Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 59-16 (2025) 230–235

A novel $\mathscr{H}_\infty$ filter design condition for discrete-time linear parameter-varying systems ⋆

Márcia L. C. Peixoto * **Márcio J. Lacerda** ** **Thierry-Marie Guerra** * **Jonathan M. Palma** *** **Reinaldo M. Palhares** ****

* *LAMIH UMR CNRS 8201, Université Polytechnique Hauts-de-France, 59313 Valenciennes, France (e-mail: marcialcpeixoto@gmail.com.br, guerra@uphf.fr).*
** *School of Computing and Digital Media, London Metropolitan University, London, UK (e-mail: m.lacerda@londonmet.ac.uk)*
*** *Department of Electrical Engineering, Faculty of Engineering, Universidad de Talca, Curicó, Chile (e-mail: jonathan.palma@utalca.cl)*
**** *Department of Electronics Engineering, Federal University of Minas Gerais, Belo Horizonte, Brazil (e-mail: rpalhares@ufmg.br)*

Abstract: This paper presents a novel condition for designing full-order parameter-dependent filters for discrete-time linear parameter-varying (LPV) systems. It is assumed that time-varying parameters are measured and can be used online. However, in scenarios where such parameters are unavailable, a robust filter can be designed. Notably, the filtering matrices are derived independently from the Lyapunov function, allowing for the use of parameter-dependent Lyapunov functions even in robust filter designs. The Lyapunov theory to design parameter-dependent filters with a guaranteed $\mathscr{H}_\infty$ performance is employed, and the proposed conditions are formulated in the form of linear matrix inequalities. A feature of the proposed method is that filtering matrices are directly recovered from synthesis conditions, eliminating the need for variable changes. Numerical experiments demonstrate the efficacy of our proposed approach.

Keywords: Linear parameter-varying systems, filter, linear matrix inequalities.

1. INTRODUCTION

Due to the high range of applications in areas such as aerospace, automotive, and chemical industries, the estimation problem experienced a rise in the last decades (Anderson and Moore, 2012). The design of filters that can properly estimate a desired combination of states from an output signal that is affected by disturbances is crucial.

The usefulness of convex formulations for filter design (Geromel, 1999; Palhares and Peres, 1999) allowed the control community to propose design conditions for different types of systems such as uncertain systems (Gonçalves et al., 2006; Duan et al., 2006; Lacerda et al., 2011; El-Amrani et al., 2017), linear parameter-varying systems (Sato, 2006; de Souza et al., 2007; Borges et al., 2010), Markov jump linear systems (Wang et al., 2024), nonlinear polynomial systems (Li et al., 2012; Lacerda et al., 2014), among others. The synthesis conditions are obtained by using the Lyapunov theory which enables deriving conditions in the form of linear matrix inequalities (LMIs) that can be solved efficiently by current computational tools.

Of particular interest in this work is the class of linear parameter-varying (LPV) systems. LPV models can be used to represent nonlinear systems and allow the formulation of design problems in a linear fashion (Mohammadpour and Scherer, 2012). Different methods have appeared in the literature to tackle the presence of time-varying parameters in the filter design problem. One may cite works dealing with inexactly scheduled parameters (Sato, 2010; Lacerda et al., 2016), time-delayed LPV systems (Mohammadpour and Grigoriadis, 2006), fault detection (Bokor and Balas, 2004), and Kalman filtering for LPV models (Delgado-Aguiñaga et al., 2021) for instance.

When there is no information about the time-varying parameters, robust filters can be designed. However, if the parameters can be estimated or measured online, parameter-dependent filters can be sought. The parameter-dependent filters can provide less conservative results in terms of the considered performance criteria. When considering time-varying parameters that can vary arbitrarily fast, the continuous-time case is limited to the use of a constant Lyapunov function (Borges et al., 2008). On the other hand, in the discrete-time framework, there are more gen-

⋆ This work was sponsored in part by ANR (National Research Agency), grant Number ANR-23-CPJ1-0116-01 (AAP Chaire HUMA^{2}IN), and in part by the National Center for Scientific Research (CNRS). This work was also supported by the Brazilian agencies CNPq (grant number: 305578/2022-7), and FAPEMIG (grant numbers: APQ-00528-24, APQ-04310-24).

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2025.10.108

eral techniques to consider the presence of time-varying parameters, for instance (Daafouz and Bernussou, 2001; Peixoto et al., 2020).

This paper presents a new condition to design a full-order parameter-dependent filter for discrete-time LPV systems. The time-varying parameters are assumed to be measured and can then be used online. However, if the time-varying parameters are not available, a robust filter can be designed. The filtering matrices are obtained independently from the Lyapunov function. In this sense, even if a robust filter is designed, one can use parameter-dependent Lyapunov functions. The Lyapunov theory is employed to design an $\mathscr{H}_\infty$ filter, and an affine Lyapunov matrix is used. The technique presented by Daafouz and Bernussou (2001) is explored to derive the conditions. An important feature of the proposed method is that the filtering matrices are recovered directly from the synthesis conditions, i.e., there is no need for any change of variables. Numerical experiments are provided to illustrate the efficacy of the proposed method.

Notation: The set of natural numbers is denoted by $\mathbb{N}$, and $\mathbb{N}_{\leq N}$ represents the set $\{1, 2, \ldots, N\}$. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbb{R}^{m\times n}$ is the set of all $m \times n$ real matrices. Symmetric blocks in a matrix are denoted by $\star$. $M > 0$ ($M < 0$) indicates that M is positive (negative) definite. The symbol $(^\top)$ represents transposition, and $\text{He}\{M\} = M + M^\top$. The identity matrix of order n is denoted by I_n and the null matrix of order $n \times m$ by $0_{n\times m}$. If the dimensions of both identity and null matrices are straightforwardly deduced, they are omitted.

2. PROBLEM FORMULATION

Consider a discrete-time LPV system given by

$$\begin{aligned} x_{k+1} &= A(\alpha_k)x_k + B_w(\alpha_k)w_k, \\ z_k &= C_z(\alpha_k)x_k + D_z(\alpha_k)w_k, \\ y_k &= C_y(\alpha_k)x_k + D_y(\alpha_k)w_k, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $w \in \mathbb{R}^{n_w}$ is the disturbance input, $y \in \mathbb{R}^{n_y}$ is the measured output, and $z \in \mathbb{R}^{n_z}$ is the signal to be estimated. $\alpha_k = [\alpha_{1,k}, \alpha_{2,k}, \ldots, \alpha_{N,k}]^\top \in \mathbb{R}^N$ is the vector of time-varying parameters, which are functions of measured exogenous signals. The parameter-dependent matrices $A(\alpha_k)$, $B_w(\alpha_k)$, $C_z(\alpha_k)$, $D_z(\alpha_k)$, $C_y(\alpha_k)$ and $D_y(\alpha_k)$ belong to a polytopic domain parameterized by the time-varying parameters $\alpha_k \in \Lambda$, defined as

$$\begin{bmatrix} A(\alpha_k) & B_w(\alpha_k) \\ C_z(\alpha_k) & D_z(\alpha_k) \\ C_y(\alpha_k) & D_y(\alpha_k) \end{bmatrix} = \sum_{i=1}^{N} \alpha_{i,k} \begin{bmatrix} A_i & B_i \\ C_{z,i} & D_{z,i} \\ C_{y,i} & D_{y,i} \end{bmatrix},$$

where N is the number of vertices of the polytopic domain, and the unit simplex Λ is given by

$$\Lambda = \left\{ \alpha_k \in \mathbb{R}^N : \sum_{i=1}^{N} \alpha_{i,k} = 1,\ \alpha_{i,k} \geq 0,\ i \in \mathbb{N}_{\leq N} \right\}.$$

This paper deals with the problem of designing a full-order parameter-dependent filter given by

$$\begin{aligned} x_{f,k+1} &= H(\alpha_k)^{-1}A_f(\alpha_k)x_{f,k} + H(\alpha_k)^{-1}B_f(\alpha_k)y_k, \\ z_f &= C_f(\alpha_k)x_{f,k} + D_f(\alpha_k)y_k, \end{aligned} \tag{2}$$

where $x_f \in \mathbb{R}^{n_x}$ is the filter state, $z_f \in \mathbb{R}^{n_z}$ is the estimated output, and the filter matrices $H(\alpha_k) \in \mathbb{R}^{n_x\times n_x}$, $A_f(\alpha_k) \in \mathbb{R}^{n_x\times n_x}$, $B_f(\alpha_k) \in \mathbb{R}^{n_x\times n_y}$, $C_f(\alpha_k) \in \mathbb{R}^{n_z\times n_x}$, and $D_f(\alpha_k) \in \mathbb{R}^{n_z\times n_y}$ are to be designed on the form

$$\begin{bmatrix} H(\alpha_k) & A_f(\alpha_k) & B_f(\alpha_k) \\ C_f(\alpha_k) & D_f(\alpha_k) & 0 \end{bmatrix} = \sum_{i=1}^{N} \alpha_{i,k} \begin{bmatrix} H_i & A_{f_i} & B_{f_i} \\ C_{f_i} & D_{f_i} & 0 \end{bmatrix}.$$

The estimation error is defined by $e_k = z_k - z_{f,k}$, and considering the LPV system in (1) and the parameter-dependent filter in (2), the following augmented system is obtained

$$\begin{aligned} \xi_{k+1} &= \mathscr{A}(\alpha_k)\xi_k + \mathscr{B}(\alpha_k)w_k, \\ e_k &= \mathscr{C}(\alpha_k)\xi_k + \mathscr{D}(\alpha_k)w_k, \end{aligned} \tag{3}$$

where $\xi_k = \begin{bmatrix} x_k^\top & x_{f,k}^\top \end{bmatrix}^\top$, and

$$\begin{aligned} \mathscr{A}(\alpha_k) &= \begin{bmatrix} A(\alpha_k) & 0 \\ H(\alpha_k)^{-1}B_f(\alpha_k)C_y(\alpha_k) & H(\alpha_k)^{-1}A_f(\alpha_k) \end{bmatrix}, \\ \mathscr{B}(\alpha_k) &= \begin{bmatrix} B_w(\alpha_k) \\ H(\alpha_k)^{-1}B_f(\alpha_k)D_y(\alpha_k) \end{bmatrix}, \\ \mathscr{C}(\alpha_k) &= \left[C_z(\alpha_k) - D_f(\alpha_k)C_y(\alpha_k) \ \ -C_f(\alpha_k) \right], \\ \mathscr{D}(\alpha_k) &= D_z(\alpha_k) - D_f(\alpha_k)D_y(\alpha_k). \end{aligned}$$

The filtering problem addressed in this paper is stated as follows.

Problem 1. Consider an LPV system as described in (1). Find the filter matrices defined in (2) such that under zero initial condition, $\xi_0 = 0$, an upper bound γ to the $\mathscr{H}_\infty$ performance index of system (3) is defined by the ℓ_2-induced gain:

$$\sup_{||w_k||_2 \neq 0} \frac{||e_k||_2}{||w_k||_2} < \gamma^2,$$

where $w_k \in \ell_2^{n_w}$ and $e_k \in \ell_2^{n_z}$.

The following lemma will be employed to convert parameter-dependent LMIs into a finite set of LMI constraints.

Lemma 1. Suppose $\Gamma_{ijl} = \Gamma_{ijl}^\top$, with $i, j, l \in \mathbb{N}_{\leq N}$ are matrices of appropriate dimensions. The parameter-dependent condition, $\forall \alpha_k, \alpha_{k+1} \in \Lambda$,

$$\Gamma(\alpha_k, \alpha_{k+1}) = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{N} \alpha_{i,k}\alpha_{j,k}\alpha_{l,k+1}\Gamma_{ijl} < 0, \tag{4}$$

is certified if the following LMIs hold for all $i, j, l \in \mathbb{N}_{\leq N}$

$$\begin{aligned} \Gamma_{iil} &< 0, \quad i = j, \\ \Gamma_{ijl} + \Gamma_{jil} &< 0, \quad i < j. \end{aligned} \tag{5}$$

Proof. The parameter-dependent matrix $\Gamma(\alpha_k, \alpha_{k+1})$ in (4) can be rewritten as

$$\begin{aligned} \Gamma(\alpha_k, \alpha_{k+1}) &= \sum_{i=1}^{N}\sum_{l=1}^{N} \alpha_{i,k}^2\alpha_{l,k+1}\Gamma_{iil} \\ &\quad + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\sum_{l=1}^{N} \alpha_{i,k}\alpha_{j,k}\alpha_{l,k+1}\left(\Gamma_{ijl} + \Gamma_{jil}\right). \end{aligned}$$

Therefore, if (5) holds, then condition (4) is guaranteed. ■

3. MAIN RESULTS

A new $\mathscr{H}_\infty$ filter design for discrete-time LPV systems is presented in the sequel in terms of LMIs.

Theorem 1. Consider the linear parameter-varying system (1) with its associate filter (2). If there exist symmetric positive-definite matrices $P_{1,i} \in \mathbb{R}^{n_x \times n_x}$, $P_{2,i} \in \mathbb{R}^{n_x \times n_x}$, and matrices $P_{3,i} \in \mathbb{R}^{n_x \times n_x}$ $X_{1,i} \in \mathbb{R}^{n_x \times n_x}$, $X_{2,i} \in \mathbb{R}^{n_x \times n_x}$, $X_{3,i} \in \mathbb{R}^{n_x \times n_x}$, $X_{4,i} \in \mathbb{R}^{n_x \times n_x}$, $X_{5,i} \in \mathbb{R}^{n_z \times n_x}$, $X_{6,i} \in \mathbb{R}^{n_w \times n_x}$, $H_i \in \mathbb{R}^{n_x \times n_x}$, $A_{f,i} \in \mathbb{R}^{n_x \times n_x}$, $B_{f,i} \in \mathbb{R}^{n_x \times n_y}$, $C_{f,i} \in \mathbb{R}^{n_z \times n_x}$ and $D_{f,i} \in \mathbb{R}^{n_z \times n_y}$, such that the following inequalities hold for a given scalar ϵ

$$\Phi_{iil} < 0, \quad i = j \tag{6}$$

$$\Phi_{ijl} + \Phi_{jil} < 0, \quad i < j \tag{7}$$

for $i, j, l \in \mathbb{N}_{\leq N}$, with

$$\Phi_{ijl} = \begin{bmatrix} \Phi_{11} & \star & \star & \star & \star & \star \\ \Phi_{21} & \Phi_{22} & \star & \star & \star & \star \\ \Phi_{31} & \Phi_{32} & \Phi_{33} & \star & \star & \star \\ \Phi_{41} & A_{f,j}^\top & \Phi_{43} & -P_{2,j} & \star & \star \\ -X_{5,j} & 0_{n_z \times n_x} & \Phi_{53} & -C_{f,j} & -\mathrm{I}_{n_z} & \star \\ \Phi_{61} & \Phi_{62} & \Phi_{63} & \Phi_{64} & \Phi_{65} & \Phi_{66} \end{bmatrix},$$

and

$$\Phi_{11} = P_{1,l} - X_{1,j} - X_{1,j}^\top,$$
$$\Phi_{21} = -X_{2,j} - \epsilon H_j^\top + P_{3,l}^\top,$$
$$\Phi_{31} = -X_{3,j} + (X_{1,j} A_i + \epsilon B_{f,j} C_{y,i})^\top,$$
$$\Phi_{41} = -X_{4,j} + \epsilon A_{f,j}^\top,$$
$$\Phi_{61} = -X_{6,j} + (X_{1,j} B_{w,i} + \epsilon B_{f,j} D_{y,i})^\top,$$
$$\Phi_{22} = P_{2l} - H_j - H_j^\top,$$
$$\Phi_{32} = -H_j + (X_{2,j} A_i + B_{f,j} C_{y,i})^\top,$$
$$\Phi_{62} = (X_{2,j} B_{w,i} + B_{f,j} D_{y,i})^\top,$$
$$\Phi_{33} = -P_{1,j} + \mathrm{He}\{X_{3,j} A_i + B_{f,j} C_{y,i}\},$$
$$\Phi_{43} = X_{4,j} A_i + A_{f,j}^\top - P_{3,j}^\top,$$
$$\Phi_{53} = X_{5,j} A_i + C_{z,i} - D_{f,j} C_{y,i},$$
$$\Phi_{63} = X_{6,j} A_i + (X_{3,j} B_{w,i} + B_{f,j} D_{y,i})^\top,$$
$$\Phi_{64} = B_{w,i}^\top X_{4,j}^\top,$$
$$\Phi_{65} = (X_{5,j} B_{w,i} - D_{f,j} D_{y,i} + D_{z,i})^\top,$$
$$\Phi_{66} = \mathrm{He}\{X_{6,j} B_{w,i}\} - \delta \mathrm{I}_{n_w}.$$

Then, there exists a filter in the form of (2), ensuring a guaranteed cost $\gamma = \sqrt{\delta}$ for the $\mathscr{H}_\infty$ performance of system (3).

Proof. Based on the convexity property of the time-varying parameters employing Lemma 1, it can be proved that inequalities (6)–(7) imply

$$\Phi(\alpha_k) = \begin{bmatrix} \overline{\Phi}_{11} & \star & \star & \star & \star & \star \\ \overline{\Phi}_{21} & \overline{\Phi}_{22} & \star & \star & \star & \star \\ \overline{\Phi}_{31} & \overline{\Phi}_{32} & \overline{\Phi}_{33} & \star & \star & \star \\ \overline{\Phi}_{41} & \overline{\Phi}_{42} & \overline{\Phi}_{43} & -P_2(\alpha_k) & \star & \star \\ \overline{\Phi}_{51} & 0 & \overline{\Phi}_{53} & -C_f(\alpha_k) & -\mathrm{I}_{n_z} & \star \\ \overline{\Phi}_{61} & \overline{\Phi}_{62} & \overline{\Phi}_{63} & \overline{\Phi}_{64} & \overline{\Phi}_{65} & \overline{\Phi}_{66} \end{bmatrix} < 0, \tag{8}$$

with

$$\overline{\Phi}_{11} = P_1(\alpha_{k+1}) - \mathrm{He}\{X_1(\alpha_k)\},$$
$$\overline{\Phi}_{21} = -X_2(\alpha_k) - \epsilon H(\alpha_k)^\top + P_3(\alpha_{k+1})^\top,$$
$$\overline{\Phi}_{31} = -X_3(\alpha_k) + (X_1(\alpha_k) A(\alpha_k) + \epsilon B_f(\alpha_k) C_y(\alpha_k))^\top,$$
$$\overline{\Phi}_{41} = -X_4(\alpha_k) + \epsilon A_f(\alpha_k)^\top,$$
$$\overline{\Phi}_{51} = -X_5(\alpha_k),$$
$$\overline{\Phi}_{61} = -X_6(\alpha_k) + (X_1(\alpha_k) B_w(\alpha_k) + \epsilon B_f(\alpha_k) D_y(\alpha_k))^\top,$$
$$\overline{\Phi}_{22} = P_2(\alpha_{k+1}) - H(\alpha_k) - H(\alpha_k)^\top,$$
$$\overline{\Phi}_{32} = -H(\alpha_k) + (X_2(\alpha_k) A(\alpha_k) + B_f(\alpha_k) C_y(\alpha_k))^\top,$$
$$\overline{\Phi}_{42} = A_f(\alpha_k)^\top,$$
$$\overline{\Phi}_{62} = (X_2(\alpha_k) B_w(\alpha_k) + B_f(\alpha_k) D_y(\alpha_k))^\top,$$
$$\overline{\Phi}_{33} = -P_1(\alpha_k) + \mathrm{He}\{X_3(\alpha_k) A(\alpha_k) + B_f(\alpha_k) C_y(\alpha_k)\},$$
$$\overline{\Phi}_{43} = X_4(\alpha_k) A(\alpha_k) + A_f(\alpha_k)^\top - P_3(\alpha_k)^\top,$$
$$\overline{\Phi}_{53} = X_5(\alpha_k) A(\alpha_k) + C_z(\alpha_k) - D_f(\alpha_k) C_y(\alpha_k),$$
$$\overline{\Phi}_{63} = X_6(\alpha_k) A(\alpha_k) + (X_3(\alpha_k) B_w(\alpha_k) + B_f(\alpha_k) D_y(\alpha_k))^\top,$$
$$\overline{\Phi}_{64} = B_w(\alpha_k)^\top X_4(\alpha_k)^\top,$$
$$\overline{\Phi}_{65} = (X_5(\alpha_k) B_w(\alpha_k) - D_f(\alpha_k) D_y(\alpha_k) + D_z(\alpha_k))^\top,$$
$$\overline{\Phi}_{66} = \mathrm{He}\{X_6(\alpha_k) B_w(\alpha_k)\} - \delta \mathrm{I}_{n_w}.$$

It follows from (8) that $H(\alpha_k)^\top + H(\alpha_k) > P_2(\alpha_{k+1})$ for all α_k and $\alpha_{k+1} \in \Lambda$. Since $P_2(\alpha_{k+1}) > 0$ for all α_k and $\alpha_{k+1} \in \Lambda$, then matrix $H(\alpha_k)$ is nonsingular for all $\alpha_k \in \Lambda$. Then, multiplying (8) by

$$\begin{bmatrix} A(\alpha_k)^\top & C_y(\alpha_k)^\top \overline{B}_f(\alpha_k)^\top & \mathrm{I} & 0 & \mathbb{C}(\alpha_k)^\top & 0 \\ B_w(\alpha_k)^\top & D_y(\alpha_k)^\top \overline{B}_f(\alpha_k)^\top & 0 & 0 & \mathbb{D}(\alpha_k)^\top & \mathrm{I} \\ 0 & A_f(\alpha_k)^\top H(\alpha_k)^\top & 0 & \mathrm{I} & -C_f(\alpha_k)^\top & 0 \end{bmatrix}$$

with

$$\overline{B}_f(\alpha_k) = H(\alpha_k)^{-1} B_f(\alpha_k),$$
$$\mathbb{C}(\alpha_k) = C_z(\alpha_k) - D_f(\alpha_k) C_y(\alpha_k),$$
$$\mathbb{D}(\alpha_k) = D_z(\alpha_k) - D_f(\alpha_k) D_y(\alpha_k),$$

on the left and by its transpose on the right, one obtains:

$$\begin{bmatrix} \Theta_{11} & \star & \star \\ \Theta_{21} & \Theta_{22} & \star \\ \Theta_{31} & \Theta_{32} & \Theta_{33} \end{bmatrix} < 0, \tag{9}$$

with

$$\begin{aligned} \Theta_{11} = & -P_1(\alpha_k) + A(\alpha_k)^\top P_1(\alpha_{k+1}) A(\alpha_k) \\ & + C_y(\alpha_k)^\top \overline{B}_f(\alpha_k)^\top P_2(\alpha_k) \overline{B}_f(\alpha_k) C_y(\alpha_k) \\ & - \mathrm{He}\{C_y(\alpha_k)^\top D_f(\alpha_k)^\top C_z(\alpha_k)\} + C_z(\alpha_k)^\top C_z(\alpha_k) \\ & + C_y(\alpha_k)^\top D_f(\alpha_k)^\top D_f(\alpha_k) C_y(\alpha_k) \\ & + \mathrm{He}\{A(\alpha_k)^\top P_3(\alpha_{k+1}) \overline{B}_f(\alpha_k) C_y(\alpha_k)\}, \end{aligned}$$

$$\begin{aligned} \Theta_{21} = & D_z(\alpha_k)^\top C_z(\alpha_k) + B_w(\alpha_k)^\top P_1(\alpha_{k+1}) A(\alpha_k) \\ & - D_y(\alpha_k)^\top D_f(\alpha_k)^\top C_z(\alpha_k) \\ & - D_z(\alpha_k)^\top D_f(\alpha_k) C_y(\alpha_k) \\ & + D_y(\alpha_k)^\top D_f(\alpha_k)^\top D_f(\alpha_k) C_y(\alpha_k) \\ & + B_w(\alpha_k) P_3(\alpha_{k+1}) \overline{B}_f(\alpha_k) C_y(\alpha_k) \\ & + D_y(\alpha_k)^\top \overline{B}_f(\alpha_k)^\top P_3(\alpha_{k+1})^\top A(\alpha_k) \\ & + D_y(\alpha_k)^\top \overline{B}_f(\alpha_k)^\top P_2(\alpha_k) \overline{B}_f(\alpha_k) C_y(\alpha_k), \end{aligned}$$

$$\begin{aligned} \Theta_{31} = & -P_3(\alpha_k)^\top + C_f(\alpha_k)^\top D_f(\alpha_k) C_y(\alpha_k) \\ & - C_f(\alpha_k)^\top C_z(\alpha_k) \\ & + A_f(\alpha_k)^\top H(\alpha_k)^{-\top} P_3(\alpha_{k+1})^\top A(\alpha_k) \\ & + A_f(\alpha_k)^\top H(\alpha_k)^{-\top} P_3(\alpha_k) \overline{B}_f(\alpha_k) C_y(\alpha_k), \end{aligned}$$

$$\Theta_{22} = -\delta \mathrm{I}_{n_w} + B_w(\alpha_k)^\top P_1(\alpha_{k+1}) B_w(\alpha_k)$$

$$
\begin{aligned}
&+ D_z(\alpha_k)^\top D_z(\alpha_k) - \mathrm{He}\{D_y(\alpha_k)^\top D_f(\alpha_k)^\top D_z(\alpha_k)\}\\
&+ D_y(\alpha_k)^\top D_f(\alpha_k)^\top D_f(\alpha_k) D_y(\alpha_k)\\
&+ \mathrm{He}\{B_w(\alpha_k)^\top P_3(\alpha_{k+1})\overline{B}_f(\alpha_k) D_y(\alpha_k)\}\\
&+ D_y(\alpha_k)^\top \overline{B}_f(\alpha_k)^\top P_2(\alpha_k)\overline{B}_f(\alpha_k) D_y(\alpha_k),\\
\Theta_{32} &= -C_f(\alpha_k)^\top D_z(\alpha_k) + C_f(\alpha_k)^\top D_f(\alpha_k) D_y(\alpha_k)\\
&+ A_f(\alpha_k)^\top H(\alpha_k)^{-\top} P_3(\alpha_{k+1})^\top B_w(\alpha_k)\\
&+ A_f(\alpha_k)^\top H(\alpha_k)^{-\top} P_2(\alpha_k)\overline{B}_f(\alpha_k) D_y(\alpha_k),\\
\Theta_{33} &= -P_2(\alpha_k) + C_f(\alpha_k)^\top C_f(\alpha_k)\\
&+ A_f(\alpha_k)^\top H(\alpha_k)^{-\top} P_2(\alpha_k) H(\alpha_k)^{-1} A_f(\alpha_k).
\end{aligned}
$$

Multiplying (9) by $\begin{bmatrix} x_k^\top & w_k^\top & x_{f,k}^\top \end{bmatrix}$ on the left its transpose on the right results in

$$V(\xi_{k+1}) - V(\xi_k) + e_k^\top e_k - \delta w_k^\top w_k < 0,$$

with $V(\xi_k) = \xi_k^T \mathscr{P}(\alpha_k)\xi_k$, being

$$\mathscr{P}(\alpha_k) = \begin{bmatrix} P_1(\alpha_k) & P_3(\alpha_k) \\ P_3(\alpha_k)^\top & P_2(\alpha_k) \end{bmatrix}.$$

If $w_k = 0\ \forall k$, it implies that $\Delta V(\xi_k) < 0$. In case of $w_k \neq 0$, taking the sum over $k \in \{0, \ldots, \tau - 1\}$, $\tau > 0$, one has

$$V(\xi_\tau) - V(\xi_0) + \sum_{k=0}^{\tau} \|e_k\|^2 - \delta \sum_{k=0}^{\tau} \|w_k\|^2 < 0.$$

Given the fact that $V(\xi_\tau) \geq 0$ and taking $\tau \to \infty$, one has

$$\sum_{k=0}^{\infty} \|e_k\|^2 < \delta \sum_{k=0}^{\infty} \|w_k\|^2 + V(\xi_0).$$

By applying the square root on both sides of the last inequality, it is possible to conclude that

$$\|e_k\|_{\ell_2} \leq \sqrt{\delta}\|w_k\|_{\ell_2} + \sqrt{V(\xi_0)}.$$

Under zero initial condition, it follows that $\sqrt{V(\xi_0)} = 0$ and $\sqrt{\delta}$ is an upper bound to the ℓ_2-induced gain. This concludes the proof. ■

If there is no information about the time-varying parameters, the conditions proposed in Theorem 1 can be adapted to design a robust filter. The following Corollary presents the conditions.

Corollary 1. If there exist symmetric positive-definite matrices $P_{1,i} \in \mathbb{R}^{n_x \times n_x}$, $P_{2,i} \in \mathbb{R}^{n_x \times n_x}$, and matrices $P_{3,i} \in \mathbb{R}^{n_x \times n_x}$ $X_{1,i} \in \mathbb{R}^{n_x \times n_x}$, $X_{2,i} \in \mathbb{R}^{n_x \times n_x}$, $X_{3,i} \in \mathbb{R}^{n_x \times n_x}$, $X_{4,i} \in \mathbb{R}^{n_x \times n_x}$, $X_{5,i} \in \mathbb{R}^{n_z \times n_x}$, $X_{6,i} \in \mathbb{R}^{n_w \times n_x}$, $H \in \mathbb{R}^{n_x \times n_x}$, $A_f \in \mathbb{R}^{n_x \times n_x}$, $B_f \in \mathbb{R}^{n_x \times n_y}$, $C_f \in \mathbb{R}^{n_z \times n_x}$ and $D_f \in \mathbb{R}^{n_z \times n_y}$, such that the inequalities (6) and (7) hold for a given scalar ϵ, then the robust filter

$$
\begin{aligned}
x_{f,k+1} &= H^{-1} A_f x_{f,k} + H^{-1} B_f y_k,\\
z_f &= C_f x_{f,k} + D_f y_k,
\end{aligned}
$$

assures a guaranteed cost given by $\gamma = \sqrt{\delta}$ for the $\mathscr{H}_\infty$ performance of the augmented system.

Remark 1. Note that both in Theorem 1 and Corollary 1 the filtering matrices are recovered independently of the Lyapunov matrices. In this sense, even when a robust filter is designed, parameter-dependent Lyapunov matrices can be used in the proposed conditions.

4. NUMERICAL EXAMPLES

This section illustrates the effectiveness of the proposed conditions when compared with existing approaches in the literature. The routines were implemented in Matlab using the parser Yalmip, and the semidefinite programming solver MOSEK.

Example 1. Consider the time-varying discrete-time system borrowed from Borges et al. (2010) with matrices

$$
\begin{aligned}
A &= \begin{bmatrix} 0.265 - 0.165\theta_k & 0.45(1+\theta_k) \\ 0.5(1-\theta_k) & 0.265 - 0.215\theta_k \end{bmatrix},\\
B_w &= \begin{bmatrix} 1.5 - 0.5\theta_k \\ 0.1 \end{bmatrix},\ C_y = [1\ 0],\ D_y = 1, \qquad (10)\\
C_z &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ D_z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned}
$$

where $-1 \leq \theta_k \leq 1$, is the time-varying parameter that can vary arbitrarily fast. Then, system (10) can be modeled as a discrete-time polytopic LPV system with 2 vertices as follows

$$
\begin{aligned}
A_1 &= \begin{bmatrix} 0.43 & 0 \\ 1 & 0.48 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.1 & 0.9 \\ 0 & 0.05 \end{bmatrix},\\
B_{w,1} &= \begin{bmatrix} 2 \\ 0.1 \end{bmatrix},\ B_{w,2} = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix},\\
C_{y,1} &= C_{y,2} = C_y,\\
D_{y,1} &= D_{y,2} = D_y,\\
C_{z,1} &= C_{z,2} = C_z,\\
D_{z,1} &= D_{z,2} = D_z.
\end{aligned}
$$

The time-varying parameters can be computed as $\alpha_{1,k} = \frac{1-\theta_k}{2}$ and $\alpha_{2,k} = 1 - \alpha_{1,k}$. In this example, the goal is to design parameter-dependent and robust filters. Table 1 shows the minimum γ obtained with Theorem 1, Corollary 1 with $\epsilon = 150$, and the approaches in (Borges et al., 2010, Theorem 4) and (Palma et al., 2020, Theorem 1). It is noteworthy that the filter design proposed by Borges et al. (2010) is expressed as bilinear matrix inequalities, with an iterative procedure provided to obtain the filter matrices. Consequently, better results can be achieved with more iterations. From Table 1, it is possible to see that the conditions proposed in Theorem 1, and in Corollary 1 yield competitive outcomes compared to other methods in the literature.

Table 1. $\mathscr{H}_\infty$ guaranteed performances ($\gamma = \sqrt{\delta}$) considering parameter-dependent (PD) and robust approaches– Example 1.

Method	Filter	γ
(Borges et al., 2010, Theorem 4) ($it = 1$)	Robust	19.41
(Borges et al., 2010, Theorem 4) ($it = 2$)	Robust	9.10
(Palma et al., 2020, Theorem 1)	Robust	8.74
Corollary 1	Robust	8.18
(Borges et al., 2010, Theorem 4) ($it = 2$)	PD	1.22
(Borges et al., 2010, Theorem 4) ($it = 2$)	PD	1.22
(Palma et al., 2020, Theorem 1)	PD	1.21
Theorem 1	PD	1.21

For illustration, consider the solution of Theorem 1 and Corollary 1 with $\epsilon = 150$. Fig. 1 shows the trajectories of e_k obtained from both Theorem 1 and Corollary 1. Fig. 2 depicts the trajectories of w_k and α_k. The noise input $w_k = \sin(0.5k)\exp(-0.1k)$ was applied from $k = 10$, and we have considered the time-varying parameter as $\theta_k = \sin(0.25k)$. It is noticeable from Figure 1, that the error e_k is smaller in the case of the parameter-dependent filter when compared with the robust filter.

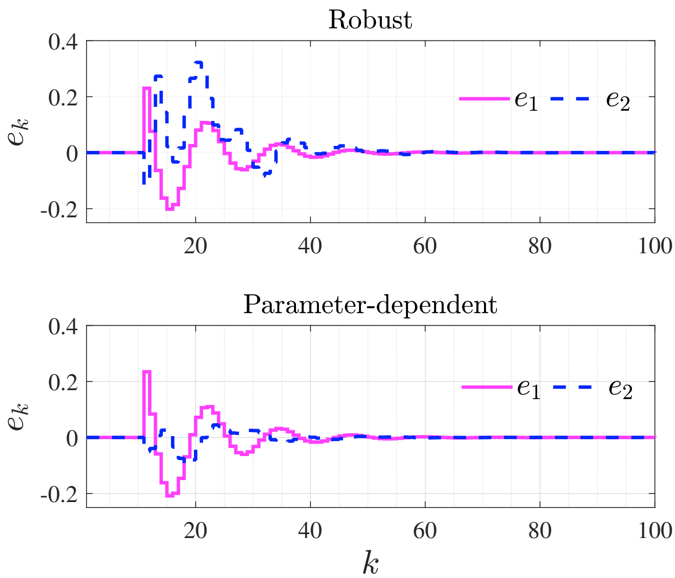

Fig. 1. Trajectories of e_k obtained by applying Corollary 1 and Theorem 1 with $\epsilon = 150$ and $\xi(0) = 0$ – Example 1.

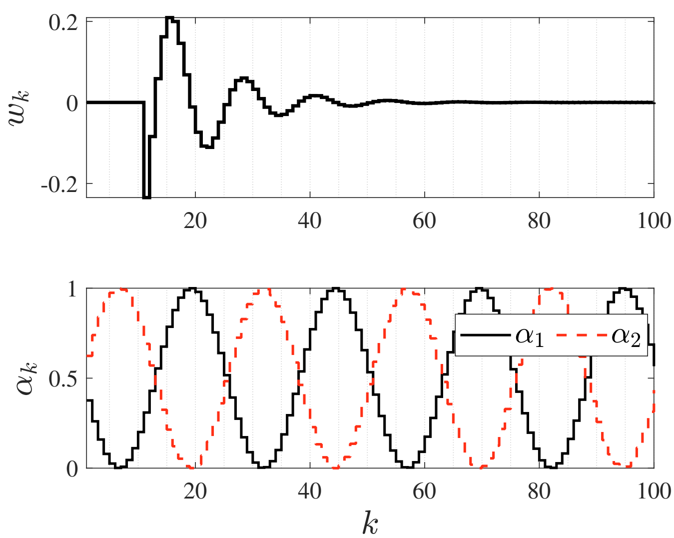

Fig. 2. Evolution of w_k, and α_k with $\epsilon = 150$ and $\xi(0) = 0$ – Example 1.

Example 2. Consider the linear parameter-varying system with matrices

$$A_1 = \begin{bmatrix} 1.05 & 0.35 \\ -0.41 & -\rho 0.05 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1.05 & 0.35 \\ -0.41 & -0.05 \end{bmatrix},$$
$$B_{w,1} = \begin{bmatrix} -0.12 \\ 0 \end{bmatrix}, \; B_{w,2} = \begin{bmatrix} 0.1 \\ -0.15 \end{bmatrix},$$
$$C_{y,1} = [1.7 \;\; 3], \; C_{y,2} = [-1.9 \;\; 2],$$
$$C_{z,1} = [1 \;\; 0], \; C_{z,2} = [1.75 \;\; 0],$$
$$D_{y,1} = 0.05, \; D_{y,2} = 0.1,$$
$$D_{z,1} = 0.1, \; D_{z,2} = 0.$$

In this example, the goal is to synthesize parameter-dependent filters with a guaranteed $\mathscr{H}_\infty$ performance bounded by $\gamma = \sqrt{\delta}$ and to evaluate the conservativeness of the approach presented in Theorem 1 when compared to one from Palma et al. (2020). The design conditions are constructed by evaluating their feasibility for $\rho \in [1, 21]$. It should be pointed out that for $\rho > 21.4$, the system is unstable. For the scalar parameter in Theorem 1, a linear line search has been performed considering $\epsilon \in [0.1, 6]$ with a grid of 0.02. Fig. 3 depicts the values γ for the considered filter design conditions. Notice that Theorem 1 achieves the lowest $\mathscr{H}_\infty$-index bound for all $\rho \in [1, 21]$ when compared with (Palma et al., 2020, Theorem1). Additionally, Fig 4 illustrates the performance improvement achieved by Theorem 1 in comparison with the condition proposed in Palma et al. (2020).

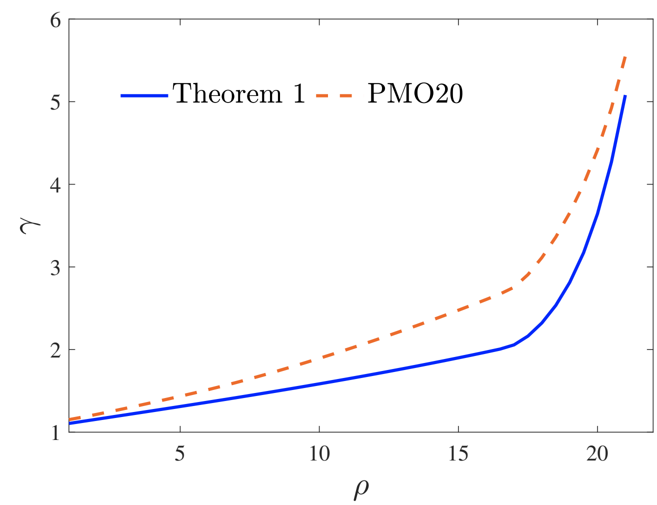

Fig. 3. $\mathscr{H}_\infty$ guaranteed performances ($\gamma = \sqrt{\delta}$) computed with Theorem 1 (straight blue line) and the condition proposed in (Palma et al., 2020, (PMO20)) (red dashed line) – Example 2.

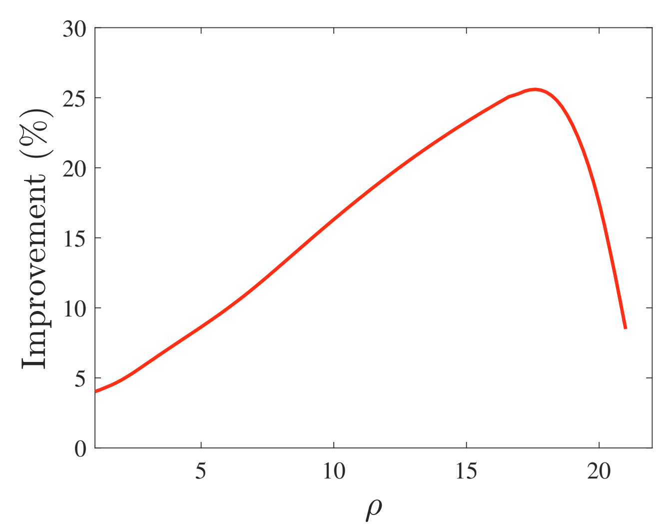

Fig. 4. Comparison of performance improvement achieved by Theorem 1 with the condition proposed in Palma et al. (2020) – Example 2.

5. CONCLUSIONS

This paper has introduced new conditions for designing full-order parameter-dependent and robust filters for discrete-time linear parameter-varying (LPV) systems. To design the parameter-dependent filter, the time-varying parameters have been assumed to be measured. However, in scenarios where such parameters are unavailable, a robust filter has been provided. Notably, the filtering matrices have been decoupled from the Lyapunov function, which enables the incorporation of parameter-dependent Lyapunov functions even in robust filter designs. The Lyapunov theory has been employed to design the filters with a guaranteed $\mathscr{H}_\infty$ performance, and the proposed conditions have been formulated as linear matrix inequalities. An important aspect of the method is that the filtering matrices have been obtained directly from synthesis conditions, eliminating the need for variable changes or iterative algorithms. The efficacy of the proposed approach has been illustrated through numerical experiments. As future research, the authors are investigating the filter design problem for fault detection.

REFERENCES

Anderson, B.D. and Moore, J.B. (2012). *Optimal filtering*. Courier Corporation.

Bokor, J. and Balas, G. (2004). Detection filter design for LPV systems — a geometric approach. *Automatica*, 40(3), 511–518.

Borges, R.A., Montagner, V.F., Oliveira, R.C.L.F., Peres, P.L.D., and Bliman, P.A. (2008). Parameter-dependent $\mathscr{H}_2$ and $\mathscr{H}_\infty$ filter design for linear systems with arbitrarily time-varying parameters in polytopic domains. *Signal Processing*, 88(7), 1801–1816.

Borges, R.A., Oliveira, R.C.L.F., Abdallah, C.T., and Peres, P.L.D. (2010). $\mathscr{H}_\infty$ filtering for discrete-time linear systems with bounded time-varying parameters. *Signal Processing*, 90(1), 282–291.

Daafouz, J. and Bernussou, J. (2001). Parameter dependent Lyapunov functions for discrete time systems with time varying parameter uncertainties. *Systems & Control Letters*, 43(5), 355–359.

de Souza, C.E., Barbosa, K.A., and Trofino, A. (2007). Robust filtering for linear systems with convex-bounded uncertain time-varying parameters. *IEEE Transactions on Automatic Control*, 52(6), 1132–1138.

Delgado-Aguiñaga, J., Puig, V., and Becerra-López, F. (2021). Leak diagnosis in pipelines based on a Kalman filter for Linear Parameter Varying systems. *Control Engineering Practice*, 115.

Duan, Z.S., Zhang, J.X., Zhang, C.S., and Mosca, E. (2006). Robust $\mathscr{H}_2$ and $\mathscr{H}_\infty$ filtering for uncertain linear systems. *Automatica*, 42(11), 1919–1926.

El-Amrani, A., Boukili, B., Hmamed, A., and Adel, E.M.E. (2017). Robust $\mathscr{H}_\infty$ filters for uncertain systems with finite frequency specifications. *Journal of Control, Automation and Electrical Systems*, 28(6), 693–706.

Geromel, J.C. (1999). Optimal linear filtering under parameter uncertainty. *IEEE Transactions on Signal Processing*, 47(1), 168–175.

Gonçalves, E.N., Palhares, R.M., and Takahashi, R.H.C. (2006). $\mathscr{H}_2/\mathscr{H}_\infty$ filter design for systems with polytope-bounded uncertainty. *IEEE Transactions on Signal Processing*, 54(9), 3620–3626.

Lacerda, M.J., Oliveira, R.C.L.F., and Peres, P.L.D. (2011). Robust $\mathscr{H}_2$ and $\mathscr{H}_\infty$ filter design for uncertain linear systems via LMIs and polynomial matrices. *Signal Processing*, 91(5), 1115–1122.

Lacerda, M.J., Tarbouriech, S., Garcia, G., and Peres, P.L.D. (2014). $\mathscr{H}_\infty$ filter design for nonlinear polynomial systems. *Systems & Control Letters*, 70, 77–84.

Lacerda, M.J., Tognetti, E.S., Oliveira, R.C.L.F., and Peres, P.L.D. (2016). A new approach to handle additive and multiplicative uncertainties in the measurement for $\mathscr{H}_\infty$ LPV filtering. *International Journal of Systems Science*, 47(5), 1042–1053.

Li, P., Lam, J., and Chesi, G. (2012). On the synthesis of linear $\mathscr{H}_\infty$ filters for polynomial systems. *Systems & Control Letters*, 61(1), 31–36.

Mohammadpour, J. and Grigoriadis, K.M. (2006). Delay-dependent $\mathscr{H}_\infty$ filtering for a class of time-delayed LPV systems. In *Proceedings of the 2006 American Control Conference*, 1523–1528. Minneapolis, MN, USA.

Mohammadpour, J. and Scherer, C.W. (eds.) (2012). *Control of Linear Parameter Varying Systems with Applications*. Springer, New York. doi:10.1007/978-1-4614-1833-7.

Palhares, R.M. and Peres, P.L.D. (1999). Robust $\mathscr{H}_\infty$ filtering design with pole placement constraint via LMIs. *Journal of Optimization Theory and Applications*, 102(2), 239–261.

Palma, J.M., Morais, C.F., and Oliveira, R.C.L.F. (2020). A less conservative approach to handle time-varying parameters in discrete-time linear parameter-varying systems with applications in networked control systems. *International Journal of Robust and Nonlinear Control*, 30(9), 3521–3546.

Peixoto, M., Pessim, P., Lacerda, M., and Palhares, R. (2020). Stability and stabilization for LPV systems based on Lyapunov functions with non-monotonic terms. *Journal of the Franklin Institute*, 357(11), 6595–6614.

Sato, M. (2006). Filter design for LPV systems using quadratically parameter-dependent Lyapunov functions. *Automatica*, 42(11), 2017–2023.

Sato, M. (2010). Gain-scheduled $\mathscr{H}_\infty$ filters using inexactly measured scheduling parameters. In *Proceedings of the 2010 American Control Conference*, 3088–3093. Baltimore, MD, USA.

Wang, Q., Peng, L., and Pan, J. (2024). Event-triggered $\mathscr{H}_\infty$ robust filtering for nonlinear semi-markov switching systems. *International Journal of Control*, 97(6), 1389–1400.