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Convex Conditions for Observer Design in Nonlinear Continuous-Time Systems Using a Spatial Discretization Procedure

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Abstract: This paper proposes convex conditions for the observer design of nonlinear continuous-time systems. A broad class of nonlinear systems can be tackled by the proposed technique. A spatial discretization is employed, and an approximate model is obtained within the error matrices that measure the difference between the nonlinear system and the approximated one. The conditions are formulated as parameter-dependent matrix inequalities and ensure that the observer can asymptotically follow the states of the original nonlinear system while guaranteeing a bound to the $\mathscr{L}_2$-gain from the disturbance input to the estimation error. Numerical experiments are used to illustrate the features of the proposed method.

Keywords: Linear parameter-varying systems, filter, linear matrix inequalities.

1. INTRODUCTION

The inherently nonlinear nature of systems cannot be neglected (Khalil, 2002). Failure to handle nonlinearities can lead to performance specifications not being met when designing controllers and filters. In some applications, linear models cannot capture the dynamics of the real systems properly. For this reason, several techniques have been developed to analyse the behaviour of nonlinear systems. Many efforts have been made to approximate nonlinear systems by a set of local linear models, instead of a single global linear model. One may cite Takagi-Sugeno fuzzy techniques (Takagi and Sugeno, 1985) and linear parameter-varying (LPV) models (Mohammadpour and Scherer, 2012). These techniques along with the Lyapunov theory, allow the formulation of control design and analysis conditions as convex optimization problems in the form of linear matrix inequalities (LMIs) (Boyd et al., 1994). Thanks to the advance of semidefinite programming algorithms the LMIs can be efficiently solved by existing solvers.

One important issue in control theory is the estimation problem. The issue of estimating the state of a system from the output can be applied in the observer-based control problem for instance Heemels et al. (2010). When disturbance is considered the problem becomes more intriguing and the $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance criteria are employed. The estimation problem has been investigated under different scenarios. One may find studies dealing with observer design (Ichihara, 2009) and filter design (Lacerda et al., 2014, 2015) for nonlinear polynomial systems, observer-based control for periodic systems (Lv and Duan, 2010) and linear time-varying periodic systems (Agulhari and Lacerda, 2016), and observer-design for affine LPV systems (Bara et al., 2001) for instance.

LPV systems have been employed for observer design considering continuous and discrete-time dynamics (Ibrir, 2009; Wang et al., 2015). However, in most of the scenarios, the polytopic description of the system is assumed to be known and no correspondence with the original nonlinear system is established. To deal with this issue, the spatial discretization introduced in Agulhari et al. (2023) will be employed in this work. With spatial discretization, a nonlinear system can be formulated in terms of an approximate polytopic model that can be used to cast convex design conditions in terms of LMIs, being such a model also known as a quasi-LPV system, since the parameters depend on the states. This avoids the use of more complex tools such as sum of squares methods for polynomial systems (Papachristodoulou and Prajna, 2005). Moreover, the system nonlinearities are not restricted to polynomial functions of the states in this scenario. To the authors' knowledge, there is no other method in the literature that considers a quasi-LPV observer directly depending on the observed states with convergence guarantees.

This paper proposes convex conditions for observer design when considering nonlinear continuous-time systems. The spatial discretization is applied to obtain a polytopic approximation of the original nonlinear model. A set of

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convex conditions is proposed to design a parameter-dependent observer that can estimate the states of the approximated system while ensuring a bound from the disturbance input to the reference output. The conditions also ensure that the same observer can estimate the states of the original nonlinear system while guaranteeing a bound from the disturbance input to the reference output of the system. Numerical experiments illustrate the behaviour of the proposed observer design technique in two academic examples. The impact of the number of samples used in the spatial discretization is explored, and it is shown that by increasing the number of samples and properly setting the position of the samples in the state space, the bound for the $\mathscr{L}_2$-gain from the disturbance input to the reference output decreases.

2. PRELIMINARIES

Consider the continuous-time state-dependent system

$$\begin{aligned} \dot{x}(t) &= A(x)x(t) + B_w(x)w(t), \\ y(t) &= C(x)x(t) + D_w(x)w(t), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state vector, $y \in \mathbb{R}^{n_y}$ is the output vector, and $w \in \mathbb{R}^{n_w}$ is the disturbance.

The system matrices are supposed to depend on the subset of states $x_1(t), \dots, x_p(t)$, $p \le n_x$, which may include all the states depending on the system. Define the subspace $\mathscr{X}_p$

$$\mathscr{X}_p \triangleq \{x_1(t), \dots, x_p(t)\} \subset \mathbb{R}^n,$$

and consider the p-tuple $\mathscr{I}$ defined as

$$\mathscr{I} \triangleq (i_1, \dots, i_p),$$

where

$$i_j \in \{1, \dots, N_j\} \triangleq \mathscr{I}_j, \quad \forall j = 1, \dots, p,$$

for any $N_j \in \mathbb{Z}^+$. For each state $x_j(t) \in \mathscr{X}_p$, let

$$\mathscr{D}_j \triangleq \{x_j^{(1)}, \dots, x_j^{(N_j)}\},$$

be a predefined real-valued grid, where $x_j^{(\ell)} > x_j^{(k)}$ if $\ell > k$. Consider also the set $\mathscr{D}_\mathscr{I} = \mathscr{D}_1 \times \dots \times \mathscr{D}_p$, being $\times$ the Cartesian product. The sets $\mathscr{D}_j$ represent a spatial discretization on the states within the subset $\mathscr{X}_p$, where N_j samples are used for each state grid.

For every index $i \in \mathscr{I}$, let $Q(x, i)$ be the real matrix-valued function described as

$$Q(x, i) = Q(x)\big|_{x_1(t)=x_1^{(i_1)}, \dots, x_p(t)=x_p^{(i_p)}},$$

being $Q(x)$ any state-dependent matrix from (1).

The following definitions are necessary for the sequence of this work (Boyd et al., 1994; Agulhari et al., 2013).

Definition 1. Let Λ_N denote the unit simplex with N vertices. It is characterized as the set of vectors $\xi(t)$ in $\mathbb{R}^N$ satisfying the following conditions:

$$\Lambda_N \triangleq \left\{ \xi(t) \in \mathbb{R}^N : \ \sum_{i=1}^{N} \xi_i(t) = 1, \ \xi_i(t) \ge 0, \ i \in \mathbb{Z}^+_{\le N} \right\}.$$

Definition 2. A multi-simplex Ω_N is the Cartesian product $\Lambda_{N_1} \times \dots \times \Lambda_{N_p}$ of a finite number p of simplexes. The dimension of Ω_N is denoted by $N = (N_1, \dots, N_p)$ and, to simplify the notation, $\mathbb{R}^N$ denotes the space $\mathbb{R}^{N_1 + \dots + N_p}$. Any element $\alpha(t)$ of Ω is decomposed as $(\alpha^{(1)}(t), \dots, \alpha^{(p)}(t))$ and, consequently, any $\alpha^{(j)}(t) \in \Lambda_{N_j}$ is decomposed as $(\alpha_1^{(j)}(t), \dots, \alpha_{N_j}^{(j)}(t))$.

Usually, the simplex and multisimplex sets are used to define polytopic domains with known matrix vertices. Define the matrix

$$\begin{aligned} \widetilde{A}(\alpha(x)) &= \sum_{i_1=1}^{N_1} \alpha_{i_1}^{(1)} \dots \sum_{i_p=1}^{N_p} \alpha_{i_p}^{(p)} A(x, i) \\ &= \sum_{i \in \mathscr{I}} \alpha_i A(x, i), \quad \alpha \in \Omega_N, \quad i \in \mathscr{I}, \end{aligned} \tag{2}$$

which is a polytope with vertices given by $A(x, i)$, for each $i \in \mathscr{I}$, and will be used in the sequence to represent a polytopic approximation of the original nonlinear system. Before, another definition is necessary.

Definition 3. The set of edges for the simplex Λ_{N_j} is defined as

$$E^{(j)} = \bigcup_{\ell=1}^{N_j-1} E_\ell^{(j)}, \ E_\ell^{(j)} \triangleq \{\alpha_\ell^{(j)} \in \Lambda_{N_j} : \alpha_\ell^{(j)} + \alpha_{\ell+1}^{(j)} = 1, \ \alpha_k^{(j)} = 0 \ \ \forall k \notin \{\ell, \ell+1\}\}.$$

The set $\partial\Omega_N$ is then defined as

$$\partial\Omega_N \triangleq E^{(1)} \times \dots \times E^{(p)}.$$

According to Definition 3, the domain of $\tilde{A}(\alpha(x))$, $\alpha \in \partial\Omega_N$ corresponds to a set of edges from the polytope Ω_N, whose vertices are given by $A(x, i)$ resultant from a spatial discretization procedure of the state subspace $\mathscr{X}_p$.

Consider the system given by

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{A}(\alpha(\tilde{x}))\tilde{x}(t) + \tilde{B}_w(\alpha(\tilde{x}))w(t) \\ \tilde{y}(t) &= \tilde{C}(\alpha(\tilde{x}))\tilde{x}(t) + \tilde{D}_w(\alpha(\tilde{x}))w(t) \end{aligned} \tag{3}$$

where all the polytopic matrices are structured such as presented in (2). System (3) is the aforementioned polytopic approximation of the original nonlinear dynamics (1), obtained from the spatial discretization methodology.

The main objective of this paper is to propose a set of conditions to guarantee that the observer denoted by

$$\begin{aligned} \dot{\hat{x}}(t) &= \tilde{A}(\alpha(\hat{x}))\hat{x}(t) + L(\alpha(\hat{x}))(y(t) - \hat{y}(t)), \\ \hat{y}(t) &= \tilde{C}(\alpha(\hat{x}))\hat{x}(t), \end{aligned} \tag{4}$$

being $L(\alpha(\hat{x}))$ the parameter-dependent observer gain, fully estimates the states of the nonlinear system (1). To assure the robustness of the proposed observer to the disturbance inputs the following reference output will be considered

$$z(t) = C_z(x(t) - \hat{x}(t)) + D_{zw}w(t).$$

To allow the application of convex procedures, the observer gain needs to be computed from the approximated dynamics (3), thus being necessary to account for the difference between both models. Such difference is described through a set of error matrices given by

$$\begin{aligned} &\delta_A(x) = A(x) - \tilde{A}(\alpha(\tilde{x})), \ \delta_B(x) = B_w(x) - \tilde{B}_w(\alpha(\tilde{x})), \\ &\delta_C(x) = C(x) - \tilde{C}(\alpha(\tilde{x})), \ \delta_D(x) = D_w(x) - \tilde{D}_w(\alpha(\tilde{x})). \end{aligned} \tag{5}$$

Remark 1. Note that the proposed modelling allows the utilization of parameter-dependent observer gains $L(\alpha(\hat{x}))$, which is not possible when using standard models of simply removing the nonlinearities of the original system by introducing uncertain time-varying parameters.

3. MAIN RESULTS

The following theorem presents a set of conditions to compute an observer gain $L(\alpha(\hat{x}))$, using the approximated model (3), assuring that the observer (4) is capable of estimating the states of the nonlinear system (1). All the parameters $\alpha(\tilde{x})$ are simply denoted by α for clarity.

Theorem 1. Given a positive scalar ρ satisfying $||x(t)|| \leq \rho||w(t)||$, if there exists a symmetric positive definite matrix $P(\alpha)$, matrices $H(\alpha)$, $Z(\alpha)$ and M, a scalar ξ and a positive scalar τ satisfying inequalities (6) and (7), being

$$\Psi_1 = M\tilde{A}(\alpha) + \tilde{A}(\alpha)^T M^T + Z(\alpha)\tilde{C}(\alpha) + \tilde{C}(\alpha)^T Z(\alpha)^T + \eta P(\alpha) + \dot{P}(\alpha),$$

$$\Psi_2 = \xi\big(M\delta_A(x) + \delta_A(x)^T M^T + Z(\alpha)\delta_C(x) + \delta_C(x)^T Z(\alpha)^T\big) - \tau I,$$

then the observer (4) with $L(\alpha) = -M^{-1}Z(\alpha)$ is capable of asymptotically follow the dynamics of the nonlinear system (1) while assuring that $||z(t)||^2 \leq (\gamma_1^2+\gamma_2^2)||w(t)||^2$.

Proof. The LMI (6) is a well-known condition for the synthesis of parameter-dependent observers for LPV systems (Yaesh and Shaked, 2009; Pipeleers et al., 2009). First, since

$$-H(\alpha)^T H(\alpha) \leq I - H(\alpha) - H(\alpha)^T,$$

then, the inequality (6) is valid if (8) is satisfied. Replacing $Z(\alpha) = -ML(\alpha)$, denoting $\tilde{A}_L(\alpha) = \tilde{A}(\alpha) - L(\alpha)\tilde{C}(\alpha)$, $\tilde{B}_L(\alpha) = \tilde{B}_w(\alpha) - L(\alpha)\tilde{D}_w(\alpha)$, and multiplying condition (8) on the left by

$$\begin{bmatrix} I & \tilde{A}_L(\alpha)^T & 0 & 0 \\ 0 & \tilde{B}_L(\alpha)^T & I & 0 \\ 0 & 0 & 0 & (H(\alpha)^T)^{-1} \end{bmatrix},$$

and on the right by its transpose results in

$$\begin{bmatrix} \Gamma & \star & \star \\ \tilde{B}_L(\alpha)^T P(\alpha) & -\gamma_1^2 I & \star \\ C_z & D_{zw} & -I \end{bmatrix} < 0,$$

where $\Gamma = \tilde{A}_L(\alpha)^T P(\alpha) + P(\alpha)\tilde{A}_L(\alpha) + \eta P(\alpha) + \dot{P}(\alpha)$. The application of the Schur complement yields

$$\begin{bmatrix} \Gamma + C_z^T C_z & \star \\ \tilde{B}_L(\alpha)^T P(\alpha) + D_{zw}^T C_z & D_{zw}^T D_{zw} - \gamma_1^2 I \end{bmatrix} < 0.$$

Multiplying the last condition by $[\tilde{e}(t)^T \;\; w(t)^T]$ on the left and by its transpose on the right, results in the $\mathscr{H}_\infty$ bound (Boyd et al., 1994) inequality

$$\dot{V}(\tilde{e}) + \bar{z}(t)^T \bar{z}(t) - \gamma_1^2 w(t)^T w(t) < -\eta\tilde{e}(t)^T P(\alpha)\tilde{e}(t), \quad (9)$$

considering that the Lyapunov function is given by

$$V(\tilde{e}) = \tilde{e}(t)^T P(\alpha)\tilde{e}(t),$$

and $\tilde{e}(t) = \tilde{x}(t) - \bar{x}(t)$ is the estimation error for the states of the approximate system (3). Therefore, it is proven that if (6) holds, then the states of the auxiliary observer

$$\begin{aligned} \dot{\bar{x}}(t) &= \tilde{A}(\alpha(\bar{x}))\bar{x}(t) + L(\alpha(\bar{x}))(\tilde{y}(t) - \bar{y}(t)), \\ \bar{y}(t) &= \tilde{C}(\alpha(\bar{x}))\bar{x}(t), \\ \bar{z}(t) &= C_z(\tilde{x}(t) - \bar{x}(t)) + D_{zw}w(t), \end{aligned} \quad (10)$$

are capable of asymptotically follow the states of the approximated system (3), while assuring that $||\bar{z}(t)||^2 \leq \gamma_1^2||w(t)||^2$. Note that the auxiliary observer (10) is only used in the proof to design the gain $L(\alpha)$ and not actually implemented.

It is now necessary to establish the conditions that assure that the observer (4), considering the same gain $L(\alpha)$, can estimate the states of the original nonlinear system (1). Consider the error signal $e(t) = x(t) - \hat{x}(t)$ between both states. The error dynamics, after applying the relations depicted in (5), is given by

$$\dot{e}(t) = \tilde{A}_L(\alpha)e(t) + \tilde{B}_L(\alpha)w(t) + (\delta_A(x) - L(\alpha)\delta_C(x))x(t) + (\delta_B(x) - L(\alpha)\delta_D(x))w(t). \quad (11)$$

In order to the proposed observer follow the nonlinear system (1) assuring also the bound $||z(t)||^2 \leq \gamma^2||w(t)||^2$, it is necessary to guarantee the condition

$$\dot{V}(e) + z(t)^T z(t) - \gamma^2 w(t)^T w(t) < 0, \quad (12)$$

being $V(e) = e(t)^T P(\alpha)e(t)$. The derivative $\dot{V}(e)$, considering the error dynamics (11), is

$$\begin{aligned} \dot{V}(e) = e(t)^T \Big(&\dot{P}(\alpha) + \Big(\tilde{A}(\alpha) - L(\alpha)\tilde{C}(\alpha)^T\Big) P(\alpha) \\ &+ P(\alpha)(\tilde{A}(\alpha) - L(\alpha)\tilde{C}(\alpha))\Big)e(t) \\ &+ 2e(t)^T P(t)(\delta_A(x) - L(\alpha)\delta_C(x))x(t) \\ &+ 2e(t)^T P(\alpha)(\delta_B(x) - L(\alpha)\delta_D(x))w(t). \end{aligned} \quad (13)$$

It is possible to show that, since (9) is verified from condition (6) and using the derivative in (13), then (12) can be rewritten as

$$\begin{aligned} &\dot{V}(e) + z(t)^T z(t) - \gamma^2 w(t)^T w(t) \\ &= \dot{V}(e) + z(t)^T z(t) - (\gamma_1^2 + \gamma_2^2)w(t)^T w(t) \\ &< -\eta e(t)^T P(\alpha)e(t) + 2e(t)^T P(\alpha)(\delta_A(x) - L(\alpha)\delta_C(x))x(t) \\ &+ 2e(t)^T P(\alpha)(\delta_B(x) - L(\alpha)\delta_D(x))w(t) - \gamma_2^2 w(t)^T w(t) < 0. \end{aligned}$$

Note that the change of variables $\gamma^2 \triangleq \gamma_1^2 + \gamma_2^2$ has been applied. Suppose now that $||x(t)||^2 \leq \rho||w(t)||^2$. Through the application of the $\mathscr{S}$-Procedure (Boyd et al., 1994) in the latter inequality, then there exists $\tau > 0$, such that

$$\begin{aligned} &-\eta e(t)^T P(\alpha)e(t) + 2e(t)^T P(t)(\delta_A(x) - L(\alpha)\delta_C(x))x(t) \\ &+ 2e(t)^T P(\alpha)(\delta_B(x) - L(\alpha)\delta_D(x))w(t) \\ &+ (\tau\rho^2 - \gamma_2^2)w(t)^T w(t) - \tau x(t)^T x(t) < 0. \end{aligned} \quad (14)$$

Finally, it remains to show that condition (14) is valid since (7) holds. Replacing $Z(\alpha) = -ML(\alpha)$ and multiplying (7) on the left by

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & \delta_A^T & I & 0 \\ 0 & \delta_B^T & 0 & I \end{bmatrix},$$

and on the right by its transpose results in

$$\begin{bmatrix} -\eta P(\alpha) & \star & \star \\ (\delta_A(x) - L(\alpha)\delta_C(x))^T P(\alpha) & -\tau I & \star \\ (\delta_B(x) - L(\alpha)\delta_D(x))^T P(\alpha) & 0 & (\tau\rho^2 - \gamma_2^2)I \end{bmatrix} < 0.$$

The multiplication of the last condition on the left by $[e(t)^T \;\; x(t)^T \;\; w(t)^T]$ and on the right by its transpose yields (14), finishing the proof. ■

Remark 2. Since the simplex presented in Definition 3 is given by $\alpha_\ell^{(j)} + \alpha_{\ell+1}^{(j)} = 1$ one may write $\dot{\alpha}_\ell^{(j)} + \dot{\alpha}_{\ell+1}^{(j)} = 0$, or simply $\left|\dot{\alpha}_\ell^{(j)}\right| = \left|\dot{\alpha}_{\ell+1}^{(j)}\right|$. In this way, following the same lines

$$\begin{bmatrix} \Psi_1 & \star & \star & \star \\ P(\alpha) - M^T + \xi(M\tilde{A}(\alpha) + Z(\alpha)\tilde{C}(\alpha)) & -\xi(M + M^T) & \star & \star \\ \tilde{B}_w(\alpha)^T M^T + \tilde{D}_w(\alpha)^T Z(\alpha)^T & \xi(\tilde{B}_w(\alpha)^T M^T + \tilde{D}_w(\alpha)^T Z(\alpha)^T) & -\gamma_1^2 I & \star \\ H(\alpha)^T C_z & 0 & H(\alpha)^T D_{zw} & I + H(\alpha) + H(\alpha)^T \end{bmatrix} < 0 \quad (6)$$

$$\begin{bmatrix} -\eta P(\alpha) & \star & \star & \star \\ P(\alpha) & -(M + M^T) & \star & \star \\ 0 & \delta_A(x)^T M^T + \delta_C(x)^T Z(\alpha)^T - \xi M & \Psi_2 & \star \\ 0 & \delta_B(x)^T M^T + \delta_D(x)^T Z(\alpha)^T & \xi(\delta_B(x)^T M^T + \delta_D(x)^T Z(\alpha)^T) & (\tau\rho^2 - \gamma_2^2)I \end{bmatrix} < 0 \quad (7)$$

$$\begin{bmatrix} \Psi & \star & \star & \star \\ P(\alpha) - M^T + \xi(M\tilde{A}(\alpha) + Z(\alpha)\tilde{C}(\alpha)) & -\xi(M + M^T) & \star & \star \\ \tilde{B}_w(\alpha)^T M^T + \tilde{D}_w(\alpha)^T Z(\alpha)^T & \xi(\tilde{B}_w(\alpha)^T M^T + \tilde{D}_w(\alpha)^T Z(\alpha)^T) & -\gamma_1^2 I & \star \\ H(\alpha)^T C_z & 0 & H(\alpha)^T D_{zw} & -H(\alpha)^T H(\alpha) \end{bmatrix} < 0 \quad (8)$$

presented in Lacerda et al. (2016)(Section 4.1), the space where the time-derivative parameters lie can be modeled and the matrix $\dot{P}(\alpha)$ can be obtained.

In the proof of Theorem 1, the bound γ^2 is rewritten as $\gamma^2 = \gamma_1^2 + \gamma_2^2$. This is a necessary step to assure the negativity of Condition (7). Therefore, if the optimization of the observer robustness is required, then it suffices to minimize $\gamma_1^2 + \gamma_2^2$ subject to (6) and (7).

Condition (7) depends on nonlinear terms $\delta_A(x)$, $\delta_B(x)$, $\delta_C(x)$ and $\delta_D(x)$. One way to solve this condition using the usual convex approaches is to consider a polytopic approach on such matrices. Suppose that these matrices can be rewritten in terms of the extreme values as

$$\delta_A(x) = \beta_1^A \underline{\delta_A} + \beta_2^A \overline{\delta_A},\ \delta_B(x) = \beta_1^B \underline{\delta_B} + \beta_2^B \overline{\delta_B},$$
$$\delta_C(x) = \beta_1^C \underline{\delta_C} + \beta_2^C \overline{\delta_C},\ \delta_D(x) = \beta_1^D \underline{\delta_D} + \beta_2^D \overline{\delta_D},$$
$$\beta^A, \beta^B, \beta^C, \beta^D \in \Lambda_2.$$

Then, the parameter-dependent inequality (7) can be solved by considering every combination of the matrix vertices of the nonlinear matrices. Another important feature is that condition (7) improves the approach to deal with the differences $\delta(x)$ between the approximated system and the original model when compared to Agulhari et al. (2023). In Agulhari et al. (2023) the robustness to such differences is assured through a norm-bounded constraint, being rather conservative when compared with condition (7).

4. NUMERICAL EXPERIMENTS

Two numerical experiments are presented to illustrate the validity of the proposed method. Experiment 1 performs a numerical analysis of the technique, while Experiment 2 depicts a comparison with another procedure from the literature. The routines are implemented in Matlab R2017a, by using the packages YALMIP (Löfberg, 2004), ROLMIP (Agulhari et al., 2019) and the solver Mosek (Andersen and Andersen, 2000).

4.1 Example 1

Consider, for this experiment, the nonlinear system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 10\mathrm{Sa}(x_1) & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t),$$
$$y(t) = [1 + 0.2\cos(x_1)\ \ 0]\, x(t),$$
$$z(t) = [1\ \ 1]\, (x(t) - \hat{x}(t)),$$

being $\mathrm{Sa}(x_1) = \sin(x_1)/x_1$.

First, the conditions from Theorem 1 are applied considering different values for the number N_1 of samples on the state x_1, and the correspondent matrices for the approximated system (3) are obtained. For instance, if $N_1 = 3$ and considering the interval $x_1 \in [-5, 5]$, matrices $\tilde{A}(\alpha)$ and $\tilde{C}(\alpha)$ are given by

$$\tilde{A}(\alpha) = \alpha_1(\tilde{x}) \begin{bmatrix} 0 & 1 \\ 10\mathrm{Sa}(-5) & -1 \end{bmatrix} + \alpha_2(\tilde{x}) \begin{bmatrix} 0 & 1 \\ 10\mathrm{Sa}(0) & -1 \end{bmatrix} + \alpha_3(\tilde{x}) \begin{bmatrix} 0 & 1 \\ 10\mathrm{Sa}(5) & -1 \end{bmatrix},$$
$$\tilde{C}(\alpha) = \alpha_1(\tilde{x})\, [1 + 0.2\cos(-5)\ \ 0] + \alpha_2(\tilde{x})\, [1.2\ \ 0] + \alpha_3(\tilde{x})\, [1 + 0.2\cos(5)\ \ 0]\,.$$

Matrices $\delta_A(x)$ and $\delta_C(x)$ applied to Condition (7) are given by

$$\delta_A(x) = \beta_1^A \begin{bmatrix} 0 & 0 \\ \min_t 10a(t) & 0 \end{bmatrix} + \beta_2^A \begin{bmatrix} 0 & 0 \\ \max_t 10a(t) & 0 \end{bmatrix},$$
$$\delta_C(x) = \beta_1^C \begin{bmatrix} \min_t c(t) & 0 \end{bmatrix} + \beta_2^C \begin{bmatrix} \max_t c(t) & 0 \end{bmatrix},$$

being

$$a = \begin{cases} \mathrm{Sa}(x_1) - (\alpha_1(\tilde{x}_1)\mathrm{Sa}(-5) + \alpha_2(\tilde{x}_1)\mathrm{Sa}(0)), \\ \qquad \text{if } x_1 \in [-5, 0],\ \alpha_3(\tilde{x}_1) = 0, \\ \mathrm{Sa}(x_1) - (\alpha_2(\tilde{x}_1)\mathrm{Sa}(0) + \alpha_3(\tilde{x}_1\mathrm{Sa}(5)), \\ \qquad \text{if } x_1 \in (0, 5],\ \alpha_1(\tilde{x}_1) = 0. \end{cases}$$

$$c = \begin{cases} \cos(x_1) - (\alpha_1(\tilde{x}_1)\cos(-5) + \alpha_2(\tilde{x}_1)), \\ \qquad \text{if } x_1 \in [-5, 0],\ \alpha_3(\tilde{x}_1) = 0, \\ \cos(x_1) - (\alpha_2(\tilde{x}_1) + \alpha_3(\tilde{x}_1)\cos(5)), \\ \qquad \text{if } x_1 \in (0, 5],\ \alpha_1(\tilde{x}_1) = 0. \end{cases}$$

$$\alpha_1(\tilde{x}_1) = \frac{-\tilde{x}_1}{5},\ \alpha_2(\tilde{x}_1) = \frac{5 - \tilde{x}_1}{5} \text{ and } \alpha_3(\tilde{x}_1) = \frac{\tilde{x}_1}{5}.$$

Table 1 presents the results obtained concerning the minimized bound $\gamma = \sqrt{\gamma_1^2 + \gamma_2^2}$, as well as the number of scalar decision variables for the LMI conditions and the uncertainty intervals for the errors $a(t)$ and $c(t)$ when a different number of spatial samples N_1 are considered.

For all the cases, Theorem 1 is applied supposing that $x_1 \in [-5, 5]$, setting $\xi = 0.01$, $\rho = 0.1$ and

$$\eta = 2 \max ||a(t)||,$$

following the suggested lines from Agulhari et al. (2023). All the parameter-dependent matrices within the LMI conditions are modeled as polynomials of unitary degree.

Table 1. Resultant bound γ, number V of scalar decision variables and the extreme values of $a(t)$ and $c(t)$ for different numbers of state samples N_1.

N_1	γ	V	$a(t)$	$c(t)$
3	0.5934	25	[-0.8763 2.6597]	[-0.0021 0.3120]
4	0.6421	31	[-4.0275 2.6351]	[-0.2191 0.2157]
5	0.3430	37	[-1.4574 1.7628]	[-0.0546 0.1146]
6	0.3064	43	[-1.5853 1.1685]	[-0.0919 0.0622]
7	0.2279	49	[-0.9014 0.8072]	[-0.0450 0.0530]
8	0.2160	55	[-0.8289 0.5825]	[-0.0489 0.0470]

From Table 1, one can see that the bound γ decreases as the number of samples increases, except when comparing $N_1 = 3$ and $N_1 = 4$. This indicates that the error reduction between the original nonlinear system and the approximated dynamics is achieved not only by increasing the number of samples but also by properly setting the position of the samples, being such analysis left for future works. Also, accompanying the reduction of γ, the uncertainty interval of the error functions also shrinks with a higher number of samples, with the exception of the analysed case.

Figure 1 illustrates the results obtained from the proposed observer with $x(0) = [1\ 1]^T$, considering $N_1 = 3$ and $N_1 = 8$. Note that the precision of the estimation increases as γ reduces, as expected. Figure 2 depicts the phase plan of the error states $e(t) = x(t) - \hat{x}(t)$ for $N_1 = 8$ showing that, for the considered interval of $x_1(t)$, the observer is indeed capable of asymptotically estimate the nonlinear states.

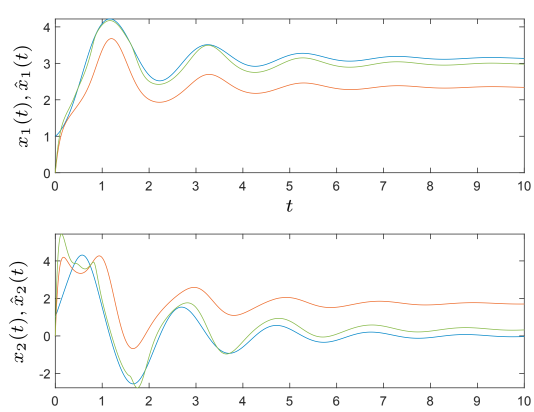

Fig. 1. States $x(t)$ (blue) and estimated states $\hat{x}(t)$ considering $N_1 = 3$ (orange) and $N_1 = 8$ (green) for Example 1.

4.2 Example 2

Consider the system adapted from Ran et al. (2021)

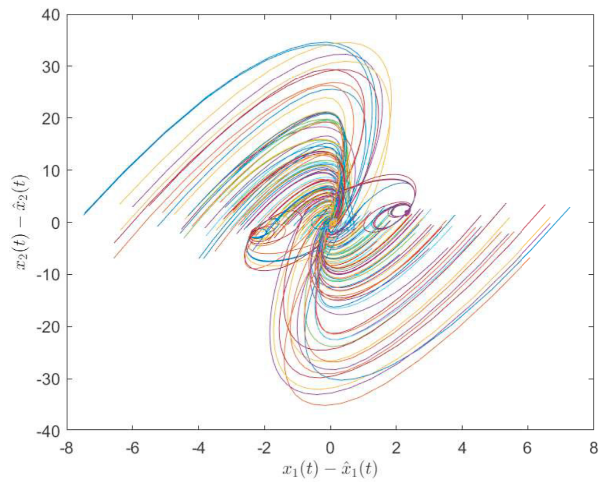

Fig. 2. Phase plan of the estimation errors $x_1(t) - \hat{x}_1(t)$ and $x_2(t) - \hat{x}_2(t)$ for Example 1.

$$\dot{x}_1(t) = -x_2^2 x_1, \quad \dot{x}_2(t) = x_3, \quad \dot{x}_3(t) = x_4,$$
$$\dot{x}_4(t) = -x_2 x_3 + x_3^2 - \sin(x_4) + x_1 + w,$$
$$y(t) = x_2.$$

To apply the proposed technique, the following state space representation is employed

$$\dot{x}(t) = \begin{bmatrix} -x_2^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & x_3 - x_2 & -Sa(x_4) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w(t).$$

Figure 3 depicts the error signals obtained from the extended observer presented in Ran et al. (2021), and for the proposed observer synthesized using the intervals $x_1(t) \in [-5, 5]$, $x_2 \in [-5, 5]$ and $x_3 \in [-5, 5]$, considering $N_1 = 4$, $N_2 = 2$, $N_3 = 4$, $\xi = 0.01$ and $\rho = 1$, resulting on the minimized bound $\gamma = 1.2511$. For the experiment, the exogenous input is given by $w(t) = 0.2 \sin(t)e^{-0.1t}$, and only the estimation errors of the states $x_3(t)$ and $x_4(t)$ are shown since they are the only states estimated in Ran et al. (2021). Note that although there is an initial peak error, the proposed method achieves a null error in less than 1 second, considerably faster than the compared technique.

5. CONCLUSIONS

This paper presented a new condition for observer design of nonlinear continuous-time systems affected by disturbances. The spatial discretization procedure was employed to obtain an approximate model. Such model, with the set of error matrices, is then used to obtain convex conditions in the form of parameter-dependent LMIs that can be used for the observer synthesis. The proposed conditions guarantee that both the approximate model states and the states of the original system can be tracked asymptotically while ensuring a bound to the $\mathscr{L}_2$-gain from the disturbance to the reference outputs. Numerical experiments illustrated the impact of the number of samples in the bounds and the error, showing the efficacy of the method. Future works include further analysis on how the selection of different system matrices in (1) affect the performance

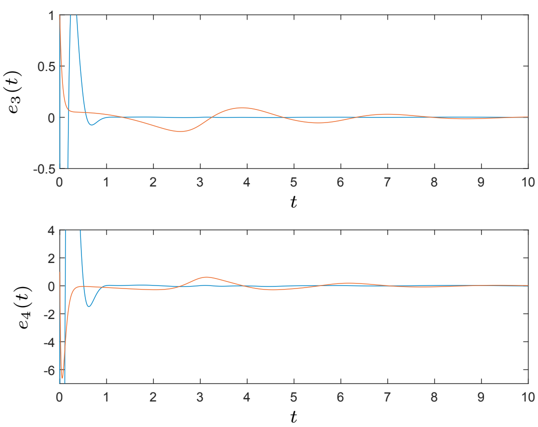

Fig. 3. Error signals $e(t)$ from the proposed observer (blue) and from the nonlinear observer presented in Ran et al. (2021) (orange) for Example 2.

of the observer, as well as non-uniform choices for the location of spatial samples.

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