq M_1 < \infty$, all operators $T_n A T_n$ belong to S^p. Therefore it follows from Theorem 4.3 that $A \in S^p$ and $\|A\|_p \leq M_1$. ■

Corollary 4.4 is partially stated in Theorem III.5.2 of [21, p.87] but only for *monotonically increasing* sequence of finite rank projections.

It should be noted that Corollary 4.4 does not hold for $p = \infty$, that is, for $S^\infty = C(H)$. Indeed, let A be a bounded non-compact operator. Then $A \notin S^\infty$.

However, as the norm $\|\cdot\|_\infty$ coincides with the usual operator norm $\|\cdot\|$, we have that (4.7) holds, since

$$\sup_n \|T_n A\|_\infty = \sup_n \|T_n A\| \leq \sup_n \|T_n\| \|A\| \leq L \|A\| .$$

Theorem 4.5 [21, Theorem III.6.3] *Let $\{T_n\}$ be a sequence of self-adjoint bounded operators on H that converges to $\mathbf{1}_H$ in the strong operator topology. Then, for each $p \in [1, \infty]$ and for each $A \in S^p$,*

$$\|A - T_n A\|_p \to 0 \text{ and } \|A - T_n A T_n\|_p \to 0, \text{ as } n \to \infty.$$

The above result means that every sequence of self-adjoint bounded operators on H that converges to $\mathbf{1}_H$ in the strong operator topology is an approximate identity in all ideals S^p, $p \in [1, \infty]$ (including $S^\infty = C(H)$).

Corollary 4.6 *Let a sequence of self-adjoint bounded operators $\{T_n\}$ on H converge to $\mathbf{1}_H$ in the strong operator topology. Suppose that $\sup_n \|T_n\| \leq 1$. Then, for each $A \in B(H)$ and each $p \in [1, \infty]$,*

$$\sup_n \|T_n A T_n\|_p = \|A\|_p \tag{4.8}$$

$$\text{and } \lim_{n\to\infty} \|T_n A T_n\| = \|A\| . \tag{4.9}$$

Proof. Let firstly $A \in S^p$. It follows from (4.3) and Theorem 4.5 that,

$$\|T_n A T_n\|_p \overset{(4.3)}{\leq} \|T_n\| \|A\|_p \|T_n\| \leq \|A\|_p \text{ and } \lim_{n\to\infty} \|T_n A T_n\|_p = \|A\|_p .$$

Hence $\sup_n \|T_n A T_n\|_p = \|A\|_p$.

Let $A \notin S^p$. Then by (4.6), $\|A\|_p = \infty$. If $\sup_n \|T_n A T_n\|_p < \infty$ then it follows from Corollary 4.4 that $A \in S^p$ and we have a contradiction. Hence $\sup_n \|T_n A T_n\|_p = \infty = \|A\|_p$ and (4.8) is proved.

Now let us prove (4.9). Given $\varepsilon > 0$, we can find $x \in H$ such that $\|x\| = 1$ and $0 \leq \|A\| - \|Ax\| < \varepsilon$. Then, as $T_n \to \mathbf{1}_H$ in the s.o.t., we have

$$\begin{aligned}\|T_n A T_n x - Ax\| &\leq \|T_n A(T_n x - x)\| + \|T_n Ax - Ax\| \\ &\leq \|T_n\| \, \|A\| \, \|T_n x - x\| + \|T_n Ax - Ax\| \to 0, \text{ as } n \to \infty,\end{aligned}$$

since $\|T_n x - x\| \to 0$ and $\|T_n Ax - Ax\| \to 0$. Choose $N \in \mathbb{N}$ such that

$$\|T_n A T_n x - Ax\| < \varepsilon, \text{ for } n > N.$$

Then, as $\|T_n A T_n x\| \leq \|T_n A T_n\| \leq \|A\|$, we have

$$\begin{aligned}0 &\leq \|A\| - \|T_n A T_n\| \leq \|A\| - \|T_n A T_n x\| \leq \|A\| - \|Ax\| + \|Ax\| - \|T_n A T_n x\| \\ &< \varepsilon + \|Ax - T_n A T_n x\| < 2\varepsilon.\end{aligned}$$

Since we can choose ε arbitrary small, we have that $\lim_{n\to\infty} \|T_n A T_n\| = \|A\|$. Thus (4.9) is proved. ■

4.2 Some minimax conditions for norms in S^p

Let a sequence $\{T_n\}$ of self-adjoint bounded operators on H converge to $\mathbf{1}_H$ in the s.o.t. For each $A \in B(H)$, consider the function $f_A(p, n) = \|T_n A T_n\|_p$, for $n \in \mathbb{N}$ and

$p \in [1, \infty)$. In this section we shall show that the function f_A satisfies the minimax condition:

$$\inf_{p\in[1,\infty)} \sup_n f_A(p,n) = \sup_n \inf_{p\in[1,\infty)} f_A(p,n) \tag{4.10}$$

in the following two cases:

1) when $A \in \cup_{p\in[1,\infty)} S^p$,

2) when $A \notin \cup_{p\in[1,\infty)} S^p$ and $T_k A T_k \notin \cup_{p\in[1,\infty)} S^p$ for some k.

We shall also show that, as a consequence of Lemma 3.2, the reversed minimax condition

$$\inf_n \sup_{p\in[1,\infty)} f_A(p,n) = \sup_{p\in[1,\infty)} \inf_n f_A(p,n) \tag{4.11}$$

holds for all operators A in $B(H)$.

The following lemma contains simple norm equalities some of which are well known.

Lemma 4.7 *Let $A \in S^q$, for some $q \in [1, \infty)$. Then*

$$\lim_{q\le p\to\infty} \|A\|_p = \|A\| . \tag{4.12}$$

Let $\{T_n\}$ be a sequence of self-adjoint bounded operators on H that converges to $\mathbf{1}_H$ in the s.o.t.. *If a sequence $\{p_n\}$ in $[q, \infty)$ satisfies $\lim_{n\to\infty} p_n = \infty$, then*

$$\lim_{n\to\infty} \|T_n A T_n\|_{p_n} = \|A\| . \tag{4.13}$$

Proof. Let $\{s_j\}$ be the non-increasing sequence of all eigenvalues of the operator $(A^*A)^{1/2}$ repeated according to multiplicity. Set $\alpha_j = \frac{s_j}{s_1}$. Then all $\alpha_j \le 1$. As

$A \in S^q$, the series $\sum_{j=1}^{\infty} s_j^q = s_1^q \sum_{j=1}^{\infty} \alpha_j^q$ converges. Hence we can find N such that $\sum_{j=N}^{\infty} \alpha_j^q < 1$. Thus $\alpha_j < 1$ for $j \geq N$, so that $\sum_{j=N}^{\infty} \alpha_j^p < \sum_{j=N}^{\infty} \alpha_j^q < 1$, for $p > q$. Therefore,

$$\sum_{j=1}^{\infty} s_j^p = s_1^p \sum_{j=1}^{\infty} \alpha_j^p = s_1^p \sum_{j=1}^{N-1} \alpha_j^p + s_1^p \sum_{j=N}^{\infty} \alpha_j^p \leq s_1^p (N-1) + s_1^p = N s_1^p.$$

Thus

$$s_1 \leq \left(\sum_{j=1}^{\infty} s_j^p \right)^{1/p} = \|A\|_p \leq s_1 N^{1/p} \to s_1, \text{ as } p \to \infty.$$

Hence

$$\lim_{p \to \infty} \|A\|_p = s_1 \overset{(4.5)}{=} \|A\| = \left\| (A^* A)^{1/2} \right\|$$

which completes the proof of (4.12).

By (4.4), we have that A belongs to all S^{p_n}. Noticing that

$$\left| \|A\| - \|T_n A T_n\|_{p_n} \right| \leq \left| \|A\| - \|A\|_{p_n} \right| + \left| \|A\|_{p_n} - \|T_n A T_n\|_{p_n} \right|,$$

we shall prove that

$$\left| \|A\| - \|T_n A T_n\|_{p_n} \right| \to 0 \text{ as } n \to \infty.$$

Indeed, by (4.12), $\lim_{n \to \infty} \left| \|A\| - \|A\|_{p_n} \right| = 0$, as $\lim_{n \to \infty} p_n = \infty$. It follows from Theorem 4.5 that $\|A - T_n A T_n\|_q \to 0$, as $n \to \infty$. Thus

$$\left| \|A\|_{p_n} - \|T_n A T_n\|_{p_n} \right| \leq \|A - T_n A T_n\|_{p_n} \overset{(4.4)}{\leq} \|A - T_n A T_n\|_q \to 0, \text{ as } n \to \infty.$$

Thus (4.13) holds. ■

In the following proposition we evaluate

$$\inf_{p\in[1,\infty)}\left(\sup_n \|T_nAT_n\|_p\right) \text{ and } \sup_n\left(\inf_{p\in[1,\infty)} \|T_nAT_n\|_p\right).$$

Suppose that $\sup_n \|T_n\| \leq 1$. Recall that $\|A\|_p = \infty$ if $A \notin S^p$. We obtain from Corollary 4.6 that

$$\sup_n \|T_nAT_n\|_p = \|A\|_p, \text{ if } A \in S^p, \tag{4.14}$$

$$\text{and } \sup_n \|T_nAT_n\|_p = \|A\|_p = \infty, \text{ if } A \notin S^p. \tag{4.15}$$

Proposition 4.8 *Let* $A \in B(H)$. *Let a sequence* $\{T_n\}$ *of self-adjoint bounded operators on* H *converge to* $\mathbf{1}_H$ *in the* s.o.t. *and* $\sup_n \|T_n\| \leq 1$.

(i) *If* A *belongs to* S^q, *for some* $q \in [1,\infty)$, *i.e.,* $A \in \cup_{p\in[1,\infty)} S^p$ *then*

$$\inf_{p\in[1,\infty)}\left(\sup_n \|T_nAT_n\|_p\right) = \|A\|.$$

(ii) *If* A *does not belong to any Schatten ideal* S^q, *for* $q \in [1,\infty)$, *i.e.,* $A \notin \cup_{p\in[1,\infty)} S^p$ *then*

$$\inf_{p\in[1,\infty)}\left(\sup_n \|T_nAT_n\|_p\right) = \infty.$$

(iii) *If, for each* n, *the operator* T_nAT_n *belongs to* $\cup_{p\in[1,\infty)} S^p$ *(for example, all* T_n *are finite rank projections, or* A *belongs to some* S^q*), then*

$$\sup_n\left(\inf_{p\in[1,\infty)} \|T_nAT_n\|_p\right) = \|A\|. \tag{4.16}$$

(iv) *If* $T_kAT_k \notin \cup_{p\in[1,\infty)} S^p$, *for some* k, *then*

$$\sup_n\left(\inf_{p\in[1,\infty)} \|T_nAT_n\|_p\right) = \infty.$$

Proof. (i) Let $A \in S^q$. Then, by (4.15),

$$\inf_{p\in[1,\infty)} \left(\sup_n \|T_nAT_n\|_p \right) = \inf \left(\|A\|_p : p \in [1,\infty),\ A \in S^p \right).$$

Taking into account (4.4), we have

$$\inf \left\{ \|A\|_p : p \in [1,\infty),\ A \in S^p \right\} = \lim_{q\leq p\to\infty} \|A\|_p \overset{(4.12)}{=} \|A\|$$

which completes the proof of (i).

(ii) If A does not belong to any Schatten ideal S^p, for $p \in [1,\infty)$, then $\|A\|_p = \infty$ and it follows from (4.15) that $\sup_n \|T_nAT_n\|_p = \infty$ for each $p \in [1,\infty)$. Hence

$$\inf_{p\in[1,\infty)} \left(\sup_n \|T_nAT_n\|_p \right) = \infty.$$

This ends the proof of (ii).

(iii) Fix n. Then T_nAT_n belongs to some $S^{q(n)}$. Hence $T_nAT_n \in S^p$, for all $p \geq q(n)$. By (4.15),

$$\inf_{p\in[1,\infty)} \|T_nAT_n\|_p = \inf \left\{ \|T_nAT_n\|_p : p \in [1,\infty),\ T_nAT_n \in S^p \right\}$$
$$\overset{(4.4)}{=} \lim_{p\to\infty} \|T_nAT_n\|_p \overset{(4.12)}{=} \|T_nAT_n\|.$$

As $\|T_nAT_n\| \leq \|T_n\| \|A\| \|T_n\| \leq \|A\|$, we have

$$\sup_n \inf_{p\in[1,\infty)} \|T_nAT_n\|_p = \sup_n \|T_nAT_n\| \leq \|A\|.$$

From (4.9) we have that $\lim_{n\to\infty} \|T_nAT_n\| = \|A\|$. Thus $\sup_n \|T_nAT_n\| \geq \|A\|$. Hence $\sup_n \|T_nAT_n\| = \|A\|$ and

$$\sup_n \inf_{p\in[1,\infty)} \|T_nAT_n\|_p = \sup_n \|T_nAT_n\| = \|A\|.$$

Thus (4.16) is proved.

(iv) If, for some k, the operator T_kAT_k does not belong to any S^p, then $\|T_kAT_k\|_p = \infty$ for all $p \in [1,\infty)$. Hence $\inf_{p\in[1,\infty)} \|T_kAT_k\|_p = \infty$. Therefore

$$\sup_n \inf_{p\in[1,\infty)} \|T_nAT_n\|_p = \infty$$

and the proof is complete. ■

Making use of Proposition 4.8, we obtain

Theorem 4.9 *Let $A \in B(H)$. Let $\{T_n\}$ be self-adjoint bounded operators on H and $T_n \overset{s.o.t.}{\to} \mathbf{1}_H$. Suppose that $\sup_n \|T_n\| \leq 1$.*

(i) *If $A \in \cup_{p\in[1,\infty)} S^p$ then the minimax condition holds:*

$$\inf_{p\in[1,\infty)} \sup_n \|T_nAT_n\|_p = \sup_n \inf_{p\in[1,\infty)} \|T_nAT_n\|_p = \|A\|.$$

(ii) *If $A \notin \cup_{p\in[1,\infty)} S^p$ and $T_kAT_k \notin \cup_{p\in[1,\infty)} S^p$, for some k, then the minimax condition trivially holds:*

$$\inf_{p\in[1,\infty)} \sup_n \|T_nAT_n\|_p = \sup_n \inf_{p\in[1,\infty)} \|T_nAT_n\|_p = \infty.$$

(iii) *If $A \notin \cup_{p\in[1,\infty)} S^p$ but each $T_nAT_n \in \cup_{p\in[1,\infty)} S^p$, then the minimax condition does not hold:*

$$\inf_{p\in[1,\infty)} \sup_n \|T_nAT_n\|_p = \infty, \text{ while } \sup_n \inf_{p\in[1,\infty)} \|T_nAT_n\|_p = \|A\|.$$

Remark 4.10 Using the same arguments as above, we obtain that the results of Theorem 4.9 hold if T_nAT_n is replaced by T_nA.

Unlike the minimax condition in Theorem 4.9 that only holds for some operators in $B(H)$, its reversed minimax, i.e., Theorem 4.11 holds for all operators in $B(H)$.

Theorem 4.11 *For a non-empty set X, let $\{A_x\}_{x\in X}$ be a family of operators in $B(H)$. Then the following minimax condition holds:*

$$\inf_{x\in X}\left(\sup_{p\in[1,\infty)}\|A_x\|_p\right)=\sup_{p\in[1,\infty)}\left(\inf_{x\in X}\|A_x\|_p\right)=\inf_{x\in X}\|A_x\|_1 .$$

In particular, if $\{T_n\}$ is a sequence of operators in $B(H)$ then, for each operator $A\in B(H)$, the following minimax condition holds:

$$\inf_{n}\left(\sup_{p\in[1,\infty)}\|T_nAT_n\|_p\right)=\sup_{p\in[1,\infty)}\left(\inf_{n}\|T_nAT_n\|_p\right)=\inf_{n}\|T_nAT_n\|_1 .$$

Proof. Set $f(x,p)=\|A_x\|_p$ for all $x\in X$ and $p\in[1,\infty)$. Then, for each $x\in X$, it follows from (4.4) that

$$\sup_{p\in[1,\infty)}f(x,p)=\sup_{p\in[1,\infty)}\|A_x\|_p=\|A_x\|_1=f(x,1).$$

Setting $\Lambda=[1,\infty)$ and $\mu=1$ in Lemma 3.2, we obtain

$$\inf_{x\in X}\left(\sup_{p\in[1,\infty)}\|A_x\|_p\right)=\sup_{p\in[1,\infty)}\left(\inf_{x\in X}\|A_x\|_p\right)=\inf_{x\in X}\|A_x\|_1 .$$

This concludes the proof. ■

Remark 4.12 Note that we can not apply Lemma 3.2 to prove Theorem 4.9. Indeed, to do this, we have to set $X=[1,\infty)$, $\Lambda=\mathbb{N}$ and $f(p,n)=\|T_nAT_n\|_p$. Then (see (3.1)) we have to find $\mu\in\mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}}\|T_nAT_n\|_p=\sup_{n\in\mathbb{N}}f(p,n)=f(p,\mu)=\|T_\mu AT_\mu\|_p\text{, for each }p\in[1,\infty).$$

From (4.8), $\sup_{n\in\mathbb{N}} \|T_nAT_n\|_p = \|A\|_p$. This means that there is $\mu \in \mathbb{N}$ such that, for each $p \in [1, \infty)$, $\|A\|_p = \|T_\mu AT_\mu\|_p$ and that is, generally speaking, not true.

Also we can not apply Lemma 3.2 using condition (3.2). Indeed, to do this we need the fact that

$$\inf_{p\in[1,\infty)} \|T_nAT_n\|_p = \|T_nAT_n\|_\infty = \|T_nAT_n\| \text{ for all } n \in \mathbb{N}.$$

However, infinity is not an element of the interval $[1, \infty)$.

4.3 Minimax condition and geometry of subspaces of Hilbert spaces

Let $\{P_n\}_{n=1}^\infty$ be a sequence of projections in $B(H)$, $P_n \neq \mathbf{1}$, and let $q \in [1, \infty)$. Suppose that a sequence $\{p_n\}_{n=1}^\infty$ in (q, ∞) satisfies $\lim_{n\to\infty} p_n = \infty$. In this section we study the minimax condition

$$\inf_{X\in S^q, \|X\|_q=1} \left(\sup_n \|P_nXP_n\|_{p_n} \right) = \sup_n \left(\inf_{X\in S^q, \|X\|_q=1} \|P_nXP_n\|_{p_n} \right) \tag{4.17}$$

and the reversed minimax condition

$$\inf_n \left(\sup_{X\in S^q, \|X\|_q=1} \|P_nXP_n\|_{p_n} \right) = \sup_{X\in S^q, \|X\|_q=1} \left(\inf_n \|P_nXP_n\|_{p_n} \right). \tag{4.18}$$

We will show that while the first minimax condition always holds, the fulfilment of the reversed minimax condition depends on an interesting geometric property of the family of subspaces $\{L_n = P_nH\}_{n=1}^\infty -$ approximate intersection of these subspaces.

Definition 4.13 *We say that a family of nonzero subspaces* $\{L_n\}_{n=1}^{\infty}$ *of* H *is approximately intersecting if, for every* $\varepsilon > 0$*, there is* $x_\varepsilon \in H$ *such that*

$$\|x_\varepsilon\| = 1 \text{ and } dist\,(x_\varepsilon, L_n) := \min_{y \in L_n} \|x_\varepsilon - y\| \le \varepsilon \text{ for all } n. \tag{4.19}$$

In particular, if there exists $0 \neq x \in H$ that belongs to all L_n, then, clearly, the family of subspaces $\{L_n\}$ is approximately intersecting - condition (4.19) holds for $x_\varepsilon \equiv \frac{x}{\|x\|}$ for all $\varepsilon > 0$.

For nonzero vectors $x, y \in H$, consider the rank one operator $x \otimes y$ on H that acts by

$$(x \otimes y)\, z = (z, x)\, y \text{ for all } z \in H. \tag{4.20}$$

Geometrically the operator $x \otimes y$ can be described as follows:

1. If $\|x\| = 1$ then the operator $x \otimes x$ is an orthogonal projection onto the one-dimensional subspace $\mathbb{C}x = \{\lambda x : \lambda \in \mathbb{C}\}$. The subspace $\ker(x \otimes x)$ consists of all vectors orthogonal to the vector x, i.e. $\ker(x \otimes x) = (\mathbb{C}x)^{\perp}$;

2. Generally, if $(x, y) \neq 0$ and $y \notin \mathbb{C}x$, then the operator $T = \frac{1}{(y,x)}(x \otimes y) = \frac{(x,y)}{\|(x,y)\|}(x \otimes y)$ is a projection (but not orthogonal projection) onto the one-dimensional subspace $\mathbb{C}y = \{\lambda y : \lambda \in \mathbb{C}\}$, i.e. $T^2 = T$. In particular, if $(y, x) = (x, y) = 1$ then $x \otimes y$ is a projection (but not orthogonal projection) onto the one-dimensional subspace $\mathbb{C}y = \{\lambda y : \lambda \in \mathbb{C}\}$. The subspace $\ker(x \otimes y)$ consists of all vectors orthogonal to the vector x, i.e. $\ker(x \otimes y) = (\mathbb{C}x)^{\perp}$.

Note that $(\ker(x\otimes y))^{\perp} = \left((\mathbb{C}x)^{\perp}\right)^{\perp} = \mathbb{C}x$. As $y \notin \mathbb{C}x$ we have that y is not perpendicular to $\ker(x\otimes y)$, i.e. the range and the null space are not perpendicular;

3. If $(x,y) = 0$ then the range of $x\otimes y$ is the subspace $\mathbb{C}y$ but the range of $(x\otimes y)^2$ consists only of the zero vector.

If $T, S \in B(H)$ and $z \in H$,

$$\begin{aligned} T(x\otimes y)Sz &= T(x\otimes y)(Sz) = T(Sz,x)y = T(z,S^*x)y \\ &= (z,S^*x)Ty = (S^*x\otimes Ty)z. \end{aligned}$$

Thus

$$T(x\otimes y)S = S^*x\otimes Ty. \tag{4.21}$$

To find the adjoint of $A = x\otimes y$, notice that for all $z, w \in H$, we have

$$((x\otimes y)z,w) = ((z,x)y,w) = (z,x)(y,w) = (z,(w,y)x) = (z,(y\otimes x)w).$$

Thus

$$A^* = (x\otimes y)^* = y\otimes x.$$

In particular, if $x = y$, we obtain

$$(x\otimes x)^* = x\otimes x. \tag{4.22}$$

For all $z \in H$, we have

$$A^*Az = (y \otimes x)(x \otimes y) z = (y \otimes x)(z, x) y = (z, x)(y \otimes x) y$$
$$= (z, x)(y, y) x = (y, y)(z, x) x = \|y\|^2 (x \otimes x) z.$$

Hence

$$A^*A = \|y\|^2 (x \otimes x). \tag{4.23}$$

In the case when $x = y$ and $\|x\| = 1$, i.e., $A^* = A = (x \otimes x)$ we obtain

$$(x \otimes x)^2 = x \otimes x. \tag{4.24}$$

To evaluate the norm $\|A\|_p = \|x \otimes y\|_p$ we need to find eigenvalues of the operator $(A^*A)^{1/2}$. These are the square roots of eigenvalues of A^*A. Note that the operator $(x \otimes x)$ has only one non-zero eigenvalue and it is $\|x\|^2$. Indeed, if $k \neq 0$ is an eigenvalue and z is the corresponding eigenvector: $(x \otimes x) z = kz$, then $(x \otimes x) z = (z, x) x = kz$, so that z and x are linearly dependent: $z = tx$, for $t = \frac{(z,x)}{k} \neq 0$. Hence

$$t = \frac{(z, x)}{k} = \frac{(tx, x)}{k} = t\frac{\|x\|^2}{k},$$

so that $k = \|x\|^2$.

It follows from (4.23) that the only non-zero eigenvalue of the operator A^*A is $\|x\|^2 \|y\|^2$. Thus the only non-zero eigenvalue of the operator $(A^*A)^{1/2}$ is $\|x\| \|y\|$. Hence

$$\|x \otimes y\|_p = \|x\| \|y\|, \text{ for all } p \in [1, \infty). \tag{4.25}$$

If $\|x\| = 1$ then the operator $x \otimes x$ is a projection on the one-dimensional subspace $\mathbb{C}x = \{\lambda x : \lambda \in \mathbb{C}\}$. Indeed (see (4.22) and (4.24)),

$$(x \otimes x)^2 = x \otimes x \text{ and } (x \otimes x)^* = x \otimes x.$$

Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal basis in H with $e_1 = x$. Then, for each $z = \sum_{n=1}^{\infty} \alpha_n e_n \in H$, where α_n are scalars, we have

$$(x \otimes x)\, z = (z, x)\, x = \left(\sum_{n=1}^{\infty} \alpha_n e_n, x \right) x = \alpha_1 \|x\|^2 x = \alpha_1 x.$$

We shall now prove a lemma that will help us to identify approximately intersecting selection of spaces.

Lemma 4.14 *Let $\{L_n\}_{n=1}^{\infty}$ be a family of nonzero subspaces of H and let P_n be the orthogonal projections onto L_n. The family $\{L_n\}_{n=1}^{\infty}$ is approximately intersecting if and only if, for each $\varepsilon > 0$, there is $x_\epsilon \in H$ such that*

$$\|x_\varepsilon\| = 1 \text{ and } \|P_n x_\varepsilon\| \geq 1 - \varepsilon \text{ for all } n. \tag{4.26}$$

Proof. For $\varepsilon \geq 1$, (4.19) and (4.26) hold trivially as $\|P_n x_\varepsilon\| \geq 0$ for all n. Thus we can assume that $0 < \varepsilon < 1$.

Let $\{L_n\}$ be approximately intersecting. Then, for each $\varepsilon \in (0,1)$, there is $x_\varepsilon \in H$ such that (4.19) holds for all n. We know from [1, pp.8-10] that $\min_{y \in L_n} \|x_\varepsilon - y\| = \|x_\varepsilon - P_n x_\varepsilon\|$. Thus $\|x_\varepsilon - P_n x_\varepsilon\| \leq \varepsilon$ and, since

$$\|x_\varepsilon\| = \|x_\varepsilon - P_n x_\varepsilon + P_n x_\varepsilon\| \leq \|x_\varepsilon - P_n x_\varepsilon\| + \|P_n x_\varepsilon\|,$$

we have

$$\|P_n x_\varepsilon\| \geq \|x_\varepsilon\| - \|x_\varepsilon - P_n x_\varepsilon\| \geq 1 - \varepsilon \text{ for all } n.$$

Conversely, let for each $\varepsilon \in (0,1)$, there be $x_\varepsilon \in H$ such that (4.26) holds for all n. Then $(\mathbf{1} - P_n)\, x_\varepsilon$ and $P_n x_\varepsilon$ are orthogonal, as

$$\begin{aligned} ((\mathbf{1} - P_n)\, x_\varepsilon, P_n x_\varepsilon) &= (x_\varepsilon, P_n x_\varepsilon) - (P_n x_\varepsilon, P_n x_\varepsilon) \\ &= (x_\varepsilon, P_n x_\varepsilon) - (x_\varepsilon, P_n^* P_n x_\varepsilon) = (x_\varepsilon, P_n x_\varepsilon) - \left(x_\varepsilon, P_n^2 x_\varepsilon\right) \\ &= (x_\varepsilon, P_n x_\varepsilon) - (x_\varepsilon, P_n x_\varepsilon) = 0. \end{aligned}$$

Since $x_\varepsilon = (\mathbf{1} - P_n)\, x_\varepsilon + P_n x_\varepsilon$, we have

$$\begin{aligned} 1 &= \|x_\varepsilon\|^2 = ((\mathbf{1} - P_n)\, x_\varepsilon + P_n x_\varepsilon, (\mathbf{1} - P_n)\, x_\varepsilon + P_n x_\varepsilon) \\ &= \|(\mathbf{1} - P_n)\, x_\varepsilon\|^2 + \|P_n x_\varepsilon\|^2 . \end{aligned}$$

Hence, by (4.26), $\|P_n x_\varepsilon\|^2 \geq (1-\varepsilon)^2$ and we have

$$\|x_\varepsilon - P_n x_\varepsilon\|^2 = 1 - \|P_n x_\varepsilon\|^2 \leq 1 - (1-\varepsilon)^2 = 2\varepsilon - \varepsilon^2 \text{ for all } n.$$

Thus, given $\varepsilon \in (0,1)$, let $x_{\varepsilon^2/2}$ be such that $\left\|P_n x_{\varepsilon^2/2}\right\|^2 \geq \left(1 - \varepsilon^2/2\right)^2$. Then

$$\min_{y \in L_n} \left\|x_{\varepsilon^2/2} - y\right\| = \left\|x_{\varepsilon^2/2} - P_n x_{\varepsilon^2/2}\right\| \leq \left(2 \times \frac{\varepsilon^2}{2} - \frac{\varepsilon^4}{4}\right)^{1/2} < \left(\varepsilon^2\right)^{1/2} = \varepsilon,$$

for all n. Hence (4.19) holds. ■

We will now verify the following minimax conditions in Schatten ideals for a family of projections.

Theorem 4.15 *Let $\{P_n\}_{n=1}^{\infty}$ be projections in $B(H)$, $P_n \neq \mathbf{1}$, and let $q \in [1, \infty)$. Suppose that a sequence $\{p_n\}_{n=1}^{\infty}$ in (q, ∞) satisfies $\lim_{n\to\infty} p_n = \infty$.*

(i) *The following minimax condition holds in all cases:*

$$\inf_{X \in S^q, \|X\|_q = 1} \left(\sup_n \|P_n X P_n\|_{p_n} \right) = \sup_n \left(\inf_{X \in S^q, \|X\|_q = 1} \|P_n X P_n\|_{p_n} \right) = 0.$$

(ii) *The reversed minimax condition*

$$\inf_n \left(\sup_{X \in S^q, \|X\|_q = 1} \|P_n X P_n\|_{p_n} \right) = \sup_{X \in S^q, \|X\|_q = 1} \left(\inf_n \|P_n X P_n\|_{p_n} \right) = 1. \qquad (4.27)$$

holds if and only if the family of subspaces $\{L_n = P_n H\}_{n=1}^{\infty}$ is approximately intersecting.

Proof. (i) As all $P_n \neq \mathbf{1}$, we can choose $X_n \in S^q$ such that $\|X_n\|_q = 1$ and $P_n X_n P_n = 0$. Then we have that RHS $= 0$.

Set

$$r = \inf\{p_n\}.$$

Since $\lim_{n\to\infty} p_n = \infty$ and all $p_n \in (q, \infty)$, we have $q < r \leq p_n \underset{n\to\infty}{\to} \infty$ and

$$\|P_n X P_n\|_{p_n} \leq \|P_n\| \, \|X\|_{p_n} \|P_n\| = \|X\|_{p_n} \overset{(4.4)}{\leq} \|X\|_r . \qquad (4.28)$$

Hence $\sup_n \|P_n X P_n\|_{p_n} \leq \|X\|_r$. Let $X_k = \{k^{-1/q}, ..., k^{-1/q}, 0, ...\}$ be the diagonal operators with first k elements equal $k^{-1/q}$ and the rest equal 0. Then

$$\|X_k\|_q = \left(\left(k^{-1/q}\right)^q + ... + \left(k^{-1/q}\right)^q \right)^{1/q} = 1 \text{ and}$$

$$\inf_k \|X_k\|_r = \inf_k \left(\frac{k}{k^{r/q}}\right)^{1/r} = \inf_k k^{\frac{1}{r}-\frac{1}{q}} = 0,$$

as $q < r$. Hence

$$\inf_{X\in S^q, \|X\|_q=1} \left(\sup_n \|P_n X P_n\|_{p_n}\right) \le \inf_k \left(\sup_n \|P_n X_k P_n\|_{p_n}\right) \overset{(4.28)}{\le} \inf_k \|X_k\|_r = 0$$

and (i) is proved.

(ii) First note that it follows from (3.4) that

$$\inf_n \left(\sup_{X\in S^q, \|X\|_q=1} \|P_n X P_n\|_{p_n}\right) \ge \sup_{X\in S^q, \|X\|_q=1} \left(\inf_n \|P_n X P_n\|_{p_n}\right)$$

always holds. As $\|P_n X P_n\|_{p_n} \le \|P_n\| \, \|X\|_{p_n} \|P_n\| = \|X\|_{p_n} \le \|X\|_q = 1$, we have

$$1 \ge \inf_n \left(\sup_{X\in S^q, \|X\|_q=1} \|P_n X P_n\|_{p_n}\right).$$

Thus in order to prove (4.27) we only need to show that

$$\sup_{X\in S^q, \|X\|_q=1} \left(\inf_n \|P_n X P_n\|_{p_n}\right) \ge 1. \tag{4.29}$$

Let the spaces $\{L_n = P_n H\}_{n=1}^{\infty}$ approximately intersect. Then, by (4.26), for each $\varepsilon > 0$, there is $x_\varepsilon \in H$ such that $\|x_\varepsilon\| = 1$ and $\|P_n x_\varepsilon\| \ge 1 - \varepsilon$ for all n. Set $X_\varepsilon = x_\varepsilon \otimes x_\varepsilon$. Then, by (4.25) and (4.21),

$$\|X_\varepsilon\|_q = \|x_\varepsilon \otimes x_\varepsilon\|_q = \|x_\varepsilon\|^2 = 1, \text{ and } P_n X_\varepsilon P_n = P_n x_\varepsilon \otimes P_n x_\varepsilon.$$

Thus

$$\|P_n X_\varepsilon P_n\|_{p_n} = \|P_n x_\varepsilon \otimes P_n x_\varepsilon\|_{p_n} \overset{(4.25)}{=} \|P_n x_\varepsilon\|^2 \ge (1-\varepsilon)^2.$$

Hence

$$\inf_n \|P_n X_\varepsilon P_n\|_{p_n} \geq (1-\varepsilon)^2$$

and

$$\sup_{X\in S^q, \|X\|_q=1} \left(\inf_n \|P_n X P_n\|_{p_n}\right) \geq \sup_\varepsilon \left(\inf_n \|P_n X_\varepsilon P_n\|_{p_n}\right) \geq \sup_\varepsilon (1-\varepsilon)^2 = 1.$$

This proves (4.29).

Conversely, let (4.27) hold. Let us prove that the spaces $\{L_n = P_n H\}_{n=1}^\infty$ approximately intersect. It follows from the last equality in (4.27) that, for each $\varepsilon > 0$, there is $X_\varepsilon \in S^q$ such that $\|X_\varepsilon\|_q = 1$ and $\|P_n X_\varepsilon P_n\|_{p_n} \geq 1-\varepsilon$, for all n. Let, as in (i), $r = \inf\{p_n\}$. Then $q < r \leq p_n$ and

$$\|X_\varepsilon\|_r \geq \|P_n X_\varepsilon P_n\|_r \geq \|P_n X_\varepsilon P_n\|_{p_n} \geq 1-\varepsilon \text{ for all } n. \tag{4.30}$$

Let $s_1(\varepsilon) \geq s_2(\varepsilon) \geq ...$ be the singular values of X_ε, that is, the eigenvalues of the operator $(X_\varepsilon^* X_\varepsilon)^{1/2}$. Then it follows from (4.2) and (4.30) that

$$\sum_{n=1}^\infty s_n^r(\varepsilon) = \|X_\varepsilon\|_r^r \geq (1-\varepsilon)^r.$$

Therefore, as $s_n^r \leq s_1^{r-q} s_n^q$ (this follows from the fact that $s_n \leq s_1$) and

$$\|X_\varepsilon\|_q = \left(\sum_{n=1}^\infty s_n^q(\varepsilon)\right)^{1/q} = 1, \tag{4.31}$$

we have

$$(1-\varepsilon)^r \leq \|X_\varepsilon\|_r^r = \sum_{n=1}^\infty s_n^r(\varepsilon) \leq s_1^{r-q}(\varepsilon) \sum_{n=1}^\infty s_n^q(\varepsilon) = s_1^{r-q}(\varepsilon) \|X_\varepsilon\|_q^q = s_1^{r-q}(\varepsilon).$$

By (4.31), $1 \geq s_1(\varepsilon)$. Hence,

$$1 \geq s_1(\varepsilon) \geq (1-\varepsilon)^{\frac{r}{r-q}}. \tag{4.32}$$

Consider the Schmidt decomposition (see [21, Chapter II.2.2.]) of the operator X_ε :

$$X_\varepsilon = \sum_k s_k(\varepsilon)\, x_k(\varepsilon) \otimes y_k(\varepsilon),$$

where $\{x_k(\varepsilon)\}_{k=1}^{\infty}$ and $\{y_k(\varepsilon)\}_{k=1}^{\infty}$ are orthonormal systems of vectors in the Hilbert space H. Then $B_\varepsilon = s_1(\varepsilon)\, x_1(\varepsilon) \otimes y_1(\varepsilon)$ is a rank one operator and,

$$\|X_\varepsilon - B_\varepsilon\|_{p_n} \leq \|X_\varepsilon - B_\varepsilon\|_q = \left(\sum_{k=2}^{\infty} s_k^q(\varepsilon)\right)^{1/q}$$

$$= \left(\sum_{k=1}^{\infty} s_k^q(\varepsilon) - s_1^q(\varepsilon)\right)^{1/q} = (1 - s_1^q(\varepsilon))^{1/q}, \tag{4.33}$$

for all n. Since

$$\|P_n X_\varepsilon P_n\|_{p_n} = \|P_n X_\varepsilon P_n - P_n B_\varepsilon P_n + P_n B_\varepsilon P_n\|_{p_n}$$

$$\leq \|P_n X_\varepsilon P_n - P_n B_\varepsilon P_n\|_{p_n} + \|P_n B_\varepsilon P_n\|_{p_n}$$

$$\leq \|P_n\| \, \|X_\varepsilon - B_\varepsilon\|_{p_n} \|P_n\| + \|P_n B_\varepsilon P_n\|_{p_n}$$

$$= \|X_\varepsilon - B_\varepsilon\|_{p_n} + \|P_n B_\varepsilon P_n\|_{p_n},$$

it follows from (4.30) and (4.33) that, for all n,

$$\|P_n B_\varepsilon P_n\|_{p_n} \geq \|P_n X_\varepsilon P_n\|_{p_n} - \|X_\varepsilon - B_\varepsilon\|_{p_n} \tag{4.34}$$

$$\geq (1-\varepsilon) - \|X_\varepsilon - B_\varepsilon\|_{p_n} \geq (1-\varepsilon) - (1 - s_1^q(\varepsilon))^{1/q}.$$

As $B_\varepsilon = s_1(\varepsilon) x_1(\varepsilon) \otimes y_1(\varepsilon)$, making use of (4.25) and (4.21), we obtain that,

$$\|P_n B_\varepsilon P_n\|_{p_n} = \|s_1(\varepsilon) P_n x_1(\varepsilon) \otimes P_n y_1(\varepsilon)\|_{p_n} = |s_1(\varepsilon)| \, \|P_n x_1(\varepsilon)\| \, \|P_n y_1(\varepsilon)\|,$$

for all n. Hence, by inequality (4.34),

$$s_1(\varepsilon) \|P_n x_1(\varepsilon)\| \, \|P_n y_1(\varepsilon)\| \geq (1-\varepsilon) - \left(1 - s_1^q(\varepsilon)\right)^{1/q}.$$

Therefore,for all n,

$$\|P_n x_1(\varepsilon)\| \geq \frac{(1-\varepsilon) - \left(1 - s_1^q(\varepsilon)\right)^{1/q}}{s_1(\varepsilon) \|P_n y_1(\varepsilon)\|} \geq (1-\varepsilon) - \left(1 - s_1^q(\varepsilon)\right)^{1/q}, \tag{4.35}$$

since, by (4.32), $s_1(\varepsilon) \leq 1$ and $\|P_n y_1(\varepsilon)\| \leq \|y_1(\varepsilon)\| = 1$.

We have

$$\|P_n x_1(\varepsilon)\| \leq \|x_1(\varepsilon)\| = 1,$$

and, by (4.32), $s_1(\varepsilon) \geq (1-\varepsilon)^{\frac{p}{p-q}}$. Hence $s_1(\varepsilon) \to 1$, as $\varepsilon \to 0$ and therefore it follows from (4.35) that

$$\|P_n x_1(\varepsilon)\| \to 1 \text{ for all } n, \text{ as } \varepsilon \to 0.$$

As $\lim_{\varepsilon \to 0} s_1(\varepsilon) = 1$, we have that given $\varepsilon_1 \in (0,1)$, there is $\delta(\varepsilon_1)$ such that

$$s_1(\varepsilon) \geq \left(1 - \left(\frac{\varepsilon_1}{2}\right)^q\right)^{1/q} \text{ for all } \varepsilon < \delta(\varepsilon_1).$$

Then

$$1 - s_1^q(\varepsilon) \leq 1 - \left(1 - \left(\frac{\varepsilon_1}{2}\right)^q\right) = \left(\frac{\varepsilon_1}{2}\right)^q. \tag{4.36}$$

Thus, it follows from (4.35) that, for $\varepsilon \in \left(0, \min\left\{\delta(\varepsilon_1), \frac{\varepsilon_1}{2}\right\}\right)$, we have

$$\|P_n x_1(\varepsilon)\| \geq (1-\varepsilon) - \left(1 - s_1^q(\varepsilon)\right)^{1/q} \overset{(4.36)}{\geq} (1-\varepsilon) - \frac{\varepsilon_1}{2} \geq 1 - \varepsilon_1,$$

for all n. Set $x_{\varepsilon_1} = x_1\left(\frac{1}{2}\min\left\{\delta(\varepsilon_1), \frac{\varepsilon_1}{2}\right\}\right)$. Then, by the above inequality, $\|P_n x_{\varepsilon_1}\| \geq 1 - \varepsilon_1$ for all n. Hence it follows from Lemma 4.14 that the family $\{L_n = P_n H\}_{n=1}^{\infty}$ approximately intersects. ■

4.4 Conclusion

The aim of this chapter is to research, identify and evaluate minimax conditions (4.10), (4.11), (4.17) and (4.18) within the theory of operators on a separable Hilbert space H. We discussed sufficient and necessary conditions in all four cases of the minimax formulae.

In fact we found that (4.11) holds unconditionally for all bounded operators on H. Similarly, (4.17) holds for all sequences of projections in $B(H)$ different from identity and (4.10) holds for Schatten class operators and, generally, for any bounded operator A, if $T_k A T_k$ is not a Schatten class operator for some k, where $\{T_k\}$ is as described in Theorem 4.9.

In section 4.3 we introduced a new concept - approximate intersection of a selection of nonzero subspaces of a Hilbert space H. We proved that the approximate intersection of subspaces $\{L_n\}$, as introduced in Theorem 4.15, is the necessary and sufficient condition for the minimax (4.27) to hold.

All the results in this chapter have been published in [19, pp.29-40].

In Chapters 5 and 6 we divert our attention from minimax theory and focus our study on inclusion of spaces $l_q(S^p)$ and $S^p(H,K)$, on analogues of Clarkson-McCarthy estimates, on inequalities for partitioned operators and Cartesian decomposition of operators.

Part II

Estimates

Chapter 5 Inclusions of spaces $l_q(S^p)$ and $S^p(H, K)$

5.1 Background

Let H be a separable complex Hilbert space and let $B(H)$ be the C^*- algebra of all bounded linear operators on H (see definitions 2.16 and 2.39). The following concepts and theorems are the main results we will need in this chapter. See also introduction to Chapter 4.

Let K be another separable Hilbert space and $B(H, K)$ be the Banach space (see Theorem 2.17) of all bounded operators from H into K. Then $B(H) = B(H, H)$. If $A \in B(H, K)$ then $A^* \in B(K, H)$ [42, page 76] and $A^*A \in B(H)$.

Definition 5.1 [30, pp.20, 99] (i) *An operator* $V \in B(H, K)$ *is an isometry if* $\|Vx\|_K = \|x\|_H$ *for all* $x \in H$*.*

(ii) *An operator* $V \in B(H, K)$ *is a partial isometry, if it is isometric on* $(\ker V)^{\perp}$*, i.e.,* $\|Vx\|_K = \|x\|_H$ *for all* $x \in (\ker V)^{\perp}$*.*

A partial isometry V is an isometry from $(\ker V)^{\perp}$ onto VH; $(\ker V)^{\perp}$ is called the *initial space* and VH the *final space* of V. [13, p.15].

We give below the proof of the following known theorem, as we could not find a reference. We will need this theorem in section 5.4.

Theorem 5.2 *Let $V \in B(H,K)$. Then*

(i) $V^*V = \mathbf{1}_H$ *if and only if* $\|Vx\|_K = \|x\|_H$ *for all* $x \in H$, *i.e.,* V *is an isometry.*

(ii) *If* V *is an isometry from* H *onto* K *then* $V^* = V^{-1}$ *is also an isometry. In this case* V *is unitary.*

Proof. (i) Suppose that $V^*V = \mathbf{1}_H$. Then

$$\|Vx\|_K = (Vx, Vx)_K^{1/2} = (x, V^*Vx)_H^{1/2} = (x,x)_H^{1/2} = \|x\|_H .$$

Conversely, suppose that $\|Vx\|_K = \|x\|_H$ for all $x \in H$. Then, using polarization [41, Theorem 1.1.1], we have, for all $x, y \in H$,

$$(V^*Vx, y)_H = (Vx, Vy)_K$$

$$= \left\|\frac{1}{2}V(x+y)\right\|_K^2 - \left\|\frac{1}{2}V(x-y)\right\|_K^2 + i\left\|\frac{1}{2}V(x+iy)\right\|_K^2 - i\left\|\frac{1}{2}V(x-iy)\right\|_K^2$$

$$= \left\|\frac{1}{2}(x+y)\right\|_H^2 - \left\|\frac{1}{2}(x-y)\right\|_H^2 + i\left\|\frac{1}{2}(x+iy)\right\|_H^2 - i\left\|\frac{1}{2}(x-iy)\right\|_H^2 = (x,y)_H .$$

Hence $(V^*Vx - x, y)_H = 0$ for all $x, y \in H$. Thus $V^*Vx - x = 0$, for all $x \in H$, i.e., $V^*V = \mathbf{1}_H$.

(ii) As V is an isometry onto K, V is invertible. Let $y \in K$ and $V^{-1}y = x \in H$. Then $\|V^{-1}y\|_H = \|x\|_H = \|Vx\|_K = \|y\|_K$. Thus V^{-1} is also an isometry. From part (i) we have $V^*V = \mathbf{1}_H$. Since also $V^{-1}V = \mathbf{1}_H$ and from the uniqueness of an inverse, we conclude that $V^* = V^{-1}$. As $V^*V = \mathbf{1}_H$ and $VV^* = VV^{-1} = \mathbf{1}_K$, V is unitary. ■

Definition 5.3 [38, p.222] *Normed linear spaces X and Y are isometrically isomorphic if there exists a one-to-one linear transformation T of X onto Y such that $\|Tx\| = \|x\|$ for all $x \in X$.*

The following theorem that considers polar decomposition of operators plays a very important role in the theory of operator algebras.

Theorem 5.4 [16, p.935] [43] *For any $T \in B(H)$, there exist a unique partial isometry U with initial space $(\ker T)^{\perp}$ and final space $\overline{R(T)}$ such that*

$$T = U|T| \text{ and } |T| = U^*T, \tag{5.1}$$

where $|T| = (T^*T)^{1/2}$. If T is invertible, then U is unitary.

If A is a bounded operator from H to K then

$$A = U|A| \tag{5.2}$$

is the polar decomposition of A, where $|A| = (A^*A)^{1/2} \in B(H)$ and U is a partial isometry from the closure $\overline{R(A^*)} = (\ker A)^{\perp}$(see Theorem 2.24) of the range of A^* onto the closure $\overline{R(A)}$ of the range of A. Indeed, it suffices to notice that if V is an isometry operator from K onto H, then

$$VA = U_1|VA| = U_1\left((VA)^* VA\right)^{1/2} = U_1|A|,$$

where U_1 is a partial isometry as per Theorem 5.4 (applied to the operator VA). Thus $A = V^{-1}U_1|A|$, where $V^{-1}U_1$ is an isometry from $(\ker A)^{\perp}$ onto $\overline{R(A)}$.

Let $T \in S^p$ be a positive operator. Then $T = |T|$ and the eigenvalues $\{s_i(T)\}_{n=1}^{\infty}$ of T, repeated according to multiplicity, are non-negative numbers. It follows from Corollary 2.36 that $\{s_i(T^2)\}_{i=1}^{\infty} = \{s_i^2(T)\}_{i=1}^{\infty}$. Hence (see (4.2)), for each $p > 0$,

$$\left\|T^2\right\|_{p/2} = \left(\sum_i \left(s_i^2(T)\right)^{p/2}\right)^{2/p} = \left(\sum_j (s_j(T))^p\right)^{2/p} = \|T\|_p^2 < \infty, \tag{5.3}$$

so that $T^2 \in S^{p/2}$.

Let $A \in S^p$. Then $A^*A = |A|^2$ and, by (4.1), $\|A\|_p = \||A|\|_p$. Hence

$$\|A^*A\|_{p/2} = \left\||A|^2\right\|_{p/2} \overset{(5.3)}{=} \||A|\|_p^2 = \|A\|_p^2. \tag{5.4}$$

Replacing A with A^* in (5.4) we have $\|AA^*\|_{p/2} = \|A^*\|_p^2$. Thus

$$\|AA^*\|_{p/2} = \|A^*\|_p^2 \overset{(4.3)}{=} \|A\|_p^2 = \left\||A|^2\right\|_{p/2} = \|A^*A\|_{p/2}, \tag{5.5}$$

for $0 < p < \infty$. Therefore

$$A^*A \in S^{p/2}(H) \iff A \in S^p(H) \iff AA^* \in S^{p/2}(H).$$

Let K be another separable Hilbert space. Then the set $C(H,K)$ of all compact operators from H to K is the closed subspace [30, p.193] of $B(H,K)$. For $0 < p < \infty$, Schatten space $S^p(H,K)$ is defined as follows:

$$S^p(H,K) = \left\{A \in C(H,K) : |A| = (A^*A)^{1/2} \in S^p(H)\right\} \tag{5.6}$$

$$\text{with norm } \|A\|_p = \||A|\|_p = \left(\sum_j s_j^p\right)^{1/p}, \text{ for } A \in S^p(H,K), \tag{5.7}$$

where s_j are eigenvalues of $|A|$, repeated according to multiplicity. Then

$$\|A\|_p = \||A|\|_p \overset{(5.4)}{=} \|A^*A\|_{p/2}^{1/2}. \tag{5.8}$$

We will need the following inequalities (see [16, Lemma XI.9.9(c)], for $p \in (0,2)$, and [21, Section III.7.2] for $p \geq 2$). If $A, B \in S^p(H)$ then $AB \in S^{p/2}(H)$ and

$$\|AB\|_{p/2} \leq 2^{2/p} \|A\|_p \|B\|_p, \text{ if } 0 < p < 2, \tag{5.9}$$

$$\|AB\|_{p/2} \leq \|A\|_p \|B\|_p, \text{ if } p \geq 2. \tag{5.10}$$

This also holds if $A \in S^p(H,K)$ and $B \in S^p(K,H)$.

If $0 < p < 1$ and $A, B \in S^p(H)$ then (see [16, Lemma XI.9.9(b)])

$$\|A+B\|_p^p \leq 2\|A\|_p^p + 2\|B\|_p^p. \tag{5.11}$$

For $1 \leq p$, we have the norm triangle inequality [16, Lemma XI.9.14(d)]

$$\|A+B\|_p \leq \|A\|_p + \|B\|_p. \tag{5.12}$$

Let S be a positive compact operator on H with eigenvalues $\{\lambda_n(S)\}$ repeated according to multiplicity. It follows from the spectral theorem (see Corollary 2.36) that $S = \sum_n \lambda_n(S)(\cdot, e_n) e_n$, where $\{e_n\}$ is an orthonormal basis of H consisting of eigenvectors of S. Then S is a diagonal operator with $\{\lambda_n(S)\}$ on the diagonal. Let g be a real-valued continuous function on $[0,\infty)$. We define (see [30, pp.180-183, 200] [5, p.5]) $g(S)$ to be a diagonal operator with the same orthornormal basis $\{e_n\}$

of H consisting of eigenvectors of S and with eigenvalues $\lambda_n(g(S)) = g(\lambda_n(S))$ on the diagonal.

We need the following results, that are probably known, but we could not find the reference.

Lemma 5.5 *Let S, T be positive compact operators on H. Let f, g be real-valued non-decreasing continuous functions on $[0,\infty)$ and $g((Sx,x)) \le f((Tx,x))$ for all $x \in H$ with $\|x\| = 1$. Then, for each p, $0 < p < \infty$,*

$$f(T) \in S^p \text{ implies } g(S) \in S^p \text{ and } \|g(S)\|_p \le \|f(T)\|_p . \tag{5.13}$$

In particular,

$$0 < S \le T \text{ and } T \in S^p \text{ implies } S \in S^p \text{ and } \|S\|_p \le \|T\|_p . \tag{5.14}$$

Proof. Let all eigenvalues of S be ordered so that $\lambda_j(S) \ge \lambda_{j+1}(S)$, $j = 1, 2, ...$ It follows from the Minimax principle (see [21, Theorem II.1] and [16, Theorem X.4.3]) that

$$\lambda_1(S) = \max_{\|x\|=1}(Sx,x) \text{ and } \lambda_{j+1}(S) = \min_{L\in\mathcal{L}_j}\left(\max_{x\in L^\perp,\|x\|=1}(Sx,x)\right), \text{ for } j \ge 1,$$

where $\mathcal{L}_j$ is the set of all j-dimensional subspaces of H. Then, as above, $g(S)$ is a diagonal operator with the same eigenvectors as S and with eigenvalues $\lambda_j(g(S)) = g(\lambda_j((S))$. The same is true for $f(T)$ and $\mu_j(f(T)) = f(\mu_j(T))$, where $\mu_j(T)$ are ordered eigenvalues of T. Since g and f are non-decreasing and continuous, we have

$$\lambda_1(g(S)) = g(\lambda_1(S)) = g\left(\max_{\|x\|=1}(Sx,x)\right) = \max_{\|x\|=1} g((Sx,x)),$$

$$\mu_1(f(T)) = f(\mu_1(T)) = f\left(\max_{\|x\|=1}(Tx,x)\right) = \max_{\|x\|=1} f((Tx,x)),$$

$$\begin{aligned}
\lambda_{j+1}(g(S)) = g(\lambda_{j+1}(S)) &= g\left(\min_{L\in\mathcal{L}_j}\left(\max_{x\in L^\perp,\|x\|=1}(Sx,x)\right)\right) \\
&= \min_{L\in\mathcal{L}_j}\left(\max_{x\in L^\perp,\|x\|=1} g((Sx,x))\right),\ j\geq 1; \\
\mu_{j+1}(f(T)) = f(\mu_{j+1}(T)) &= f\left(\min_{L\in\mathcal{L}_j}\left(\max_{x\in L^\perp,\|x\|=1}(Tx,x)\right)\right) \\
&= \min_{L\in\mathcal{L}_j}\left(\max_{x\in L^\perp,\|x\|=1} f((Tx,x))\right),\ j\geq 1.
\end{aligned}$$

From this and from the condition of the lemma it follows that $\lambda_j(g(S)) \leq \mu_j(f(T))$ for all j. Since S and T are positive, we have $(S^*S)^{1/2} = S$ and $(T^*T)^{1/2} = T$. Thus $s_j(S) = \lambda_j(S)$ and $s_j(T) = \mu_j(T)$, and condition (5.13) follows from (4.1).

Let $f(t) = g(t) = t$ for $0 \leq t < \infty$. Then $g(S) = S$, $f(T) = T$, and (5.14) follows from (5.13). ■

Lemma 5.6 *Let S, T be positive operators on H and $0 < p < \infty$. Then*

$$0 < S \leq T \text{ and } T \in S^p, \text{ implies } S \in S^p \text{ and } \|S\|_p \leq \|T\|_p. \tag{5.15}$$

Proof. We only need to verify that S is a compact operator and to apply Lemma 5.5. As T is a positive compact operator, we have that $T^{1/2}$ is also a positive compact operator. Indeed, by spectral theorem there is an orthonormal basis $\{e_n\}$ of H consisting of eigenvectors for T such that

$$Tx = \sum_n \lambda_n (x, e_n) e_n, \text{ for each } x \in H,$$

where λ_n is the eigenvalue of T corresponding to the eigenvector e_n and $\lambda_n \to 0$, as $n \to \infty$. As T is positive, all the λ_n are nonnegative. Thus

$$T^{1/2}x = \sum_n \lambda_n^{1/2} (x, e_n) e_n, \text{ for each } x \in H.$$

We have $\lambda_n^{1/2} \to 0$, as $n \to \infty$ and thus the operator $T^{1/2}$ is also a compact operator (see [43, Proposition 1.3.10]). By Theorem 2.26, for all $x \in H$, we have

$$\left\|S^{1/2}x\right\|^2 = (S^{1/2}x, S^{1/2}x) = (Sx, x) \leq (Tx, x) = \left\|T^{1/2}x\right\|^2. \tag{5.16}$$

Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in H. Then there is a subsequence $\left\{T^{1/2}x_{n_k}\right\}_{n=1}^{\infty}$ such that $T^{1/2}x_{n_k} \to x$, as $k \to \infty$, for some $x \in H$. Hence $\left\{T^{1/2}x_{n_k}\right\}_{n=1}^{\infty}$ is a Cauchy sequence. As

$$\left\|S^{1/2}x_{n_k} - S^{1/2}x_{n_l}\right\| = \left\|S^{1/2}\left(x_{n_k} - x_{n_l}\right)\right\| \overset{(5.16)}{\leq} \left\|T^{1/2}\left(x_{n_k} - x_{n_l}\right)\right\| = \left\|T^{1/2}x_{n_k} - T^{1/2}x_{n_l}\right\|,$$

$\left\{S^{1/2}x_{n_k}\right\}_{n=1}^{\infty}$ is also a Cauchy sequence. Hence $S^{1/2}$ is a compact operator. Thus $S = S^{1/2}S^{1/2}$ is also compact. Hence (5.15) follows from (5.14). ■

Definition 5.7 [25, p.3] (i) *A family $\{P_n\}_{n=1}^{N}$, for $N \leq \infty$, of mutually orthogonal projections on H, i.e. $P_iP_j = 0$ if $i \neq j$, is a partition of $\mathbf{1}_H$ if*

$$\sum_{n=1}^{N} P_n = \mathbf{1}_H \text{ for } N < \infty; \text{ and } \sum_{n=1}^{m} P_n \overset{s.o.t.}{\underset{m\to\infty}{\to}} \mathbf{1}_H \text{ for } N = \infty.$$

We denote by $\mathcal{P}_N$ the set of all partitions $P = \{P_n\}_{n=1}^{N}$ of N elements of $\mathbf{1}_H$.

(ii) *For two such partitions $\{P_n\}_{n=1}^{N} \in \mathcal{P}_N, \{Q_m\}_{m=1}^{M} \in \mathcal{P}_M$, $N, M \leq \infty$, and an operator $A \in B(H)$, the set*

$$\mathcal{A} = \{P_nAQ_m\}_{n=1,\ldots,N, m=1,\ldots,M}$$

is called a partition of A.

5.2 The spaces $B(H, H^{\infty})$, $S^p(H, H^{\infty})$ and $l_2(S^p)$

In this section we prove some important norm inequalities that we will use later. We proved in Chapter 2 (see Theorem 2.4) that if X is a Banach space, then the space $l_2(X)$ of sequences $x = (x_1, ..., x_n, ...)$, all $x_n \in X$, with

$$\|x\|_{l_2(X)} = \left(\sum_{n=1}^{\infty} \|x_n\|_p^2\right)^{1/2} < \infty$$

is a Banach space.

Let $H^{\infty} = H \oplus ... \oplus H \oplus ...$ be the infinite orthogonal sum of H, i.e.,

$$H^{\infty} = l_2(H) \tag{5.17}$$

We shall use H^{∞} and $l_2(H)$ and also $\|\cdot\|_{H^{\infty}}$ and $\|\cdot\|_{l_2(H)}$ interchangeably. Thus H^{∞} is a Hilbert space with inner product

$$(x, y)_{H^{\infty}} = \sum_{n=1}^{\infty} (x_n, y_n) \text{ ,for } x, y \in H^{\infty}. \tag{5.18}$$

We omit details of the proof that (5.18) defines inner product on H^{∞}.

Let A be a bounded operator from H into H^{∞}, i.e., $A \in B(H, H^{\infty})$. Then A has form $A = (A_1, ..., A_n, ...)$, where all $A_n \in B(H)$. For $x \in H$, we have $Ax = (A_1x, ..., A_nx, ...)$ and

$$\|A\|_{B(H,H^{\infty})} = \sup_{\|x\|=1} \{\|A_1x, ..., A_nx, ...\|_{H^{\infty}}\} \tag{5.19}$$

$$= \sup_{\|x\|=1} \left\{ \left(\sum_{n=1}^{\infty} \|A_nx\|^2\right)^{1/2} \right\} \leq \left(\sum_{n=1}^{\infty} \|A_n\|^2\right)^{1/2}.$$

Hence each $A = (A_n)_{n=1}^{\infty} \in l_2(B(H))$ also belongs to $B(H, H^{\infty})$, so that

$$\begin{aligned} l_2(B(H)) &\subseteq B(H, H^{\infty}), \\ \|A\|_{B(H,H^{\infty})} &\leq \left(\sum_{n=1}^{\infty} \|A_n\|^2\right)^{1/2} = \|A\|_{l_2(B(H))} \end{aligned} \quad (5.20)$$

and $B(H, H^{\infty})$ is a Banach operator space [38, p.221] with respect to pointwise addition and scalar multiplication and the above norm $\|\cdot\|_{B(H,H^{\infty})}$.

Since H^{∞} is a Hilbert space, we have additional structure - the adjoint operation $A \to A^*$ such that

$$(Ax, y)_{H^{\infty}} = (x, A^*y)_H \text{ for all } x \in H, \ y = (y_n)_{n=1}^{\infty} \in H^{\infty}.$$

Noticing that

$$(Ax, y)_{H^{\infty}} = \sum_{n=1}^{\infty} (A_n x, y_n)_H = \sum_{n=1}^{\infty} (x, A_n^* y_n)_H = \left(x, \sum_{n=1}^{\infty} A_n^* y_n\right)_H$$

we have that if $A = (A_n)_{n=1}^{\infty}$, $y = (y_n)_{n=1}^{\infty} \in H^{\infty}$, then $A^* = (A_n^*)_{n=1}^{\infty}$ and

$$A^*y = \sum_{n=1}^{\infty} A_n^* y_n \in H \text{ where the series converges in w.o.t..} \quad (5.21)$$

For all n, consider the subspaces

$$H_n = \left\{\sum_{n=1}^{\infty} \oplus x_n = (x_1, ..., x_n, ...) \in H^{\infty} : x_n \in H \text{ and } x_k = 0 \text{ if } k \neq n\right\}$$

of H^{∞} isomorphic to H and let Q_n be the projections on H_n, i.e.,

$$Q_n x = (0, ..., 0, x_n, 0, ...) \text{ for all } x = (x_1, ..., x_n, ...) \in l_2(H). \quad (5.22)$$

For all n, let U_n be isometry operators from H_n onto H, such that

$$U_n (0, ..., 0, x_n, 0, ...) = x_n. \tag{5.23}$$

We will identify H_n with H. Consider also the projections

$$P_m = \sum_{n=1}^{m} \oplus Q_n, \text{ i.e., } P_m x = (x_1, ..., x_m, 0, ...) \textit{ for all } m = 1, 2, ... \tag{5.24}$$

Let $A = (A_n)_{n=1}^{\infty} \in l_\infty (B(H))$. Then Q_n and P_m act on $l_\infty (B(H))$ by

$$Q_n A = (0, ..., 0, A_n, 0, ...), \ P_m A = P_m x = (A_1, ..., A_m, 0, ...),$$

so that $Q_n A$, $P_m A \in B(H, H^\infty)$, for all $A \in l_\infty (B(H))$ and $m, n \in N$, and $A_n = U_n Q_n A$. Indeed,

$$\begin{aligned} \|Q_n A\|_{B(H,H^\infty)} &= \sup_{\|x\|=1} \left\{ \left(\|A_n x\|^2 \right)^{1/2} \right\} \\ &= \|A_n\| \leq \sup_k \|A_k\| = \|A\|_{l_\infty(B(H))} \end{aligned}$$

and

$$\begin{aligned} \|P_m A\|^2_{B(H,H^\infty)} &= \sup_{\|x\|=1} \left\{ \sum_{n=1}^{m} \|A_n x\|^2 \right\} \leq \sum_{n=1}^{m} \left(\sup_{\|x\|=1} \|A_n x\|^2 \right) \\ &\leq \sum_{n=1}^{m} \|A_n\|^2_{B(H)} \leq m \|A\|^2_{l_\infty(B(H))} < \infty. \end{aligned} \tag{5.25}$$

We have

$$P_m \overset{\text{s.o.t.}}{\to} \mathbf{1}_{H^\infty} \tag{5.26}$$

since, for each $x \in H^\infty$,

$$\|P_m x - x\|_{H^\infty} = \|(0, ..., 0, x_{m+1}, ...)\|_{H^\infty} = \left(\sum_{n=m+1}^{\infty} \|x_n\|^2 \right)^{1/2} \underset{m \to \infty}{\to} 0.$$

Thus, for each $A \in B(H, H^{\infty})$,

$$P_m A \overset{\text{s.o.t.}}{\to} A, \tag{5.27}$$

since $Ax \in H^{\infty}$ for all $x \in H$. This implies that $P_m A \overset{\text{w.o.t.}}{\to} A$ for each $A \in B(H, H^{\infty})$. Hence $(P_m A)^* (P_m A)$ converge to $A^* A \in B(H)$ in w.o.t.:

$$\begin{aligned} ((P_m A)^* (P_m A)x, y) &= (A^* P_m Ax, y) \\ &= (P_m Ax, Ay) \underset{m\to\infty}{\to} (Ax, Ay) = (A^* Ax, y) \end{aligned}$$

for all $x, y \in H$. Thus we have

$$(P_m A)^* (P_m A) = A^* P_m A = A^* \begin{pmatrix} A_1 \\ \vdots \\ A_m \\ 0 \\ \vdots \end{pmatrix} \overset{(5.21)}{=} \sum_{n=1}^{m} A_n^* A_n \overset{\text{w.o.t.}}{\to} A^* A. \tag{5.28}$$

If $P_m A \in S^p(H, H^{\infty})$, for some $0 < p < \infty$ and some m, then

$$\|P_m A\|_p^2 \overset{(5.54)}{=} \|(P_m A)^* (P_m A)\|_{p/2} \overset{(5.28)}{=} \left\| \sum_{n=1}^{m} A_n^* A_n \right\|_{p/2}. \tag{5.29}$$

We shall now prove some norm inequalities for operators in the space $l_{\infty}(S^p)$. We shall need these results at the end of this chapter when proving inclusions of spaces $S^p(H, H^{\infty})$ and $l_p(S^p)$.

McCarthy and Simon (see [39, Theorem 1.22]) proved that if A and B are positive operators in S^p then

$$\|A\|_p^p + \|B\|_p^p \leq \|A + B\|_p^p \text{ for } 1 \leq p < \infty.$$

Let $\{T_n\}_{n=1}^{\infty}$ be positive operators in S^p. We can prove by mathematical induction that, for each $m < \infty$,

$$\sum_{n=1}^{m} \|T_n\|_p^p \le \left\| \sum_{n=1}^{m} T_n \right\|_p^p \text{ for } 1 \le p < \infty. \tag{5.30}$$

For $0 < p \le 1$, Bhatia and Kittaneh [8, pp.111-112] showed in Lemma 1 (the first inequality) and in Theorem 1 formula (7) (the second inequality) that

$$\left(\|A\|_p + \|B\|_p\right)^p \le \|A+B\|_p^p \le \|A\|_p^p + \|B\|_p^p \text{ for } 0 < p < 1.$$

Similarly to (5.30) we could extend this result to all $m < \infty$:

$$\left(\sum_{n=1}^{m} \|T_n\|_p \right)^p \le \left\| \sum_{n=1}^{m} T_n \right\|_p^p \le \sum_{n=1}^{m} \|T_n\|_p^p \text{ for } 0 < p < 1. \tag{5.31}$$

We can see that if we add the norm triangle inequality to (5.30) and reverse all the inequality signs, then we would obtain from it the inequalities (5.31).

Proposition 5.8 *Let $A = (A_n)_{n=1}^{\infty} \in B(H, H^{\infty})$ and all $A_n \in S^p(H)$. If $1 \le p < 2$ then, for each m,*

$$\left(\sum_{n=1}^{m} \|A_n\|_p^2 \right)^{p/2} \le \|P_m A\|_p^p = \left\| \sum_{n=1}^{m} A_n^* A_n \right\|_{p/2}^{p/2} \le \sum_{n=1}^{m} \|A_n\|_p^p . \tag{5.32}$$

If $2 \le p < \infty$ then, for each m,

$$\sum_{n=1}^{m} \|A_n\|_p^p \le \|P_m A\|_p^p = \left\| \sum_{n=1}^{m} A_n^* A_n \right\|_{p/2}^{p/2} \le \left(\sum_{n=1}^{m} \|A_n\|_p^2 \right)^{p/2} . \tag{5.33}$$

Proof. We have $P_m A = (A_1, ..., A_m, 0, ...)$ and

$$\|P_m A\|_p^p \overset{(5.4)}{=} \|(P_m A)^*(P_m A)\|_{p/2}^{p/2} \overset{(5.28)}{=} \left\| \sum_{n=1}^{m} A_n^* A_n \right\|_{p/2}^{p/2} . \tag{5.34}$$

If $1 \le p < 2$ then $\frac{p}{2} < 1$. Replacing T_n by $A_n^* A_n$ and p by $\frac{p}{2}$ in (5.31),

$$\left(\sum_{n=1}^{m} \|A_n\|_p^2\right)^{p/2} \overset{(5.4)}{=} \left(\sum_{n=1}^{m} \|A_n^* A_n\|_{p/2}\right)^{p/2} \overset{(5.31)}{\le} \left\|\sum_{n=1}^{m} A_n^* A_n\right\|_{p/2}^{p/2}$$

$$\overset{(5.34)}{=} \|P_m A\|_p^p \overset{(5.31)}{\le} \sum_{n=1}^{m} \|A_n^* A_n\|_{p/2}^{p/2} \overset{(5.4)}{=} \sum_{n=1}^{m} \|A_n\|_p^p.$$

Let $2 \le p$. Then $1 \le \frac{p}{2}$ and $S^{p/2}$ is a Banach space. Using the triangle inequality for norms, replacing T_n by $A_n^* A_n$ and p by $\frac{p}{2}$ in (5.31),we obtain

$$\sum_{n=1}^{m} \|A_n\|_p^p \overset{(5.4)}{=} \sum_{n=1}^{m} \|A_n^* A_n\|_{p/2}^{p/2} \overset{(5.30)}{\le} \left\|\sum_{n=1}^{m} A_n^* A_n\right\|_{p/2}^{p/2} \overset{(5.34)}{=} \|P_m A\|_p^p$$

$$\overset{(5.12)}{\le} \left(\sum_{n=1}^{m} \|A_n^* A_n\|_{p/2}\right)^{p/2} \overset{(5.4)}{=} \left(\sum_{n=1}^{m} \|A_n\|_p^2\right)^{p/2}.$$

This completes the proof. ■

As $S^p = S^p(H)$, $p \in [1, \infty]$, is a Banach space, we have that the space $l_2(S^p)$ of sequences $A = (A_n)_{n=1}^{\infty}$, all $A_n \in S^p$, with

$$\|A\|_{l_2(S^p)} = \left(\sum_{n=1}^{\infty} \|A_n\|_p^2\right)^{1/2} < \infty$$

is a Banach space. For convenience, we set

$$\|A\|_{l_2(S^p)} = \infty \text{ if } A \notin l_2(S^p).$$

Let $A = (A_n)_{n=1}^{\infty} \in B(H, H^{\infty})$. As $P_m A \overset{\text{s.o.t.}}{\to} A$ (see (5.27)), we have

$$\left(\sum_{n=1}^{m} A_n^* A_n x, y\right) = ((P_m A)^*(P_m A)x, y) = (A^* P_m A x, y)$$

$$= (P_m A x, A y) \to (Ax, Ay) = (A^* A x, y),$$

for all $x, y \in H$. Therefore

$$A^*A = \sum_{n=1}^{\infty} A_n^* A_n \in B(H), \tag{5.35}$$

where the series converges in the weak operator topology.

Recall (see (5.6), (5.7)) that a compact operator $A = (A_n)_{n=1}^{\infty}$ belongs to $S^p(H, H^\infty)$ if and only if $|A| = (A^*A)^{1/2} \in S^p(H)$. Then

$$\|A\|_p = \||A|\|_p = \left(\sum_j s_j^p\right)^{1/p}$$

where s_j are eigenvalues of $|A|$ in non increasing order.

Since all infinite dimensional separable Hilbert spaces are isometrically isomorphic, similarly to Theorems 4.3 and 4.5, we have:

Proposition 5.9 (i) *Let operators $\{A_n\}$ from $S^p(H, K)$, $p \in [1, \infty)$, converge to $A \in B(H, K)$ in* w.o.t. *If $\sup_n \|A_n\|_p = M < \infty$, then $A \in S^p(H, K)$ and $\|A\|_p \leq M$.*

(ii) *Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of projections in $B(K)$ that converges to $\mathbf{1}_K$ in* s.o.t. *For each $p \in [1, \infty]$ and for each $A \in S^p(H, K)$,*

$$\|A - P_n A\|_p \to 0 \text{ as } n \to \infty. \tag{5.36}$$

Proposition 5.10 *Let $A = (A_n)_{n=1}^{\infty} \in B(H, H^\infty)$.*

(i) *If $A \in S^p(H, H^\infty)$, for some $p \in [1, \infty)$, then*

$$\lim_{m \to \infty} \left\| A^*A - \sum_{n=1}^{m} A_n^* A_n \right\|_{p/2} = 0, \tag{5.37}$$

so that $\sum_{n=1}^{m} A_n^* A_n$ *is a Cauchy sequence with respect to the norm* $\|\cdot\|_{p/2}$.

(ii) *Let* $1 \le p < 2$. *If* $A \in S^p(H, H^\infty)$ *then* $A \in l_2(S^p)$ *and*

$$\|A\|_{l_2(S^p)} \le \|A\|_p = \|A^*A\|_{p/2}^{1/2} = \lim_{m\to\infty} \left\| \sum_{n=1}^{m} A_n^* A_n \right\|_{p/2}^{1/2} \le \left(\sum_{n=1}^{\infty} \|A_n\|_p^p \right)^{1/p}, \quad (5.38)$$

where the last term in (5.38) *could diverge.*

(iii) *Let* $2 \le p < \infty$. *If* $A \in l_2(S^p)$ *then* $A \in S^p(H, H^\infty)$ *and*

$$\left(\sum_{n=1}^{\infty} \|A_n\|_p^p \right)^{1/p} \le \|A\|_p = \|A^*A\|_{p/2}^{1/2} = \lim_{m\to\infty} \left\| \sum_{n=1}^{m} A_n^* A_n \right\|_{p/2}^{1/2} \le \|A\|_{l_2(S^p)}. \quad (5.39)$$

(iv) $A \in l_2(S^2)$ *if and only if* $A \in S^2(H, H^\infty)$. *In this case* $\|A\|_{l_2(S^2)} = \|A\|_2$.

Proof. Let $A \in S^p(H, H^\infty)$. By (5.36), $\|A - P_m A\|_p \to 0$ as $m \to \infty$. Hence

$$\|P_m A\|_p \to \|A\|_p. \quad (5.40)$$

(i) Let $1 \le p < 2$. Then $\frac{p}{2} < 1$ and

$$\left\| A^*A - \sum_{n=1}^{m} A_n^* A_n \right\|_{p/2} \overset{(5.28)}{=} \|A^*A - (P_m A)^*(P_m A)\|_{p/2}$$

$$= \|A^*A - A^*P_m A\|_{p/2} = \|A^*(A - P_m A)\|_{p/2}$$

$$\overset{(5.9)}{\le} 2^{2/p} \|A^*\|_p \|A - P_m A\|_p \underset{m\to\infty}{\longrightarrow} 0.$$

Let $2 \le p$. Then

$$\left\| A^*A - \sum_{n=1}^{m} A_n^* A_n \right\|_{p/2} \overset{(5.28)}{=} \|A^*A - (P_m A)^*(P_m A)\|_{p/2}$$

$$= \|A^*A - A^*P_m A\|_{p/2} = \|A^*(A - P_m A)\|_{p/2}$$

$$\overset{(5.10)}{\le} \|A^*\|_p \|A - P_m A\|_p \underset{m\to\infty}{\longrightarrow} 0.$$

This completes the proof of (i).

(ii) Let $1 \le p < 2$. It follows from Proposition 5.8 and (i) that

$$\|A\|_{l_2(S^p)} = \lim_{m\to\infty} \left(\sum_{n=1}^{m} \|A_n\|_p^2 \right)^{1/2} \le \lim_{m\to\infty} \|P_m A\|_p \overset{(5.40)}{=} \|A\|_p \overset{(5.8)}{=} \|A^*A\|_{p/2}^{1/2}$$

$$\overset{\text{part } (i)}{=} \lim_{m\to\infty} \left\| \sum_{n=1}^{m} A_n^* A_n \right\|_{p/2}^{1/2} \le \lim_{m\to\infty} \left(\sum_{n=1}^{m} \|A_n\|_p^p \right)^{1/p} \le \left(\sum_{n=1}^{\infty} \|A_n\|_p^p \right)^{1/p},$$

where the last term could diverge.

(iii) Let $2 \le p < \infty$. It follow from Proposition 5.8 and part (i) that

$$\left(\sum_{n=1}^{\infty} \|A_n\|_p^p \right)^{1/p} \le \lim_{m\to\infty} \|P_m A\|_p \overset{(5.40)}{=} \|A\|_p \overset{(5.8)}{=} \|A^*A\|_{p/2}^{1/2}$$

$$\overset{\text{part } (i)}{=} \lim_{m\to\infty} \left\| \sum_{n=1}^{m} A_n^* A_n \right\|_{p/2}^{1/2} \overset{(5.12)}{\le} \lim_{m\to\infty} \left(\sum_{n=1}^{m} \|A_n^* A_n\|_{p/2} \right)^{1/2}$$

$$\overset{(5.8)}{=} \lim_{m\to\infty} \left(\sum_{n=1}^{m} \|A\|_p^2 \right)^{1/2} = \|A\|_{l_2(S^p)}.$$

Part (iv) is evident from (iii) in case when $2 = p$. ■

Proposition 5.10 extends the results of Lemma 6 in [25, p.4] to infinite families of operators.

5.3 Action of operators on $l_2(S^p)$

In this section we introduce a subset $\mathcal{B}(l_2(S^p))$ of $B(H^\infty)$. We show connections between norms of operators $R \in \mathcal{B}(l_2(S^p))$, $A \in l_2(S^p) \cup l_p(S^p)$ and $B = RA$.

Recall that $H^\infty = H \oplus ... \oplus H \oplus ...$ is a Hilbert space with respect to scalar product $(x, y)_{H^\infty} = \sum_{n=1}^{\infty} (x_n, y_n)_H$ (see (5.18)). Thus $B(H^\infty)$ is a C^*-algebra. The space

$B(H, H^\infty)$ is a left $B(H^\infty)$-module, that is, if $R \in B(H^\infty)$ and $A \in B(H, H^\infty)$ then $RA \in B(H, H^\infty)$, since

$$\begin{aligned} \|RA\|_{B(H,H^\infty)} &= \sup_{\|x\|=1} \|(RA)\,x\|_{H^\infty} = \sup_{\|x\|=1} \|R\,(Ax)\|_{H^\infty} \\ &\leq \sup_{\|x\|=1} \|R\|_{B(H^\infty)} \|Ax\|_{H^\infty} = \|R\|_{B(H^\infty)} \|A\|_{B(H,H^\infty)} < \infty. \end{aligned}$$

The operator R has block-operator form $R = (R_{ij})$ where all $R_{ij} \in B(H)$. It follows from (5.20) that $l_2(S^p) \subseteq B(H, H^\infty)$. Set

$$\mathcal{B}(l_2(S^p)) = \{R \in B(H^\infty) : RA \in l_2(S^p) \text{ for all } A \in l_2(S^p)\}.$$

The following theorem is the main result of this section. It shows connections between norms of operators: $R \in \mathcal{B}(l_2(S^p))$, $A \in l_2(S^p) \cup l_p(S^p)$ and $B = RA$. It extends the results of Corollary 7 [25, p.5] to infinite sets of operators.

Theorem 5.11 *Let $R \in \mathcal{B}(l_2(S^p))$. Set $\beta = \|R\|_{B(H^\infty)}$. For $A = (A_n)_{n=1}^\infty \in B(H, H^\infty)$, set $B = RA$.*

(i) *Let $p \in [1, 2)$, let $A \in S^p(H, H^\infty)$ and $A \in l_p(S^p)$. Then $B \in l_2(S^p)$ and*

$$\|B\|_{l_2(S^p)} = \left(\sum_{n=1}^\infty \|B_n\|_p^2\right)^{1/2} \leq \beta \left(\sum_{n=1}^\infty \|A_n\|_p^p\right)^{1/p} = \beta \|A\|_{l_p(S^p)}.$$

(ii) *Let $p \in [2, \infty)$ and $A \in l_2(S^p)$. Then $B \in l_p(S^p)$ and*

$$\|B\|_{l_p(S^p)} = \left(\sum_{n=1}^\infty \|B_n\|_p^p\right)^{1/p} \leq \beta \left(\sum_{n=1}^\infty \|A_n\|_p^2\right)^{1/2} = \beta \|A\|_{l_2(S^p)}.$$

(iii) *Let* $p = 2$ *and* $A \in l_2(S^2)$. *Let* R *be invertible in* $\mathcal{B}(l_2(S^p))$ *and* $\alpha = \|R^{-1}\|_{B(H^\infty)}$. *Then*

$$\alpha^{-1}\left(\sum_{n=1}^{\infty}\|A_n\|_2^2\right)^{1/2} \le \left(\sum_{n=1}^{\infty}\|B_n\|_2^2\right)^{1/2} \le \beta\left(\sum_{n=1}^{\infty}\|A_n\|_2^2\right)^{1/2}.$$

Proof. The operator $K = \beta^2\mathbf{1}_{H^\infty} - R^*R \in B(H^\infty)$ is positive, as

$$(Kx, x) = \beta^2\|x\|^2 - (R^*Rx, x) = \|R\|_{B(H^\infty)}^2\|x\|^2 - \|Rx\|^2 \ge 0 \text{ for all } x \in H^\infty.$$

Since $A^* \in B(H^\infty, H)$, the operator A^*KA is positive in $B(H)$, as $(A^*KAy, y) = (KAy, Ay) \ge 0$ for all $y \in H$. Therefore we have $A^*KA = A^*\beta^2\mathbf{1}_{H^\infty}A - A^*R^*RA$. Rearranging it, we obtain

$$\beta^2A^*A = A^*(R^*R + K)A = B^*B + A^*KA \ge B^*B. \tag{5.41}$$

(i) Let $p \in [1, 2)$ and $A \in S^p(H, H^\infty)$. By Proposition 5.10 (ii), $A \in l_2(S^p)$. Hence $B \in l_2(S^p)$. We know from (5.8) that $A \in S^p(H, H^\infty)$ if and only if $A^*A \in S^{p/2}(H)$ and $\|A^*A\|_{p/2} = \|A\|_p^2$.

We also know that β^2A^*A and B^*B are positive operators. Therefore it follows from (5.41) and (5.15) that

$$\|B^*B\|_{p/2} \le \beta^2\|A^*A\|_{p/2}. \tag{5.42}$$

Thus $B^*B \in S^{p/2}(H)$ and therefore $B = RA \in S^p(H, H^\infty)$.

We also have from (5.38) that

$$\|B\|_{l_2(S^p)} = \left(\sum_{n=1}^{\infty} \|B_n\|_p^2\right)^{1/2} \leq \|B\|_p = \|B^*B\|_{p/2}^{1/2},$$

$$\text{and } \|A\|_p = \|A^*A\|_{p/2}^{1/2} \leq \left(\sum_{n=1}^{\infty} \|A_n\|_p^p\right)^{1/p}.$$

Combining this with (5.42) yields

$$\|B\|_{l_2(S^p)} = \left(\sum_{n=1}^{\infty} \|B_n\|_p^2\right)^{1/2} \leq \beta \left(\sum_{n=1}^{\infty} \|A_n\|_p^p\right)^{1/p} = \beta \|A\|_{l_p(S^p)}.$$

(ii) Let $p \in [2, \infty)$ and $A \in l_2(S^p)$. By Proposition 5.10(iii), $A \in S^p(H, H^\infty)$. Then, for the same reasons as in part (i), (5.42) holds and $B = RA \in S^p(H, H^\infty)$. As $A \in l_2(S^p)$ and $R \in B(l_2(S^p))$, we have from definition of $B(l_2(S^p))$ that $B \in l_2(S^p)$.We also have from (5.39) that

$$\left(\sum_{n=1}^{\infty} \|B_n\|_p^p\right)^{1/p} \leq \|B\|_p = \|B^*B\|_{p/2}^{1/2},$$

$$\text{and } \|A\|_p = \|A^*A\|_{p/2}^{1/2} \leq \left(\sum_{n=1}^{\infty} \|A_n\|_p^2\right)^{1/2}.$$

Combining this with (5.42) yields

$$\|B\|_{l_p(S^p)} = \left(\sum_{n=1}^{\infty} \|B_n\|_p^p\right)^{1/p} \leq \beta \left(\sum_{n=1}^{\infty} \|A_n\|_p^2\right)^{1/2} = \beta \|A\|_{l_2(S^p)}.$$

Part (iii) follows from part (ii). Indeed, the second inequality follows immediately from part (ii) by substituting $p = 2$. The proof of the first inequality is as follows. Let $R^{-1} \in B(l_2(S^p))$. Then $A = R^{-1}B \in l_2(S^p)$ and, applying part (ii) to $B \in l_2(S^p)$, i.e., swapping B and A, we have

$$\left(\sum_{n=1}^{\infty} \|A_n\|_2^2\right)^{1/2} \leq \left\|R^{-1}\right\|_{B(H^\infty)} \left(\sum_{n=1}^{\infty} \|B_n\|_2^2\right)^{1/2}$$

Thus

$$\left\|R^{-1}\right\|_{B(H^\infty)}^{-1}\left(\sum_{n=1}^{\infty}\|A_n\|_2^2\right)^{1/2} \leq \left(\sum_{n=1}^{\infty}\|B_n\|_2^2\right)^{1/2}$$

and the proof is complete. ■

5.4 The spaces $l_q\left(S^p\right), l_\infty\left(B\left(H\right)\right)$ and $S^p\left(H,K\right)$

In this section, unless otherwise stated, we assume that $1 \leq p < \infty$. We shall prove several results such as: equivalent definitions of the spaces $S^p\left(H,K\right)$, that $S^p\left(H,K\right)$ are Banach spaces and $S^p\left(H,K\right)$ are left $B\left(K\right)$-module. Towards the end of this section we will develop an approach that enables us to identify which operators from $l_\infty\left(B\left(H\right)\right)$ belong to $B\left(H,H^\infty\right)$ and $S^p\left(H,H^\infty\right)$. We will prove a lemma that studies the inclusion of spaces $l_2\left(B\left(H\right)\right), B\left(H,H^\infty\right)$ and $l_q\left(B\left(H\right)\right)$ for $q \in [1,2)$. In addition, the lemma states that for $q > 2$ and all p, the spaces $l_q\left(S^p\right)$ are not subsets of $B\left(H,H^\infty\right)$. We shall need these results in the subsequent section when proving inclusions of spaces $S^p\left(H,H^\infty\right)$ and $l_p\left(S^p\right)$.

For a Banach space $(X,\|\cdot\|)$ and $n \in \mathbb{N}$, let $X^n = X \oplus ... \oplus X$ be the direct sum of n copies of X. That is, X^n consists of sequences $x = (x_1, ..., x_n)$ with all $x_k \in X$. For $1 \leq q \leq \infty$, denote by $l_q^n(X)$ the space X^n, endowed with the norm

$$\|x\|_{l_q^n(X)} \stackrel{\text{def}}{=} \left(\sum_{i=1}^{n}\|x_i\|^q\right)^{1/q} \quad \text{and} \quad \|x\|_{l_\infty^n(X)} \stackrel{\text{def}}{=} \sup\|x_i\|. \tag{5.43}$$

Similarly, for $n = \infty$, the space $l_q^\infty(X) = l_q(X)$ consists of infinite sequences

$x = (x_n)_{n=1}^{\infty}$, all $x_n \in X$, endowed with the norm.$\|\cdot\|_{l_q(X)}$. That is

$$\|x\|_{l_q(X)} = \left(\sum_{n=1}^{\infty} \|x_n\|^q\right)^{1/q} < \infty, \text{ for } q \in [1, \infty), \tag{5.44}$$
$$\text{and } \|x\|_{l_\infty(X)} = \sup \|x_n\| < \infty, \text{ for } q = \infty.$$

In Theorem 2.4 we proved that all $l_q(X)$ are Banach spaces.

As the function $f(q) = (\sum_{i=1}^{\infty} t_i^q)^{1/q}$ is decreasing [16, Lemma 9 (a)], we have for $x \in l_p(X)$ and $p < q$,

$$\|x\|_{l_\infty(X)} \leq \|x\|_{l_q(X)} \leq \|x\|_{l_p(X)} \text{ and } l_p(X) \subsetneqq l_q(X). \tag{5.45}$$

Definition 5.12 [27, Definition 4.1] *Norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a linear space X are equivalent if and only if there are positive numbers k_1 and k_2 such that*

$$\|x\|_1 \leq k_1 \|x\|_2 \text{ and } \|x\|_2 \leq k_2 \|x\|_1 \text{ for all } x \in X.$$

For $N < \infty$, the norms $\|\cdot\|_{l_q^N(X)}$ are equivalent for all q. Indeed, we know that if $p < q$ then $\|x\|_{l_q^N(X)} \leq \|x\|_{l_p^N(X)}$ and, on the other hand,

$$\|x\|_{l_p^N(X)} \leq N^{1/p} \max_{i=1,\ldots,N} \{\|x_i\|\} \leq N^{1/p} \|x\|_{l_q^N(X)}.$$

For $N = \infty$, and $1 \leq p < q$ the norms $\|\cdot\|_{l_q(X)}$ and $\|\cdot\|_{l_p(X)}$ on the space $l_q(X)$ are not equivalent. Indeed, fix $x \in X$, $\|x\| = 1$. Consider $x^m = (x_n)_{n=1}^{\infty}$, such that $x_1 = ... = x_m = x$ and $x_n = 0$ for $n > m$. Then all $x^m \in l_p(X) \subseteq l_q(X)$, $\|x^m\|_{l_q(X)} = m^{1/q}$ and $\|x^m\|_{l_p(X)} = m^{1/p}$. Clearly, there does not exist a constant $K > 0$ such that

$$\|x^m\|_{l_p(X)} = m^{1/p} \leq K \|x^m\|_{l_q(X)} = K m^{1/q} \text{ for all } m.$$

Let H and K be Hilbert spaces. Let $C(H,K)$ be the Banach spaces of all compact operators from H to K. Recall that the subspace $S^p(H,K)$, $1 \le p < \infty$, of $C(H,K)$ is defined by

$$S^p(H,K) = \left\{ \begin{array}{l} A \in C(H,K):\ |A| \in S^p(H), \\ \|A\|_p = \||A|\|_p = \left(\sum_j s_j^p\right)^{1/p} \end{array} \right\}, \tag{5.46}$$

where s_j are eigenvalues of $|A|$. The above definition is equivalent to the following definition (see Proposition 5.13 below):

$$S^p(H,K) = \left\{ A \in B(H,K):\ UA \in S^p(H) \text{ and } \|A\|_p \overset{\text{def}}{=} \|UA\|_p \right\}. \tag{5.47}$$

where U is an isometry operator from K onto H.

Proposition 5.13 (i) *Definition* (5.47) *does not depend on the choice of the isometry operator* U.

(ii) *Definitions* (5.46) *and* (5.47) *of the space* $S^p(H,K)$ *are equivalent.*

Proof. (i) We know that the adjoint of isometry is its inverse and the inverse is also an isometry (see theorem 5.2). Suppose that V is also an isometry operator from K onto H. Thus UV^{-1} and VU^{-1} are unitary operators on H, as UV^{-1} and VU^{-1} are onto and

$$\left\|UV^{-1}x\right\| = \left\|V^{-1}x\right\| = \|x\| \text{ and } \left\|VU^{-1}x\right\| = \left\|U^{-1}x\right\| = \|x\|.$$

If $VA \in S^p(H)$, then $UA = UV^{-1}(VA) \in S^p(H)$, since $S^p(H)$ is an ideal in $B(H)$. On the other hand, if $UA \in S^p(H)$, then $VA = VU^{-1}(UA) \in S^p(H)$, as $S^p(H)$ is an ideal in $B(H)$. To show that $\|UA\|_p = \|VA\|_p$ use (4.3):

$$\begin{aligned} \|UA\|_p &= \left\|UV^{-1}(VA)\right\|_p \le \left\|UV^{-1}\right\| \|VA\|_p = \|VA\|_p \\ \text{and } \|VA\|_p &= \left\|VU^{-1}(UA)\right\|_p \le \left\|VU^{-1}\right\| \|UA\|_p = \|UA\|_p . \end{aligned}$$

Hence $\|UA\|_p = \|VA\|_p$. Thus $VA \in S^p(H)$ if and only if $UA \in S^p(H)$ and $\|UA\|_p = \|VA\|_p$ and therefore the definition (5.47) does not depend on the choice of the isometry operator U from K onto H.

(ii) Let $S^p_{|\cdot|}(H,K)$ and $S^p_U(H,K)$ denote the set of operators defined in (5.46) and in (5.47) respectively. Suppose that $A \in B(H,K)$. Let us show that $S^p_U(H,K) = S^p_{|\cdot|}(H,K)$ and $\||A|\|_p = \|UA\|_p$,where U is an isometry operator from K onto H. We have that $A \in S^p_U(H,K)$ if and only if

$$A \in B(H,K) \text{ and } \|A\|_p = \|UA\|_p < \infty,$$

if and only if

$$\begin{aligned} A &= U^*(UA) \in C(H,K) \text{ and} \\ \||A|\|_p &= \left\|(A^*A)^{1/2}\right\|_p = \left\|((UA)^*(UA))^{1/2}\right\|_p = \|UA\|_p < \infty, \end{aligned}$$

if and only if $A \in S^p_{|\cdot|}(H,K)$. This ends the proof. ■

Corollary 5.14 *The spaces $S^p(H,K)$ are Banach spaces for all $1 \le p < \infty$.*

Proof. We know that the Schatten ideals $S^p(H)$, $1 \le p < \infty$, are Banach spaces [43]. Let U be an isometry from H onto K. It follows from Proposition 5.13 that

$S^p(H,K)$ and $S^p(H)$ spaces are isometrically isomorphic. Indeed, the operator T that maps every $A \in S^p(H)$ to UA is one-to-one, onto and $\|T(A)\|_p = \|UA\|_p = \|A\|_p$. Hence, as $S^p(H)$ is a Banach space, $S^p(H,K)$ is also a Banach space. ■

Definition 5.15 [11] *Let R be a Banach algebra. We say that a Banach space M is a left R-module if M is endowed with an exterior left multiplication by elements from R that is associative and distributive and*

$$\|ra\|_M \leq \|r\| \, \|a\|_M \text{, for all } r \in R \text{ and } a \in M.$$

Lemma 5.16 *$S^p(H,K)$ is a left $B(K)$-module (multiplication by composition) for all $1 \leq p < \infty$. If $B \in B(K)$ and $A \in S^p(H,K)$ then*

$$\|BA\|_p \leq \|B\| \, \|A\|_p . \tag{5.48}$$

Proof. We have that $S^p(H,K)$ is a Banach space. Let U be an isometry operator from K onto H. If $B \in B(K)$ and $A \in S^p(H,K)$ then $UBU^* \in B(H)$, so that $UBA = UBU^*(UA) \in S^p(H)$, since $S^p(H)$ is an ideal in $B(H)$. Hence $BA \in S^p(H,K)$ and

$$\|BA\|_p = \|UBA\|_p = \|UBU^*(UA)\|_p \overset{(4.3)}{\leq} \|UBU^*\| \, \|UA\|_p = \|B\| \, \|A\|_p .$$

The proof is complete. ■

To prove that $A \in S^p(H,K)$ implies $A^* \in S^p(K,H)$, we need the following lemma and another yet equivalent definition of the spaces $S^p(H,K)$.

Lemma 5.17 *For $A \in S^p(H,K)$, the nonzero eigenvalues of AA^* and of A^*A are the same and the multiplicities are also the same.*

Proof. Let V be an isometry from K onto H. Then $VA \in B(H)$. Let $VA = U_1|VA|$ be the polar decomposition of VA, where U_1 is a partial isometry with initial space $(\ker VA)^{\perp} = (\ker A)^{\perp}$ and final space $\overline{R(VA)}$. Hence,

$$A = V^*U_1|VA| = V^*U_1\left((VA)^* VA\right)^{1/2} = V^*U_1|A|.$$

Set $U = V^*U_1$. Then $A = U|A|$, where U is a partial isometry with initial space $(\ker A)^{\perp}$ and final space $\overline{R(V^*VA)} = \overline{R(A)}$.

Let us assume to begin with that U is unitary, i.e. $U^*U = UU^* = I$. Suppose that λ is an eigenvalue of $|A|^2$, i.e., $|A|^2 x = \lambda x$, for some $x \in H$. Consider $z = Ux$. Then

$$AA^*z = U|A|(U|A|)^* z = U|A|^2 U^*Ux = U|A|^2 x = U\lambda x = \lambda Ux = \lambda z.$$

Thus λ is an eigenvalue of AA^*. In general, when U is a partial isometry, for each $x \in H$, we have $x = x_1 + x_2$, where $x_1 \in \ker(A)$ and $x_2 \in (\ker(A))^{\perp}$. Then

$$\lambda x = |A|^2 x = |A|^2(x_1 + x_2) = A^*A(x_1 + x_2) = A^*Ax_2 = |A|^2 x_2. \tag{5.49}$$

Hence, as U^*U is a projection onto $(\ker(A))^{\perp}$ [14, p.88],

$$AA^*z = U|A|(U|A|)^* z = U|A|^2 U^*Ux = U|A|^2 x_2 \overset{(5.49)}{=} U\lambda x = \lambda Ux = \lambda z.$$

Hence λ is an eigenvalue of AA^*.

Let us assume now that λ is an eigenvalue of AA^* i.e., $AA^*x = \lambda x$ for some $x \in K$. Suppose that U is unitary. Consider the vector $z = U^*x \in H$. Then, since $AA^* = U|A|^2 U^*$, we have that $U^*AA^*U = |A|^2$ and

$$|A|^2 z = U^*AA^*Uz = U^*AA^*UU^*x = U^*AA^*x = U^*\lambda x = \lambda U^*x = \lambda z.$$

Thus λ is an eigenvalue of $|A|^2$. In general, when U is a partial isometry, $U^*x = z_1 + z_2$, where $z_1 \in \ker(A)$ and $z_2 \in (\ker(A))^{\perp}$. Since $AA^*x = U|A|^2 U^*x$,

$$U^*AA^*x = U^*U|A|^2 U^*x = U^*U|A|^2 (z_1 + z_2) = U^*U|A|^2 z_2. \tag{5.50}$$

We have $|A|^2 z_2 = A^*Az_2 \in \overline{R(A^*)}$ and $\overline{R(A^*)} = (\ker(A))^{\perp}$ (see Theorem 2.24). As U^*U is a projection onto $(\ker(A))^{\perp}$ [14, p.88], we obtain that $U^*U|A|^2 z_2 = |A|^2 z_2$. Thus

$$U^*AA^*x \overset{(5.50)}{=} U^*U|A|^2 z_2 = |A|^2 z_2 = |A|^2 z$$

On the other hand,

$$U^*AA^*x = U^*(\lambda x) = \lambda U^*x = \lambda z.$$

Hence $|A|^2 z = \lambda z$. This ends the proof. ■

Consider the following definition of the space $S^p(H, K)$:

$$S^p(H,K) = \left\{ \begin{array}{c} A \in B(H,K)\colon\ AV \in S^p(K) \\ \text{with norm } \|A\|_p \overset{\text{def}}{=} \|AV\|_p \end{array} \right\}, \tag{5.51}$$

where V is an isometry operator from K onto H.

Proposition 5.18 (i) *Definition* (5.51) *does not depend on the choice of the isometry operator* V.

(ii) *Definitions* (5.46) *and* (5.51) *of the space* $S^p(H,K)$ *are equivalent.*

Proof. (i) We omit the proof as it is similar to the proof of Proposition 5.13(i).

(ii) Let $S^p_{|\cdot|}(H,K)$ and $S^p_V(H,K)$ denote the set of operators defined in (5.46) and in (5.51) respectively. Suppose $A \in B(H,K)$. Let us show that $S^p_V(H,K) = S^p_{|\cdot|}(H,K)$ and $\||A|\|_p = \|AV\|_p$, where V is an isometry operator from K onto H. We have that $A \in S^p_V(H,K)$ if and only if

$$A \in B(H,K) \text{ and } \|A\|_{S^p_V} = \|AV\|_p < \infty,$$

if and only if

$$A = (AV)V^* \in C(H,K) \text{ and } \|AV\|_p^2 = \left\|((AV)^*(AV))^{1/2}\right\|_p^2$$

$$\overset{(5.3)}{=} \|(AV)^*(AV)\|_{p/2} \overset{(5.5)}{=} \|(AV)(AV)^*\|_{p/2}$$

$$= \|AVV^*A^*\|_{p/2} = \|AA^*\|_{p/2}$$

$$\overset{\text{Lemma 5.17}}{=} \|A^*A\|_{p/2} \overset{(5.3)}{=} \||A|\|_p^2 < \infty,$$

if and only if $A \in S^p_{|\cdot|}(H,K)$. We also proved that $\|AV\|_p = \||A|\|_p$. This ends the proof. ■

Corollary 5.19 *If* $A \in S^p(H,K)$ *then* $A^* \in S^p(K,H)$ *and*

$$\|A\|_p = \|A^*\|_p. \tag{5.52}$$

Proof. We have

$$\|A\|_p \overset{(5.47)}{=} \|UA\|_p \overset{(4.3)}{=} \|(UA)^*\|_p = \|A^*U^*\|_p \overset{(5.51)}{=} \|A^*\|_p\,,$$

where U is an isometry operator from K onto H. ■

Corollary 5.20 *Let $2 \le p$. If $A, B \in l_2(S^p)$ then $B^*A \in S^{p/2}(H)$ and*

$$\|B^*A\|_{p/2} \le \|A\|_p \|B\|_p \le \|A\|_{l_2(S^p)} \|B\|_{l_2(S^p)}\,. \tag{5.53}$$

Proof. As $A, B \in l_2(S^p)$, it follows from (5.20) that $A, B \in B(H, H^\infty)$. Hence, from Proposition 5.10(iii), we have that $A, B \in S^p(H, H^\infty)$. Hence, by Corollary 5.19, $B^* \in S^p(H^\infty, H)$. It follows from (5.10) that $B^*A \in S^{p/2}(H)$ and $\|B^*A\|_{p/2} \le \|A\|_p \|B\|_p$. Therefore, (5.53) follows from (5.39):

$$\|B^*A\|_{p/2} \overset{(5.10)}{\le} \|A\|_p \|B\|_p \overset{(5.39)}{\le} \|A\|_{l_2(S^p)} \|B\|_{l_2(S^p)}\,.$$

This completes the proof. ■

The following is a generalization of (5.5) for $S^p(H, K)$. Let $A \in S^p(H, K)$ and $1 \le p < \infty$. We have from Lemma 5.17 that

$$\|AA^*\|_{p/2} = \|A^*A\|_{p/2} \overset{(5.3)}{=} \||A|\|_p^2 \overset{(5.46)}{=} \|A\|_p^2 \overset{(5.52)}{=} \|A^*\|_p^2\,. \tag{5.54}$$

Therefore

$$A^*A \in S^{p/2}(H) \iff A \in S^p(H, K) \iff AA^* \in S^{p/2}(K)$$

The following lemma gives conditions when $A \in l_\infty(B(H))$ belongs to $B(H, H^\infty)$ and $S^p(H, H^\infty)$. It also provides some information about inclusion of spaces $l_2(B(H))$,

$B(H, H^\infty)$ and $l_q(B(H))$, $q \in [1,2)$. Additionally, we find that for $q > 2$ and all p, the spaces $l_q(S^p)$ are not subspaces of $B(H, H^\infty)$.According to (5.44), for $X = B(H)$ and for $X = H$, we have

$$l_\infty(B(H)) = \left\{ \begin{array}{c} A = (A_n)_{n=1}^\infty : \text{ all } A_n \in B(H), \\ \|A\|_{l_\infty(B(H))} = \sup_n \|A_n\| < \infty. \end{array} \right\},$$

$$l_\infty(H) = \left\{ \begin{array}{c} x = (x_n)_{n=1}^\infty : \text{ all } x_n \in H \\ \text{and } \|x\|_{l_\infty(H)} = \sup_n \|x_n\| < \infty. \end{array} \right\}.$$

Each $A = (A_n)_{n=1}^\infty \in l_\infty(B(H))$ acts as an operator from H into $l_\infty(H)$:

$$Ax = (A_n x)_{n=1}^\infty \in l_\infty(H), \text{ for each } x \in H,$$

since

$$\|Ax\|_{l_\infty(H)} = \sup_n \|A_n x\| \le \sup_n \|A_n\| \|x\| = \|x\| \sup_n \|A_n\| = \|x\| \|A\|_{l_\infty(B(H))} < \infty.$$

The Hilbert space $H^\infty = l_2(H)$ is a linear subspace of $l_\infty(H) : H^\infty \subseteq l_\infty(H)$,as

$$\|x\|_{l_\infty(H)} = \sup \|x_n\| \le \left(\sum_{n=1}^\infty \|x_n\|_H^2 \right)^{1/2} = \|x\|_{H^\infty}.$$

For all operators $A = (A_n)_{n=1}^\infty \in B(H, H^\infty)$ we have (see (5.22))

$$\|A\|_{l_\infty(B(H))} = \sup_n \|A_n\| = \sup_n \|Q_n A\| \le \sup_n \|Q_n\| \|A\|_{B(H,H^\infty)} = \|A\|_{B(H,H^\infty)} < \infty.$$

Therefore

$$B(H, H^\infty) \subseteq l_\infty(B(H)). \tag{5.55}$$

On the other hand if $A \in l_\infty(B(H))$, $Ax \in H^\infty$ and $\|Ax\|_{H^\infty} \le C\|x\|_H$, for some $C > 0$ and all $x \in H$, then A is a bounded operator from H into H^∞. We shall now consider some necessary and sufficient condition when $A \in l_\infty(B(H))$ belongs to $B(H, H^\infty)$.

Lemma 5.21 *Let $A \in l_\infty(B(H))$. Then*

(i) $A \in B(H, H^\infty)$ *if and only if* $\{P_m A\}$ *converges in* the w.o.t. to an operator from $B(H, H^\infty)$.

(ii) $A \in S^p(H, H^\infty)$, *for some* $p \in [1, \infty)$, *if and only if there is* $M > 0$ *such that* $\|P_m A\|_p \le M$ *for all* m. *Moreover,* $\|A\|_p \le M$.

(iii) $l_q(B(H)) \subset l_2(B(H)) \subset B(H, H^\infty)$ *for* $q \in [1, 2)$, *and*

$$\|A\|^2_{B(H,H^\infty)} \le \sum_{n=1}^{\infty} \|A_n\|^2 = \|A\|^2_{l_2(B(H))} \text{ for } A \in l_2(B(H)). \tag{5.56}$$

(iv) *For* $q > 2$ *and all* $p \in [1, \infty)$, $l_q(S^p)$ is not contained in $B(H, H^\infty)$.

Proof. (i) Clearly, for each m, $P_m A \in B(H, H^\infty)$ (see (5.25)). Let, for each $x \in H$, $\{P_m A x\} \overset{w.o.t.}{\to} z_x \in H^\infty$, i.e., for each $y \in H^\infty$, $(P_m Ax, y)_{H^\infty} \underset{m\to\infty}{\to} (z_x, y)_{H^\infty}$. Let $\varepsilon = 1$. Then, there is N such that, for all $m > N$, $|(P_m Ax, y)_{H^\infty} - (z_x, y)_{H^\infty}| < 1$. Then $|(P_m Ax, y)_{H^\infty}| < |(z_x, y)_{H^\infty}| + 1$. Thus

$$\sup_m |(P_m Ax, y)_{H^\infty}| < \infty \text{ for all } x \in H, y \in H^\infty. \tag{5.57}$$

Recall [38, page 261] that, for each bounded functional f on H^∞, there is a unique $y_f \in H^\infty$ such that $f(x) = (x, y_f)$ for $x \in H^\infty$. Thus for any arbitrary functional f

on H^∞ we have $f(P_m Ax) = (P_m Ax, y_f)_{H^\infty}$. Thus

$$\sup_m |f(P_m Ax)| = \sup_m \left|(P_m Ax, y_f)_{H^\infty}\right| \overset{(5.57)}{<} \infty \text{ for all } x \in H \text{ and } f \in (H^\infty)^*.$$

Applying now the uniform boundedness principle (see [16, Volume 1, Chapter II, §3, Corollary 21]), we obtain that the set $\{P_m A\}_{m=1}^\infty$ is bounded in $B(H, H^\infty)$:

$$C = \sup_m \|P_m A\|_{B(H,H^\infty)} < \infty,$$

for some $C > 0$. Therefore, for all m and each $x \in H$, we have

$$\|P_m Ax\|_{H^\infty} = \left(\sum_{n=1}^m \|A_n x\|^2\right)^{1/2} \le C\|x\|.$$

Hence, for every $x \in H$,

$$\|Ax\|_{H^\infty} = \left(\sum_{n=1}^\infty \|A_n x\|^2\right)^{1/2} = \lim_{m\to\infty} \left(\sum_{n=1}^m \|A_n x\|^2\right)^{1/2} \le C\|x\|.$$

Thus $A \in B(H, H^\infty)$. The part only if follows from (5.27) and the fact that convergence in s.o.t. implies convergence in w.o.t.

(ii) Let $A \in S^p(H, H^\infty)$ for some $p \in [1, \infty)$. For all m, let P_m be the projections given in (5.24). Then $P_m \in B(H^\infty)$. Setting $M = \|A\|_p$, we have

$$\|P_m A\|_p \overset{(5.48)}{\le} \|P_m\| \|A\|_p = \|A\|_p.$$

Conversely, let there exist $M > 0$, such that $\|P_m A\|_p \le M$ for all m. Hence, all $P_m A \in S^p(H, H^\infty)$ and it follows from (5.47) that

$$\|P_m A\|_{B(H,H^\infty)} = \|UP_m A\|_{B(H)} \overset{(4.4)}{\le} \|UP_m A\|_p \overset{(5.47)}{=} \|P_m A\|_p \le M$$

for all m,where U is an isometry operator from H^{∞} onto H. Hence

$$\sup_{m} \|P_m A\|_{B(H,H^{\infty})} \leq M < \infty.$$

Thus $A \in B(H, H^{\infty})$ (see (i)) and $\|A\|_{B(H,H^{\infty})} \leq M$. As $P_m A \overset{s.o.t.}{\to} A$ (see (5.27)),.we have from Theorem 5.9(i) that $A \in S^p(H, H^{\infty})$ and $\|A\|_p \leq M$.

(iii) $l_q(B(H)) \subseteq l_2(B(H))$ for $q \in [1, 2)$ follows from the fact that if $A = (A_n)_{n=1}^{\infty} \in l_q(B(H))$, then

$$\|A\|_{l_2(B(H))} = \left(\sum_{n=1}^{\infty} \|A_n\|^2\right)^{1/2} \overset{(4.4)}{\leq} \left(\sum_{n=1}^{\infty} \|A_n\|^q\right)^{1/q} = \|A\|_{l_q(B(H))}.$$

To prove that $l_q(B(H))$ is a proper subset of $l_2(B(H))$ we consider an example. Let $A = (A_n)_{n=1}^{\infty}$, where $A_n = (n^{-1})^{1/q} \mathbf{1}_H$ for each n. Then $A \in l_2(B(H))$, as

$$\begin{aligned}
\|A\|_{l_2(B(H))} &= \left(\sum_{n=1}^{\infty} \|A_n\|^2\right)^{1/2} \\
&= \left(\sum_{n=1}^{\infty} \left(\left(n^{-1}\right)^{1/q}\right)^2\right)^{1/2} = \left(\sum_{n=1}^{\infty} n^{-2/q}\right)^{1/2} < \infty,
\end{aligned}$$

since $\frac{2}{q} > 1$. However, $A \notin l_q(B(H))$, since

$$\|A\|_{l_q(B(H))} = \left(\sum_{n=1}^{\infty} \|A_n\|^q\right)^{1/q} = \left(\sum_{n=1}^{\infty} n^{-1}\right)^{1/q} \text{ diverges.}$$

We have $l_2(B(H)) \subseteq B(H, H^{\infty})$ (see (5.20)). To prove that $l_2(B(H))$ is a proper subset of $B(H, H^{\infty})$ we consider the following example. Let $(e_n)_{n=1}^{\infty}$ be an orthonormal basis in H. For $x \in H, x = \sum_{n=1}^{\infty} (x, e_n) e_n$ and $\|x\|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2$. Let

$A = (A_n)_{n=1}^{\infty}$, where $A_n x = \left(\frac{1}{n}\right)^{1/2}(x, e_n)\, e_n$. Then

$$\begin{aligned} \|A\|_{B(H,H^\infty)} &= \sup_{\|x\|=1} \left(\sum_{n=1}^{\infty} \|A_n x\|^2 \right)^{1/2} = \sup_{\|x\|=1} \left(\sum_{n=1}^{\infty} \frac{1}{n} |(x, e_n)|^2 \right)^{1/2} \\ &\leq \sup_{\|x\|=1} \left(\sum_{n=1}^{\infty} |(x, e_n)|^2 \right)^{1/2} = 1. \end{aligned}$$

Hence $A \in B(H, H^\infty)$. However, $A \notin l_2(B(H))$, since

$$\|A_n\| = \sup_{\|x\|=1} \|A_n x\| = \sup_{\|x\|=1} \left\| |(x, e_n)| / n^{1/2} \right\| = n^{-1/2} \text{ and}$$

$$\|A\|_{l_2(B(H))}^2 = \sum_{n=1}^{\infty} \|A_n\|^2 = \sum_{n=1}^{\infty} n^{-1} \text{ - diverges.}$$

The estimate (5.56) follows from the following reasoning

$$\begin{aligned} \|A\|_{B(H,H^\infty)}^2 &= \sup_{\|x\|=1} \left(\sum_{n=1}^{\infty} \|A_n x\|^2 \right) \leq \sum_{n=1}^{\infty} \left(\sup_{\|x\|=1} \|A_n x\|^2 \right) \\ &= \sum_{n=1}^{\infty} \|A_n\|^2 = \|A\|_{l_2(B(H))}^2, \text{ if } A = (A_n)_{n=1}^{\infty} \in l_2(B(H)). \end{aligned}$$

(iv) Let $q > 2$ and $p \in [1, \infty)$. We are going to construct an operator A such that $A \in l_q(S^p)$ and $A \notin B(H, H^\infty)$. Let $\alpha = \frac{2q}{2+q}$. Then $\alpha > 1$, as $q > 2$. For some $0 \neq T \in S^p$, let $A = (A_n)_{n=1}^{\infty}$, where $A_n = n^{-\frac{\alpha}{q}} T$ for each n. Then

$$\begin{aligned} \|A\|_{l_q(S^p)} = \left(\sum_{n=1}^{\infty} \|A_n\|_p^q \right)^{1/q} &= \left(\sum_{n=1}^{\infty} \left\| n^{-\frac{\alpha}{q}} T \right\|_p^q \right)^{1/q} = \|T\|_p \left(\sum_{n=1}^{\infty} \left(n^{-\frac{\alpha}{q}} \right)^q \right)^{1/q} \\ = \|T\|_p \left(\sum_{n=1}^{\infty} n^{-\alpha} \right)^{1/q} &< \infty, \text{ as } \sum_{n=1}^{\infty} n^{-\alpha} \text{ converges.} \end{aligned}$$

Thus $A \in l_q(S^p)$. On the other hand, as $\frac{2\alpha}{q} = \frac{4}{2+q} < 1$, we have

$$\begin{aligned} \|A\|_{B(H,H^\infty)}^2 &= \sup_{\|x\|=1} \|Ax\|_{H^\infty}^2 = \sup_{\|x\|=1} \sum_{n=1}^{\infty} \|A_n x\|^2 = \sup_{\|x\|=1} \sum_{n=1}^{\infty} \left\| n^{-\frac{\alpha}{q}} T x \right\|^2 \\ &= \sup_{\|x\|=1} \left(\|Tx\|_H^2 \right) \sum_{n=1}^{\infty} n^{-\frac{2\alpha}{q}} = \|T\|^2 \sum_{n=1}^{\infty} n^{-\frac{2\alpha}{q}} \text{ - diverges,} \end{aligned}$$

as $\sum_{n=1}^{\infty} n^{-\frac{2\alpha}{q}}$ diverges. Hence $A \notin B(H, H^{\infty})$. Thus, for $q > 2$ and $p \in [1, \infty)$, none of the spaces $l_q(S^p)$ is contained in $B(H, H^{\infty})$. ∎

5.5 Inclusions of spaces $S^p(H, H^{\infty})$ and $l_p(S^p)$

In this section we prove the main result of this chapter - inclusions that hold for the spaces $S^p(H, H^{\infty})$ and $l_p(S^p)$. Let $C(H, H^{\infty})$ be the subspace of all compact operators in $B(H, H^{\infty})$. Recall that, for $p, q \in [1, \infty)$, the Banach space $l_q(S^p)$ consists of all sequences $A = (A_n)_{n=1}^{\infty}$, such that all $A_n \in S^p$ and

$$\|A\|_{l_q(S^p)} = \left(\sum_{n=1}^{\infty} \|A_n\|_p^q\right)^{1/q} < \infty. \tag{5.58}$$

Similar to the rank one operator on H in (4.20), we define a rank one operator in $B(H, K)$. For $x \in H$ and $u \in K$, denote by $x \otimes u$ the rank one operator in $B(H, K)$ that acts by

$$(x \otimes u) z = (z, x) u \text{ for each } z \in H. \tag{5.59}$$

All finite dimensional operators on H belong to $S^p(H)$ for all $p \in [1, \infty)$ (see [21, Calkin Theorem]). Therefore $x \otimes u \in S^p(H, K)$ and

$$\begin{aligned} \|x \otimes u\|_{S^p(H,K)} &= \|U(x \otimes u)\|_{S^p(H)} \\ &= \|x \otimes Uu\|_{S^p(H)} \overset{(4.25)}{=} \|x\| \|Uu\| = \|x\| \|u\| \end{aligned} \tag{5.60}$$

where U is an isometry from K onto H.

Theorem 5.22 (i) *Let* $1 \leq p < 2$. *Then*

$$l_p(S^p) \subsetneqq S^p(H, H^{\infty}) \subsetneqq l_2(S^p) \subsetneqq l_2(C(H)) \subsetneqq C(H, H^{\infty}).$$

For $A \in S^p(H, H^\infty)$,

$$\|A\|_{l_2(S^p)} \le \|A\|_p \le \|A\|_{l_p(S^p)} \ \textit{where} \ \|A\|_{l_p(S^p)} = \infty \ \textit{if} \ A \notin l_p(S^p). \tag{5.61}$$

For $q \in (p, 2)$, *the space* $l_q(S^p)$ *neither contains, nor is contained in* $S^p(H, H^\infty)$.

(ii) *Let* $p \in (2, \infty)$. *Then*

$$l_2(S^p) \subsetneqq S^p(H, H^\infty) \subsetneqq l_p(S^p) \nsubseteq B(H, H^\infty).$$

For $A \in S^p(H, H^\infty)$,

$$\|A\|_{l_p(S^p)} \le \|A\|_p \le \|A\|_{l_2(S^p)}, \ \textit{where} \ \|A\|_{l_2(S^p)} = \infty \ \textit{if} \ A \notin l_2(S^p). \tag{5.62}$$

Moreover, $S^p(H, H^\infty)$ *is not contained in* $l_q(S^p)$, *for any* $q \in [2, p)$ *and* $l_q(S^p)$ *is not contained in* $B(H, H^\infty)$, *for all* $q > 2$ *and* $p \in [1, \infty)$.

(iii) $l_2(S^2) = S^2(H, H^\infty)$ *and* $\|A\|_{l_2(S^2)} = \|A\|_2$ *for each* $A \in S^2(H, H^\infty)$.

Proof. (i) Let $1 \le p < 2$. We begin by showing that $l_2(C(H)) \subseteq C(H, H^\infty)$. Let $A = (A_n)_{n=1}^\infty \in l_2(C(H))$. Then all $A_n \in C(H)$ and $P_m A \in l_2(C(H))$ for all m. By Lemma 5.21(iii), $A \in B(H, H^\infty)$. Since $P_m A = (A_1, ..., A_m, 0, ...)$ is a sum of finite number of compact operators, it is compact, i.e. $P_m A \in C(H, H^\infty)$ for all m (see [30, p.193.]). We also have

$$\|A - P_m A\|_{B(H,H^\infty)} \overset{(5.56)}{\le} \|A - P_m A\|_{l_2(B(H))} \overset{(4.5)}{=} \|A - P_m A\|_{l_2(C(H))} \underset{m\to\infty}{\to} 0.$$

Since the set of all compact operators $C(H, H^\infty)$ is closed (see [42, Theorem 8.3]), we have that $A \in C(H, H^\infty)$. Thus $l_2(C(H)) \subseteq C(H, H^\infty)$.

We shall now prove the inclusions

$$l_p(S^p) \subseteq S^p(H, H^\infty) \subseteq l_2(S^p) \subseteq l_2(C(H)).$$

The inclusion $l_2(S^p) \subseteq l_2(C(H))$ is obvious.

Suppose that $A \in l_p(S^p)$. Since $A \in l_p(S^p) \subseteq l_\infty(S^p)$, we have

$$\|P_m A\|_p \overset{(5.32)}{\leq} \left(\sum_{n=1}^{m} \|A_n\|_p^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty} \|A_n\|_p^p\right)^{1/p} = \|A\|_{l_p(S^p)},$$

for all m. By Lemma 5.21(ii), $A \in S^p(H, H^\infty)$. Thus $l_p(S^p) \subseteq S^p(H, H^\infty)$.

Suppose that $A \in S^p(H, H^\infty)$. For all n, let Q_n be the projection given in (5.24) and U_n be the isometry given in (5.23). Then, for all n, $A_n = U_n Q_n A$. Hence $U_n^{-1} A_n = Q_n A$ and we obtain that for all n

$$\begin{aligned} \|A_n\|_p &= \left\|U_n U_n^{-1} A_n\right\|_p \overset{(5.47)}{=} \left\|U_n^{-1} A_n\right\|_p \\ &= \|Q_n A\|_p \overset{(5.48)}{\leq} \|Q_n\| \, \|A\|_p = \|A\|_p. \end{aligned}$$

Thus $\sup_n \|A_n\|_p < \infty$ and therefore $A \in l_\infty(S^p)$. Applying (5.32), we have

$$\begin{aligned} \|A\|_{l_2(S^p)}^p &\overset{(5.58)}{=} \left(\sum_{n=1}^{\infty} \|A_n\|_p^2\right)^{p/2} = \lim_{m\to\infty} \left(\sum_{n=1}^{m} \|A_n\|_p^2\right)^{p/2} \\ &\overset{(5.32)}{\leq} \lim_{m\to\infty} \|P_m A\|_p^p \overset{(5.36)}{=} \|A\|_p^p. \end{aligned}$$

Hence $A \in l_2(S^p)$, so that $S^p(H, H^\infty) \subseteq l_2(S^p)$. Additionally,

$$\|A\|_p^p \overset{(5.36)}{=} \lim_{m\to\infty} \|P_m A\|_p^p \overset{(5.32)}{\leq} \lim_{m\to\infty} \sum_{n=1}^{m} \|A_n\|_p^p = \|A\|_{l_p(S^p)}^p.$$

Thus $\|A\|_p \leq \|A\|_{l_p(S^p)}$. This ends the proof of (5.61).

To complete the proof of (i), we shall construct examples of operators that will show proper inclusions, i.e.,

$$l_p(S^p) \neq S^p(H, H^\infty) \neq l_2(S^p) \neq l_2(C(H)) \neq C(H, H^\infty).$$

We begin with $l_2(S^p) \neq l_2(C(H))$. Let $A_1 \in C(H)$ and $A_1 \notin S^p$. Then

$$A = \begin{pmatrix} A_1 \\ \\ 0 \\ \vdots \end{pmatrix}, \in l_2(C(H)) \text{ but } A \notin l_2(S^p).$$

Before proving the other proper inclusions, we construct some operators. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis in H and P_{e_n} be projections on $\mathbb{C}e_n = \{\lambda e_n : \lambda \in \mathbb{C}\}$. Then $\|P_{e_n}\|_p = 1$ for all n and $p \in [1, \infty)$. Let $\{\lambda_n\}_{n=1}^\infty$, $\{\mu_n\}_{n=1}^\infty$ be nonincreasing sequences in $(0, 1]$ and

$$A = (A_n)_{n=1}^\infty, \qquad B = (B_n)_{n=1}^\infty \text{ with } A_n = \lambda_n P_{e_n}, \quad B_n = \mu_n P_{e_1}. \tag{5.63}$$

Then $\|A_n\|_p = \lambda_n$, $\|B_n\|_p = \mu_n$. Hence A and B belong to $l_\infty(S^p)$, since $\sup_n \|A_n\|_p = \sup_n \lambda_n \leq 1$ and $\sup_n \|B_n\|_p = \sup_n \mu_n \leq 1$. Let $x = \sum_{n=1}^\infty \alpha_n e_n \in H$. Then $\sum_{n=1}^\infty |\alpha_n|^2 = \|x\|^2$ and $Ax = \sum_{n=1}^\infty \oplus \lambda_n \alpha_n e_n$, where $\lambda_n \alpha_n e_n$ belongs to the n-th component of H^∞. As

$$\|Ax\|_{H^\infty} = \left\| \sum_{n=1}^\infty \oplus \lambda_n \alpha_n e_n \right\|_{H^\infty} \leq \left(\sum_{n=1}^\infty |\alpha_n|^2 \right)^{1/2} = \|x\| < \infty,$$

we have $Ax \in H^\infty$, so that $A \in B(H, H^\infty)$ and $.\|A\| \leq 1$. Then

$$|A|^2 = A^*A \overset{(5.35)}{=} \sum_{n=1}^\infty A_n^* A_n = \sum_{n=1}^\infty \lambda_n^2 P_{e_n}$$

is a diagonal operator (see Example 2.19) with diagonal sequence $\{\lambda_n^2\}$ and Theorem 2.26 implies that $|A| = \sum_{n=1}^{\infty} \lambda_n P_{e_n}$. Hence, by (4.2)

$$\|A\|_p^p = \sum_{n=1}^{\infty} \lambda_n^p \text{ for } p \in [1, \infty). \tag{5.64}$$

We also have that $Bx = \alpha_1 \sum_{n=1}^{\infty} \oplus \mu_n e_1$, where $\mu_n e_1$ belongs to the n-th component of H^∞, and

$$\|Bx\|_{H^\infty}^2 = \alpha_1^2 \sum_{n=1}^{\infty} \mu_n^2 \leq \|x\|^2 \sum_{n=1}^{\infty} \mu_n^2.$$

Hence B is bounded if and only if $\sum_{n=1}^{\infty} \mu_n^2 < \infty$. Setting $x = e_1$, we have

$$\|B\|_{B(H,H^\infty)}^2 = \sum_{n=1}^{\infty} \mu_n^2. \tag{5.65}$$

Moreover, if B is bounded then $B = e_1 \oplus u$, where $u = \sum_{n=1}^{\infty} \oplus \mu_n e_1 \in H^\infty$, is a rank one operator in $B(H, H^\infty)$. Indeed, $(x, e_1) = \alpha_1$, so that

$$(e_1 \oplus u)\, x = (x, e_1)\, u = \alpha_1 \sum_{n=1}^{\infty} \oplus \mu_n e_1 = Bx.$$

Thus, as every rank one operator, $B \in S^p(H, H^\infty)$ for all $p \in [1, \infty)$.

We shall now prove that $S^p(H, H^\infty) \neq l_2(S^p)$. Let $1 \leq p < q \leq 2$. Set in (5.63) $\lambda_n = n^{-1/p}$. Then

$$\|A\|_{l_q(S^p)}^q = \sum_{n=1}^{\infty} \|A_n\|_p^q = \sum_{n=1}^{\infty} \lambda_n^q = \sum_{n=1}^{\infty} n^{-q/p} < \infty.$$

On the other hand, by (5.64), $\|A\|_p^p = \sum_{n=1}^{\infty} \lambda_n^p = \sum_{n=1}^{\infty} n^{-1}$ - diverges. Hence $A \in l_q(S^p)$ and $A \notin S^p(H, H^\infty)$. Thus $l_q(S^p)$ is not contained in $S^p(H, H^\infty)$. In particular, when $q = 2$ we have $S^p(H, H^\infty) \neq l_2(S^p)$. This ends the proof that $S^p(H, H^\infty) \subsetneqq l_2(S^p)$.

Now we shall prove that $l_p(S^p) \neq S^p(H, H^\infty)$. Let $1 \leq p \leq q < 2$. Set in (5.63) $\mu_n = n^{-1/q}$. Then

$$\|B\|_{l_q(S^p)}^q = \sum_{n=1}^{\infty} \|B_n\|_p^q = \sum_{n=1}^{\infty} \mu_n^q = \sum_{n=1}^{\infty} n^{-1} \text{ - diverges,}$$

and thus $B \notin l_q(S^p)$. On the other hand, $\sum_{n=1}^{\infty} \mu_n^2 = \sum_{n=1}^{\infty} n^{-2/q} < \infty$, as $2/q > 1$. Hence, as above, $B \in S^p(H, H^\infty)$ for all $p \in [1, \infty)$. Thus the space $l_q(S^p)$ does not contain $S^p(H, H^\infty)$. In case when $p = q$ we have that $l_p(S^p) \neq S^p(H, H^\infty)$ and therefore $l_p(S^p) \subsetneqq S^p(H, H^\infty)$.

Finally, to prove that $l_2(C(H)) \neq C(H, H^\infty)$, set $\lambda_n = n^{-1/2}$ in (5.63). Then $A_n = n^{-1/2} P_{e_n}$ and $A \notin l_2(C(H))$, since

$$\|A\|_{l_2(C(H))}^2 = \sum_{n=1}^{\infty} \|A_n\|^2 = \sum_{n=1}^{\infty} \lambda_n^2 = \sum_{n=1}^{\infty} n^{-1} - \text{diverges.}$$

On the other hand, $|A|^2 = A^*A = \sum_{n=1}^{\infty} A_n^* A_n = \sum_{n=1}^{\infty} n^{-1} P_{e_n}$ and, by Theorem 2.26, $|A| = \sum_{n=1}^{\infty} n^{-1/2} P_{e_n}$. Thus $|A|$ is a compact operator. Indeed, for each m, the operator $T_m = \sum_{n=1}^{m} n^{-1/2} P_{e_n}$ is compact, as it is a finite sum of rank one operators, and the operators T_m converge uniformly to $|A|$:

$$\||A| - T_m\| = \left\| \sum_{n=m+1}^{\infty} n^{-1/2} P_{e_n} \right\| = \sup_{n>m} \left\{ n^{-1/2} \right\} = (m+1)^{-1/2} \underset{m\to\infty}{\to} 0.$$

Thus, by Theorem 2.33, $|A| \in C(H)$. Hence $A = U|A| \in C(H, H^\infty)$. Since $A \notin l_2(C(H))$ and $A \in C(H, H^\infty)$, we have $l_2(C(H)) \subsetneqq C(H, H^\infty)$.

(ii) Let $p > 2$. We shall begin by proving the inclusion $l_2(S^p) \subseteq S^p(H, H^\infty)$ and the RHS inequality in (5.62). Let $A \in l_2(S^p)$. As $l_2(S^p) \subseteq l_2(B(H)) \subseteq B(H, H^\infty)$ (see (5.20)), it follows from Proposition 5.8 that, for all m,

$$\|P_m A\|_p \le \left(\sum_{n=1}^{m} \|A_n\|_p^2\right)^{1/2} \le \|A\|_{l_2(S^p)}.$$

As $A \in l_2(S^p) \subseteq l_\infty(S^p) \subseteq l_\infty(B(H))$, we can apply Lemma 5.21(ii) to obtain that $A \in S^p(H, H^\infty)$ and $\|A\|_p \le \|A\|_{l_2(S^p)}$. Thus $l_2(S^p) \subseteq S^p(H, H^\infty)$ and the RHS inequality in (5.62) holds for all $A \in l_2(S^p)$.

We shall now prove the inclusion $S^p(H, H^\infty) \subseteq l_p(S^p)$ and LHS in (5.62). Let $A = \{A_n\}_{n=1}^\infty \in S^p(H, H^\infty)$. By (5.36), $\|P_m A\|_p \underset{m\to\infty}{\to} \|A\|_p$. Applying now Proposition 5.8 we obtain that, for each m,

$$\sum_{n=1}^{m} \|A_n\|_p^p \le \|P_m A\|_p^p \overset{(5.36)}{\to} \|A\|_p^p, \text{ as } m \to \infty.$$

Hence, $\left(\sum_{n=1}^\infty \|A_n\|_p^p\right)^{1/p} < \infty$, i.e., $A \in l_p(S^p)$. Thus $S^p(H, H^\infty) \subseteq l_p(S^p)$ and LHS in (5.62) holds.

To complete the proof of (ii), consider some examples that show that

$$S^p(H, H^\infty) \nsubseteq l_q(S^p) \text{ for } 2 \le q < p,$$

$$l_q(S^p) \nsubseteq B(H, H^\infty), \text{ for } q > 2 \text{ and } p \in [1, \infty),$$

$$l_2(S^p) \ne S^p(H, H^\infty) \ne l_p(S^p), \text{ for } p > 2. \tag{5.66}$$

Let $2 \le q < p$. Set $\lambda_n = n^{-1/q}$ in (5.63). Then $A \notin l_q(S^p)$, since

$$\|A\|_{l_q(S^p)}^q = \sum_{n=1}^\infty \|A_n\|_p^q = \sum_{n=1}^\infty \lambda_n^q = \sum_{n=1}^\infty n^{-1} \text{ diverges.}$$

On the other hand, $A \in S^p(H, H^\infty)$ since, applying (5.64), we have $\|A\|_p^p = \sum_{n=1}^\infty \lambda_n^p = \sum_{n=1}^\infty n^{-p/q} < \infty$. Thus, (5.66) holds. In particular, when $q = 2$, we have $S^p(H, H^\infty) \neq l_2(S^p)$.

Set now $\mu_n = n^{-1/2}$ in (5.63). Then, for $2 < q$ and $p \in [1, \infty)$,

$$\|B\|_{l_q(S^p)}^q = \sum_{n=1}^\infty \|\mu_n P_{e_1}\|_p^q = \sum_{n=1}^\infty \mu_n^q = \sum_{n=1}^\infty n^{-q/2} < \infty.$$

Thus $B \in l_q(S^p)$. On the other hand, since

$$\|B\|_{B(H,H^\infty)} \overset{(5.65)}{=} \left(\sum_{n=1}^\infty \mu_n^2\right)^{1/2} = \left(\sum_{n=1}^\infty n^{-1}\right)^{1/2} \text{ diverges,}$$

B is not bounded, i.e., $B \notin B(H, H^\infty)$. Hence $l_q(S^p) \nsubseteq B(H, H^\infty)$, for $2 < q$ and $p \in [1, \infty)$. In particular, if $p = q > 2$, we have $S^p(H, H^\infty) \neq l_p(S^p)$.

(iii) Repeating the proof of (ii) for $p = 2$, we obtain that

$$l_2\left(S^2\right) \subseteq S^2(H, H^\infty) \subseteq l_2\left(S^2\right),$$

$$\|A\|_{l_2(S^2)}^2 \leq \|A\|_2^2 \leq \|A\|_{l_2(S^2)}^2, \text{ for } A \in S^2(H, H^\infty).$$

Thus $l_2(S^2) = S^2(H, H^\infty)$ and $\|A\|_{l_2(S^2)} = \|A\|_2$ for each $A \in S^2(H, H^\infty)$. The proof is complete. ■

5.6 Conclusion

The main results in this chapter are Propositions 5.8 and 5.10, Lemma 5.21 and Theorems 5.11 and 5.22. In Proposition 5.8 we prove some norm estimates for an operator from $B(H, H^{\infty})$ with all its components from $S^p(H)$. This is an auxiliary result that we use in the proof of Theorem 5.22.

In Proposition 5.10 we prove some norm inequalities for operators from the spaces $S^p(H.H^{\infty})$ and $l_2(S^p)$. In fact, this proposition and Theorem 5.11 extend, respectively, the results of Lemma 6 and Corollary 7 of [25, p.4] to infinite families of operators.

In Lemma 5.21 we find and prove necessary and sufficient condition when an operator A from $l_{\infty}(B(H))$ belongs to $B(H, H^{\infty})$ and to $S^p(H.H^{\infty})$. We also prove inclusions that hold for spaces $l_{\infty}(B(H))$, $B(H, H^{\infty})$ and $l_q(B(H))$. In addition, we find that for $q > 2$ and all p, the spaces $l_q(S^p)$ do not lie in $B(H, H^{\infty})$.

In Theorem 5.22 we prove some inclusions that hold for spaces $S^p(H.H^{\infty})$ and $l_p(S^p)$. We will use the results of this chapter in the subsequent chapter.

Chapter 6 is dedicated to generalized Clarkson-McCarthy inequalities, convexity of spaces $l_p(S^p)$, partitions of operators from S^p, Cartesian decomposition.

Chapter 6 Analogues of Clarkson-McCarthy inequalities. Partitioned operators and Cartesian decomposition.

This chapter is mainly devoted to generalized Clarkson-McCarthy inequalities for vector l_q-spaces $l_q(S^p)$ of operators from Schatten ideals S^p. We show that all Clarkson-McCarthy type inequalities are, in fact, some estimates on the norms of operators acting on the spaces $l_q(S^p)$ or from one such space into another. The first section is dedicated to known analogues of McCarthy inequalities. In the second section we analyse actions of operators from $B(H^\infty)$ on $l_q(S^p)$ spaces. We obtain a further generalization of McCarthy estimates in section 6.3. In the fourth section we study the convexity of spaces $l_p(S^p)$. In the fifth section we study partitioned operators from S^p and the sixth section is about Cartesian decomposition and Schatten norms. Finally, in the last section we summarize the results in this chapter.

6.1 Background on analogues of McCarthy inequalities

Clarkson [12, Theorem 2] proved the following estimates for spaces L_p and l_p. If $p \geq 2, q = p/(p-1)$ and $x, y \in L_p$, or $x, y \in l_p$, then

$$2^{1/p}\left(\|x\|^p + \|y\|^p\right)^{1/p} \leq \left(\|x+y\|^p + \|x-y\|^p\right)^{1/p} \leq 2^{1-1/p}\left(\|x\|^p + \|y\|^p\right)^{1/p};$$

$$2^{1/q}\left(\|x\|^p+\|y\|^p\right)^{1/p}\le\left(\|x+y\|^q+\|x-y\|^q\right)^{1/q};$$

$$\|x+y\|^p+\|x-y\|^p\le 2\left(\|x\|^q+\|y\|^q\right)^{p-1}.$$

For $1<p\le 2$ these inequalities hold in reversed order.

The algebras S^p are non-commutative, i.e., $T_1T_2\ne T_2T_1$ in general for $T_1,T_2\in S^p$. McCarthy [28, Theorem 2.7] stated that the non-commutativity of S^p spaces complicates the proofs of estimates for these spaces. However, he obtained the following non-commutative analogues of Clarkson estimates.

For $A,B\in S^p$, $2\le p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}2^{1/p}\left(\|A\|_p^p+\|B\|_p^p\right)^{1/p}&\le\left(\|A+B\|_p^p+\|A-B\|_p^p\right)^{1/p}\\&\le 2^{1-1/p}\left(\|A\|_p^p+\|B\|_p^p\right)^{1/p};\end{aligned}\tag{6.1}$$

$$2^{1/q}\left(\|A\|_p^p+\|B\|_p^p\right)^{1/p}\le\left(\|A+B\|_p^q+\|A-B\|_p^q\right)^{1/q}.\tag{6.2}$$

For $1<p\le 2$, inequalities in (6.1) and (6.2) are reversed.

We will consider now these and some other inequalities from the perspective of $l_p\left(S^p\right)$-spaces of operators from Schatten ideals.

Let H^n be the orthogonal sum of n copies of H. Each $R\in B\left(H^n\right)$ has the block-matrix form $R=\left(R_{ij}\right)$, $1\le i,j\le n$, with all $R_{ij}\in B\left(H\right)$. It generates a bounded operator (we also call it R) on each space $l_q^n\left(S^p\right)$ that acts in the following way

$$RA = (R_{ij}) \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n R_{1i} A_i \\ \vdots \\ \sum_{i=1}^n R_{ni} A_i \end{pmatrix} \in l_q^n (S^p), \tag{6.3}$$

for $A = (A_i)_{i=1}^n \in l_q^n (S^p)$. Clearly, $RA \in l_q^n (S^p)$, since

$$\begin{aligned} \|RA\|_{l_q^n(S^p)} &= \left(\sum_{j=1}^n \left\| \sum_{i=1}^n R_{ji} A_i \right\|_p^q \right)^{1/q} \\ &\leq \left(n \max_{j,i=1,\ldots,n} \|R_{ji}\|^q \, n^q \max_{i=1,\ldots,n} \|A_i\|_p^q \right)^{1/q} \\ &= n^{1+1/q} \max_{j,i=1,\ldots,n} \|R_{ji}\| \max_{i=1,\ldots,n} \|A_i\|_p < \infty. \end{aligned}$$

In particular, each $n \times n$ matrix $a = (a_{ij})$ generates an operator

$$R_a = (a_{ij} \mathbf{1}_H) \text{ on } l_q^n (S^p). \tag{6.4}$$

Consider the unitary matrix (the conjugate of the transpose is its inverse) $u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. By (6.4) and (6.3), the operator R_u acts as

$$R_u A = \frac{1}{\sqrt{2}} \begin{pmatrix} A_1 + A_2 \\ A_1 - A_2 \end{pmatrix}, \text{ for } A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \in l_p^2 (S^p).$$

We will show now that Clarkson-McCarthy inequalities (6.1) and (6.2) could be transformed to the form

$$2^{-\left|\frac{1}{2} - \frac{1}{p}\right|} \|A\|_{l_p^2(S^p)} \leq \|R_u A\|_{l_p^2(S^p)} \leq 2^{\left|\frac{1}{2} - \frac{1}{p}\right|} \|A\|_{l_p^2(S^p)}, \text{ for } p \in [1, \infty); \tag{6.5}$$

$$\|R_uA\|_{l_p^2(S^p)} \le 2^{\left(\frac{1}{2}-\frac{1}{q}\right)} \|A\|_{l_q^2(S^p)}, \text{ where } \frac{1}{p}+\frac{1}{q}=1 \text{ and } p\in[2,\infty). \tag{6.6}$$

The inequality (6.6) is reversed for $1<p\le 2$.

Indeed, substituting A_1 for A and A_2 for B in (6.1) and (6.2), we obtain

$$\begin{aligned} 2^{1/p}\left(\|A_1\|_p^p+\|A_2\|_p^p\right)^{1/p} &\le \left(\|A_1+A_2\|_p^p+\|A_1-A_2\|_p^p\right)^{1/p} \qquad (6.7)\\ &\le 2^{1-1/p}\left(\|A_1\|_p^p+\|A_2\|_p^p\right)^{1/p}; \end{aligned}$$

$$2^{1/q}\left(\|A_1\|_p^p+\|A_2\|_p^p\right)^{1/p} \le \left(\|A_1+A_2\|_p^q+\|A_1-A_2\|_p^q\right)^{1/q}, \tag{6.8}$$

for $2\le p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. For $1<p\le 2$, inequalities in (6.7) and (6.8) are reversed. Note that we can extend (6.7) to $p=1$. Indeed, substituting $p=1$ in (6.7) we get

$$2\left(\|A_1\|_1+\|A_2\|_1\right) \ge \|A_1+A_2\|_1+\|A_1-A_2\|_1 \ge \|A_1\|_1+\|A_2\|_1.$$

We can verify this using the norm triangle inequality:

$$\|A_1+A_2\|_1+\|A_1-A_2\|_1 \le 2\left(\|A_1\|_1+\|A_2\|_1\right) \text{ and } \|A_1\|_1+\|A_2\|_1$$

$$=\frac{1}{2}\left(\|A_1+A_2+A_1-A_2\|_1+\|A_2+A_1-A_1+A_2\|_1\right) \le \|A_1+A_2\|_1+\|A_1-A_2\|_1.$$

Let $A=\begin{pmatrix}A_1\\ A_2\end{pmatrix}$. Then $A\in l_p^2(S^p)$,

$$\begin{aligned} \|A\|_{l_p^2(S^p)} &= \left(\|A_1\|_p^p+\|A_2\|_p^p\right)^{1/p}\\ \text{and } \|R_uA\|_{l_p^2(S^p)} &= \frac{1}{\sqrt{2}}\left(\|A_1+A_2\|_p^p+\|A_1-A_2\|_p^p\right)^{1/p}. \end{aligned}$$

Thus we can transfer the first inequality to

$$2^{1/p} \|A\|_{l_p^2(S^p)} \leq 2^{1/2} \|R_u A\|_{l_p^2(S^p)} \leq 2^{1-1/p} \|A\|_{l_p^2(S^p)} \text{ for } 2 \leq p < \infty \text{ and}$$

$$2^{1/p} \|A\|_{l_p^2(S^p)} \geq 2^{1/2} \|R_u A\|_{l_p^2(S^p)} \geq 2^{1-1/p} \|A\|_{l_p^2(S^p)} \text{ for } 1 \leq p \leq 2.$$

Simplifying and rearranging, we get

$$2^{\frac{1}{p}-\frac{1}{2}} \|A\|_{l_p^2(S^p)} \leq \|R_u A\|_{l_p^2(S^p)} \leq 2^{\frac{1}{2}-\frac{1}{p}} \|A\|_{l_p^2(S^p)} \text{ for } 2 \leq p < \infty \text{ and}$$

$$2^{\frac{1}{2}-\frac{1}{p}} \|A\|_{l_p^2(S^p)} \leq \|R_u A\|_{l_p^2(S^p)} \leq 2^{\frac{1}{p}-\frac{1}{2}} \|A\|_{l_p^2(S^p)} \text{ for } 1 \leq p \leq 2.$$

Taking into account that $\frac{1}{p} - \frac{1}{2} \leq 0$ and $\frac{1}{2} - \frac{1}{p} \geq 0$, for $2 \leq p < \infty$, and $\frac{1}{2} - \frac{1}{p} \leq 0$ and $\frac{1}{p} - \frac{1}{2} \geq 0$, for $1 \leq p \leq 2$, we obtain the transformed Clarkson-McCarthy inequality (6.5).

Similar procedure shows that we could transform the inequality (6.8) to

$$2^{\frac{1}{q}-\frac{1}{2}} \|A\|_{l_p^2(S^p)} \leq \|R_u A\|_{l_q^2(S^p)}, \text{ where } A = \begin{pmatrix} A_1 \\ \\ A_2 \end{pmatrix} \in l_p^2(S^p),$$

$\frac{1}{p} + \frac{1}{q} = 1$ and $2 \leq p < \infty$. We could rearrange it, by substituting $T = A_1 + A_2$ and $S = A_1 - A_2$, to obtain (6.6)

$$\|R_u A\|_{l_p^2(S^p)} \leq 2^{\frac{1}{2}-\frac{1}{q}} \|A\|_{l_q^2(S^p)}, \text{ where } A = \begin{pmatrix} T \\ \\ S \end{pmatrix} \in l_p^2(S^p).$$

For $1 < p \leq 2$ the above inequality is reversed.

Ball, Carlen and Lieb proved in [3, Theorem 1(b)] the following inequality for $X, Y \in S^p$ and $2 \leq p \leq \infty$:

$$\left(\frac{\|X+Y\|_p^p + \|X-Y\|_p^p}{2} \right)^{2/p} \leq \|X\|_p^2 + (p-1) \|Y\|_p^2. \tag{6.9}$$

For $1 \leq p \leq 2$, the estimate (6.9) is reversed.

Set $\lambda = (p-1)^{-1/2}$. Then (6.9) could be transformed to the form

$$\|R_a A\|_{l_p^2(S^p)} \leq 2^{1/p} \|A\|_{l_2^2(S^p)}, \text{ where } a = \begin{pmatrix} 1 & \lambda \\ 1 & -\lambda \end{pmatrix}, A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}. \tag{6.10}$$

For $1 \leq p \leq 2$, the inequality is reversed.

Indeed, set $X = A_1$ and $Y = \lambda A_2$. Then

$$\begin{aligned} R_a A &= \begin{pmatrix} A_1 + \lambda A_2 \\ A_1 - \lambda A_2 \end{pmatrix} = \begin{pmatrix} X+Y \\ X-Y \end{pmatrix}, \\ \|R_a A\|_{l_p^2(S^p)} &= \left(\|X+Y\|_p^p + \|X-Y\|_p^p\right)^{1/p} \end{aligned}$$

and

$$\|A\|_{l_2^2(S^p)} = \left(\|A_1\|_p^2 + \|A_2\|_p^2\right)^{1/2} = \left(\|X\|_p^2 + (p-1)\|Y\|_p^2\right)^{1/2}.$$

Substituting the above formulae for $\|R_a A\|_{l_p^2(S^p)}$ and $\|A\|_{l_2^2(S^p)}$ into (6.9) we obtain $\|R_a A\|_{l_p^2(S^p)}^2 \times 2^{-2/p} \leq \|A\|_{l_2^2(S^p)}^2$. Rearranging it, we get (6.10).

For $n \geq 2$, let $a = (a_{kj})_{k,j=1}^n$ be the $n \times n$ matrix with entries

$$a_{kj} = n^{-1/2} \exp\left(i\frac{2\pi (j-1)(k-1)}{n}\right) = n^{-1/2}\omega_{j-1}^{k-1},$$

where $\omega_{j-1} = e^{2\pi i(j-1)/n}$, $j = 1, 2, ..., n$, are the n-th roots of unity. For $1 \leq p \leq \infty$, let, as before, q be the conjugate index defined by $\frac{1}{p} + \frac{1}{q} = 1$. Bhatia and Kittaneh [9, Theorems 1, 2, 4] obtained an analogue of Clarkson-McCarthy inequalities (6.1) and (6.2) for n operators $A_0, ..., A_{n-1}$ in S^p:

$$n^{\frac{2}{p}} \sum_{j=0}^{n-1} \|A_j\|_p^2 \leq \sum_{k=0}^{n-1} \left\| \sum_{j=0}^{n-1} \omega_j^k A_j \right\|_p^2 \leq n^{2-2/p} \sum_{j=0}^{n-1} \|A_j\|_p^2, \tag{6.11}$$

$$n\sum_{j=0}^{n-1}\|A_j\|_p^p \le \sum_{k=0}^{n-1}\left\|\sum_{j=0}^{n-1}\omega_j^k A_j\right\|_p^p \le n^{p-1}\sum_{j=0}^{n-1}\|A_j\|_p^2\,, \tag{6.12}$$

$$n\left(\sum_{j=0}^{n-1}\|A_j\|_p^p\right)^{q/p} \le \sum_{k=0}^{n-1}\left\|\sum_{j=0}^{n-1}\omega_j^k A_j\right\|_p^q \text{ for } 2\le p\le\infty. \tag{6.13}$$

For $1 \le p \le 2$, inequalities (6.11) and (6.12) and (6.13) are reversed.

Let $A = \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}$ and all $A_i \in S^p$. Inequalities (6.11) and (6.12) can be interpreted as a particular case ($q = p$ or 2) of the following inequalities in $l_q^n\left(S^p\right)$, for $1 \le p < \infty$:

$$n^{-\left|\frac{1}{2}-\frac{1}{p}\right|}\|A\|_{l_q^n(S^p)} \le \|R_a A\|_{l_q^n(S^p)} \le n^{\left|\frac{1}{2}-\frac{1}{p}\right|}\|A\|_{l_q^n(S^p)}\,. \tag{6.14}$$

Inequality (6.13) can be interpreted as inequality in $l_q^n\left(S^p\right)$:

$$n^{\left(\frac{1}{q}-\frac{1}{2}\right)}\|A\|_{l_p^n(S^p)} \le \|R_a A\|_{l_q^n(S^p)} \text{ for } p\in[2,\infty)\,, \tag{6.15}$$

and reversed for $1 < p \le 2$. Indeed, we have

$$\|R_a A\|_{l_q^n(S^p)} = n^{-1/2}\left(\sum_{k=1}^{n}\left\|\sum_{j=1}^{n}\omega_{j-1}^{k-1}A_j\right\|_p^q\right)^{1/q}, \tag{6.16}$$

$$\|A\|_{l_q^n(S^p)} = \left(\sum_{j=1}^{n}\|A_j\|_p^q\right)^{1/q}. \tag{6.17}$$

Set $q = 2$ and $q = p$. Substituting (6.16) and (6.17) in (6.11) and (6.12) and changing the index of summation, we get

$$n^{\frac{1}{p}-\frac{1}{2}}\|A\|_{l_q^n(S^p)} \le \|R_a A\|_{l_q^n(S^p)} \le n^{\frac{1}{2}-\frac{1}{p}}\|A\|_{l_q^n(S^p)}\,, \text{ for } 2\le p<\infty,$$

$$n^{\frac{1}{2}-\frac{1}{p}}\|A\|_{l_q^n(S^p)} \le \|R_a A\|_{l_q^n(S^p)} \le n^{\frac{1}{p}-\frac{1}{2}}\|A\|_{l_q^n(S^p)}\,, \text{ for } 1\le p\le 2.$$

Noticing that $\frac{1}{p}-\frac{1}{2}\leq 0$ for $2\leq p<\infty$ and $\frac{1}{2}-\frac{1}{p}\leq 0$ for $1\leq p\leq 2$ we obtain (6.14). Substituting (6.16) and (6.17) for $q=p$ and changing the index of summation in (6.13), we get (6.15).

In [25, Theorems 1 and 2] Kissin extended the above results to all invertible operators $R=(R_{ij})_{i,j=1}^{n}\in B(H^n)$ with all $R_{ij}\in B(H)$. Set $r=\max_{i,j=1,\ldots,n}\|R_{ij}\|$, $\rho=\max_{i,j=1,\ldots,n}\left\|R_{ij}^{-1}\right\|$, $\alpha=\|R^{-1}\|$, $\beta=\|R\|$ and let $A=(A_j)_{j=1}^{n}\in l_p^n(S^p)$ and $B=(B_j)_{j=1}^{n}=RA$. He proved that

1) if $2\leq p<\infty$ and $\lambda,\mu\in[2,p]$, or if $1<p\leq 2$ and $\lambda,\mu\in[p,2]$, then

$$n^{-\left|\frac{1}{p}-\frac{1}{2}\right|}\alpha^{-1}\left(\frac{1}{n}\sum_{j=1}^{n}\|A_j\|_p^{\mu}\right)^{\frac{1}{\mu}}\leq\left(\frac{1}{n}\sum_{j=1}^{n}\|B_j\|_p^{\lambda}\right)^{\frac{1}{\lambda}}\leq n^{\left|\frac{1}{p}-\frac{1}{2}\right|}\beta\left(\frac{1}{n}\sum_{j=1}^{n}\|A_j\|_p^{\mu}\right)^{\frac{1}{\mu}},\tag{6.18}$$

2) if $\frac{1}{p}+\frac{1}{q}=1$ and $2\leq p<\infty$, then

$$\left(\sum_{j=1}^{n}\|A_j\|_p^{p}\right)^{\frac{1}{p}}\leq\rho^{1-\frac{2}{p}}\alpha^{\frac{2}{p}}\left(\sum_{j=1}^{n}\|B_j\|_p^{q}\right)^{\frac{1}{q}},\tag{6.19}$$

and if $1<p\leq 2$, then

$$\left(\sum_{j=1}^{n}\|B_j\|_p^{q}\right)^{\frac{1}{q}}\leq r^{\frac{2}{p}-1}\beta^{\frac{2}{q}}\left(\sum_{j=1}^{n}\|A_j\|_p^{p}\right)^{\frac{1}{p}}.\tag{6.20}$$

Replacing λ by t and μ by s and using the fact that

$$\|RA\|_{l_t^n(S^p)}=\|B\|_{l_t^n(S^p)}=\left(\sum_{j=1}^{n}\|B_j\|_p^{t}\right)^{1/t}\text{ and }\|A\|_{l_s^n(S^p)}=\left(\sum_{j=1}^{n}\|A_j\|_p^{s}\right)^{1/s},$$

we can interpret (6.18) as inequalities in $l_q^n(S^p)$ in the following way:

$$n^{-\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{1}{s}+\frac{1}{t}}\alpha^{-1}\|A\|_{l_s^n(S^p)}\leq\|RA\|_{l_t^n(S^p)}\leq n^{\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{1}{s}+\frac{1}{t}}\beta\|A\|_{l_s^n(S^p)},\tag{6.21}$$

for $1 \leq p < \infty$, where $t, s \in [\min(p,2), \max(p,2)]$.

Similarly, we can interpret (6.19) and (6.20) as inequalities in $l_q^n(S^p)$:

$$\|RA\|_{l_q^n(S^p)} \leq r^{\frac{2}{p}-1} \beta^{2/q} \|A\|_{l_p^n(S^p)} \text{ for } 1 < p \leq 2, \tag{6.22}$$

$$\|A\|_{l_p^n(S^p)} \leq \rho^{1-\frac{2}{p}} \alpha^{2/p} \|RA\|_{l_q^n(S^p)} \text{ for } p \geq 2, \tag{6.23}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In turn, inequalities (6.21) could be interpreted as estimates of the norm of the operator R acting from $l_s^n(S^p)$ into $l_t^n(S^p)$ for $1 \leq p < \infty$:

$$n^{-\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{1}{s}+\frac{1}{t}} \left\|R^{-1}\right\|^{-1} \leq \|R\|_{l_s^n(S^p)\to l_t^n(S^p)} \leq n^{\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{1}{s}+\frac{1}{t}} \|R\|. \tag{6.24}$$

Inequalities (6.22) and (6.23) could be interpreted as estimates of the norm of the operator R acting from $l_p^n(S^p)$ into $l_q^n(S^p)$, where $\frac{1}{p} + \frac{1}{q} = 1$:

$$\|RA\|_{l_p^n(S^p)\to l_q^n(S^p)} \leq r^{\frac{2}{p}-1} \|R\|^{2/q} \text{ for } 1 < p \leq 2, \tag{6.25}$$

$$\|RA\|_{l_p^n(S^p)\to l_q^n(S^p)} \geq \rho^{-1+\frac{2}{p}} \left\|R^{-1}\right\|^{-2/p} \text{ for } p \geq 2. \tag{6.26}$$

We call inequalities (6.24) - (6.26) the generalized Clarkson-McCarthy inequalities.

6.2 Action of operators from $B(H^\infty)$ on $l_q(S^p)$ spaces

In this section we analyse action of operators from $B(H^\infty)$ on $l_q(S^p)$ spaces.

By Theorem 5.22(iii), $l_2(S^2) = S^2(H, H^\infty)$. By Lemma 5.16, $S^p(H, K)$ is a left $B(K)$-module for all $p \in [1, \infty)$. Thus, in case $p = 2$ and $K = H^\infty$, we have $l_2(S^2) = S^2(H, H^\infty)$ is a left $B(H^\infty)$-module. In this section we assume that H is a separable infinite dimensional complex Hilbert space. We show that, apart from $l_2(S^2)$, the Banach spaces $l_q(S^p)$ are not left $B(H^\infty)$-modules. We also establish the following inequalities. Let $R \in B(H^\infty)$.

(i) if $1 \le p \le 2$ and $A \in l_p(S^p)$, then

$$RA \in l_2(S^p) \text{ and } \|RA\|_{l_2(S^p)} \le \|R\|_{B(H^\infty)} \|A\|_{l_p(S^p)};$$

(ii) if $p \ge 2$ and $A \in l_2(S^p)$, then

$$RA \in l_p(S^p) \text{ and } \|RA\|_{l_p(S^p)} \le \|R\|_{B(H^\infty)} \|A\|_{l_2(S^p)}.$$

We will need these estimates to prove the main result in this chapter, namely, the analogue of McCarthy inequality (6.1) for $l_q(S^p)$ spaces.

Each operator $R \in B(H^\infty)$ has the block matrix form $R = (R_{kn})_{k,n=1}^\infty$ where $R_{kn} \in B(H)$. It acts on a subspace $D(R)$ of $l_\infty(B(H))$ - the domain of R - defined as follows:

$$D(R) = \left\{ \begin{array}{c} A = (A_n)_{n=1}^\infty \in l_\infty(B(H)) : \\ B_k := \sum_{n=1}^\infty R_{kn}A_n \in B(H) \text{ for all } k = 1, 2, ... \\ \text{and } B := RA = (B_n)_{n=1}^\infty \in l_\infty(B(H)) \end{array} \right\},$$

where all $\sum_{n=1}^\infty R_{kn}A_n$ converge in the w.o.t. Thus, for $A = (A_n)_{n=1}^\infty \in D(R)$,

$$B = RA = R \begin{pmatrix} A_1 \\ \vdots \\ A_n \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^\infty R_{1n}A_n \\ \vdots \\ \sum_{n=1}^\infty R_{kn}A_n \\ \vdots \end{pmatrix} \in l_\infty(B(H)),$$

where all $B_k = \sum_{n=1}^{\infty} R_{kn} A_n \in B(H)$. We also have

$$\|RA\|_{l_\infty(B(H))} = \|B\|_{l_\infty(B(H))} = \sup_{k=1,2,\ldots} \|B_k\|_{B(H)}\,.$$

Proposition 6.1 (i) $\cap_{R \in B(H^\infty)} D(R) = B(H, H^\infty)$

(ii) *If* $(p,q) \neq (2,2)$ *then the space* $l_q(S^p)$ *is not a left* $B(H^\infty)$*-module.*

Proof. (i) Let $D = \cap_{R \in B(H^\infty)} D(R)$. First let us prove that $D \subseteq B(H, H^\infty)$. Let $A = (A_n)_{n=1}^{\infty} \in D$. Then, $A \in l_\infty(B(H))$ and, for all $R \in B(H^\infty)$, we have $A \in D(R)$. In particular, $A \in D(R)$ where R is an operator that we are about to construct. Since all separable infinite dimensional Hilbert spaces are isometrically isomorphic to the complex sequence space l_2 (see for example [32, p.26]), there exists an isometry L from H^∞ onto H. It is of a form $L = (L_1, \ldots, L_n, \ldots)$ with all $L_n \in B(H)$, $\|L\|_{B(H^\infty, H)} = 1$ and

$$Lx = (L_1, \ldots, L_n, \ldots) \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \vdots \end{pmatrix} = \sum_{n=1}^{\infty} L_n x_n \in H, \text{ for all } x = (x_n)_{n=1}^{\infty} \in H^\infty,$$

where all $x_n \in H$, and the series converges in H. Let $R = (R_{ij})_{i,j=1}^{\infty} \in B(H^\infty)$ be such that all $R_{1n} = L_n$, for all n, and $R_{in} = 0$ for all $i \geq 2$ and all n. Then

$$R = \begin{pmatrix} L_1 & \cdots & L_n & \cdots \\ 0 & \cdots & 0 & \cdots \\ \vdots & & \vdots & \end{pmatrix} \text{ and } Rx = y = (y_n)_{n=1}^{\infty} = \begin{pmatrix} Lx \\ 0 \\ \vdots \end{pmatrix},$$

for all $x = (x_n)_{n=1}^{\infty} \in H^{\infty}$. Thus $R \in B(H^{\infty})$, as

$$\|R\|_{B(H^{\infty})} = \sup_{\|x\|=1} \|Rx\|_{H^{\infty}} = \sup_{\|x\|=1} \|Lx\|_{H} = \|L\|_{B(H^{\infty},H)} = 1.$$

Since R maps $D(R)$ into $l_{\infty}(B(H))$, we have

$$RA = \begin{pmatrix} L_1 & \cdots & L_n & \cdots \\ & & & \\ 0 & \cdots & 0 & \cdots \\ \vdots & & \vdots & \end{pmatrix} \begin{pmatrix} A_1 \\ \vdots \\ A_n \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{\infty} L_n A_n \\ \\ 0 \\ \vdots \end{pmatrix} \in l_{\infty}(B(H)).$$

Let P_m be the projections defined in (5.24). As $RA \in l_{\infty}(B(H))$, we have $LP_mA = \sum_{n=1}^{m} L_n A_n \overset{w.o.t.}{\rightarrow} B_1 \in B(H)$. Since L is invertible isometry, $L^{-1} \in B(H, H^{\infty})$ and $L^{-1} = L^*$. Thus, for all $x \in H$, $y \in H^{\infty}$

$$\left(L^{-1}(LP_mA)x, y\right) = ((LP_mA)x, Ly) \underset{m\to\infty}{\rightarrow} (B_1x, Ly) = \left(L^{-1}B_1x, y\right).$$

Hence

$$P_mA = L^{-1}(LP_mA) \overset{w.o.t.}{\rightarrow} L^{-1}B_1 \in B(H, H^{\infty}).$$

By Lemma 5.21(i), $A \in B(H, H^{\infty})$. Thus $D \subseteq B(H, H^{\infty})$.

Let us now prove that $B(H, H^{\infty}) \subseteq D$. Clearly, for all $A \in B(H, H^{\infty}) \subseteq l_{\infty}(B(H))$ (see (5.55)) and $R \in B(H^{\infty})$, we have $RA \in B(H, H^{\infty}) \subseteq l_{\infty}(B(H))$. Thus $B(H, H^{\infty}) \subseteq D(R)$. Hence $B(H, H^{\infty}) \subseteq D$.

Combining these inclusions, we have $D = B(H, H^{\infty})$.

(ii) (1) First consider the case when $q \in (2, \infty)$ and $p \in [1, \infty)$.

We see that $l_q(S^p) \subseteq l_{\infty}(B(H))$. Indeed, if $A = (A_n)_{n=1}^{\infty} \in l_q(S^p)$ then

$$\|A\|_{l_{\infty}(B(H))} = \sup_n \|A_n\| \leq \sup_n \|A_n\|_p \leq \left(\sum_{n=1}^{\infty} \|A_n\|_p^q\right)^{1/q} = \|A\|_{l_q(S^p)}.$$

If $l_q(S^p)$ is a left $B(H^\infty)$-module then $RA \in l_q(S^p) \subseteq l_\infty(B(H))$, for each $A \in l_q(S^p)$ and each $R = (R_{kn})_{k,n=1}^\infty \in B(H^\infty)$. Hence $l_q(S^p) \subseteq \cap_{R \in B(H^\infty)} D(R)$. Therefore, by (i), $l_q(S^p) \subseteq B(H, H^\infty)$. This contradicts Theorem 5.22(ii). Thus $l_q(S^p)$, for $q \in (2, \infty)$ and $p \in [1, \infty)$, is not a left $B(H^\infty)$-module.

(2) Consider now the case when $q \in [1, 2)$ and $p \in [1, \infty)$.

Let $R = (R_{nk})_{n,k=1}^\infty \in B(H^\infty)$ be such that $R_{nk} = 0$ for all $k > 1$. Then, for $A = (A_n)_{n=1}^\infty \in l_q(S^p)$ and $x = (x_n)_{n=1}^\infty \in H^\infty$,

$$R = \begin{pmatrix} R_{11} & 0 & \cdots \\ \vdots & \vdots & \\ R_{n1} & 0 & \cdots \\ \vdots & \vdots & \end{pmatrix}, RA = \begin{pmatrix} R_{11}A_1 \\ \vdots \\ R_{n1}A_1 \\ \vdots \end{pmatrix}, Rx = \begin{pmatrix} R_{11}x_1 \\ \vdots \\ R_{n1}x_1 \\ \vdots \end{pmatrix}. \tag{6.27}$$

Let $R_{n1} = \alpha_n \mathbf{1}_H$, where all $\alpha_n > 0$, $\sum_{n=1}^\infty \alpha_n^2 = 1$ and $\sum_{n=1}^\infty \alpha_n^q = \infty$ (for example, $\alpha_n = n^{-1/q}\sigma$, where $\sigma = \left(\sum_{j=1}^\infty j^{-2/q}\right)^{-1/2}$). Then, by (6.27),

$$\|Rx\|^2 = \sum_{n=1}^\infty \|\alpha_n x_1\|^2 = \|x_1\|^2 \sum_{n=1}^\infty \alpha_n^2 = \|x_1\|^2 \leq \|x\|^2, \text{ for all } x \in H^\infty.$$

Hence $R \in B(H^\infty)$. If $A \in l_q(S^p)$ and $A_1 \neq 0$, then, by (6.27),

$$\begin{aligned} \|RA\|_{l_q(S^p)} &= \left(\sum_{n=1}^\infty \|R_{n1}A_1\|_p^q\right)^{1/q} = \left(\sum_{n=1}^\infty \|\alpha_n A_1\|_p^q\right)^{1/q} \\ &= \left(\sum_{n=1}^\infty |\alpha_n|^q \|A_1\|_p^q\right)^{1/q} = \|A_1\|_p \left(\sum_{n=1}^\infty \alpha_n^q\right)^{1/q} = \infty. \end{aligned}$$

Therefore $RA \notin l_q(S^p)$. Thus $l_q(S^p)$ is not a left $B(H^\infty)$-module for $q < 2$.

(3) We only have to consider the case when $q = 2$ and $p \in [1, 2) \cup (2, \infty)$.

Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in H, let P_{e_n} be the projections on $\mathbb{C}e_n$, $n \in \mathbb{N}$ and $\{V_n\}_{n=1}^{\infty}$ be partial isometries that map $\mathbb{C}e_n$ onto $\mathbb{C}e_1$, i.e.,

$$V_n e_n = e_1 \text{ for all } n \text{ and } V_n e_j = 0, \text{ for all } j \neq n.$$

Noticing that $(V_n e_n, e_1) = 1 = (e_n, V_n^* e_1)$, for all n, and

$$(V_n e_n, e_k) = (e_1, e_k) = 0 = (e_n, V_n^* e_k)\,, \text{ for all } n \text{ and all } k \neq 1,$$

we have that $V_n^* e_1 = e_n$ and $V_n^* e_k = 0$, for all $k \neq 1$. Thus

$$P_{e_n} = V_n^* V_n \text{ for all } n.$$

Set now $R_{n1} = V_n$ in (6.27). Let $x = (x_n)_{n=1}^{\infty} \in H^{\infty}$ and $x_1 = \sum_{k=1}^{\infty} \alpha_k e_k \in H$. As $R_{n1} x_1 = V_n x_1 = \alpha_n e_1$, we have that $R \in B\left(H^{\infty}\right)$, since

$$\begin{aligned} \|R\|_{B(H^{\infty})}^2 &= \sup_{\|x\|=1} \|Rx\|_{H^{\infty}}^2 \overset{(6.27)}{=} \sup_{\|x\|=1} \sum_{n=1}^{\infty} \|R_{n1} x_1\|_H^2 = \\ &= \sup_{\|x\|=1} \sum_{n=1}^{\infty} \|\alpha_n e_1\|_H^2 = \sup_{\|x\|=1} \sum_{n=1}^{\infty} \alpha_n^2 = 1. \end{aligned}$$

(3a) Let $p \in [1,2)$. Set $A_n = n^{-1/p} V_n$ and $A = (A_n)_{n=1}^{\infty}$. Since $A_n^* A_n = n^{-2/p} V_n^* V_n = n^{-2/p} P_{e_n}$ and $\|V_n^* V_n\|_{p/2} = \|P_{e_n}\|_{p/2} = 1$, we have

$$\begin{aligned} \|A\|_{l_2(S^p)} &= \left(\sum_{n=1}^{\infty} \|A_n\|_p^2 \right)^{1/2} \overset{(5.5)}{=} \left(\sum_{n=1}^{\infty} \|A_n^* A_n\|_{p/2} \right)^{1/2} \\ &= \left(\sum_{n=1}^{\infty} n^{-2/p} \|P_{e_n}\|_{p/2} \right)^{1/2} = \left(\sum_{n=1}^{\infty} n^{-2/p} \right)^{1/2} < \infty. \end{aligned}$$

Thus $A \in l_2(S^p)$. Since $R \in B(H^\infty)$, the operator

$$R^* = \begin{pmatrix} R_{11}^* & \cdots & R_{n1}^* & \cdots \\ 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \in B(H^\infty) \text{ and } R^*A = \begin{pmatrix} \sum_{n=1}^{\infty} R_{n1}^* A_n \\ 0 \\ \vdots \end{pmatrix}.$$

The operator

$$B = \sum_{n=1}^{\infty} R_{n1}^* A_n = \sum_{n=1}^{\infty} V_n^* \left(n^{-1/p} V_n\right) = \sum_{n=1}^{\infty} n^{-1/p} P_{e_n}$$

is self-adjoint and positive. Thus its s-numbers are the eigenvalues $\left\{n^{-1/p}\right\}_{n=1}^{\infty}$. Hence $B \notin S^p$ and $R^*A \notin l_2(S^p)$. Thus $l_2(S^p)$ is not a left $B(H^\infty)$-module.

(3b) Assume now that $p \in (2, \infty)$. Set $A_n = 0$, for all $n \geq 2$, and $A_1 = \sum_{n=1}^{\infty} n^{-1/2} P_{e_n}$. Then A_1 is self-adjoint and positive. Thus, the s-numbers of A_1 are its eigenvalues $\left\{n^{-1/2}\right\}_{n=1}^{\infty}$. Hence $A_1 \in S^p(H)$, since

$$\|A_1\|_p = \left(\sum_{n=1}^{\infty} n^{-p/2}\right)^{1/p} < \infty$$

and $A \in l_2(S^p)$, since

$$\|A\|_{l_2(S^p)} = \left(\sum_{n=1}^{\infty} \|A_n\|_p^2\right)^{1/2} = \|A_1\|_p < \infty.$$

For each n, $A_1^* P_{e_n} A_1 = A_1 P_{e_n} A_1 = A_1 n^{-1/2} P_{e_n} = n^{-1} P_{e_n}$. Therefore

$$\|RA\|_{l_2(S^p)} = \left\| \begin{pmatrix} R_{11}A_1 \\ \vdots \\ R_{n1}A_1 \\ \vdots \end{pmatrix} \right\|_{l_2(S^p)} = \left\| \begin{pmatrix} V_1 A_1 \\ \vdots \\ V_n A_1 \\ \vdots \end{pmatrix} \right\|_{l_2(S^p)} = \left(\sum_{n=1}^{\infty} \|V_n A_1\|_p^2 \right)^{1/2}$$

$$\overset{(5.5)}{=} \left(\sum_{n=1}^{\infty} \|(V_n A_1)^* V_n A_1\|_{p/2} \right)^{1/2} = \left(\sum_{n=1}^{\infty} \|A_1^* V_n^* V_n A_1\|_{p/2} \right)^{1/2}$$

$$= \left(\sum_{n=1}^{\infty} \|A_1^* P_{e_n} A_1\|_{p/2} \right)^{1/2} = \left(\sum_{n=1}^{\infty} \left\|n^{-1} P_{e_n}\right\|_{p/2} \right)^{1/2} = \left(\sum_{n=1}^{\infty} n^{-1} \right)^{1/2}$$

diverges. Thus $RA \notin l_2(S^p)$, so that $l_2(S^p)$ is not a left $B(H^\infty)$-module. ■

Making use of Theorem 5.22, we obtain the following theorem.

Theorem 6.2 *Let $R \in B(H^\infty)$.*

(i) *Let $p \in [1,2]$ and $A \in l_p(S^p)$. Then*

$$RA \in l_2(S^p) \text{ and } \|RA\|_{l_2(S^p)} \leq \|R\|_{B(H^\infty)} \|A\|_{l_p(S^p)}.$$

(ii) *Let $p \in [2,\infty)$ and $A \in l_2(S^p)$. Then*

$$RA \in l_p(S^p) \text{ and } \|RA\|_{l_p(S^p)} \leq \|R\|_{B(H^\infty)} \|A\|_{l_2(S^p)}.$$

Proof. (i) Let $p \in [1,2)$ and $A \in l_p(S^p)$. Applying Theorem 5.22(i), we obtain that $A \in S^p(H, H^\infty)$. Hence, by Lemma 5.16, $RA \in S^p(H, H^\infty)$ and $\|RA\|_p \leq \|R\|_{B(H^\infty)} \|A\|_p$. Since $p \in [1,2)$, we have from Theorem 5.22(i) that $RA \in l_2(S^p)$ and $\|RA\|_{l_2(S^p)} \leq \|RA\|_p$. We also have from Theorem 5.22(i) that $\|A\|_p \leq \|A\|_{l_p(S^p)}$. Combining these inequalities yields

$$\|RA\|_{l_2(S^p)} \leq \|RA\|_p \overset{(5.48)}{\leq} \|R\|_{B(H^\infty)} \|A\|_p \leq \|R\|_{B(H^\infty)} \|A\|_{l_p(S^p)}.$$

(ii) Let $p \in (2, \infty)$ and $A \in l_2(S^p)$. Applying Theorem 5.22(ii), we have that $A \in S^p(H, H^\infty)$. Hence, by Lemma 5.16, $RA \in S^p(H, H^\infty)$ and $\|RA\|_p \le \|R\|_{B(H^\infty)} \|A\|_p$. We have from Theorem 5.22(ii) that $RA \in l_p(S^p)$, $\|RA\|_{l_p(S^p)} \le \|RA\|_p$ and $\|A\|_p \le \|A\|_{l_2(S^p)}$. Combining these inequalities yields

$$\|RA\|_{l_p(S^p)} \le \|RA\|_p \overset{(5.48)}{\le} \|R\|_{B(H^\infty)} \|A\|_p \le \|R\|_{B(H^\infty)} \|A\|_{l_2(S^p)}.$$

For $p = 2$, it follows from Theorem 5.22(iii) and Lemma 5.16 that $\|A\|_{l_2(S^2)} = \|A\|_2$ and $\|RA\|_{l_2(S^2)} = \|RA\|_2$. Thus

$$\|RA\|_{l_2(S^2)} = \|RA\|_2 \overset{(5.48)}{\le} \|R\|_{B(H^\infty)} \|A\|_2 = \|R\|_{B(H^\infty)} \|A\|_{l_2(S^2)}.$$

This completes the proof. ■

6.3 The main result: The case of $l_q(S^p)$ spaces

For operators $R \in B(H^\infty)$ of a particular form, we can use inequality (6.21) to obtain some further analogues of McCarthy inequality (6.1) for $l_q(S^p)$ spaces. Let $\{n_k\}_{k=1}^\infty$ be a sequence of positive integers. For $A = (A_n)_{n=1}^\infty \in l_q(S^p)$, let

$$\widehat{A_1} = \begin{pmatrix} A_1 \\ \vdots \\ A_{n_1} \end{pmatrix}, \quad \widehat{A_2} = \begin{pmatrix} A_{n_1+1} \\ \vdots \\ A_{n_1+n_2} \end{pmatrix}, \ldots, \quad \widehat{A_k} = \begin{pmatrix} A_{n_1+\ldots+n_{k-1}+1} \\ \vdots \\ A_{n_1+\ldots+n_k} \end{pmatrix}, \ldots.$$

Then each $\widehat{A_k}$ belongs to $l_q^{n_k}(S^p)$, and we have

$$A = \begin{pmatrix} \widehat{A_1} \\ \vdots \\ \widehat{A_k} \\ \vdots \end{pmatrix} \text{ and } \|A\|_{l_q(S^p)} = \left(\sum_{k=1}^\infty \left\| \widehat{A_k} \right\|_{l_q^{n_k}(S^p)}^q \right)^{1/q}. \tag{6.28}$$

For each k, let $H^{n_k} = H \oplus ... \oplus H$ be the orthogonal sum of n_k copies of H and $H^\infty = \oplus_{k=1}^\infty H^{n_k}$ Let $R_k \in B(H^{n_k})$. Then R_k is an $n_k \times n_k$ block-operator. Consider a block-diagonal operator $R = \{R_k\}_{k=1}^\infty \in B(H^\infty)$ such that the operators R_k are on the diagonal and off the diagonal there are all 0, i.e.,

$$R = \begin{pmatrix} R_1 & 0 & \cdots & \cdots \\ 0 & \ddots & 0 & \cdots \\ 0 & 0 & R_k & 0 \\ \vdots & \vdots & 0 & \ddots \end{pmatrix} \tag{6.29}$$

Theorem 6.3 *Let $p \in [1,\infty)$ and $q \in [\min(p,2), \max(p,2)]$. Let $R = \{R_k\}_{k=1}^\infty \in B(H^\infty)$ be a block-diagonal operator on H^∞ described in (6.29). Assume that*

$$\omega = \sup_{k=1,...} n_k^{\left|\frac{1}{p}-\frac{1}{2}\right|} \|R_k\| < \infty \text{ and let } \zeta = \inf_k n_k^{-\left|\frac{1}{p}-\frac{1}{2}\right|} \left\|R_k^{-1}\right\|^{-1}.$$

Then

$$\zeta \|A\|_{l_q(S^p)} \le \|RA\|_{l_q(S^p)} \le \omega \|A\|_{l_q(S^p)} \text{ for } A \in l_q(S^p). \tag{6.30}$$

Proof. It follows from (6.28) and the block-diagonal structure of R that

$$RA = \begin{pmatrix} R_1\widehat{A_1} \\ \vdots \\ R_k\widehat{A_k} \\ \vdots \end{pmatrix} \text{ and } \|RA\|_{l_q(S^p)} = \left(\sum_{k=1}^\infty \left\|R_k\widehat{A_k}\right\|_{l_q^{n_k}(S^p)}^q\right)^{1/q}. \tag{6.31}$$

Replacing s and t with q and n with n_k, we have from (6.21) that, for all k,

$$n_k^{-\left|\frac{1}{p}-\frac{1}{2}\right|} \left\|R_k^{-1}\right\|^{-1} \left\|\widehat{A_k}\right\|_{l_q^{n_k}(S^p)} \le \left\|R_k\widehat{A_k}\right\|_{l_q^{n_k}(S^p)} \le n_k^{\left|\frac{1}{p}-\frac{1}{2}\right|} \|R_k\| \left\|\widehat{A_k}\right\|_{l_q^{n_k}(S^p)}. \tag{6.32}$$

Applying RHS of (6.32) to formula (6.31), we have

$$\begin{aligned}\|RA\|_{l_q(S^p)} &\leq \left(\sum_{k=1}^{\infty} n_k^{q\left|\frac{1}{p}-\frac{1}{2}\right|} \|R_k\|^q \left\|\widehat{A_k}\right\|_{l_q^{n_k}(S^p)}^q\right)^{1/q} \\ &\leq \left(\sum_{k=1}^{\infty} \omega^q \left\|\widehat{A_k}\right\|_{l_q^{n_k}(S^p)}\right)^{1/q} \stackrel{(6.28)}{=} \omega \|A\|_{l_q(S^p)}.\end{aligned}$$

Applying LHS of (6.32) to formula (6.31), we have

$$\begin{aligned}\|RA\|_{l_q(S^p)} &\geq \left(\sum_{k=1}^{\infty} n_k^{-q\left|\frac{1}{p}-\frac{1}{2}\right|} \left\|R_k^{-1}\right\|^{-q} \left\|\widehat{A_k}\right\|_{l_q^{n_k}(S^p)}^q\right)^{1/q} \\ &\geq \left(\sum_{k=1}^{\infty} \zeta^q \left\|\widehat{A_k}\right\|_{l_q^{n_k}(S^p)}\right)^{1/q} \stackrel{(6.28)}{=} \zeta \|A\|_{l_q(S^p)}.\end{aligned}$$

This completes the proof. ■

Let $n = 2$. Consider the block-diagonal operator $R = \{R_k\}_{k=1}^{\infty}$, where all $R_k = 2^{-1/2}\begin{pmatrix} \mathbf{1}_H & \mathbf{1}_H \\ \mathbf{1}_H & -\mathbf{1}_H \end{pmatrix}$ are unitary operators on H^2, as $R_k^* = R_k$ and $R_k^* R_k = R_k^2 = \mathbf{1}_{H^2}$. The operator R is also unitary, as $R^* = R$ and $R^*R = R^2 = \{R_k^2\}_{k=1}^{\infty} = \{\mathbf{1}_{H^2}\}_{k=1}^{\infty} = \mathbf{1}_{H^\infty}$. Then $R \in B(H^\infty)$, since

$$\begin{aligned}\|R\|_{B(H^\infty)} &= \sup_{\|x\|=1} \|Rx\|_{H^\infty} = \sup_{\|x\|=1} 2^{-1/2} \left\| \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ \vdots \\ x_{2n-1} + x_{2n} \\ x_{2n-1} - x_{2n} \\ \vdots \end{pmatrix} \right\|_{H^\infty} \qquad (6.33) \\ &= \sup_{\|x\|=1} 2^{-1/2} \left(\sum_{n=1}^{\infty} (x_{2n-1} + x_{2n})^2 + \sum_{n=1}^{\infty} (x_{2n-1} - x_{2n})^2 \right)^{1/2} \\ &= \sup_{\|x\|=1} \|x\| = 1.\end{aligned}$$

For $A=(A_n)_{n=1}^{\infty}\in l_q(S^p)$, consider $X=(X_n)_{n=1}^{\infty}$ and $Y=(Y_n)_{n=1}^{\infty}$ such that $X_n=A_{2n-1}$ and $Y_n=A_{2n}$. Then $X,Y\in l_q(S^p)$ and

$$\|A\|_{l_q(S^p)}=\left(\sum_{n=1}^{\infty}\|A_{2n-1}\|_p^q+\sum_{n=1}^{\infty}\|A_{2n}\|_p^q\right)^{1/q}=\left(\|X\|_{l_q(S^p)}^q+\|Y\|_{l_q(S^p)}^q\right)^{1/q}.$$

Then

$$RA=2^{-1/2}\begin{pmatrix}A_1+A_2\\ A_1-A_2\\ \vdots\\ A_{2n-1}+A_{2n}\\ A_{2n-1}-A_{2n}\\ \vdots\end{pmatrix}\tag{6.34}$$

and

$$\|RA\|_{l_q(S^p)}=2^{-1/2}\left(\sum_{n=1}^{\infty}\|A_{2n-1}+A_{2n}\|_p^q+\sum_{n=1}^{\infty}\|A_{2n-1}-A_{2n}\|_p^q\right)^{1/q}=$$
$$=2^{-1/2}\left(\|X+Y\|_{l_q(S^p)}^q+\|X-Y\|_{l_q(S^p)}^q\right)^{1/q}.$$

Using Theorems 6.2 and 6.3, we obtain the following analogue of McCarthy inequalities (6.1) and (6.2) for $l_q(S^p)$ spaces.

Corollary 6.4 (i) *Let $p\in[1,2]$ and $X,Y\in l_p(S^p)$. Then*

$$\left(\|X+Y\|_{l_2(S^p)}^2+\|X-Y\|_{l_2(S^p)}^2\right)^{1/2}\le 2^{1/2}\left(\|X\|_{l_p(S^p)}^p+\|Y\|_{l_p(S^p)}^p\right)^{1/p}.\tag{6.35}$$

Let $p\in[2,\infty)$ and $X,Y\in l_2(S^p)$. Then

$$\left(\|X+Y\|_{l_p(S^p)}^p+\|X-Y\|_{l_p(S^p)}^p\right)^{1/p}\le 2^{1/2}\left(\|X\|_{l_2(S^p)}^2+\|Y\|_{l_2(S^p)}^2\right)^{1/2}.\tag{6.36}$$

(ii) *Let* $p \in [1, \infty)$*,* $q \in [\min(p, 2), \max(p, 2)]$ *and* $X, Y \in l_q(S^p)$*. Then*

$$
\begin{aligned}
& 2^{-\left|\frac{1}{p}-\frac{1}{2}\right|+\frac{1}{2}} \left(\|X\|_{l_q(S^p)}^q + \|Y\|_{l_q(S^p)}^q\right)^{1/q} \qquad (6.37) \\
\leq\ & \left(\|X+Y\|_{l_q(S^p)}^q + \|X-Y\|_{l_q(S^p)}^q\right)^{1/q} \\
\leq\ & 2^{\left|\frac{1}{p}-\frac{1}{2}\right|+\frac{1}{2}} \left(\|X\|_{l_q(S^p)}^q + \|Y\|_{l_q(S^p)}^q\right)^{1/q}.
\end{aligned}
$$

Proof. We proved in (6.33) that $\|R\|_{B(H^\infty)} = 1$. The rest is simply substitution of

$$
\|RA\|_{l_q(S^p)} = 2^{-1/2} \left(\|X+Y\|_{l_q(S^p)}^q + \|X-Y\|_{l_q(S^p)}^q\right)^{1/q}
$$

and

$$
\|A\|_{l_q(S^p)} = \left(\|X\|_{l_q(S^p)}^q + \|Y\|_{l_q(S^p)}^q\right)^{1/q}
$$

in Theorems 6.2 and 6.3. Additionally, we need to see that, in this case in Theorem 6.3, $\omega = 2^{\left|\frac{1}{p}-\frac{1}{2}\right|}$ and $\zeta = 2^{-\left|\frac{1}{p}-\frac{1}{2}\right|}$ as $n = 2$ and all operators R_k,R_k^{-1} are unitary. ■

6.4 Uniform convexity of spaces $l_p(S^p)$

This section is dedicated to the proof that the spaces $l_p(S^p)$, for $p \in [2, \infty)$, are p-uniformly convex.

Definition 6.5 (i) [39, p.23] *A Banach space* B *is called uniformly convex if and only if, for all* $0 < \varepsilon \leq 2$*, the modulus of convexity*

$$
\delta_B(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\|_B : x, y \in B, \|x\|_B = \|y\|_B = 1, \|x-y\|_B \geq \varepsilon\right\}
$$

is strictly positive.

(ii) [3, p.464] *A Banach space B is r-uniformly convex, if its modulus of convexity $\delta_B(\varepsilon)$ is such that*

$$\delta_B(\varepsilon) \geq (\varepsilon/C)^r \text{ for } 0 < \varepsilon \leq 2 \text{ where } C > 0 \text{ is some constant.}$$

Lemma 6.6 *For all $x \in [0,1]$ and $p \geq 1$,*

$$1 - x^p \leq \left(1 - \frac{x^p}{p}\right)^p.$$

Proof. Fix p and consider the function

$$f(x) = \left(1 - \frac{x^p}{p}\right)^p - 1 + x^p.$$

Then $f(0) = 0$ and

$$f'(x) = p\left(1 - \frac{x^p}{p}\right)^{p-1}\left(-x^{p-1}\right) + px^{p-1} = px^{p-1}\left(1 - \left(1 - \frac{x^p}{p}\right)^{p-1}\right) \geq 0.$$

Hence f increases, so that $f(x) \geq 0$, for all $x \in [0,1]$. ■

Theorem 6.7 *The space $l_p(S^p)$, for $p \in [2,\infty)$, is p-uniformly convex.*

Proof. Let $2 \leq p = q < \infty$. Then $\left|\frac{1}{p} - \frac{1}{2}\right| + \frac{1}{2} = 1 - \frac{1}{p}$ and, for $X, Y \in l_p(S^p)$, it follows from Corollary 6.4(ii) that

$$\left(\|X+Y\|_{l_p(S^p)}^p + \|X-Y\|_{l_p(S^p)}^p\right)^{1/p} \leq 2^{1-\frac{1}{p}}\left(\|X\|_{l_p(S^p)}^p + \|Y\|_{l_p(S^p)}^p\right)^{1/p} \qquad (6.38)$$

To apply the definition of p-uniformly convex space, we set $\|X\|_{l_p(S^p)} = \|Y\|_{l_p(S^p)} = 1$ and $\|X-Y\|_{l_p(S^p)} \geq \varepsilon$. Substituting into (6.38) and rearranging we have

$$\|X+Y\|_{l_p(S^p)}^p \leq \left(2^{1-\frac{1}{p}}2^{\frac{1}{p}}\right)^p - \|X-Y\|_{l_p(S^p)}^p = 2^p - \|X-Y\|_{l_p(S^p)}^p \leq 2^p - \varepsilon^p.$$

Thus, making use of Lemma 6.6, we have

$$\left\|\frac{X+Y}{2}\right\|_{l_p(S^p)}^p \le 1-\left(\frac{\varepsilon}{2}\right)^p \le \left(1-\frac{\left(\frac{\varepsilon}{2}\right)^p}{p}\right)^p.$$

Hence

$$\left\|\frac{X+Y}{2}\right\|_{l_p(S^p)} \le 1-\frac{\left(\frac{\varepsilon}{2}\right)^p}{p}$$

and we obtain that

$$1-\left\|\frac{X+Y}{2}\right\|_{l_p(S^p)} \ge 1-\left(1-\frac{\left(\frac{\varepsilon}{2}\right)^p}{p}\right) = \left(\frac{\varepsilon}{2p^{1/p}}\right)^p. \tag{6.39}$$

Thus

$$\begin{aligned} & \delta_{l_p(S^p)}(\varepsilon) \\ = & \inf\left\{1-\frac{1}{2}\|X+Y\|_{l_p(S^p)} : \|X\|_{l_p(S^p)} = \|Y\|_{l_p(S^p)} = 1, \|X-Y\|_{l_p(S^p)} \ge \varepsilon\right\} \\ & \overset{(6.39)}{\ge} \left(\frac{\varepsilon}{2p^{1/p}}\right)^p. \end{aligned}$$

The proof is complete. ■

Problem 6.8 *Are the spaces $l_p(S^p)$, for $p \in [1,2)$, p-uniformly convex?*

6.5 Estimates for partitions of operators from S^p

Let $\{P_n\}_{n=1}^N \in \mathcal{P}_N$, be a partition of $\mathbf{1}_H$. Then (see [21, Theorem III.4.2])

$$\left\|\sum_{n=1}^N P_n A P_n\right\|_p \le \|A\|_p.$$

Let $\{Q_m\}_{m=1}^M \in \mathcal{P}_M$ be another partition and $\mathcal{U} = \{P_n A Q_m\}_{n,m=1}^{N,M}$ be a partition of $A \in S^p(H)$, $1 \le p < \infty$ (see Definition 5.7). For $M, N < \infty$, it was proved in [25, Theorem 4] that, for $2 \leq q \leq p < \infty$,

$$(NM)^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{n,m} \|P_n A Q_m\|_p^q \right)^{1/q} \leq \|A\|_p \leq (NM)^{\frac{1}{2}-\frac{1}{q}} \left(\sum_{n,m} \|P_n A Q_m\|_p^q \right)^{1/q}.$$

In other words, $\mathcal{U} = \{P_n A Q_m\}_{n,m=1}^{N,M}$ belongs to $l_q^{NM}(S^p)$ and

$$(NM)^{\frac{1}{p}-\frac{1}{q}} \|\mathcal{U}\|_{l_q^{mn}(S^p)} \leq \|A\|_p \leq (NM)^{\frac{1}{2}-\frac{1}{q}} \|\mathcal{U}\|_{l_q^{mn}(S^p)}.$$

For $0 < p \leq q \leq 2$, the inequalities are reversed.

For $N = M$, $P_n = Q_n$ and $q = 2$ or p, these inequalities were proved by Bhatia and Kittaneh [7, Theorem 1 for $q = 2$ and Theorem 2 for $q = p$.]. They used them to prove that symmetrically normed ideals of $B(H)$ corresponding to Q^*-norms have Radon-Riesz property.

In this section we consider the case when $M = N = \infty$ and prove that, if $A \in S^p(H)$ and $2 \leq p < \infty$, then the partition $\mathcal{U} = \{P_n A Q_m\}_{n,m=1}^{\infty}$ belongs to $l_p(S^p)$ and

$$\left(\sum_{n,m=1}^{\infty} \|P_n A Q_m\|_p^p \right)^{1/p} \leq \|A\|_p \leq \left(\sum_{n,m=1}^{\infty} \|P_n A Q_m\|_p^2 \right)^{1/2}.$$

In other words,

$$\|\mathcal{U}\|_{l_p(S^p)} \leq \|A\|_p \leq \|\mathcal{U}\|_{l_2(S^p)}.$$

Note that, for $\mathcal{U} \notin l_2(S^p)$, we set $\|\mathcal{U}\|_{l_2(S^p)} = \infty$. For $1 \leq p \leq 2$, $\mathcal{U}$ belongs to $l_2(S^p)$ and satisfies the reversed inequalities.

We start with the following proposition.

Proposition 6.9 *Let* $A = (A_n)_{n=1}^{\infty} \in S^p(H, H^{\infty})$, $1 \leq p < \infty$ *and let* $\{A_n\}_{n=1}^{\infty}$ *have mutually orthogonal ranges, i.e.,*

$$A_k^* A_n = 0 \text{ if } k \neq n. \tag{6.40}$$

Let P_n *be the projections on the closure of the ranges* $\overline{A_n H}$ *of the operators* A_n. *Then the series* $\sum_{n=1}^{\infty} A_n$ *converges in* $\|\cdot\|_p$ *to some operator* $\widetilde{A} \in S^p(H)$ *such that* $\left\|\widetilde{A}\right\|_p = \|A\|_p$ *and all* $A_n = P_n \widetilde{A}$.

Proof. We have

$$\left(\sum_{n=m+1}^{m+k} A_n\right)^* = \left(\sum_{n=m+1}^{m+k} A_n^*\right)$$

It follows from (6.40) that, for all $m = 0, 1, ...$ and $k = 1, 2, ...$,

$$\begin{aligned}\left\|\sum_{n=m+1}^{m+k} A_n^* A_n\right\|_{p/2} &\overset{(6.40)}{=} \left\|\left(\sum_{n=m+1}^{m+k} A_n^*\right)\left(\sum_{n=m+1}^{m+k} A_n\right)\right\|_{p/2} \\ &= \left\|\left(\sum_{n=m+1}^{m+k} A_n\right)^*\left(\sum_{n=m+1}^{m+k} A_n\right)\right\|_{p/2} \overset{(5.4)}{=} \left\|\left(\sum_{n=m+1}^{m+k} A_n\right)\right\|_p^2.\end{aligned} \tag{6.41}$$

As all A_n belong to $S^p(H)$, the operators $A_n^* A_n$ belong to $S^{p/2}(H)$. Hence all operators $\sum_{n=1}^{m} A_n^* A_n$ also belong to $S^{p/2}(H)$. As $A \in S^p(H, H^{\infty})$, we have $A^* A \in S^{p/2}(H)$ (see (5.8)). Hence we obtain from (6.41)

$$\begin{aligned}\left\|\sum_{n=1}^{m+k} A_n - \sum_{n=1}^{m} A_n\right\|_p^2 &= \left\|\left(\sum_{n=m+1}^{m+k} A_n\right)\right\|_p^2 \overset{(6.41)}{=} \left\|\sum_{n=m+1}^{m+k} A_n^* A_n\right\|_{p/2} \\ &= \left\|\sum_{n=1}^{m+k} A_n^* A_n - A^* A + A^* A - \sum_{n=1}^{m} A_n^* A_n\right\|_{p/2}.\end{aligned} \tag{6.42}$$

If $1 \leq p \leq 2$ then $\frac{p}{2} \leq 1$. It follows from (5.11), (6.42) and (5.37) that

$$\left\| \sum_{n=1}^{m+k} A_n - \sum_{n=1}^{m} A_n \right\|_p^p \overset{(6.42)}{=} \left\| \sum_{n=1}^{m+k} A_n^* A_n - A^*A + A^*A - \sum_{n=1}^{m} A_n^* A_n \right\|_{p/2}^{p/2}$$

$$\overset{(5.11)}{\leq} 2 \left\| A^*A - \sum_{n=1}^{m+k} A_n^* A_n \right\|_{p/2}^{p/2} + 2 \left\| A^*A - \sum_{n=1}^{m} A_n^* A_n \right\|_{p/2}^{p/2} \overset{(5.37)}{\to} 0.$$

If $2 \leq p$ then $1 \leq \frac{p}{2}$ and we have

$$\left\| \sum_{n=1}^{m+k} A_n - \sum_{n=1}^{m} A_n \right\|_p^2 \overset{(6.42)}{=} \left\| \sum_{n=1}^{m+k} A_n^* A_n - A^*A + A^*A - \sum_{n=1}^{m} A_n^* A_n \right\|_{p/2}$$

$$\overset{(5.12)}{\leq} \left\| A^*A - \sum_{n=1}^{m+k} A_n^* A_n \right\|_{p/2} + \left\| A^*A - \sum_{n=1}^{m} A_n^* A_n \right\|_{p/2} \overset{(5.37)}{\to} 0.$$

We conclude that $\{\sum_{n=1}^{m} A_n\}_{m=1}^{\infty}$ is a Cauchy sequence in $S^p(H)$. Thus it converges in $\|\cdot\|_p$ to some operator $\widetilde{A} \in S^p(H)$:

$$\lim_{m\to\infty} \left\| \sum_{n=1}^{m} A_n - \widetilde{A} \right\|_p = 0, \tag{6.43}$$

so that

$$\left\| \widetilde{A} \right\|_p = \lim_{m\to\infty} \left\| \sum_{n=1}^{m} A_n \right\|_p . \tag{6.44}$$

Hence $\left\| \widetilde{A} \right\|_p = \|A\|_p$, since

$$\left\| \widetilde{A} \right\|_p^2 \overset{(6.44)}{=} \lim_{m\to\infty} \left\| \sum_{n=1}^{m} A_n \right\|_p^2 \overset{(6.41)}{=} \lim_{m\to\infty} \left\| \sum_{n=1}^{m} A_n^* A_n \right\|_{p/2}$$

$$\overset{(5.37)}{=} \|A^*A\|_{p/2} \overset{(5.54)}{=} \|A\|_p^2 .$$

It follows from (6.40) that all projections P_n are mutually orthogonal and that $P_n A_k = P_n P_k A_k = 0$ if $n \neq k$. Fix $n \in \mathbb{N}$. Then, for $m \geq n$,

$$\left\| P_n \widetilde{A} - A_n \right\|_p = \left\| P_n \left(\widetilde{A} - \sum_{k=1}^{m} A_k \right) \right\|_p \overset{(4.3)}{\leq} \|P_n\| \left\| \widetilde{A} - \sum_{k=1}^{m} A_k \right\|_p \underset{m\to\infty}{\overset{(6.43)}{\to}} 0.$$

Thus $\left\|P_n\widetilde{A} - A_n\right\|_p = 0$. Hence $P_n\widetilde{A} - A_n = 0$, so that $P_n\widetilde{A} = A_n$. ■

Theorem 6.10 *Let $\{P_n\}_{n=1}^{\infty}$ be a partition of $\mathbf{1}_H$. Let $A \in S^p(H)$. Then*

$$\left(\sum_{n=1}^{\infty} \|P_n A\|_p^2\right)^{1/2} \leq \|A\|_p \leq \left(\sum_{n=1}^{\infty} \|P_n A\|_p^p\right)^{1/p} \text{ for } 1 \leq p \leq 2,$$

where the last series could diverge. For $2 \leq p < \infty$, the above inequalities are reversed.

Proof. It follows from Definition 5.7 that $Q_m = \sum_{n=1}^{m} P_n \stackrel{\text{s.o.t.}}{\to} \mathbf{1}_H$, as $m \to \infty$:

$$\|x - Q_m x\| \to 0 \text{ for all } x \in H. \tag{6.45}$$

We have $Q_m^* = \sum_{n=1}^{m} P_n^* = \sum_{n=1}^{m} P_n = Q_m$. As $P_n P_k = 0$, if $k \neq n$, we also have $Q_m^2 = \left(\sum_{n=1}^{m} P_n\right)^2 = \sum_{n=1}^{m} P_n = Q_m$. Thus Q_m are projections and

$$\begin{aligned} \|Q_m x\|^2 &= (Q_m x, Q_m x) = (Q_m^* Q_m x, x) = (Q_m x, x) \\ &= \sum_{n=1}^{m} (P_n x, x) = \sum_{n=1}^{m} (P_n x, P_n x) = \sum_{n=1}^{m} \|P_n x\|^2 . \end{aligned}$$

Hence, by (6.45), $\sum_{n=1}^{m} \|P_n x\|^2 = \|Q_m x\|^2 \to \|x\|^2$, as $m \to \infty$. Thus

$$\|x\|^2 = \sum_{n=1}^{\infty} \|P_n x\|^2 \text{ for } x \in H. \tag{6.46}$$

Set $A_n = P_n A$. As $A \in S^p(H)$, all A_n belong to $S^p(H)$, since $S^p(H)$ is an ideal of $B(H)$, and all A_n satisfy (6.40):

$$A_k^* A_n = A^* P_k^* P_n A = A^* 0 A = 0 \text{ if } k \neq n. \tag{6.47}$$

Consider the operator $\bar{A} = (A_n)_{n=1}^{\infty}$ from H to H^{∞}. For $x \in H$,

$$\left\|\bar{A}x\right\|_{H^{\infty}}^{2} = \sum_{n=1}^{\infty} \|A_n x\|^2 = \sum_{n=1}^{\infty} \|P_n A x\|^2 \overset{(6.46)}{=} \|Ax\|^2 .$$

Hence $\bar{A} \in B(H, H^{\infty})$ and $\left\|\bar{A}\right\|_{B(H,H^{\infty})} = \|A\|_{B(H)}$.

We shall now prove that $\bar{A}^*\bar{A} = A^*A$. For all $x, y \in H$, we have

$$\left(\bar{A}^*\bar{A}x - \sum_{n=1}^{m} A_n^* A_n x, y\right) \overset{(5.28)}{\to} 0$$

as $m \to \infty$. We also have

$$\begin{aligned} \left|\left(\sum_{n=1}^{m} A_n^* A_n x - A^*Ax, y\right)\right| &= \left|\left(\sum_{n=1}^{m} (P_n A)^* P_n A x - A^*Ax, y\right)\right| \\ &= \left|\left(\sum_{n=1}^{m} A^* P_n A x - A^*Ax, y\right)\right| \\ &= \left|(Q_m A x - Ax, Ay)\right| \overset{(6.45)}{\to} 0, \end{aligned}$$

as $m \to \infty$. Thus

$$\begin{aligned} &\left|\left(\bar{A}^*\bar{A}x - A^*Ax, y\right)\right| \\ &\leq \left|\left(\bar{A}^*\bar{A}x - \sum_{n=1}^{m} A_n^* A_n x, y\right)\right| + \left|\left(\sum_{n=1}^{m} A_n^* A_n x - A^*Ax, y\right)\right| \underset{m\to\infty}{\to} 0. \end{aligned}$$

Hence $\left(\bar{A}^*\bar{A}x, y\right) = (A^*Ax, y)$ for all $x, y \in H$, so that $\bar{A}^*\bar{A} = A^*A$.

Applying (5.5) we have that $\bar{A}^*\bar{A} \in S^{p/2}(H)$. It follows from (5.54) that $\bar{A} \in S^p(H, H^{\infty})$ and $\left\|\bar{A}\right\|_p = \|A\|_p$.

Let $1 \le p < 2$. It follows from theorem 5.22(i) that $\bar{A} \in l_2(S^p)$ and

$$\left(\sum_{n=1}^{\infty} \|P_n A\|_p^2\right)^{1/2} = \left(\sum_{n=1}^{\infty} \|A_n\|_p^2\right)^{1/2} \overset{(5.58)}{=} \|\bar{A}\|_{l_2(S^p)} \overset{(5.61)}{\le} \|\bar{A}\|_p$$

$$= \|A\|_p \overset{(5.61)}{\le} \|\bar{A}\|_{l_p(S^p)} \overset{(5.58)}{=} \left(\sum_{n=1}^{\infty} \|P_n A\|_p^p\right)^{1/p},$$

where the last series could diverge if $\bar{A} \notin l_p(S^p)$.

Let $2 \le p < \infty$. By Theorem 5.22(ii) and (iii), $\bar{A} \in l_p(S^p)$ and

$$\left(\sum_{n=1}^{\infty} \|P_n A\|_p^p\right)^{1/p} = \left(\sum_{n=1}^{\infty} \|A_n\|_p^p\right)^{1/p} \overset{(5.58)}{=} \|\bar{A}\|_{l_p(S^p)}$$

$$\overset{(5.62)}{\le} \|\bar{A}\|_p = \|A\|_p \overset{(5.62)}{\le} \|\bar{A}\|_{l_2(S^p)} \overset{(5.58)}{=} \left(\sum_{n=1}^{\infty} \|P_n A\|_p^2\right)^{1/2},$$

where the last series could diverge if $\bar{A} \notin l_2(S^p)$. The proof is complete. ■

Consider now partitions of operators. If $A \notin l_q(S^p)$, we set $\|A\|_{l_q(S^p)} = \infty$.

Theorem 6.11 *Let $\{P_n\}_{n=1}^{\infty}$ and $\{Q_k\}_{k=1}^{\infty}$ be partitions of $\mathbf{1}_H$. Let $A \in S^p(H)$ and $\mathcal{U} = \{P_n A Q_k\}_{n,k=1}^{\infty}$ be the partition of A.*

(i) *If $1 \le p \le 2$, then $\mathcal{U} \in l_2(S^p)$ and*

$$\|\mathcal{U}\|_{l_2(S^p)} = \left(\sum_{n,k=1}^{\infty} \|P_n A Q_k\|_p^2\right)^{1/2} \le \|A\|_p \le \left(\sum_{n,k=1}^{\infty} \|P_n A Q_k\|_p^p\right)^{1/p} = \|\mathcal{U}\|_{l_p(S^p)}.$$

(ii) *Let $2 \le p < \infty$. Then $\mathcal{U} \in l_p(S^p)$ and*

$$\|\mathcal{U}\|_{l_p(S^p)} = \left(\sum_{n,k=1}^{\infty} \|P_n A Q_k\|_p^p\right)^{1/p} \le \|A\|_p \le \left(\sum_{n,k=1}^{\infty} \|P_n A Q_k\|_p^2\right)^{1/2} = \|\mathcal{U}\|_{l_2(S^p)}.$$

Proof. (i) Let $1 \le p \le 2$. It follows from Theorem 6.10 that

$$\left(\sum_{n=1}^{\infty} \|P_n A\|_p^2\right)^{1/2} \le \|A\|_p \le \left(\sum_{n=1}^{\infty} \|P_n A\|_p^p\right)^{1/p}, \tag{6.48}$$

where the last series above could diverge.

Fix n and set $B_n = A^* P_n$. As $A \in S^p(H)$ and $S^p(H)$ is a s.n. ideal of $B(H)$, we have $B_n \in S^p(H)$. Replacing in (6.48), A by B_n and $\{P_n\}_{n=1}^{\infty}$ by $\{Q_k\}_{k=1}^{\infty}$ we obtain

$$\left(\sum_{k=1}^{\infty} \|Q_k B_n\|_p^2\right)^{1/2} \leq \|B_n\|_p \leq \left(\sum_{k=1}^{\infty} \|Q_k B_n\|_p^p\right)^{1/p}, \tag{6.49}$$

where the last series above could diverge. Since, by (4.3), $\|B_n\|_p = \|B_n^*\|_p = \|P_n A\|_p$ and

$$\|Q_k B_n\|_p = \|Q_k A^* P_n\|_p \overset{(4.3)}{=} \|(Q_k A^* P_n)^*\|_p = \|P_n A Q_k\|_p,$$

we can rewrite (6.49) as follows:

$$\left(\sum_{k=1}^{\infty} \|P_n A Q_k\|_p^2\right)^{1/2} \leq \|P_n A\|_p \leq \left(\sum_{k=1}^{\infty} \|P_n A Q_k\|_p^p\right)^{1/p}, \text{ for each } n,$$

and obtain

$$\sum_{k=1}^{\infty} \|P_n A Q_k\|_p^2 \leq \|P_n A\|_p^2 \text{ and } \|P_n A\|_p^p \leq \sum_{k=1}^{\infty} \|P_n A Q_k\|_p^p, \text{ for each } n.$$

Thus summing up for n, we get

$$\left(\sum_{n,k=1}^{\infty} \|P_n A Q_k\|_p^2\right)^{1/2} \leq \left(\sum_{n=1}^{\infty} \|P_n A\|_p^2\right)^{1/2}$$

and

$$\left(\sum_{n=1}^{\infty} \|P_n A\|_p^p\right)^{1/p} \leq \left(\sum_{n,k=1}^{\infty} \|P_n A Q_k\|_p^p\right)^{1/p}.$$

We can now apply (6.48) to obtain

$$\begin{aligned}\left(\sum_{n,k=1}^{\infty}\|P_nAQ_k\|_p^2\right)^{1/2} &\leq \left(\sum_{n=1}^{\infty}\|P_nA\|_p^2\right)^{1/2} \leq \|A\|_p \\ &\leq \left(\sum_{n=1}^{\infty}\|P_nA\|_p^p\right)^{1/p} \leq \left(\sum_{n,k=1}^{\infty}\|P_nAQ_k\|_p^p\right)^{1/p}.\end{aligned}$$

Clearly, on the left we have $\|\mathcal{U}\|_{l_2(S^p)} = \left(\sum_{n,k=1}^{\infty}\|P_nAQ_k\|_p^2\right)^{1/2}$ and on the right we have $\left(\sum_{n,k=1}^{\infty}\|P_nAQ_k\|_p^p\right)^{1/p} = \|\mathcal{U}\|_{l_p(S^p)}$. This ends the proof of (i).

(ii) Let $2 \leq p < \infty$. It follows from Theorem 6.10 that

$$\left(\sum_{n=1}^{\infty}\|P_nA\|_p^p\right)^{1/p} \leq \|A\|_p \leq \left(\sum_{n=1}^{\infty}\|P_nA\|_p^2\right)^{1/2}, \tag{6.50}$$

where the last series could diverge. Proceeding now, as in (i), fix n and set $B_n = A^*P_n$. Then $B_n \in S^p(H)$. Replacing in (6.50) A by B_n and $\{P_n\}_{n=1}^{\infty}$ by $\{Q_k\}_{k=1}^{\infty}$ we get

$$\left(\sum_{k=1}^{\infty}\|Q_kB_n\|_p^p\right)^{1/p} \leq \|B_n\|_p \leq \left(\sum_{k=1}^{\infty}\|Q_kB_n\|_p^2\right)^{1/2}, \tag{6.51}$$

where the last series can diverge. Since, as in (i), $\|B_n\|_p = \|B_n^*\|_p = \|P_nA\|_p$ and $\|Q_kB_n\|_p = \|P_nAQ_k\|_p$, we can rewrite (6.51) as

$$\left(\sum_{k=1}^{\infty}\|P_nAQ_k\|_p^p\right)^{1/p} \leq \|P_nA\|_p \leq \left(\sum_{k=1}^{\infty}\|P_nAQ_k\|_p^2\right)^{1/2}, \text{ for each } n.$$

Hence $\sum_{k=1}^{\infty}\|P_nAQ_k\|_p^p \leq \|P_nA\|_p^p$ and $\|P_nA\|_p^2 \leq \sum_{k=1}^{\infty}\|P_nAQ_k\|_p^2$, for each n. Thus summing up for n, we get

$$\left(\sum_{n,k=1}^{\infty}\|P_nAQ_k\|_p^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty}\|P_nA\|_p^p\right)^{1/p}$$

and

$$\left(\sum_{n=1}^{\infty}\|P_nA\|_p^2\right)^{1/2} \leq \left(\sum_{n,k=1}^{\infty}\|P_nAQ_k\|_p^2\right)^{1/2}.$$

We can now apply (6.50) to obtain

$$\begin{aligned}\left(\sum_{n,k=1}^{\infty}\|P_nAQ_k\|_p^p\right)^{1/p} &\leq \left(\sum_{n=1}^{\infty}\|P_nA\|_p^p\right)^{1/p} \leq \|A\|_p \\ &\leq \left(\sum_{n=1}^{\infty}\|P_nA\|_p^2\right)^{1/2} \leq \left(\sum_{n,k=1}^{\infty}\|P_nAQ_k\|_p^2\right)^{1/2}.\end{aligned}$$

Clearly, on the right we have $\|\mathcal{U}\|_{l_2(S^p)} = \left(\sum_{n,k=1}^{\infty}\|P_nAQ_k\|_p^2\right)^{1/2}$ and on the left we have $\left(\sum_{n,k=1}^{\infty}\|P_nAQ_k\|_p^p\right)^{1/p} = \|\mathcal{U}\|_{l_p(S^p)}$. The proof is complete. ■

The case of finite families $\{P_n\}$ and $\{Q_k\}$ was studied in [7] and [25].

6.6 Cartesian decomposition of operators

In this chapter we analyse the following natural involution on $l_\infty(B(H))$:

$$A^{\#} = (A_n^*)_{n=1}^{\infty} = \begin{pmatrix} A_1^* \\ \vdots \\ A_n^* \\ \vdots \end{pmatrix}, \text{ for each } A = (A_n)_{n=1}^{\infty} = \begin{pmatrix} A_1 \\ \vdots \\ A_n \\ \vdots \end{pmatrix} \in l_\infty(B(H)).$$

Then $\left(A^{\#}\right)^{\#} = A$ and $\#$ preserves all spaces $l_q(S^p)$, since, by (4.3),

$$\|A\|_{l_q(S^p)} = \left\|A^{\#}\right\|_{l_q(S^p)}, \text{ for all } A \in l_q(S^p). \tag{6.52}$$

However, if $A = (A_n)_{n=1}^{\infty} \in S^p(H, H^\infty)$ then $A^{\#}$ does not necessarily belong to $S^p(H, H^\infty)$. We will now construct an example to show this.

Example 6.12 Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis in H. Let $A = (A_n)_{n=1}^{\infty}$, where each A_n has matrix form $A_n = (a_n^{km})$ with respect to $\{e_k\}_{k=1}^{\infty}$ such that $a_n^{n1} > 0$ for all n, and all other $a_n^{km} = 0$, i.e.,.

$$A_n = \begin{pmatrix} 0 & 0 & \cdots \\ \vdots & \vdots & \\ 0 & 0 & \cdots \\ & & \\ a_n^{n1} & 0 & \cdots \\ & & \\ 0 & 0 & \cdots \\ \vdots & \vdots & \end{pmatrix}, \text{ and } A_n^* = \begin{pmatrix} 0 & \cdots & 0 & a_n^{n1} & 0 & \cdots \\ & & & & & \\ 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Then $A_n^*A_n = (c_n^{km}) \in B(H)$ and $A_nA_n^* = (d_n^{km}) \in B(H)$, where $c_n^{11} = (a_n^{n1})^2$ and all other $c_n^{km} = 0$, and $d_n^{nn} = (a_n^{n1})^2$ and all other $d_n^{km} = 0$, i.e.,

$$A_n^*A_n = \begin{pmatrix} (a_n^{n1})^2 & 0 & \cdots \\ & & \\ 0 & 0 & \cdots \\ \vdots & \vdots & \end{pmatrix} \text{ and } A_nA_n^* = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \\ 0 & \cdots & 0 & 0 & 0 & \cdots \\ & & & & & \\ 0 & \ldots & 0 & (a_n^{n1})^2 & 0 & \cdots \\ & & & & & \\ 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \end{pmatrix}.$$

1) Let $p \in [1, 2)$. Set $a_n^{n1} = \frac{1}{n^{1/p}}$ for all n. Then $A^*A = \sum A_n^*A_n = \left(c^{km}\right) \in B(H)$, where $c^{11} = \sum_{n=1}^{\infty} (a_n^{n1})^2 = \sum_{n=1}^{\infty} \frac{1}{n^{2/p}} < \infty$, as $p < 2$, and all other $c^{km} = 0$. Thus $A^*A = c^{11}P_{e_1}$ and $|A| = (c^{11})^{1/2} P_{e_1}$ is a multiple of the projection P_{e_1} on the subspace Ce_1, so that $|A| \in S^p(H)$. Hence $A \in S^p(H, H^{\infty})$.

On the other hand, $\left(A^{\#}\right)^* A^{\#} = \sum A_nA_n^* = \left(d^{km}\right) \in B(H)$ is a diagonal operator with $d^{nn} = \left(a_n^{n1}\right)^2 = \frac{1}{n^{2/p}}$, for all n, and all other $d^{km} = 0$, i.e.,

$$\left(A^{\#}\right)^* A^{\#} = \begin{pmatrix} \frac{1}{1^{2/p}} & 0 & \cdots \\ 0 & \frac{1}{2^{2/p}} & \cdots \\ \vdots & \vdots & \cdots \end{pmatrix}.$$

Therefore $s_n\left(A^{\#}\right) = \frac{1}{n^{1/p}}$ and $A^{\#} \notin S^p(H, H^\infty)$, since

$$\left\|A^{\#}\right\|_p^p = \sum_{n=1}^{\infty} \left(n^{-1/p}\right)^p = \sum_{n=1}^{\infty} n^{-1} \text{ diverges.}$$

2) Let $p \in [2, \infty)$. Set $a_n^{n1} = n^{-1/2}$ for all n. Then $A^*A = \sum A_n^* A_n = \left(c^{km}\right) \notin B(H)$, since $c^{11} = \sum_{n=1}^{\infty} \left(a_n^{n1}\right)^2 = \sum_{n=1}^{\infty} n^{-1}$ diverges, as $p > 2$. Hence $A \notin S^p(H, H^\infty)$.

On the other hand, $\left(A^{\#}\right)^* A^{\#}$ (see above) is a diagonal operator with $d^{nn} = \left(a_n^{n1}\right)^2 = n^{-1}$, for all n, and all other $d^{km} = 0$, i.e.,

$$\left(A^{\#}\right)^* A^{\#} = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & \frac{1}{2} & \cdots \\ \vdots & \vdots & \cdots \end{pmatrix}.$$

Therefore $s_n\left(A^{\#}\right) = \frac{1}{n^{1/2}}$ and $A^{\#} \in S^p(H, H^\infty)$, since

$$\left\|A^{\#}\right\|_p^p = \sum_{n=1}^{\infty} \frac{1}{n^{p/2}} < \infty, \text{ as } p > 2.$$

Thus $B = A^{\#} \in S^p(H, H^\infty)$, while $B^{\#} = \left(A^{\#}\right)^{\#} = A \notin S^p(H, H^\infty)$.

All $l_q(S^p)$ spaces are symmetrically normed ideals of the Banach*-algebra $l_\infty(B(H))$ with involution # and multiplication

$$AB = \begin{pmatrix} A_1B_1 \\ \vdots \\ A_nB_n \\ \vdots \end{pmatrix}, \text{ for } A = (A_n)_{n=1}^{\infty}, B = (B_n)_{n=1}^{\infty} \in l_\infty(B(H)).$$

Indeed, let $A \in l_q(S^p)$ and $T, B \in l_\infty(B(H))$, then

$$\|TAB\|_{l_q(S^p)} = \left\| \begin{pmatrix} T_1A_1B_1 \\ \vdots \\ T_nA_nB_n \\ \vdots \end{pmatrix} \right\|_{l_q(S^p)}$$

$$= \left(\sum_{n=1}^{\infty} \|T_nA_nB_n\|_p^q \right)^{1/q} \overset{(4.3)}{\leq} \left(\sum_{n=1}^{\infty} \|T_n\|^q \|A_n\|_p^q \|B_n\|^q \right)^{1/q}$$

$$\leq \left(\sup_n \|T_n\|^q \sup_n \|B_n\|^q \sum_{n=1}^{\infty} \|A_n\|_p^q \right)^{1/q} = \|T\|_{l_\infty(B(H))} \|B\|_{l_\infty(B(H))} \|A\|_{l_q(S^p)}.$$

Let $A_n = X_n + iY_n$ be the "Cartesian decomposition" of all A_n in A, where

$$X_n = \frac{1}{2}(A_n + A_n^*) \text{ and } Y_n = \frac{1}{2i}(A_n - A_n^*)$$

are self-adjoint operators. Indeed,

$$X_n^* = ((A_n + A_n^*)/2)^* = (A_n^* + A_n)/2 = X_n$$

and, similarly, $Y_n^* = Y_n$. Set $X = (X_n)_{n=1}^{\infty}$ and $Y = (Y_n)_{n=1}^{\infty}$, so that

$$X = \frac{1}{2}\left(A + A^{\#}\right) \text{ and } Y = \frac{1}{2i}\left(A - A^{\#}\right). \tag{6.53}$$

Then $X, Y \in l_q(S^p)$. Indeed as $\|X_n\|_p^q \le \frac{1}{2}\left(\|A_n\|_p + \|A_n^*\|_p\right) = \|A_n\|_p$, we have

$$\|X\|_{l_q(S^p)} = \left(\sum_{n=1}^{\infty} \|X_n\|_p^q\right)^{1/q} \le \left(\sum_{n=1}^{\infty} \|A_n\|_p^q\right)^{1/q} = \|A\|_{l_q(S^p)} < \infty.$$

Similarly, $\|Y\|_{l_q(S^p)} \le \|A\|_{l_q(S^p)} < \infty$.

Theorem 6.13 *Let $A \in l_q(S^p)$, where $p \in [1, \infty)$ and $q \in [\min(p, 2), \max(p, 2)]$, and let $X = \frac{1}{2}\left(A + A^{\#}\right)$ and $Y = \frac{1}{2i}\left(A - A^{\#}\right)$. Then*

$$2^{\frac{1}{q}-\frac{1}{2}-\left|\frac{1}{p}-\frac{1}{2}\right|} \|A\|_{l_q(S^p)} \le \left(\|X\|_{l_q(S^p)}^q + \|Y\|_{l_q(S^p)}^q\right)^{1/q} \le 2^{\frac{1}{q}-\frac{1}{2}+\left|\frac{1}{p}-\frac{1}{2}\right|} \|A\|_{l_q(S^p)}.$$

Proof. Replacing Y by iY in Corollary 6.4(ii), we have

$$\left(\|X + iY\|_{l_q(S^p)}^q + \|X - iY\|_{l_q(S^p)}^q\right)^{1/q} \le 2^{\left|\frac{1}{p}-\frac{1}{2}\right|+\frac{1}{2}} \left(\|X\|_{l_q(S^p)}^q + \|iY\|_{l_q(S^p)}^q\right)^{1/q}.$$

Replacing now $X + iY$ by A and $X - iY$ by $A^{\#}$, we have

$$\left(\|A\|_{l_q(S^p)}^q + \left\|A^{\#}\right\|_{l_q(S^p)}^q\right)^{1/q} \le 2^{\left|\frac{1}{p}-\frac{1}{2}\right|+\frac{1}{2}} \left(\|X\|_{l_q(S^p)}^q + \|iY\|_{l_q(S^p)}^q\right)^{1/q}.$$

Then, applying (6.52), we obtain

$$2^{\frac{1}{q}} \|A\|_{l_q(S^p)} \le 2^{\left|\frac{1}{p}-\frac{1}{2}\right|+\frac{1}{2}} \left(\|X\|_{l_q(S^p)}^q + \|iY\|_{l_q(S^p)}^q\right)^{1/q}.$$

Thus, the left-hand side inequality holds:

$$2^{\frac{1}{q}-\frac{1}{2}-\left|\frac{1}{p}-\frac{1}{2}\right|} \|A\|_{l_q(S^p)} \le \left(\|X\|_{l_q(S^p)}^q + \|Y\|_{l_q(S^p)}^q\right)^{1/q}.$$

Replacing now X by A and Y by $A^{\#}$ in Corollary 6.4(ii), we have

$$\left(\left\|A + A^{\#}\right\|_{l_q(S^p)}^q + \left\|A - A^{\#}\right\|_{l_q(S^p)}^q\right)^{1/q} \le 2^{\left|\frac{1}{p}-\frac{1}{2}\right|+\frac{1}{2}} \left(\|A\|_{l_q(S^p)}^q + \left\|A^{\#}\right\|_{l_q(S^p)}^q\right)^{1/q}.$$

Applying (6.53) and (6.52), and rearranging, we obtain the right-hand side inequality:

$$\left(\|X\|_{l_q(S^p)}^q + \|Y\|_{l_q(S^p)}^q\right)^{1/q} \leq 2^{\frac{1}{q}-\frac{1}{2}+\left|\frac{1}{p}-\frac{1}{2}\right|} \|A\|_{l_q(S^p)} .$$

The proof is complete. ■

In [25, Theorem 5 (ii)] Kissin proved a result for $l_q^n(S^p)$ spaces similar to Theorem 6.13.

Remark 6.14 Doing the same replacements in Corollary 6.4(i) instead of (ii), as we did in the proof of Theorem 6.13, we get the following inequalities: for $p \in [1,2]$ and $A \in l_p(S^p)$,

$$\|A\|_{l_2(S^p)} \leq \left(\|X\|_{l_p(S^p)}^p + \|Y\|_{l_p(S^p)}^p\right)^{1/p}, \tag{6.54}$$

$$\left(\|X\|_{l_2(S^p)}^2 + \|Y\|_{l_2(S^p)}^2\right)^{1/2} \leq 2^{\frac{1}{p}-\frac{1}{2}} \|A\|_{l_p(S^p)} ; \tag{6.55}$$

for $p \in [2,\infty)$ and $A \in l_2(S^p)$,

$$\|A\|_{l_p(S^p)} \leq 2^{\frac{1}{2}-\frac{1}{p}} \left(\|X\|_{l_2(S^p)}^2 + \|Y\|_{l_2(S^p)}^2\right)^{1/2}, \tag{6.56}$$

$$\left(\|X\|_{l_p(S^p)}^p + \|Y\|_{l_p(S^p)}^p\right)^{1/p} \leq \|A\|_{l_2(S^p)} . \tag{6.57}$$

To get (6.54), replace in (6.35) Y by iY and, consequently, replace $X+iY$ by A and $X-iY$ by $A^{\#}$. To get (6.55), we replace X by A and Y by $A^{\#}$.

To get (6.56), replace in (6.36) Y by iY and, consequently, replace $X+iY$ by A and $X-iY$ by $A^{\#}$. To get (6.57), we replace X by A and Y by $A^{\#}$.

However, these estimates could be deduced from Theorem 6.13. Indeed, let $p \in [1,2]$ and $A \in l_p(S^p)$. Then $\left|\frac{1}{p}-\frac{1}{2}\right| = \frac{1}{p}-\frac{1}{2}$ and $\|A\|_{l_2(S^p)} \leq \|A\|_{l_p(S^p)}$. Setting $q = p$ in the LHS of the inequality in Theorem 6.13, we obtain (6.54):

$$\|A\|_{l_2(S^p)} \leq \|A\|_{l_p(S^p)} \leq \left(\|X\|_{l_p(S^p)}^p + \|Y\|_{l_p(S^p)}^p\right)^{1/p}.$$

If we set $q = 2$ in the RHS inequality Theorem 6.13, we get (6.55):

$$\left(\|X\|_{l_2(S^p)}^2 + \|Y\|_{l_2(S^p)}^2\right)^{1/2} \leq 2^{\frac{1}{p}-\frac{1}{2}} \|A\|_{l_2(S^p)} \leq 2^{\frac{1}{p}-\frac{1}{2}} \|A\|_{l_p(S^p)}.$$

Let now $p \in [2,\infty)$ and $A \in l_2(S^p)$. Then $\left|\frac{1}{p}-\frac{1}{2}\right| = \frac{1}{2}-\frac{1}{p}$ and by (5.62) $\|A\|_{l_p(S^p)} \leq \|A\|_{l_2(S^p)}$. Setting $q = 2$ in the LHS inequality in Theorem 6.13, we obtain (6.56):

$$\|A\|_{l_p(S^p)} \leq \|A\|_{l_2(S^p)} \leq 2^{\frac{1}{2}-\frac{1}{p}} \left(\|X\|_{l_2(S^p)}^2 + \|Y\|_{l_2(S^p)}^2\right)^{1/2}.$$

Setting $q = p$ in the RHS inequality Theorem 6.13,we obtain (6.57):

$$\left(\|X\|_{l_p(S^p)}^p + \|Y\|_{l_p(S^p)}^p\right)^{1/p} \leq \|A\|_{l_p(S^p)} \leq \|A\|_{l_2(S^p)}.$$

Although the involution $\#$ preserves all spaces $l_q(S^p)$, it does not preserve $S^p(H, H^\infty)$, if $p \neq 2$ (see Example 6.12). Since, by Theorem 5.22(iii), $S^2(H, H^\infty) = l_2(S^2)$, the involution $\#$ preserves $S^2(H, H^\infty)$.

Set $S^b(H, H^\infty) = B(H, H^\infty)$ and $S^\infty(H, H^\infty) = C(H, H^\infty)$ - the space of all compact operators from H to H^∞. For each $p \in [1,\infty] \cup \{b\}$, set

$$D_p(\#) = \left\{A \in S^p(H, H^\infty) : A^\# \in S^p(H, H^\infty)\right\}.$$

Note that $D_p(\#)$ is the maximal linear subspace of $S^p(H,H^\infty)$ preserved by $\#$. Indeed, if $A \in D_p(\#)$ then $A \in S^p(H,H^\infty)$ and $A^\# \in S^p(H,H^\infty)$. Thus $A^\# \in D_p(\#)$, as $A^\# \in S^p(H,H^\infty)$ and $\left(A^\#\right)^\# = A \in S^p(H,H^\infty)$.

If $A, B \in D_p(\#)$ then $A, B \in S^p(H,H^\infty)$ and $A^\#, B^\# \in S^p(H,H^\infty)$. Hence $\alpha A + \beta B \in S^p(H,H^\infty)$ and $(\alpha A + \beta B)^\# = \overline{\alpha} A^\# + \overline{\beta} B^\# \in S^p(H,H^\infty)$. Thus $\alpha A + \beta B \in D_p(\#)$, so that $D_p(\#)$ is a linear subspace of $S^p(H,H^\infty)$. To see that $D_p(\#)$ is maximal, assume that there is a linear subspace D of $S^p(H,H^\infty)$ preserved by $\#$ such that $D_p(\#) \subsetneqq D$. If $A \in D \setminus D_p(\#)$ then $A^\# \in D \subseteq S^p(H,H^\infty)$. Hence $A \in D_p(\#)$, so that $D_p(\#) = D$. Thus $D_p(\#)$ is the maximal linear subspace of $S^p(H,H^\infty)$ preserved by $\#$.

Proposition 6.15 (i) *If* $1 \le p < 2$, *then* $l_p(S^p) \subsetneqq D_p(\#) \subsetneqq S^p(H,H^\infty)$.

(ii) $S^2(H,H^\infty) = D_2(\#)$.

(iii) *If* $2 < p \le \infty$ *then* $D_p(\#) \subsetneqq S^p(H,H^\infty) \nsubseteq D_b(\#)$.

Proof. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis in H, P_{e_n} be the projections on C_{e_n} and let $\{V_n\}_{n=1}^\infty$ be partial isometries from C_{e_n} on C_{e_1}: $V_n e_n = e_1$ and $V_n e_j = 0$, for $j \neq n$. Then, for all n, $V_n^* e_1 = e_n$ and $V_n^* e_k = 0$ if $k \neq 1$. Thus

$$P_{e_n} = V_n^* V_n \text{ and } P_{e_1} = V_n V_n^*. \tag{6.58}$$

(i) Let $1 \le p < 2$. By Theorem 5.22(i), $l_p(S^p) \subset S^p(H,H^\infty)$. As $\#$ preserves all $l_q(S^p)$, we have $l_p(S^p) \subseteq D_p(\#) \subseteq S^p(H,H^\infty)$ It follows from Example 6.12

that $D_p(\#) \neq S^p(H, H^\infty)$. Thus to finish the proof of (i), we need to show that $l_p(S^p) \neq D_p(\#)$.

Let $A = (A_n)_{n=1}^\infty \in l_\infty(B(H))$ where $A_n = n^{-\frac{1}{p}} P_{e_1}$. As $\|P_{e_1}\|_p = 1$,

$$\|A\|_{l_p(S^p)}^p = \sum_{n=1}^\infty \|A_n\|_p^p = \sum_{n=1}^\infty \left\| n^{-\frac{1}{p}} P_{e_1} \right\|_p^p = \sum_{n=1}^\infty n^{-1} \text{ diverges.}$$

Hence $A \notin l_p(S^p)$.

Let $u = \sum_{n=1}^\infty \oplus n^{-\frac{1}{p}} e_1$, where each $n^{-\frac{1}{p}} e_1$ lies in the n-th component of $H^\infty = l_2(H)$. Then $\|u\|^2 = \sum_{n=1}^\infty n^{-\frac{2}{p}} < \infty$. Thus $u \in H^\infty$. For each $x = \sum_{n=1}^\infty \alpha_n e_n \in H$, we have $A_n x = n^{-\frac{1}{p}} P_{e_1} x = n^{-\frac{1}{p}} \alpha_1 e_1$, so that

$$Ax = \sum_{n=1}^\infty \oplus A_n x = \sum_{n=1}^\infty \oplus n^{-\frac{1}{p}} \alpha_1 e_1 = \alpha_1 \sum_{n=1}^\infty \oplus n^{-\frac{1}{p}} e_1 = \alpha_1 u.$$

Hence $A = e_1 \oplus u$ is a rank one operator in $B(H, H^\infty)$, i.e., $Ax = (e_1 \oplus u)x = (x, e_1)u = \alpha_1 u$. Thus $A \in S^p(H, H^\infty)$ (see (5.59) and (5.60)). Moreover, since for each n, $A_n^* = A_n$, we have that $A^\# = A$. Thus $A \in D_p(\#)$. We proved earlier that $A \notin l_p(S^p)$. Therefore $l_p(S^p) \neq D_p(\#)$.

(ii) From Theorem 5.22(iii) it follows that $S^2(H, H^\infty) = l_2(S^2)$.As $\#$ preserves $l_2(S^2)$, we have $S^2(H, H^\infty) = D_2(\#)$.

(iii) Let $2 < p \leq \infty$. It follows from Example 6.12 that $D_p(\#) \neq S^p(H, H^\infty)$. Thus we only need to show that $S^p(H, H^\infty) \nsubseteq D_b(\#)$. To prove this, we shall construct an operator $A = (A_n)_{n=1}^\infty \in S^p(H, H^\infty)$ such that $A \notin D_b(\#)$.

Set $A_n = n^{-\frac{1}{2}} V_n$, for $n \in \mathbb{N}$. The operator $\sum_{n=1}^m n^{-1} P_{e_n} = \sum_{n=1}^m n^{-1} (\cdot, e_n) e_n$ is self-adjoint, for all m, and its eigenvalues are exactly all n^{-1} (see Corollary 2.36).

Hence

$$\|P_m A\|_p^2 \overset{(5.54)}{=} \|(P_m A)^* P_m A\|_{p/2} \overset{(5.28)}{=} \left\|\sum_{n=1}^{m} A_n^* A_n\right\|_{p/2}$$

$$= \left\|\sum_{n=1}^{m} n^{-1} V_n^* V_n\right\|_{p/2} \overset{(6.58)}{=} \left\|\sum_{n=1}^{m} n^{-1} P_{e_n}\right\|_{p/2}$$

$$= \left(\sum_{n=1}^{m} \left(n^{-1}\right)^{p/2}\right)^{2/p} = \left(\sum_{n=1}^{m} n^{-\frac{p}{2}}\right)^{2/p} < \infty, \text{ as } p > 2.$$

Therefore, by Lemma 5.21(ii), $A \in S^p(H, H^\infty)$ for all $2 < p < \infty$. As $S^p(H, H^\infty) \subset S^\infty(H, H^\infty) = C(H, H^\infty)$, we have $A \in S^\infty(H, H^\infty)$. On the other hand, $A^\#$ does not belong to $B(H, H^\infty)$, since

$$A^\# e_1 = (A_n^* e_1)_{n=1}^\infty = \left(n^{-\frac{1}{2}} V_n^* e_1\right)_{n=1}^\infty = \left(n^{-\frac{1}{2}} e_n\right)_{n=1}^\infty$$

and

$$\begin{aligned} \left\|A^\#\right\|_{B(H,H^\infty)} &= \sup_{\|x\|=1} \left\|A^\# x\right\|_{H^\infty} \geq \left\|A^\# e_1\right\|_{H^\infty} \\ &= \left(\sum_{n=1}^{\infty} \left\|n^{-\frac{1}{2}} e_n\right\|^2\right)^{1/2} = \left(\sum_{n=1}^{\infty} n^{-1}\right)^{1/2} \text{-diverges.} \end{aligned}$$

Thus $A \notin D_b(\#)$, so that $S^p(H, H^\infty) \nsubseteq D_b(\#)$ for $2 < p \leq \infty$. The proof is complete. ■

Let $A = (A_n)_{n=1}^\infty \in D_p(\#)$. Then $A^\# \in D_p(\#)$. As in (6.53), let $X = \frac{1}{2}\left(A + A^\#\right)$ and $Y = \frac{1}{2i}\left(A - A^\#\right)$ be the 'Cartesian decomposition' of A. Since $D_p(\#)$ is a linear subspace of $S^p(H, H^\infty)$, we have $X, Y \in D_p(\#)$. Since $A_n =$

$X_n + iY_n$ and all X_n, Y_n are self-adjoint, we have

$$A_n^* A_n + A_n A_n^* = (X_n - iY_n)(X_n + iY_n) + (X_n + iY_n)(X_n - iY_n) = 2\left(X_n^2 + Y_n^2\right). \tag{6.59}$$

Therefore

$$|A|^2 + \left|A^{\#}\right|^2 = A^*A + \left(A^{\#}\right)^* A^{\#} \overset{(5.21)}{=} \sum_{n=1}^{\infty} \left(A_n^* A_n + A_n A_n^*\right) = 2 \lim_{m\to\infty} \sum_{n=1}^{m} \left(X_n^2 + Y_n^2\right) = 2\sum_{n=1}^{\infty} \left(X_n^2 + Y_n^2\right) \overset{(5.21)}{=} 2\left(X^*X + Y^*Y\right). \tag{6.60}$$

Hence

$$\left\| \left(|A|^2 + \left|A^{\#}\right|^2\right)^{1/2} \right\|_p = 2^{\frac{1}{2}} \left\| \lim_{m\to\infty} \left(\sum_{n=1}^{m} \left(X_n^2 + Y_n^2\right) \right)^{1/2} \right\|_p = 2^{\frac{1}{2}} \lim_{m\to\infty} \left\| \left(\sum_{n=1}^{m} \left(X_n^2 + Y_n^2\right) \right)^{1/2} \right\|_p . \tag{6.61}$$

Theorem 6.16 *Let $A = (A_n)_{n=1}^{\infty} \in D_p(\#)$, $A_n = X_n + iY_n$. For $1 \le p \le 2$,*

$$\|A\|_{l_2(S^p)} \le \lim_{m\to\infty} \left\| \left(\sum_{n=1}^{m} \left(X_n^2 + Y_n^2\right) \right)^{1/2} \right\|_p = 2^{-\frac{1}{2}} \left\| \left(|A|^2 + \left|A^{\#}\right|^2\right)^{1/2} \right\|_p \le 2^{\frac{1}{p}-\frac{1}{2}} \|A\|_{l_p(S^p)},$$

where $\|A\|_{l_p(S^p)} = \infty$ if $A \notin l_p(S^p)$. For $2 \le p < \infty$, the inequalities are reversed.

Proof. By our assumption, $A \in D_p(\#)$. Hence $A, A^{\#} \in S^p(H, H^{\infty})$.Consider the operator $B = (B_n)_{n=1}^{\infty}$ such that $B_{2j} = A_j$and $B_{2j-1} = A_j^*$. Let us show that

$B, B^{\#} \in S^p(H, H^\infty)$. Indeed,

$$B = \begin{pmatrix} A_1^* \\ A_1 \\ \vdots \\ A_n^* \\ A_1 \\ \vdots \end{pmatrix} \text{ and } B^*Bx = B^* \begin{pmatrix} B_1 x \\ \vdots \\ B_n x \\ \vdots \end{pmatrix} = \sum_{n=1}^{\infty} B_n^* B_n x$$

$$= \sum_{j=1}^{\infty} B_{2j-1}^* B_{2j-1} x + \sum_{j=1}^{\infty} B_{2j}^* B_{2j} x$$

$$= \sum_{j=1}^{\infty} \left(A_j^*\right)^* A_j^* x + \sum_{j=1}^{\infty} A_j^* A_j x = \left(\left(A^{\#}\right)^* A^{\#} + A^*A\right) x.$$

for each $x \in H$. Thus $B^*B = \left(A^{\#}\right)^* A^{\#} + A^*A$. Applying (5.8), we obtain that $\left(A^{\#}\right)^* A^{\#}, A^*A \in S^{p/2}(H)$. Since $S^{p/2}(H)$ is a linear space (see [16, Lemma XI.9.9 (b)]), we have $B^*B = \left(A^{\#}\right)^* A^{\#} + A^*A \in S^{p/2}(H)$. Applying (5.8) again, we have that $B \in S^p(H, H^\infty)$.

Similarly, we have that, for each $x \in H$,

$$\left(B^{\#}\right)^* B^{\#} x = \sum_{n=1}^{\infty} \left(B_n^*\right)^* B_n^* x = \sum_{j=1}^{\infty} B_{2j-1} B_{2j-1}^* x + \sum_{j=1}^{\infty} B_{2j} B_{2j}^* x$$

$$= \sum_{j=1}^{\infty} A_j^* A_j x + \sum_{j=1}^{\infty} \left(A_j^*\right)^* A_j^* x = A^*Ax + \left(A^{\#}\right)^* A^{\#} x.$$

Thus $\left(B^{\#}\right)^* B^{\#} = A^*A + \left(A^{\#}\right)^* A^{\#} \in S^{p/2}(H)$, so that $B^{\#} \in S^p(H, H^\infty)$. Thus $B, B^{\#} \in S^p(H, H^\infty)$, so that $B \in D_p(\#)$.

We also have

$$B_{2j-1}^* B_{2j-1} + B_{2j}^* B_{2j} = A_j A_j^* + A_j^* A_j \overset{(6.59)}{=} 2\left(X_j^2 + Y_j^2\right). \tag{6.62}$$

From the considerations just above the Theorem 6.16 and from Proposition 6.15 we know that $X, Y \in D_p(\#) \subset S^p(H, H^\infty)$. By Theorem 5.22, $S^p(H, H^\infty) \subseteq l_2(S^p)$, for $1 \le p \le 2$, and $S^p(H, H^\infty) \subseteq l_p(S^p)$, for $2 \le p < \infty$. Thus all $X_n, Y_n \in S^p(H)$, so that X_j^2, $Y_j^2 \in S^{p/2}(H)$.

Set $T_m = \left(\sum_{j=1}^m \left(X_j^2 + Y_j^2\right)\right)^{1/2}$. As $S^{p/2}(H)$ is a linear space,

$$T_m^2 = \sum_{j=1}^m \left(X_j^2 + Y_j^2\right) \in S^{p/2}(H)$$

and is self-adjoint. Then $T_m \in S^p(H)$ and

$$\left\|T_m^2\right\|_{p/2} = \left\|T_m^* T_m\right\|_{p/2} \overset{(5.3)}{=} \left\|T_m\right\|_p^2. \tag{6.63}$$

Thus

$$\|B\|_p \overset{(5.54)}{=} \|B^*B\|_{p/2}^{1/2} \overset{(5.37)}{=} \lim_{m\to\infty} \left\|\sum_{n=1}^m B_n^* B_n\right\|_{p/2}^{1/2} \tag{6.64}$$

$$= \lim_{m\to\infty} \left\|\sum_{j=1}^m \left(B_{2j-1}^* B_{2j-1} + B_{2j}^* B_{2j}\right)\right\|_{p/2}^{1/2} \overset{(6.62)}{=} 2^{1/2} \lim_{m\to\infty} \left\|\sum_{j=1}^m \left(X_j^2 + Y_j^2\right)\right\|_{p/2}^{1/2}$$

$$\overset{(6.63)}{=} 2^{1/2} \lim_{m\to\infty} \left\|\left(\sum_{j=1}^m \left(X_j^2 + Y_j^2\right)\right)^{1/2}\right\|_p.$$

As $\|B_{2j}\|_p = \|A_j\|_p = \left\|A_j^*\right\|_p = \|B_{2j-1}\|_p$, we have, for each q,

$$\|B\|_{l_q(S^p)}^q = \sum_{j=1}^\infty \left(\|B_{2j}\|_p^q + \|B_{2j-1}\|_p^q\right) \tag{6.65}$$

$$= \sum_{j=1}^\infty \|A_j\|_p^q + \sum_{j=1}^\infty \left\|A_j^*\right\|_p^q = 2\,\|A\|_{l_q(S^p)}^q.$$

Let $1 \le p \le 2$. Then

$$\|B\|_{l_2(S^p)} \overset{(5.61)}{\le} \|B\|_p \overset{(6.64)}{=} 2^{1/2} \lim_{m\to\infty} \left\|\left(\sum_{j=1}^m \left(X_j^2 + Y_j^2\right)\right)^{1/2}\right\|_p \overset{(5.61)}{\le} \|B\|_{l_p(S^p)}.$$

By (6.65), $\|B\|_{l_p(S^p)} = 2^{1/p} \|A\|_{l_p(S^p)}$ and $\|B\|_{l_2(S^p)} = 2^{1/2} \|A\|_{l_2(S^p)}$. Hence

$$2^{1/2} \|A\|_{l_2(S^p)} \leq 2^{1/2} \lim_{m\to\infty} \left\| \left(\sum_{j=1}^{m} \left(X_j^2 + Y_j^2 \right) \right)^{1/2} \right\|_p \leq 2^{1/p} \|A\|_{l_p(S^p)} .$$

Making use of (6.61), we complete the proof of the case when $1 \leq p \leq 2$:

$$\begin{aligned} \|A\|_{l_2(S^p)} &\leq \lim_{m\to\infty} \left\| \left(\sum_{j=1}^{m} \left(X_j^2 + Y_j^2 \right) \right)^{1/2} \right\|_p \\ &= 2^{-\frac{1}{2}} \left\| \left(|A|^2 + \left|A^{\#}\right|^2 \right)^{1/2} \right\|_p \leq 2^{\frac{1}{p}-\frac{1}{2}} \|A\|_{l_p(S^p)} . \end{aligned}$$

To prove the estimate in the case $2 \leq p < \infty$, we use inequality (5.62) instead of (5.61). We have

$$\|B\|_{l_2(S^p)} \overset{(5.62)}{\geq} \|B\|_p \overset{(6.64)}{=} 2^{1/2} \lim_{m\to\infty} \left\| \left(\sum_{j=1}^{m} \left(X_j^2 + Y_j^2 \right) \right)^{1/2} \right\|_p \overset{(5.62)}{\geq} \|B\|_{l_p(S^p)} .$$

By (6.65), $\|B\|_{l_p(S^p)} = 2^{1/p} \|A\|_{l_p(S^p)}$ and $\|B\|_{l_2(S^p)} = 2^{1/2} \|A\|_{l_2(S^p)}$. Hence

$$2^{1/2} \|A\|_{l_2(S^p)} \geq 2^{1/2} \lim_{m\to\infty} \left\| \left(\sum_{j=1}^{m} \left(X_j^2 + Y_j^2 \right) \right)^{1/2} \right\|_p \geq 2^{1/p} \|A\|_{l_p(S^p)} .$$

Making use of (6.61), we obtain

$$\begin{aligned} \|A\|_{l_2(S^p)} &\geq \lim_{m\to\infty} \left\| \left(\sum_{j=1}^{m} \left(X_j^2 + Y_j^2 \right) \right)^{1/2} \right\|_p \\ &= 2^{-\frac{1}{2}} \left\| \left(|A|^2 + \left|A^{\#}\right|^2 \right)^{1/2} \right\|_p \geq 2^{\frac{1}{p}-\frac{1}{2}} \|A\|_{l_p(S^p)} . \end{aligned}$$

The theorem is proved. ■

6.7 Conclusion

The main aim of this chapter is to find a generalization of Clarkson-McCarthy inequalities (6.21) to infinite families of operators. The inspiration came from studying actions of operators from $B(H^{\infty})$ on $l_q(S^p)$ spaces.

In Proposition 6.1 we prove that, apart from $l_2(S^2) = S^2(H, H^{\infty})$, the Banach spaces $l_q(S^p)$ are not left $B(H^{\infty})$-modules. By applying results from Chapter 5, namely Theorem 5.22 and Lemma 5.16, we obtain, in Theorem 6.2, important inequalities involving operators from $B(H^{\infty})$, $l_2(S^p)$ and $l_p(S^p)$. Using interpretation (6.21) we prove Theorem 6.3, that gives us estimate (6.30) involving a block-diagonal operator on H^{∞} (see (6.29)) and operator from the space $l_q(S^p)$. We know that a similar estimate would not work for all bounded operators on H^{∞} as the spaces $l_q(S^p)$ other than $l_2(S^p)$ are not left $B(H^{\infty})$-modules. Applying Theorems 6.2 and 6.3 we obtain Corollary 6.4. It gives us an analogue of McCarthy inequalities (6.1) and (6.2) for $l_q(S^p)$ spaces. In Theorem 6.7 we prove that for $2 \leq p < \infty$, the space $l_p(S^p)$ is p-uniformly convex.

Next, we concentrate on infinite partitions of operators from S^p. We prove estimates for partitions in Theorem 6.11. The case when the partitions were finite was studied in [25] and [7]. In Theorem 6.13 we prove estimates for Cartesian decomposition of operators from $l_q(S^p)$. A similar result for $l_q^n(S^p)$ spaces was proved in [25, Theorem 5(ii)]. We also prove Proposition 6.15 that shows inclusions of spaces $l_p(S^p)$, $D_p(\#)$, $S^p(H, H^{\infty})$ and $l_2(S^p)$. Our last Theorem in this thesis is Theorem 6.16. It proves estimates for Cartesian decomposition of operator $A \in S^p(H, H^\infty)$ such that $A^\# \in S^p(H, H^\infty)$. It is similar to [25, Theorem 5(i)].

Chapter 7 Conclusion

This thesis had two aims. The first aim was to identify and prove a number of minimax conditions that arise in the context of the theory of Hilbert spaces and linear operators on Hilbert spaces. The second aim was to analyze l_q-spaces $l_q(S^p)$ of operators from Schatten ideals S^p. In this chapter we summarize the results we have achieved until now and indicate possible future research in this area.

We began our research in Chapter 3 by considering sequences of bounded seminorms on Hilbert spaces and obtaining minimax theorems for them. We established two minimax formulae for bounded seminorms on Hilbert spaces, namely Proposition 3.6 and Theorem 3.8. We consider a sequence $\{g_k\}_{k=1}^{\infty}$ of bounded seminorms on a Hilbert space H that is bounded at each point $x \in H$. We find that one of the above minimax formulae holds for such a sequence and its value is zero, if the bounded seminorm $g(x) = \sup_n g_n(x)$ is not equivalent to the norm $\|\cdot\|$ of the Hilbert space H. We prove that the condition does not hold when g is equivalent to $\|\cdot\|$ but all g_n are not equivalent to $\|\cdot\|$. Generally, if g is equivalent to $\|\cdot\|$ then this minimax condition holds if and only if, for each $\varepsilon > 0$, there exists n_ε such that g_{n_ε} is equivalent to $\|\cdot\|$ and $\inf_{\|x\|=1} g_{n_\varepsilon}(x) \geq \inf_{\|x\|=1} g(x) - \varepsilon$. We also showed that the reversed minimax condition, as stated in Theorem 3.8, holds for all sequences of seminorms. The restrictions imposed on the sequences of seminorms are different for the reversed version. In Theorem 3.8 we require that the sequence $\{g_k\}_{k=1}^{\infty}$ of seminorms on H is such that $g_m(x) = \inf_n g_n(x)$ for all $x \in H$ and some $m \in \mathbb{N}$

(for example, $\{g_k\}_{k=1}^{\infty}$ could be monotone increasing, i.e. $g_k(x) \leq g_{k+1}(x)$ for all $x \in H$). We illustrate Proposition 3.6 and Theorem 3.8 with examples of seminorms on the Hilbert space l_2.

By replacing sequences of seminorms with sequences of operators $\{A_k\}_{k=1}^{\infty}$ on H and evaluating their norms, we obtain the following version of the minimax condition:

$$\inf_{\|x\|=1} \sup_{n} \|A_n x\| = \sup_{n} \inf_{\|x\|=1} \|A_n x\|,$$

$$\inf_{n} \sup_{\|x\|=1} \|A_n x\| = \sup_{\|x\|=1} \inf_{n} \|A_n x\|.$$

Perhaps, it would be interesting to find necessary and sufficient conditions for the minimax to hold and to evaluate the left and right hand sides of the above minimax formulae.

At the end of Chapter 3, we divert our attention from seminorms and concentrate on finding minimax theorems that hold for bounded operators on Hilbert spaces. In Theorem 3.12, we obtain certain minimax conditions for bounded operators on H. We evaluate this minimax formula as zero if the bounded operator A is not invertible and find that the minimax condition does not hold if A is invertible and $\dim H > 1$. We discuss application of this minimax formula to a bounded bilinear functional Ω on H in Corollary 3.15.

In Chapter 4 we study the validity of various types of minimax conditions for operators in Schatten ideals of compact operators. Our work has been inspired by reading the theory of linear nonselfadjoint operators in [21]. In Theorem 4.9 we consider minimax conditions that involve a bounded operator A on a Hilbert space H and a sequence of self-adjoint bounded operators on H that converges to $\mathbf{1}_H$ in the s.o.t. We were able to identify exactly for which bounded operators A this minimax condition holds. We proved that the reversed minimax condition holds for all operators $A \in B(H)$. The most important theorem in this chapter is, in our opinion, Theorem 4.15. It evaluates and verifies minimax conditions in Schatten ideals for a family of projections. We discovered that the first formula in this theorem holds in all cases and is equal to zero. However, the validity of the second minimax condition depends on a new interesting property - approximate intersection of a family of subspaces. Details of this notion and the results are explained in Definition 4.13 and Theorem 4.15(ii).

A possibility of future research in this direction lies in the further attempts to identify and verify some other minimax conditions for various classes of bounded operators on Hilbert spaces. Another avenue which is worth, perhaps, pursuing is investigating whether the minimax conditions could be generalized and then applied to the operator theory.

In Chapter 5 we study $l_q(S^p)$ spaces of operators from Schatten ideals S^p and the spaces $S^p(H, H^\infty)$ of Schatten operators from Hilbert space H into H^∞. In Theorem 5.22 we establish the inclusion of these spaces in each other and obtain various estimates for norms of operators from these spaces. In particular, we found that the spaces $l_2(S^2)$ and $S^2(H, H^\infty)$ coincide. Lemma 6 [25], gives some estimates for norms of n-tuples $A = (A_1, ..., A_n)$ of operators from $l_p^n(S^p)$. In Proposition 5.10, we extended these estimates to infinite families of operators. In Theorem 5.11 we establish a connection between the norms of an operator, $A \in l_2(S^p) \cup l_p(S^p)$ and the operator $B = RA$, where R is a bounded operator on H^∞. This, in fact, extends the results of Corollary 7 [25], which proved this estimate for the norms of an n-tuple of operators $A = (A_1, ..., A_n)$ and the n-tuple of operators $B = RA$, where $R \in B(H^n)$.

We obtained further generalization of Clarkson-McCarthy estimates in Chapter 6 in Corollary 6.4. We apply this result to prove that the spaces $l_p(S^p)$ are p-uniformly convex for $p \in [2, \infty)$. This, in turn, implies that the spaces $l_p(S^p)$, for $p \in [2, \infty)$, are reflexive (see [39, p.23]). Partitions of operators were studied in section 6.5. We established inequalities for infinite partitions of operators from S^p in Theorem 6.11. This result builds on estimates achieved in Theorem 4 [25] for finite partitions of operators from S^p. In Theorem 6.13 we consider the Cartesian decomposition $A = X + iY$ of infinite sequences $A = (A_n)_{n=1}^\infty$ of operators from $l_q(S^p)$, for $p \in [1, \infty)$ and $q \in [\min(p, 2), \max(p, 2)]$. We obtain a certain estimates that link the norms $\|X\|_{l_q(S^p)}$, $\|Y\|_{l_q(S^p)}$ and $\|A\|_{l_q(S^p)}$. These results extend Theorem 5(ii) [25], where this decomposition was investigated for n-tuples $A = (A_1, ..., A_n)$ of operators from S^p. We also study special type of operators from $S^p(H, H^\infty)$ and obtain some inequalities for Cartesian decomposition of these operators in Theorem

6.16.

As we stated in Problem 6.8, the question about p-uniformly convexity of the spaces $l_p(S^p)$, for $p \in [1,2)$, is still open. This question is a subject for our future research.

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