Reference Tracking in Sampled-Data Systems: A DLMI-Based Purely-Discrete PID Framework

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***Abstract*—This paper presents a tracking reference design condition based on a purely discrete-time PID controller aimed at sampled-data linear systems operating over hybrid time domains. The proposed method formulates the PID tuning problem as a convex optimization task based on Differential Linear Matrix Inequalities (DLMIs), with the specific objective of minimizing the $\mathcal{H}_2$ guaranteed cost of the tracking error signal. Purely discrete PID gains are obtained automatically without requiring model reduction or transformation to continuous-time domains. The resulting controller ensures stability and performance by imposing an $\mathcal{H}_2$ upper limit cost for hybrid systems. Numerical examples are used to show the effectiveness and practicality of the proposed method.**

***Keywords*—DLMI, Hybrid Linear System, Reference Tracking, Sampled-data PID.**

I. Introduction

REAL-WORLD physical systems commonly used in industry are typically modeled by continuous-time differential equations. In early automation processes for these systems, mechanisms employing mechanical or electronic components were used to generate continuous-time control signals, enabling the design of systems and controllers operating in the same temporal domain [1]. However, these controllers were difficult to modify due to their analog construction. With the transition to digital controllers, two main approaches emerged: the use of high-performance hardware capable of emulating continuous-time controllers, which is usually expansive for the industry, or costless digital controllers used in addition to analog-to-digital-to-analog converters (usually zero-order holders) and classical approaches of discretization [2]. However, the last one, although cheaper than the other, tends to fail or experience poor performance when dealing with fast dynamics or non-linear behavior [1], [3]. This is primarily because classical control models cannot adequately capture the inter-sampling information of the continuous-time system. Other problem that arises from this scenario is the pathological frequencies that insert into the system uncontrollable unstable poles for some selected sampling frequencies [4].

To address the aforementioned problems, modern control theory literature proposes classes of systems capable of describing the continuous-time physical dynamics, while control actions are applied at discrete instants. Among others, the impulsive or hybrid linear system (HLS) approach is based mainly on the definition of an augmented system composed of two dynamics: the continuous- and discrete-time evolutions of the system. As a consequence, the design of filters and controllers, in the hybrid approach, requires to solve a two-point boundary value problem (TPBVP) where the differential equation is associated with the continuous-time dynamic, and boundary conditions corresponding to the discrete-time behavior of the system, [3], [5], [6]. Besides the challenge of obtaining linear conditions to be solved by the tools available in the literature to date, a great challenge of this approach is to solve the resulting TPBVP. To address this problem within a convex optimization framework, Differential Linear Matrix Inequalities (DLMIs) have gained prominence in recent years [5], [7], offering a mathematical tool to lead the DLMI to a convex and tractable formulation to address common control-related challenges such as system uncertainties, delays, and performance specifications in the discrete- or continuous-time domain. Due to the novelty of the method, many of these problems are still open in the literature. Furthermore, few papers have been published considering the solution of DLMIs. In [8] three methods have been proposed: the piecewise linear approximation and the approximations considering truncated Taylor and Fourier series. Other papers, instead of considering DLMI conditions, propose LMI-based ones that include approximations in the problem formulation, such as the piecewise linear approximation, as in [6], or the polynomial approximation for the Lyapunov matrices as in [9].

Despite this significant development in control theory, classical approaches are still commonly found in most industries. Specifically, the families of proportional derivative integral (PID) action-based control are widely used for reference-tracking structures, [10], [11], due to its practicality and simplicity to obtain effective designs. For this reason, a recent effort is in the direction of designing PID controllers through modern sampled-data approaches. In [12] and [13], a sampled-

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data PID-based controller is formulated. Therein, the authors consider the measured output signal as the feedback signal instead of the error signal. In addition, the derivative action introduces a delay-based approximation that allows them to obtain convex conditions for a purely discrete-time controller design. In [14], a similar approach is used, but considering only the last two measurements. In this paper, uncertain systems are also considered through an LMI formulation. In [15], a nonlinear system is controlled considering an error-based sampled-data structure for the controller. For the design, the time-delay and uncertain parameters are considered. In addition, this approach admits a decay rate constraint, which can be generalizable as a $\mathcal{H}_2$ performance condition. Finally, [16] tackled the sampled-data reference tracking control with integrative terms.

Considering the PID structure applied to systems with a performance optimization, [17] a PID structure is designed considering the performance otimization based on $\mathcal{H}_\infty$ norm considering the continuous-time domain. To the knowledge of the authors, there is no work in the literature that considers the optimization performance based on the $\mathcal{H}_2$ norm considering a PID structure and the hybrid time domain. In this context, we propose a purely discrete-time PID controller design that minimizes the $\mathcal{H}_2$ norm for sampled-data system. The design conditions are presented through the DLMIs. The main contribution lies in the minimization of an upper bound of the $\mathcal{H}_2$ norm of the output tracking error. Unlike traditional tuning methods, which often neglect the combined effects of sampling, delays, and hybrid dynamics, the proposed approach provides convex conditions that ensure quadratic stability and robust performance under digital constraints. The design procedure automatically computes the discrete PID gains K_P, K_I, and K_D by explicitly solving a convex optimization problem in terms of DLMIs, avoiding the need for continuous-time approximations or heuristic search strategies. The effectiveness and computational efficiency of the method are validated through numerical simulations.

Notation: The notation is standard. $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$ represent the set of natural, real, and non-negative real numbers, respectively. Matrices are represented by capital letters, and vectors and scalars are represented by small letters. Given a matrix M and a vector v, their transpositions are $M^\top$ and $v^\top$, respectively. For block matrices, the symmetric block is represented by $\star$. Let $M \in \mathbb{R}^{n_m \times n_m}$ a square matrix, the symmetric sum is given by $\mathrm{Her}[M] = M + M^\top$. Let S a symmetric square matrix, it is denoted by $S \in \mathbb{S}^n$. Block diagonal matrices are represented by $\mathrm{diag}(M, N)$. Positive (negative) definiteness conditions of symmetric matrices $P \in \mathbb{R}^{n_p \times n_p}$ are denoted by $P \succ 0$ ($P \prec 0$). For any continuous-time function $f(t)$, we set $f[k] = f(t_k)$ for $k \in \mathbb{N}$. The symbol $1_{n\times m}$ denotes a matrix full of ones of dimension n by m.

II. Problem Description

Consider an uncertain continuous-time system in the form

$$\dot{x}(t) \quad = A(\alpha)x(t) + B(\alpha)u(t) + E(\alpha)w(t), \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the regulated input vector, and $w(t) \in \mathbb{R}^{n_w}$ is the exogenous input vector. The system matrices belong to an uncertain domain represented by the simplex $\Lambda_\alpha \in \mathbb{R}^N$ such that

$$(A, B, E)(\alpha) = \sum_{i=1}^{N} \alpha_i (A_i, B_i, E_i), \tag{2}$$

where $\Lambda_\alpha := \{\alpha = [\alpha_1, \dots, \alpha_N] \ : \ \sum_{i=1}^N \alpha_i = 1, \ \alpha_i \geq 0\}$. This system is one of the blocks of the sampled-data controlled system illustrated in Fig. 1, which aims to track a constant reference $r[k] \in \mathbb{R}^{n_r}, n_r \geq n_x$. Consider that the state vector $x(t)$ is submitted to an ideal sampler, which produces $x[k] = x(t_k)$ for all $t \in [t_k, t_{k+1}), k \in \mathbb{N}$, where $t_{k+1} - t_k = h > 0$ is called the sampling period, with $t_0 = 0$, generating a sequence of sampling instants $\{t_k\}_{\forall k \in \mathbb{N}}$.

Assuming that the state vector is available for measurement, we define

$$e[k] = r[k] - C_r x[k], \tag{3}$$

as the tracking error signal, which assumes that the signal to be tracked is in the form of $\zeta(t) = C_r x[k]$, for all $t \in [t_k, t_{k+1})$, for each $k \in \mathbb{N}$. In this definition, the y-th row of $C_r \in \mathbb{R}^{n_r \times n_x}$ corresponds to the j-th row of the identity matrix of dimension n_x. Moreover, the controlled input signal $u[k] = u(t_k)$ for all $t \in [t_k, t_{k+1}), k \in \mathbb{N}$ is given by a general PID controller in the form,

$$\begin{aligned} u[k] =& K_x x[k] + K_u u[k-1] \\ &+ K_p e[k] + K_d e_\Delta[k] + K_i e_\Sigma[k], \end{aligned} \tag{4}$$

where $e_\Delta[k] \in \mathbb{R}^{n_r}$ and $e_\Sigma[k] \in \mathbb{R}^{n_r}$ are the difference and the accumulation of the error signal, defined as

$$e_\Delta[k] = e[k] - e[k-1], \tag{5}$$

$$e_\Sigma[k] = e[k] + e_\Sigma[k-1]. \tag{6}$$

As a consequence of the discrete behavior of the discrete-time signals, the system in Fig. 1 can be modeled as an impulsive system (HLS) with an augmented state vector $\xi(\cdot)^\top = \begin{bmatrix} x(\cdot)^\top & u(\cdot)^\top & e(\cdot)^\top & e_\Delta(\cdot)^\top & e_\Sigma(\cdot)^\top \end{bmatrix}$, $\xi(\cdot) \in \mathbb{R}^{n_\xi}$, with state-space representation

$$\mathcal{G} : \begin{cases} \dot{\xi}(t) = & F(\alpha)\xi(t) + J_c(\alpha)w(t), \ \forall t \notin \mathcal{T}, \\ \xi(t_k) = & H\xi(t_k^-) + J_d r[k], \ \forall t \in \mathcal{T}, \\ z(t) = & G\xi(t), \end{cases} \tag{7}$$

and matrices

$$F(\alpha) = \begin{bmatrix} A(\alpha) & B(\alpha) & 0_{n_x \times 3n_r} \\ 0_{n_u \times n_x} & 0_{n_u} & 0_{n_u \times 3n_r} \\ \star & \star & 0_{3n_r} \end{bmatrix},$$

$$J_c(\alpha)^T = \begin{bmatrix} E(\alpha)^T & 0_{3n_r \times (nu + 3n_r)} \end{bmatrix},$$

along with

$$H = \begin{bmatrix} I_{n_x} & 0_{n_x \times n_u} & 0_{n_x \times n_r} & 0_{n_x \times n_r} & 0_{n_x \times n_r} \\ K_x & K_u & K_p & K_d & K_i \\ -C_r & 0_{n_r \times n_u} & 0_{n_r \times n_r} & 0_{n_r \times n_r} & 0_{n_r \times n_r} \\ -C_r & 0_{n_r \times n_u} & -I_{n_r \times n_r} & 0_{n_r \times n_r} & 0_{n_r \times n_r} \\ -C_r & 0_{n_r \times n_u} & 0_{n_r \times n_r} & 0_{n_r \times n_r} & I_{n_r \times n_r} \end{bmatrix},$$

$$J_d = \begin{bmatrix} 0_{(n_x + n_u) \times n_r} \\ 1_{3n_r \times n_r} \end{bmatrix}, \ G = \begin{bmatrix} C_z & D_z & C_{ze} \end{bmatrix}$$

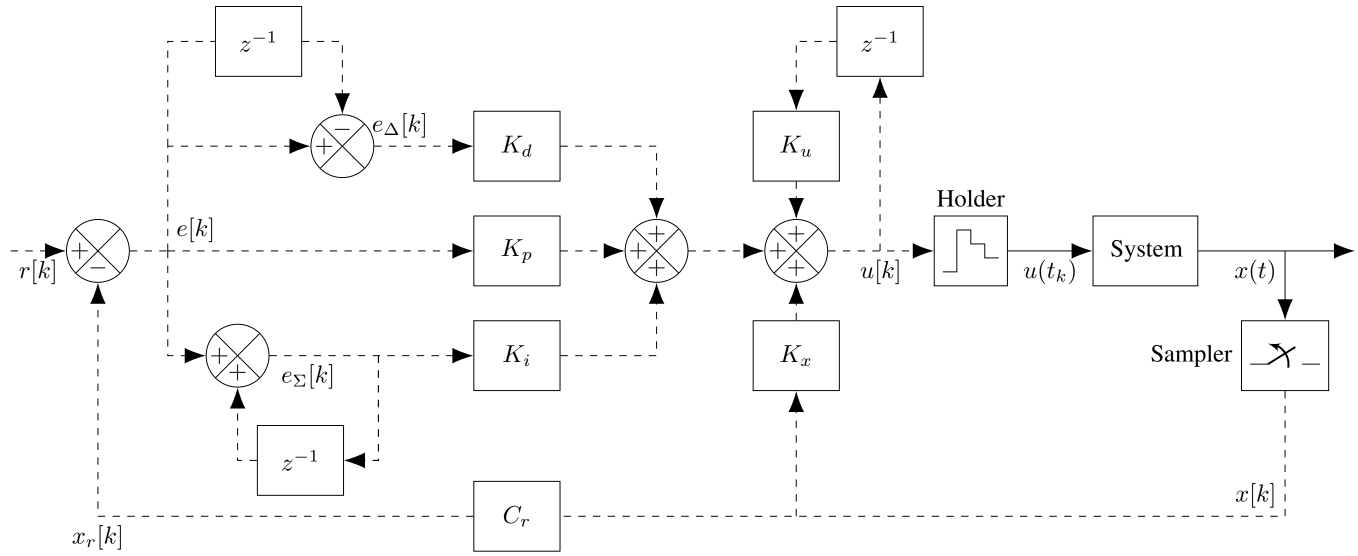

Figure 1: Control structure diagram. The dashed line corresponds to the discrete-time signals, while the solid line represents the continuous-time signals.

where $z(t) \in \mathbb{R}^{n_z}$ is the regulated output vector, whose matrices C_z, D_z, and C_{ze} establish the weights for the state, the control input, and the error signals, respectively, in the optimization function.

Remark 1: It is important to notice that, from a continuous-time perspective, a signal obtained from a discrete-time sequence via a zero-order holder can be viewed as a continuous piecewise function. Specifically, within the interval $t \in [t_k, t_{k+1})$, its derivative can be defined as zero. For example, when discussing the error signal, it implies that $\frac{\mathrm{d}e(t)}{\mathrm{d}t} = 0$ for all $t \in [t_k, t_{k+1})$ by setting $e(t) = e(t_k)$. This leads to an interesting outcome when t approaches t_{k+1}, where a discontinuous jump occurs, instantaneously updating the signal. By examining the behavior of the error signal, we observe that

$$e(t_k) = e(t_{k+1}^-) \neq e(t_{k+1}). \tag{8}$$

A reference tracking structure is designed to align some system states with a reference signal. In the context of a PID structure, the internal model principle indicates that, when a constant signal is used as the reference, the error signals converges to zero. Consequently, a key challenge is to minimize the convergence time as much as possible. This performance is closely related to the $\mathcal{H}_2$ cost of a dynamic system, which aims to reduce the time taken for the system to converge from its initial state to the origin, assuming no external inputs are present. In this context, we define the following objective function.

Definition 1: For a stable HLS system in the form (7), the $\mathcal{H}_2$ performance index is defined as

$$\mathcal{J}_2(\alpha) = \sum_{j=1}^{n_w} ||z_j||_2^2 + \sum_{i=1}^{n_r} ||z_i||_2^2, \tag{9}$$

where $z_j(t),\ j \leq n_w$ corresponds to the regulated output when the system is submitted to a continuous-time impulse in the j-th channel of $w(\cdot)$, while $z_i(t),\ i \leq n_r$ indicates the regulated output of the system for the effect of applying a discrete-time impulse in the i-th channel of $r[\cdot]$.

As a consequence of Definition 1, a guaranteed cost for the system in (7) for all $\alpha \in \Lambda_\alpha$ is obtained by calculating the supremum of $J_2(\alpha)$ for all $\alpha \in \Lambda_\alpha$. Moreover, it is important to stress that the $\mathcal{H}_2$ optimization problem is a natural regulation problem for which the system evolves from any initial condition towards the equilibrium point, usually the origin of the state space. Moreover, the classical tracking control problem in an optimization context can be recovered by enforcing $C_z = 0$, as well as $K_x = 0$ and $K_u = 0$ in (4). In this case, for the $\mathcal{H}_2$ tracking control design problem, it is necessary to apply an impulsive signal in each channel of the exogenous input w.

Given the previous definition, we can now state the main goal of this work, that is, finding a controller as in (4) that stabilizes the closed-loop system, imposes a guaranteed cost $\gamma > 0$ on the $\mathcal{H}_2$ performance index in (9), and guarantees that the tracking error converges so that

$$\lim_{t\to\infty} e(t) = 0. \tag{10}$$

This formulation is similar to the one given in [18] for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ LTI case. In the next section, we will provide the auxiliary tools for solving this problem.

III. $\mathcal{H}_2$ Performance Preliminary Results

First, consider the abuse of the notation $F_\alpha = F(\alpha)$ for all $\alpha \in \Lambda_\alpha$ to simplify the readability of the proof in this section, which seeks to adapt the current results regarding stability analysis of the HLS in the context of sampled-data systems. Some works to refer to are [5], [19], [20] and the references herein, where the optimality in sampled-data systems is studied considering different contexts. In these works, the authors stated a stability analysis condition based on LMIs, which is adapted below.

Lemma 1: Given $h > 0$, if there exists a symmetric positive matrix $P : [0, h) \mapsto \mathbb{S}^{n_\xi}$ that satisfies

$$F_\alpha^\top P(t) + P(t)F_\alpha + \frac{\partial P(t)}{\partial t} + G_\alpha^\top G_\alpha \prec 0, \tag{11}$$

with boundary conditions

$$P(h) \succeq H^\top P(0)H, \; P(0) \succ 0, \tag{12}$$

for all $t \in [0, h)$, and for all $\alpha \in \Lambda_\alpha$, then the next statements hold true.

(i) For with $w \equiv 0$ and $r \equiv 0$, the system (7) is globally asymptotically stable for every $\alpha \in \Lambda_\alpha$.

(ii) For $w(\cdot)$ and $r[\cdot]$ given as in Definition 1, it follows that

$$\mathcal{J}_2(\alpha) \leq \sup_{\alpha\in\Lambda_\alpha} \left\{ \text{tr}(J_{\alpha c}^\top P(h)J_{\alpha c}) + \text{tr}(J_d^\top P(0)J_d) \right\} \tag{13}$$

subject to (11)-(12), for every $\alpha \in \Lambda_\alpha$.

(iii) By taking $r[\cdot]$ as a constant reference, we get that the tracking condition (10) holds for every $\alpha \in \Lambda_\alpha$.

Proof 1: Let $h > 0$ be given.

(i): Consider each subsystem parametrized by $\alpha \in \Lambda_\alpha$ and its corresponding system in (7) with $w \equiv 0$ and $r \equiv 0$. Then the asymptotic stability is a consequence of the feasibility of (11) and (12) as proved in [5] for a unique Lyapunov functional for all subsystems.

(ii): Considering the performance index in Definition 1 which corresponds to apply a continuous-time impulse on the j-th channel of $w(\cdot)$, $j \leq n_w$ occurring at t_0^-, and a discrete-time impulse in the i-th channel of $r[\cdot]$, $i \leq n_r$, it follows from Corollary 4.1 of [5] that,

$$\mathcal{J}_2(\alpha) \leq \text{tr}(J_{\alpha c}^\top P(h)J_{\alpha c}) + \text{tr}(J_d^\top P(0)J_d) \tag{14}$$

for all $\alpha \in \Lambda_\alpha$. Therefore, by taking the supremum over α, we get (13).

(iii): Finally considering $r[k]$ as a constant reference, we get that (10) holds as a consequence of the system closed-loop stability, as discussed in [16]. □

Therefore, the goal of this problem can be rewritten as follows:

$$\inf_{P(\cdot),K,\gamma} \gamma \tag{15}$$

subject to (11)-(12), along with

$$\text{tr}(J_{\alpha c}^\top P(h)J_{\alpha c}) + \text{tr}(J_d^\top P(0)J_d) \prec \gamma \tag{16}$$

for all $\alpha \in \Lambda_\alpha$, where $K = (K_x, K_u, K_p, K_d, K_i)$, and $\gamma > 0$ is a guaranteed cost. We note that (15) is hard to solve due to the products involving variables $P(\cdot)$ and K, so that the resulting problem is non-convex. In the next section, we provide a convex formulation that can be solved with off-the-shelf software.

IV. $\mathcal{H}_2$ Control Design Conditions

The design condition for obtaining a minimized $\mathcal{H}_2$ guaranteed cost for a sampled-data system with control law as in (4) can be performed by the following theorem.

Theorem 1: Given $h > 0$, if there exists a time-varying symmetric positive matrix $W : [0, h) \mapsto \mathbb{R}^{n_\xi}$, positive definite matrices M_c and M_d, and matrix $Y \in \mathbb{R}^{n_u\times 3n_r}$ that satisfies

$$\begin{bmatrix} W(t)F_\alpha^\top + F_\alpha W(t) - \frac{\partial W(t)}{\partial t} & \star \\ G_\alpha W(t) & -I_{n_r} \end{bmatrix} \prec 0, \tag{17}$$

for all $t \in [0, h)$, and for all $\alpha \in \Lambda_\alpha$, with the boundary conditions

$$\begin{bmatrix} W(h) & \star \\ \bar{H}W(h) + \bar{C}Y & W(0) \end{bmatrix} \succ 0, \tag{18}$$

along with

$$\begin{bmatrix} M_c & \star \\ J_{c\alpha} & W(h) \end{bmatrix} \succ 0, \begin{bmatrix} M_d & \star \\ J_d & W(0) \end{bmatrix} \succ 0 \tag{19}$$

and

$$\text{tr}(M_c) + \text{tr}(M_d) \prec \gamma \tag{20}$$

where

$$\bar{C} = \begin{bmatrix} 0_{n_u\times n_x} & I_{n_u} & 0_{n_u\times 3n_r} \end{bmatrix}^\top \tag{21}$$

$$\bar{H} = \begin{bmatrix} I_{n_x} & 0_{n_x\times n_u} & 0_{n_x\times n_r} & 0_{n_x\times n_r} & 0_{n_x\times n_r} \\ 0_{n_u\times n_x} & 0_{n_u} & 0_{n_u\times n_r} & 0_{n_u\times n_r} & 0_{n_u\times n_r} \\ -C_r & 0_{n_r\times n_u} & 0_{n_r\times n_r} & 0_{n_r\times n_r} & 0_{n_r\times n_r} \\ -C_r & 0_{n_r\times n_u} & -I_{n_r\times n_r} & 0_{n_r\times n_r} & 0_{n_r\times n_r} \\ -C_r & 0_{n_r\times n_u} & 0_{n_r\times n_r} & 0_{n_r\times n_r} & I_{n_r\times n_r} \end{bmatrix}, \tag{22}$$

then by setting

$$K = \begin{bmatrix} K_x & K_u & K_p & K_d & K_i \end{bmatrix} = YW(h)^{-1}, \tag{23}$$

the system (7) is globally asymptotically stable with $\mathcal{J}_2(\alpha) < \gamma$ and (10) holds.

Proof 2: First, by applying Schur's complement on (17) one obtains

$$W(t)F_\alpha^\top + F_\alpha W(t) - \frac{\partial W(t)}{\partial t} + W(t)G_\alpha^\top G_\alpha W(t) \prec 0. \tag{24}$$

Pre- and post-multiplying (24) by $W^{-1}(t)$ yields

$$F_\alpha^\top W^{-1}(t) + W^{-1}(t)F_\alpha - W^{-1}(t)\frac{\partial W(t)}{\partial t}W^{-1}(t) \prec 0, \tag{25}$$

which becomes (12) by defining $W^{-1} = P(t)$ noting that $\frac{\partial P(t)}{\partial t} = -W^{-1}(t)\frac{\partial W(t)}{\partial t}W^{-1}(t)$.

Concerning to (18), called that $Y = KW(h)$, the lower-left term (and its symmetric) can be re-written as $\left(\bar{H} + \bar{C}K\right)W(h) = HW(h)$, so that

$$\begin{bmatrix} W(h) & \star \\ HW(h) & W(0) \end{bmatrix} \succ 0. \tag{26}$$

Apply Schur's complement and later pre- and post-multiplying it by $W^{-1}(h)$, lead to

$$W^{-1}(h)W(h)W^{-1}(h) - H^\top W^{-1}(0)H \succ 0, \tag{27}$$

which match (12) by considering $P(h) = W^{-1}(h)$. Finally, by applying the Schur complement to (19), the trace operator in the resulting inequality, and considering (20), we get (16), concluding the proof. □

Note that the variable change $Y = KW(h)$ is the key tool for retrieving the controller gains. This change variable is responsible for incorporating conservatism, as it is necessary to apply the inverse of $W(t)$ to recover the control vector. However, thanks to this formulation, the extension from a robust gains vector to a gains-scheduled control vector depends on the definition of $Y \to Y_\alpha$, which can be useful for certain applications.

At this moment, it is worth noting that the control law given in (4) is not a classical formulation of a PID because this does not employ the integral and derivative actions directly; instead, it uses the cumulative and difference actions. A closer-to-classical formulation can be represented by the following control law

$$u[k] = K_x x[k] + K_u u[k-1] + K_p e[k] + K_d \frac{1}{h} e_\Delta[k] + K_i h e_\Sigma[k], \tag{28}$$

by considering the integral and derivative approximation as $\hat{e}_\Delta[k] = \frac{e[k+1]-e[k]}{h}$ and $\hat{e}_\Sigma[k] = he[k] + \hat{e}_\Sigma[k-1]$, respectively. The next corollary considers this case.

Corollary 1: Given $h > 0$, the system (7) is globally asymptotically stable by applying the control law (28), if there exists a symmetric positive matrix $W : [0, h) \mapsto \mathbb{R}^{n_\xi}$ and the matrix $Y \in \mathbb{R}^{n_u \times 3n_r}$ that satisfies (11)-(12) with re-definition of $\bar{H}$ by $\hat{H}$ as

$$\hat{H} = \begin{bmatrix} I_{n_x} & 0_{n_x \times n_u} & 0_{n_x \times n_r} & 0_{n_x \times n_r} & 0_{n_x \times n_r} \\ 0_{n_u \times n_x} & 0_{n_u} & 0_{n_u \times n_r} & 0_{n_u \times n_r} & 0_{n_u \times n_r} \\ -C_r & 0_{n_r \times n_u} & 0_{n_r \times n_r} & 0_{n_r \times n_r} & 0_{n_r \times n_r} \\ -\frac{1}{h} C_r & 0_{n_r \times n_u} & -\frac{1}{h} I_{n_r \times n_r} & 0_{n_r \times n_r} & 0_{n_r \times n_r} \\ -hC_r & 0_{n_r \times n_u} & 0_{n_r \times n_r} & 0_{n_r \times n_r} & I_{n_r \times n_r} \end{bmatrix}, \tag{29}$$

and $J_d^\top$ by $\hat{J}_d^\top = \begin{bmatrix} 0_{n_r \times (n_x+n_u)} & 1_{n_r} & \frac{1}{h} 1_{n_r} & h 1_{n_r} \end{bmatrix}$, with guaranteed cost given by (13).

This formulation can yield different results compared to Theorem 1. In particular, since Corollary 1 explicitly includes the sampling period, the results may encounter numerical issues for sufficiently high values of the sampling period; therefore, it must be used with caution.

V. Numerical Examples

To illustrate the performance of the proposal, a numerical example is presented, which was adapted from [15]. The model corresponds to a mechanical system represented by the following matrices

$$A_\alpha = \begin{bmatrix} 0 & 1 \\ -\frac{m_\alpha l g}{2G} & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{G} \end{bmatrix}, C_r = \begin{bmatrix} 1 & 0 \end{bmatrix}, E_c = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \tag{30}$$

$$C_z = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0 \end{bmatrix}, \quad D_z = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C_{rz} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{31}$$

where $m_\alpha \in \begin{bmatrix} 1.5 & 2.5 \end{bmatrix}$ kg, $l = 1$ m, $G = 1$, and $g = 9.8$ m/s^2. Note that the regulated output prioritizes performance

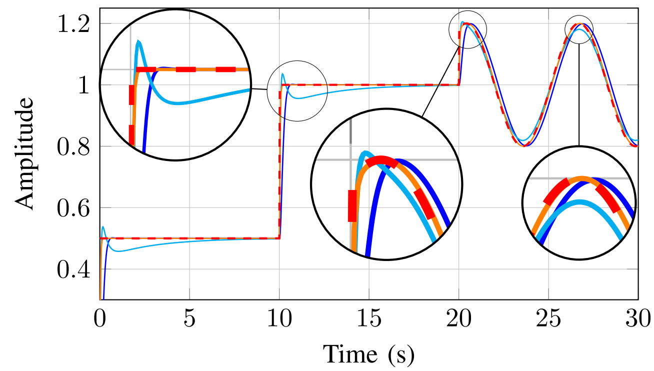

Figure 2: Temporal response in a time interval from 0 s to 30 s. The dashed red line (- -) is the reference signal $r(t_k)$. In contrast, the solid lines are the trajectory of the objective state when the proposed control is applied (orange line — for Theorem 1, the blue line — for Corollary 1) and the comparison control is used (cyan line — for [15]).

in terms of accumulated error, as determined by the choice of C_{zr}, and the selection of C_z helps to avoid overshoot, offering a trade-off between tracking convergence time and overshoot in the states. By using Theorem 1 with a sampling period $h = 0.01$s, the obtained control gain matrices are

$$K_x = \begin{bmatrix} -1180.2 & -45.2311 \end{bmatrix}, \ K_u = \begin{bmatrix} 0.2470 \end{bmatrix},$$
$$K_p = \begin{bmatrix} 0.0428 \end{bmatrix}, \ K_d = \begin{bmatrix} -0.00711 \end{bmatrix}, \ K_i = \begin{bmatrix} 138.5139 \end{bmatrix},$$

with $\mathcal{H}_2$ guaranteed cost $\gamma = 0.0868$. On the other hand, by solving Corollary 1, the controller obtained is

$$K_x = \begin{bmatrix} -55.4464 & -19.7997 \end{bmatrix}, \ K_u = \begin{bmatrix} 0.0330 \end{bmatrix},$$
$$K_p = \begin{bmatrix} 16.7307 \end{bmatrix}, \ K_d = \begin{bmatrix} -0.0145 \end{bmatrix}, \ K_i = \begin{bmatrix} 155.1681 \end{bmatrix},$$

with $\mathcal{H}_2$ guaranteed cost $\gamma = 0.5255$. For comparison, the controller offered in [15] is considered, where the numerical values are $K_x = K_u = 0$, and

$$K_p = \begin{bmatrix} 59.0900 \end{bmatrix}, \ K_d = \begin{bmatrix} 19.7540 \end{bmatrix}, \ K_i = \begin{bmatrix} 22.6660 \end{bmatrix}.$$

This controller does not appear to offer a guaranteed cost, but it employs a decay rate constraint as a performance metric, which enables comparison with $\mathcal{H}_2$ performance. By considering the controller in [15] as input in Lemma 1, and solving (13), a guaranteed cost $\gamma = 2.8659$ was obtained, which is a higher cost than that obtained from the proposal. A temporal simulation is used to show the behavior along the time when a disturbance signal $w(t) = 0.5$, a reference signal $r(t_k) = 0.5$ for $t \le 10$s, $r(t_k) = 1$ for 10s $< t \le 20$s and $r(t_k) = 1 + \sin(t_k)$ for 20s $< t$, and initial condition $\xi(0) = 0$ were employed. For the construction of the simulation, the nominal model ($\alpha_1 = \alpha_2 = 0.5$) is used for a fair comparison with [15]. As illustrated Fig. 2, the proposed structure offers better temporal performance than the alternatives in the literature, apparently with zero error in steady-state, at least for a constant reference signal (as illustrated until $t = 20$ s). Although for sinusoidal-like reference, the zero error cannot be guaranteed, it is apparent that the tracking error is lower for the proposal, such that is shown from $t > 30$ s. This better transient behavior of the proposal can be explained since it incorporates the measurement of the states and the previous input signal in the computation of the control signal, which is information that other alternatives do not include. To quantify this improvement, a Γ cost of the regulated output concerning the temporal simulation was obtained, where

$$\Gamma = \int_0^{30\,\mathrm{s}} ||z(t)||_2^2 \mathrm{d}t,$$

where $z(t)$ is associated with each temporal signal in Fig. 2 under the same condition and reference signal. The cost associated with the Theorem 1 is 30.51, while the cost that corresponds with Corollary 1 is 64.11, and the cost for the literature controller is 84.53.

VI. Conclusion

The proposed design conditions provide a method for obtaining a sampled-data PID controller that minimizes the $\mathcal{H}_2$ guaranteed cost using a DLMI-based formulation. In the numerical example, the resulting controllers exhibited improved temporal performance, as indicated by a lower cost. Specifically, the performance associated with Theorem 1 was 64% better than the PID controllers found in the literature and 52% better than those based on Corollary 1. The latter offered an improvement of 24% over the option available in the existing literature.

The next step in this line of research could involve formulating a sampled-data PID controller that does not rely on state variables or previous control signals, making it more suitable for direct implementation in conventional devices.

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