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Quantum Field Theory and Statistical Systems

Non-Abelian elastic collisions, associated difference systems of equations and discrete analytic functions

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ABSTRACT

We extend the equations of motion that describe non-relativistic elastic collision of two particles in one dimension to an arbitrary associative algebra. Relativistic elastic collision equations turn out to be a particular case of these generic equations. Furthermore, we show that these equations can be reinterpreted as difference systems defined on the $\mathbb{Z}^2$ graph and this reinterpretation relates (unifies) the linear and the non-linear approach of discrete analytic functions.

1. Introduction

The physical phenomenon of the non-relativistic elastic collisions of particles is governed by the equations that give rise to a map with the Yang–Baxter property (see Section 3.1) and after a suitable reinterpretation leads to a system of linear difference equations [1,2]. In this article, we show that abandoning the assumption of commutativity of variables, even the simple linear integrable difference system of equations associated with elastic collisions, yields nonlinear, non-abelian integrable difference system of equations. From this non-abelian difference system we can recover the abelian one as a special case. Actually, in general we obtain several abelian systems with different features. In that respect, the non-abelian system serves as a "top" system since it includes (unifies) the abelian ones. In other words, if we consider two or several systems of equations where the dependent variables are elements of a field, that is the multiplication of variables is abelian (commutative), then one can try to unify these systems in the following sense:

First, extend one of the systems to the non-abelian setting i.e. allow all of its variables to be elements of an algebra $\mathcal{A}$ rather than a field. Second, show that the original abelian systems are special cases of the non-abelian one, which are obtained by specifying the algebra (or some of its subspaces) in question.

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In this article we present two examples of such a procedure. In the first illustrative example, we unify, in the aforementioned sense, the equations that describe non-relativistic elastic collisions of point-mass particles with a system of equations that includes the equations of relativistic elastic collisions as a subsystem. In the second example (that can be considered as an extension and a reinterpretation of the first one) we unify the linear and the nonlinear approach of discrete analytic functions. In the classical article on discrete analytic functions [3], the authors show that Duffin's linear theory of discrete analytic functions [4] can be viewed as a linearisation of the nonlinear theory that "is based on the ideas by Thurston [5] and declares circle patterns to be natural discrete analogs of analytic functions". Here instead we present the two theories as special abelian cases of the non-abelian theory.

There is not a unique procedure to extend an abelian system to its non-abelian counterpart. To resolve this non-uniqueness issue, we perform the non-abelian extension on the level of the underlying Lax formulation known from the theory of integrable systems [6]. The procedure of using the Lax pair formulation to lift an abelian system to its non-abelian version has already been considered in the literature, see e.g. [7].

The following system of non-abelian equations serves as a central point of this article

$$v^{i\prime} - v^{j\prime} = v^j - v^i, \tag{1a}$$

$$\mu^{i\prime} + \mu^{j\prime} = \mu^j + \mu^i, \tag{1b}$$

$$\mu^{i\prime}\mu^{j\prime} = \mu^j\mu^i, \tag{1c}$$

$$\mu^{i\prime}v^{j\prime} + v^{i\prime}\mu^{j\prime} = \mu^j v^i + v^j\mu^i. \tag{1d}$$

where $i \neq j \in \{1, \ldots, N\}$, $N \geq 2 \in \mathbb{N}$ and we assume that both the primed variables $\mu^{i\prime}$, $v^{i\prime}$ and the non-primed ones μ^i, v^i belong to a unital associative algebra $\mathcal{A}$ over a field $\mathbb{F}$.

System (1) is equivalent to the matrix refactorization problem,

$$L(v^{i\prime}, \mu^{i\prime}; \lambda)L(v^{j\prime}, \mu^{j\prime}; \lambda) = L(v^j, \mu^j; \lambda)L(v^i, \mu^i; \lambda)\,, \tag{2}$$

where L is the following Lax matrix

$$L(v^i, \mu^i; \lambda) := \begin{pmatrix} \mu^i + \lambda & \lambda v^i \\ 0 & \mu^i - \lambda \end{pmatrix}, \tag{3}$$

and λ stands for a spectral parameter, which is assumed to belong to the centre of the algebra. The refactorization problem (2) in components reads

$$(\mu^{i\prime} + \lambda)(\mu^{j\prime} + \lambda) = (\mu^j + \lambda)(\mu^i + \lambda), \tag{4a}$$

$$(\mu^{i\prime} - \lambda)(\mu^{j\prime} - \lambda) = (\mu^j - \lambda)(\mu^i - \lambda), \tag{4b}$$

$$(\mu^{i\prime} + \lambda)v^{j\prime} + v^{i\prime}(\mu^{j\prime} - \lambda) = (\mu^j + \lambda)v^i + v^j(\mu^i - \lambda). \tag{4c}$$

Demanding that (4) hold for every λ, (4a) is equivalent to (4b) and together with (4c) we obtain exactly the system of non-abelian equations (1). The sub-system, (1b), (1c) could be re-interpreted as a difference system in edge variables associated with *the discrete nonlinear chiral field (or σ-model) equation* [8] (see Section 4). Various reductions of (1b), (1c) have been considered in the literature. In detail, in [9,10] where the algebra $\mathcal{A}$ was considered as the algebra of the $n \times n$ matrices over $\mathbb{C}$, the solution of (1b), (1c) with respect to $\mu^{i\prime}, \mu^{j\prime}$ has been explicitly presented as a multi-component Yang-Baxter map. The Lax matrix (3) with entries in a commutative ring and in connection with the linear theory of discrete analytic functions can be found in [11].

The article is organized as follows.

In Section 2, we consider two choices of the underlying algebra $\mathcal{A}$. The first choice, is to take the field of real numbers $\mathbb{R}$ as the algebra $\mathcal{A}$. This choice leads to the equations that describe the elastic collisions between the i-th and the j-th non-relativistic particles, where $m^i := \frac{1}{\mu^i}$, $m^j := \frac{1}{\mu^j}$ and v^i, v^j respectively denote the masses and the velocities of the particles before collision, while the primed variables denote these quantities after collision. The second choice is to consider all μ's as off-diagonal 2×2 matrices over the field $\mathbb{R}$ and all v's as diagonal 2×2 matrices over the same field. This choice results in the equations that describe the elastic collisions of relativistic particles.

In Section 3, first we assume that $\mathcal{A}$ is a division ring and then we show that the system (1) can be solved for four (out of eight variables) and give rise to maps $\mathcal{A}^2 \times \mathcal{A}^2 \to \mathcal{A}^2 \times \mathcal{A}^2$. We prove that these maps are multidimensional compatible, while their *companion* maps satisfy the Yang-Baxter property.

In Section 4, we reinterpret (1) as a system of difference equations defined on edges of the $\mathbb{Z}^N$ graph and furthermore we associate difference systems of equations defined on vertices of the same graph. Specifically, the system of vertex equations on the functions $\chi, \psi, \omega, \sigma, \phi : \mathbb{Z}^N \to \mathcal{A}$, where $\mathcal{A}$ a division ring, reads

$$\begin{aligned} \psi_i - \psi &= \phi_i\phi^{-1}, \\ \sigma_i - \sigma &= \phi_i^{-1}\phi, \\ \chi_i + \chi &= \phi_i(\omega_i - \omega)\phi^{-1}, \end{aligned} \tag{5}$$

where $i \in \{1, \ldots, N\}$ and the subscripts denote forward shifts in the i-th direction (see Fig. 2 and Section 4.1 for details). In particular, having eliminated ψ, ω, and σ from (5), we obtain *the discrete nonlinear chiral field (or σ-model) equation* [8] for the function ϕ that reads

$$\phi_{ij}\left(\phi_j^{-1} - \phi_i^{-1}\right) = \left(\phi_i - \phi_j\right)\phi^{-1}, \tag{6}$$

which is coupled to the following linear equation for the function χ

$$\phi_{ij}\phi_j^{-1}(\chi_j + \chi) + (\chi_{ij} + \chi_j)\phi_j\phi^{-1} = \phi_{ij}\phi_i^{-1}(\chi_i + \chi) + (\chi_{ij} + \chi_i)\phi_i\phi^{-1}. \tag{7}$$

In Section 5, we show that the solutions of the difference systems of equations defined on vertices of quads on the $\mathbb{Z}^N$ graph, satisfy systems of three-dimensional vertex equations. For example, we show that the solutions of (6) and (7) satisfy the following three-dimensional system

$$\begin{gathered}\phi_{ij}\phi_{ik}^{-1}\phi_{jk} = \phi_{jk}\phi_{ik}^{-1}\phi_{ij},\\ \left(\phi_{ik}^{-1}\left(\chi_{ik} + \chi_k\right) - \phi_{jk}^{-1}\left(\chi_{jk} + \chi_k\right)\right)\phi_k + \left(\phi_{ij}^{-1}\left(\chi_{ij} + \chi_i\right) - \phi_{ik}^{-1}\left(\chi_{ik} + \chi_i\right)\right)\phi_i\\ + \left(\phi_{jk}^{-1}\left(\chi_{jk} + \chi_j\right) - \phi_{ij}^{-1}\left(\chi_{ij} + \chi_j\right)\right)\phi_j = 0,\end{gathered} \tag{8}$$

where $i \neq j \neq k \in \{1, \ldots, N\}$.

In the final Section 6, we show how the linear and the nonlinear approach of discrete analytic functions are related via system (5). When $\mathcal{A} = \mathbb{C}$, system (5) can be treated as the basic equations of the theory of discrete analytic functions [4]. While, when as $\mathcal{A}$ we choose the subspace of 2×2 off-diagonal matrices over a commutative ring, system (5) can be reinterpreted as the nonlinear approach to discrete analytic functions [5]. Therefore, we relate (unify) in the sense mentioned in the beginning of this introduction, the linear and the nonlinear approach of discrete analytic functions.

2. Reductions to classical and relativistic elastic collisions of point-mass particles

In this Section we show that in the abelian case, system (1) describes the elastic collisions of non-relativistic particles. Furthermore we show that the relativistic elastic collisions, with an additional degree of freedom, are obtained by a suitable choice of the underlying algebra $\mathcal{A}$.

2.1. Classical head-on elastic collisions of point-mass particles

Let the point-mass particles labelled by the index $i \in \mathbb{N}$, with initial respective masses and velocities m^i and v^i, undergo pairwise one-dimensional elastic collisions. The conservation of kinetic energy and the conservation of momentum in the event of collision of two particles respectively read

$$\begin{aligned} m^i\left(v^i\right)^2 + m^j\left(v^j\right)^2 &= m^{i\prime}\left(v^{i\prime}\right)^2 + m^{j\prime}\left(v^{j\prime}\right)^2,\\ m^i v^i + m^j v^j &= m^{i\prime} v^{i\prime} + m^{j\prime} v^{j\prime}, \end{aligned} \qquad i \neq j \in \mathbb{N}, \tag{9}$$

where $m^{i\prime}, m^{j\prime}$ and $v^{i\prime}, v^{j\prime}$, denote the respective masses and velocities after the collision. Demanding that momentum is preserved in any Galilean frame leads to the conservation of mass

$$m^i + m^j = m^{i\prime} + m^{j\prime}. \tag{10}$$

In the standard elastic collision process, it is assumed that there is no transfer of mass between the particles, i.e. we have

$$m^{i\prime} = m^i, \qquad m^{j\prime} = m^j. \tag{11}$$

Then (9) equivalently reads

$$\begin{aligned} m^i\left(v^i + v^{i\prime}\right)\left(v^i - v^{i\prime}\right) + m^j\left(v^j + v^{j\prime}\right)\left(v^j - v^{j\prime}\right) &= 0,\\ m^i\left(v^i - v^{i\prime}\right) + m^j\left(v^j - v^{j\prime}\right) &= 0, \end{aligned}$$

or when $m^i\left(v^i - v^{i\prime}\right) \neq 0$,

$$\begin{aligned} v^i + v^{i\prime} &= v^j + v^{j\prime},\\ m^i\left(v^i - v^{i\prime}\right) + m^j\left(v^j - v^{j\prime}\right) &= 0. \end{aligned} \tag{12}$$

In the following Proposition we show how to obtain the equations that describe classical elastic collisions of point-mass particles from an appropriate reduction of system (1).

Proposition 2.1. *When we consider the underlying algebra $\mathcal{A}$ to be abelian, e.g. $\mathcal{A} = \mathbb{R}$, the system of equations (1), for $\mu^{i\prime} = \mu^i = (m^i)^{-1}$ and $\mu^{j\prime} = \mu^j = (m^j)^{-1}$, reduces to*

$$m^{i\prime}=m^{j},\qquad m^{j\prime}=m^{i},\qquad v^{i\prime}-v^{j\prime}=v^{j}-v^{i},\qquad m^{i}(v^{j\prime}-v^{j})=m^{j}(v^{i}-v^{i\prime}),\tag{13}$$

or to

$$m^{i\prime}=m^{i},\qquad m^{j\prime}=m^{j},\qquad v^{i\prime}-v^{j\prime}=v^{j}-v^{i},\qquad m^{i}(v^{j\prime}-v^{j})=m^{j}(v^{i}-v^{i\prime}).\tag{14}$$

The latter coincides with (11), (12), that is the equations of classical elastic collisions of two point-mass particles with masses $m^i, m^j > 0$, with velocities v^i, v^j before the collision and velocities $v^{i\prime}, v^{j\prime}$ after the collision.

Proof. For $\mathcal{A}=\mathbb{R}$, equations (1b) and (1c) yield $\mu^{i\prime}=\mu^j$ and $\mu^{j\prime}=\mu^i$ or $\mu^{i\prime}=\mu^i=:(m^i)^{-1}$ and $\mu^{j\prime}=\mu^j=:(m^j)^{-1}$. In the latter case system (1) becomes exactly (14) that coincides with (11) and (12). □

2.2. Head-on elastic collisions of relativistic point-mass particles

Let the relativistic point-mass particles labelled by the index $i\in\mathbb{N}$, with initial respective rest masses m_0^i and m_0^j and respective velocities u^i and u^j before the collision and velocities $u^{i\prime}, u^{j\prime}$ after the collision. The conservation of relativistic momentum and energy under the elastic collision respectively read

$$\begin{aligned}\frac{m_0^i u^{i\prime}}{\sqrt{1-\left(\frac{u^{i\prime}}{c}\right)^2}}+\frac{m_0^j u^{j\prime}}{\sqrt{1-\left(\frac{u^{j\prime}}{c}\right)^2}}&=\frac{m_0^i u^{i}}{\sqrt{1-\left(\frac{u^{i}}{c}\right)^2}}+\frac{m_0^j u^{j}}{\sqrt{1-\left(\frac{u^{j}}{c}\right)^2}},\\ \frac{m_0^i}{\sqrt{1-\left(\frac{u^{i\prime}}{c}\right)^2}}+\frac{m_0^j}{\sqrt{1-\left(\frac{u^{j\prime}}{c}\right)^2}}&=\frac{m_0^i}{\sqrt{1-\left(\frac{u^{i}}{c}\right)^2}}+\frac{m_0^j}{\sqrt{1-\left(\frac{u^{j}}{c}\right)^2}},\end{aligned}\tag{15}$$

where c the speed of light. Introducing the change of variables

$$\begin{aligned}&\mathbb{R}^+\ni x^i\mapsto u^i=c\frac{(x^i)^2-1}{(x^i)^2+1}\in(-c,c),\\ &\mathbb{R}^+\ni x^{i\prime}\mapsto u^{i\prime}=c\frac{(x^{i\prime})^2-1}{(x^i)^2+1}\in(-c,c),\end{aligned}\qquad i=1,\dots,N,$$

that is a bijection, equations (15) become

$$\begin{aligned}m_0^i\left(x^{i\prime}-\frac{1}{x^{i\prime}}\right)+m_0^j\left(x^{j\prime}-\frac{1}{x^{j\prime}}\right)&=m_0^i\left(x^{i}-\frac{1}{x^{i}}\right)+m_0^j\left(x^{j}-\frac{1}{x^{j}}\right),\\ m_0^i\left(x^{i\prime}+\frac{1}{x^{i\prime}}\right)+m_0^j\left(x^{j\prime}+\frac{1}{x^{j\prime}}\right)&=m_0^i\left(x^{i}+\frac{1}{x^{i}}\right)+m_0^j\left(x^{j}+\frac{1}{x^{j}}\right).\end{aligned}$$

By adding and subtracting the equations above we obtain the conservation laws

$$m_0^i\left(x^{i\prime}-x^i\right)=m_0^j\left(x^j-x^{j\prime}\right),\tag{16}$$

$$m_0^i\left(\frac{1}{x^{i\prime}}-\frac{1}{x^i}\right)=m_0^j\left(\frac{1}{x^j}-\frac{1}{x^{j\prime}}\right),\tag{17}$$

which are equivalent to the conservation of relativistic momentum and energy.

It turns out that the system of equations (1) under a suitable choice of the underlying algebra $\mathcal{A}$ includes the relativistic elastic collisions, henceforth it relates the Newtonian and the relativistic approaches.

Lemma 2.2. *Let $\mathcal{A}=\mathcal{A}_o\bigoplus\mathcal{A}_e$ be a $\mathbb{Z}_2$-graded algebra over $\mathbb{C}$. Let the odd subspace $\mathcal{A}_o$ of the algebra $\mathcal{A}$ be spanned by the 2×2 off-diagonal matrices*

$$\mu^i=\begin{pmatrix}0 & x^i\\ X^i & 0\end{pmatrix},\qquad \mu^{i\prime}=\begin{pmatrix}0 & x^{i\prime}\\ X^{i\prime} & 0\end{pmatrix},$$

and the even subspace $\mathcal{A}_e$ be spanned by the 2×2 diagonal matrices

$$v^i=\begin{pmatrix}y^i & 0\\ 0 & Y^i\end{pmatrix},\qquad v^{i\prime}=\begin{pmatrix}y^{i\prime} & 0\\ 0 & Y^{i\prime}\end{pmatrix}.$$

Then the system of equations (1a)-(1d), restricted on these subspaces respectively reads

$$y^{i\prime}-y^{j\prime}=y^j-y^i,\qquad Y^{i\prime}-Y^{j\prime}=Y^j-Y^i,\tag{18a}$$

$$x^{i\prime}+x^{j\prime}=x^i+x^j,\qquad X^{i\prime}+X^{j\prime}=X^i+X^j,\tag{18b}$$

$$x^{i\prime}X^{j\prime}=X^ix^j, \qquad X^{i\prime}x^{j\prime}=x^iX^j, \tag{18c}$$

$$x^{j\prime}y^{i\prime}+x^{i\prime}Y^{j\prime}=x^iy^j+x^jY^i, \qquad X^{j\prime}Y^{i\prime}+X^{i\prime}y^{j\prime}=X^iY^j+X^jy^i, \tag{18d}$$

and it satisfies the following invariant relations

$$x^{i\prime}X^{i\prime}=x^iX^i, \qquad x^{j\prime}X^{j\prime}=x^jX^j, \qquad y^{i\prime}-Y^{i\prime}=Y^i-y^i, \qquad y^{j\prime}-Y^{j\prime}=Y^j-y^j. \tag{19}$$

Proof. By restricting (1) on the odd and the even subspaces of the $\mathbb{Z}_2$-graded algebra $\mathcal{A}$, we obtain exactly (18).

The system of equations (18) admits two solutions with respect to the primed variables which they read

$$x^{i\prime}=x^j, \qquad X^{i\prime}=X^j, \qquad x^{j\prime}=x^i, \qquad X^{j\prime}=X^i,$$
$$y^{i\prime}=y^j, \qquad Y^{i\prime}=Y^j, \qquad y^{j\prime}=y^i, \qquad Y^{j\prime}=Y^i,$$

or

$$x^{i\prime}=X^i\frac{x^i+x^j}{X^i+X^j}, \qquad X^{i\prime}=x^i\frac{X^i+X^j}{x^i+x^j}, \tag{20a}$$

$$x^{j\prime}=X^j\frac{x^i+x^j}{X^i+X^j}, \qquad X^{j\prime}=x^j\frac{X^i+X^j}{x^i+x^j}, \tag{20b}$$

$$y^{i\prime}=X^i\frac{Y^j-y^i}{X^i+X^j}+\frac{x^jY^i+x^iy^j}{x^i+x^j}, \qquad Y^{i\prime}=x^i\frac{y^j-Y^i}{x^i+x^j}+\frac{X^jy^i+X^iY^j}{X^i+X^j}, \tag{20c}$$

$$y^{j\prime}=x^j\frac{Y^i-y^j}{x^i+x^j}+\frac{X^jy^i+X^iY^j}{X^i+X^j}, \qquad Y^{j\prime}=X^j\frac{y^i-Y^j}{X^i+X^j}+\frac{x^jY^i+x^iy^j}{x^i+x^j}. \tag{20d}$$

By substituting (20) into (19), we prove that the invariant relations hold. □

Proposition 2.3. *The system of equations (18b), (18c) is equivalent to (16), (17), hence it describes elastic collisions of relativistic pairs of particles.*

Proof. From Lemma 2.2 we have that (18b), (18c) respect the invariant conditions

$$x^{i\prime}X^{i\prime}=x^iX^i:=\alpha^i, \qquad x^{j\prime}X^{j\prime}=x^jX^j:=\alpha^j.$$

Under the change of variables $(x^i,X^i,x^j,X^j)\mapsto(x^i,\alpha^i,x^j,\alpha^j)$, the invariant conditions read

$$\alpha^{i\prime}=\alpha^i, \qquad \alpha^{j\prime}=\alpha^j,$$

while (18b), (18c) become

$$\begin{aligned} x^{i\prime}-x^i&=x^j-x^{j\prime},\\ \alpha^i\left(\frac{1}{x^{i\prime}}-\frac{1}{x^i}\right)&=\alpha^j\left(\frac{1}{x^j}-\frac{1}{x^{j\prime}}\right),\\ \alpha^jx^{i\prime}x^i&=\alpha^ix^jx^{j\prime}. \end{aligned} \tag{21}$$

Under the re-scaling $(x^i,x^j)\mapsto\left(\sqrt{\alpha^i}x^i,\sqrt{\alpha^j}x^j\right)$ followed by the re-parametrization $\alpha^i=(m_0^i)^2$, $\alpha^j=(m_0^j)^2$, where m_0^i,m_0^j represent the rest masses of the particles, (21) reads

$$m_0^i(x^{i\prime}-x^i)=m_0^j(x^j-x^{j\prime}), \tag{22}$$

$$m_0^i\left(\frac{1}{x^{i\prime}}-\frac{1}{x^i}\right)=m_0^j\left(\frac{1}{x^j}-\frac{1}{x^{j\prime}}\right), \tag{23}$$

$$x^{i\prime}x^i=x^jx^{j\prime}. \tag{24}$$

Note that from the three equations above, as expected, only two of them are functionally independent. Indeed, by multiplying (24) with (22) we obtain (23). It is clear that equations (22), (23) coincide with (16), (17) and that completes the proof. □

Remark 2.4. In Proposition 2.3 we proved that the sub-system of equations (18b), (18c) is equivalent to the description of elastic collisions of relativistic pairs of particles. At this point it is not clear to us if there is an underlying physical phenomenon that the whole system of equations (18a)-(18d) describes. We would like here to mention that using (19) we re-write the equations (18a)-(18d) (or (20) in the new variables

$$(x^i,X^i,x^j,X^j,y^i,Y^i,y^j,Y^j)\mapsto(x^i,(m_0^i)^2,x^j,(m_0^j)^2,y^i,p^i,y^j,p^j),$$

where $(m_0^i)^2:=x^iX^i$, $(m_0^j)^2:=x^jX^j$, $p^i:=Y^i-y^i$, $p^j:=Y^j-y^j$, to obtain

$$
\begin{aligned}
x^{i\prime} &= x^j \frac{m_0^i x^i + m_0^j x^j}{m_0^j x^i + m_0^i x^j}, & x^{j\prime} &= x^i \frac{m_0^i x^i + m_0^j x^j}{m_0^j x^i + m_0^i x^j},\\
m_0^{i\prime} &= m_0^i, & m_0^{j\prime} &= m_0^j,\\
y^{i\prime} &= y^j + A, & y^{j\prime} &= y^i + A,\\
p^{i\prime} &= -p^i, & p^{j\prime} &= -p^j
\end{aligned}
\tag{25}
$$

where

$$
A := -\left((m_0^i)^2 - (m_0^j)^2\right) \frac{x^i x^j (y^i - y^j)}{(m_0^i x^i + m_0^j x^j)(m_0^j x^i + m_0^i x^j)} + x^j \left(\frac{m_0^j p^i}{m_0^i x^i + m_0^j x^j} + \frac{m_0^i p^j}{m_0^j x^i + m_0^i x^j} \right).
$$

Note that the variable y could be considered as an additional degree of freedom.

Remark 2.5. An equivalent description of the relativistic elastic collisions of particles is obtained by

$$
\mu^i = \begin{pmatrix} 0 & z^{i,0} + z^{i,1} \\ z^{i,0} - z^{i,1} & 0 \end{pmatrix}, \qquad v^i = v^{i\prime} = v^j = v^{j\prime} =: c_v \text{ (constant)},
$$

where $z^{i,0}, z^{i,1} \in \mathbb{C}$. Then (1) reads

$$
\mathbf{z^{i\prime}} = \mathbf{z^j} + k(\mathbf{z^i}, \mathbf{z^j})(\mathbf{z^i} + \mathbf{z^j}), \qquad \mathbf{z^{j\prime}} = \mathbf{z^i} - k(\mathbf{z^i}, \mathbf{z^j})(\mathbf{z^i} + \mathbf{z^j}), \tag{26}
$$

where $\mathbf{z^i} = (z^{i,0}, z^{i,1})$, $\mathbf{z^{i\prime}} = (z^{i,0\prime}, z^{i,1\prime})$,

$$
k(\mathbf{z^i}, \mathbf{z^j}) = \frac{\langle \mathbf{z^i}, \mathbf{z^j} \rangle - \langle \mathbf{z^j}, \mathbf{z^j} \rangle}{\langle \mathbf{z^i} + \mathbf{z^j}, \mathbf{z^i} + \mathbf{z^j} \rangle},
$$

and $\langle\ ,\ \rangle$ denotes the bilinear form $\langle \mathbf{z^i}, \mathbf{z^j} \rangle := z^{i,0} z^{j,0} - z^{i,1} z^{j,1}$. The map $\mathbf{R} : (\mathbf{z^i}, \mathbf{z^j}) \mapsto (\mathbf{z^{i\prime}}, \mathbf{z^{j\prime}})$ represents the transformation of the momentum-energy vectors of two particles under relativistic collision [2]. In this setting, the vectors $\mathbf{z^i}$ and $\mathbf{z^{i\prime}}$ correspond to the momentum-energy vectors of the two particles before and after collision, that is

$$
\mathbf{z^i} = (z^{i,0}, z^{i,1}) := \left(\frac{E^i}{c}, p^i\right), \quad \mathbf{z^{i\prime}} = (z^{i,0}{}_{i'}, z^{i,1\prime}) := \left(\frac{E^{i\prime}}{c}, p^{i\prime}\right), \ i \in \mathbb{N},
$$

where $E^i, E^{i\prime}$ denote the relativistic energy of the two particles and $p^i, p^{i\prime}$ their momenta before and after collision respectively.

3. Elastic collisions as maps

In the abelian, case system (1) can be solved uniquely with respect to the primed variables $v^{i\prime}$, $v^{j\prime}$ $\mu^{i\prime}$, $\mu^{j\prime}$ in terms of the un-primed ones. Provided that $m^i + m^j \neq 0$ we obtain the maps

$$
R_{ij} : (v^i, m^i; v^j, m^j) \mapsto \left(v^{i\prime}, m^{i\prime}; v^{j\prime}, m^{j\prime}\right), \tag{27}
$$

where

$$
\begin{aligned}
v^{i\prime} &= v^j + \frac{m^i - m^j}{m^i + m^j}\left(v^i - v^j\right), & m^{i\prime} &= m^i,\\
v^{j\prime} &= v^i + \frac{m^i - m^j}{m^i + m^j}\left(v^i - v^j\right), & m^{j\prime} &= m^j.
\end{aligned}
$$

Maps R_{ij} will be referred to as *elastic collision maps*. The elastic collision maps have the Yang-Baxter property, that is if we take three particles with given velocities v^1, v^2 and v^3, after the interaction of the $1st$ particle with the $2nd$, followed by the interaction of the $1st$ with the $3rd$ and finally of the $2nd$ particle with the $3rd$, the outgoing velocities are the same if the order of the three interactions is reversed (see Fig. 1).

Alternatively, the system (1) can be solved for $v^{i\prime}$, $m^{i\prime}$ v^j, m^j in terms of v^i, m^i $v^{j\prime}$, $m^{j\prime}$ provided that $m^i - m^j \neq 0$ to obtain

$$
Q_{ij} : (v^i, m^i; v^{j\prime}, m^{j\prime}) \mapsto \left(v^{i\prime}, m^{i\prime}; v^j, m^j\right), \tag{28}
$$

where

$$
\begin{aligned}
v^{i\prime} &= v^{j\prime} + \frac{m^i + m^{j\prime}}{m^i - m^{j\prime}}\left(v^i - v^{j\prime}\right), & m^{i\prime} &= m^i,\\
v^j &= v^i + \frac{m^i + m^{j\prime}}{m^i - m^{j\prime}}\left(v^i - v^{j\prime}\right), & m^j &= m^{j\prime},
\end{aligned}
$$

Maps Q_{ij}, will be called *companion elastic collision maps*.

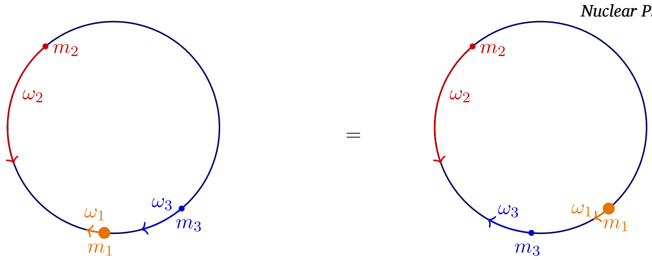


Fig. 1. The Yang-Baxter relation, realised as elastic collision of three particles moving on a circle. That is the outgoing velocities after the interaction of the 1st particle with the 2nd, followed by the interaction of the 1st with the 3rd and finally of the 2nd particle with the 3rd (left figure), are the same if the order of the three interactions is reversed (right figure).

Note that in the non-abelian case the problem of expressing some variables of the system (1) as functions (maps) of the remaining ones is more subtle. In detail, in order to avoid indeterminacies we have to consider that the velocities and the masses of particles that participate in the collisions to be elements of a division ring. Also, as we shall see, in order to obtain the non-abelian elastic collision map, the solution of a system of Sylvester equations is required. In this Section, we address all these just mentioned issues and we investigate the properties of the obtained maps. First we recall the definition of Yang-Baxter maps and the definition of $3D$-compatible maps and we show how these notions are mutually related. Next, we show that the maps (28) and their non-abelian versions are $3D$-compatible, whereas the maps (27) and their non-abelian versions are Yang-Baxter.

We would like to mention that the conservation relations (22)-(24), which are equivalent to the conservation of the relativistic momentum and energy, can be considered as the defining relations of the following map

$$R_{ij}:(x^i,x^j)\mapsto(x^{i\prime},x^{j\prime})=\left(x^j\frac{m_0^ix^i+m_0^jx^j}{m_0^jx^i+m_0^ix^j},x^i\frac{m_0^ix^i+m_0^jx^j}{m_0^jx^i+m_0^ix^j}\right).$$

This map is equivalent with a Yang–Baxter map that was referred to as H_{III}^A in [12]. While relations (25) can be considered as the defining relations of a two-component Yang–Baxter map that extends H_{III}^A. This two-component map explicitly reads

$$\begin{aligned}\hat{R}_{ij}:(x^i,y^i,p^i;x^j,y^j,p^j)\mapsto&(x^{i\prime},y^{i\prime},p^{i\prime};x^{j\prime},y^{j\prime},p^{j\prime})\\&=\left(x^j\frac{m_0^ix^i+m_0^jx^j}{m_0^jx^i+m_0^ix^j},y^j+A,-p^i;x^i\frac{m_0^ix^i+m_0^jx^j}{m_0^jx^i+m_0^ix^j},y^i+A,-p^j\right),\end{aligned}$$

where

$$A:=-\left((m_0^i)^2-(m_0^j)^2\right)\frac{x^ix^j(y^i-y^j)}{(m_0^ix^i+m_0^jx^j)(m_0^jx^i+m_0^ix^j)}+x^j\left(\frac{m_0^jp^i}{m_0^ix^i+m_0^jx^j}+\frac{m_0^ip^j}{m_0^jx^i+m_0^ix^j}\right).$$

3.1. Yang-Baxter and $3D$-compatible maps

Let $\mathcal{X}$ be any set.

Definition 1 *($3D$-compatible maps [13]).* Let $Q:\mathcal{X}\times\mathcal{X}\ni(\mathbf{x},\mathbf{y})\mapsto(\mathbf{u},\mathbf{v})=(f(\mathbf{x},\mathbf{y}),g(\mathbf{x},\mathbf{y}))\in\mathcal{X}\times\mathcal{X}$, be a map and Q_{ij} $i\neq j\in\{1,2,3\}$, be the maps that act as Q on the i-th and j-th factor of $\mathcal{X}\times\mathcal{X}\times\mathcal{X}$ and as identity to the remaining factor. In detail we have

$$\begin{aligned}&Q_{12}:(\mathbf{x},\mathbf{y},\mathbf{z})\mapsto(\mathbf{x}_2,\mathbf{y}_1,\mathbf{z})=(f(\mathbf{x},\mathbf{y}),g(\mathbf{x},\mathbf{y}),\mathbf{z}),\\&Q_{13}:(\mathbf{x},\mathbf{y},\mathbf{z})\mapsto(\mathbf{x}_3,\mathbf{y},\mathbf{z}_1)=(f(\mathbf{x},\mathbf{z}),\mathbf{y},g(\mathbf{x},\mathbf{z})),\\&Q_{23}:(\mathbf{x},\mathbf{y},\mathbf{z})\mapsto(\mathbf{x},\mathbf{y}_3,\mathbf{z}_2)=(\mathbf{x},f(\mathbf{y},\mathbf{z}),g(\mathbf{y},\mathbf{z})).\end{aligned}$$

The map $Q:\mathcal{X}\times\mathcal{X}\to\mathcal{X}\times\mathcal{X}$ will be called *3D-compatible* or *3D-consistent map* if it holds $\mathbf{x}_{23}=\mathbf{x}_{32}$, $\mathbf{y}_{13}=\mathbf{y}_{31}$, $\mathbf{z}_{12}=\mathbf{z}_{21}$, that is

$$f(\mathbf{x}_3,\mathbf{y}_3)=f(\mathbf{x}_2,\mathbf{z}_2),\qquad g(\mathbf{x}_3,\mathbf{y}_3)=f(\mathbf{y}_1,\mathbf{z}_1),\qquad g(\mathbf{x}_2,\mathbf{z}_2)=g(\mathbf{y}_1,\mathbf{z}_1).\tag{29}$$

Definition 2 *(Yang-Baxter maps [14,15]).* A map $R:\mathcal{X}\times\mathcal{X}\ni(\mathbf{x},\mathbf{y})\mapsto(\mathbf{u},\mathbf{v})=(s(\mathbf{x},\mathbf{y}),t(\mathbf{x},\mathbf{y}))\in\mathcal{X}\times\mathcal{X}$, will be called a *Yang-Baxter map* if it satisfies

$$R_{12}\circ R_{13}\circ R_{23}=R_{23}\circ R_{13}\circ R_{12},\tag{30}$$

where R_{ij} $i\neq j\in\{1,2,3\}$, denotes the maps that act as R on the i-th and the j-th factor of $\mathcal{X}\times\mathcal{X}\times\mathcal{X}$, and as identity to the remaining factor.

The first instances of Yang-Baxter maps appeared in [14,15]. The term *Yang-Baxter maps* was introduced in [16,17] to refer to maps that serve as set-theoretical solutions of the Yang–Baxter equation. The Yang–Baxter equation originally appeared in quantum physics [18,19] and in statistical mechanics [20,21] and it was further noticed that it is equivalent to the braid group relations [22,23]. Within this identification, the Yang-Baxter theory emerged in various areas of mathematics such as quantum groups (Hopf algebras, von Neumann algebras), knot theory and, most importantly from the point of view of this article, the theory of discrete integrable systems [24].

Definition 3 *(Birational maps).* An invertible map $R : \mathcal{X}\times\mathcal{X} \ni (\mathbf{x},\mathbf{y}) \mapsto (\mathbf{u},\mathbf{v}) \in \mathcal{X}\times\mathcal{X}$ will be called *birational*, if both the map R and its inverse $R^{-1} : \mathcal{X}\times\mathcal{X} \ni (\mathbf{u},\mathbf{v}) \mapsto (\mathbf{x},\mathbf{y}) \in \mathcal{X}\times\mathcal{X}$, are rational maps.

Definition 4 *(Quadrirational maps and their companion maps [25,13]).* A map $R : \mathcal{X}\times\mathcal{X} \ni (\mathbf{x},\mathbf{y}) \mapsto (\mathbf{u},\mathbf{v}) \in \mathcal{X}\times\mathcal{X}$ will be called *quadrirational*, if both the map R and the so-called *companion map* $R^c : \mathcal{X}\times\mathcal{X} \ni (\mathbf{x},\mathbf{v}) \mapsto (\mathbf{u},\mathbf{y}) \in \mathcal{X}\times\mathcal{X}$, are birational maps.

Proposition 3.1. *[13] Map $Q : (\mathbf{x},\mathbf{y}) \mapsto (f(\mathbf{x},\mathbf{y}), g(\mathbf{x},\mathbf{y}))$ is a $3D$-compatible map, iff its companion map is Yang Baxter map.*

The definitions above can be easily extended to N-dimensions with $N > 3$.

Definition 5 *(Multidimensionally compatible maps [13]).* Let $Q : \mathcal{X}\times\mathcal{X} \ni (\mathbf{x},\mathbf{y}) \mapsto (\mathbf{u},\mathbf{v}) = (f(\mathbf{x},\mathbf{y}), g(\mathbf{x},\mathbf{y})) \in \mathcal{X}\times\mathcal{X}$, be a map and Q_{ij} $i \neq j \in \{1,\dots,N\}$, be the maps that act as Q on the i-th and j-th factor of $\mathcal{X}^n$ and as identity to the remaining factor. In detail we have

$$\begin{aligned} Q_{ij} :&(\mathbf{x}^1,\dots,\mathbf{x}^i,\dots,\mathbf{x}^j,\dots,\mathbf{x}^n) \mapsto (\mathbf{x}^1,\dots,\mathbf{x}^i_j,\dots,\mathbf{x}^j_i,\dots,\mathbf{x}^n) \\ &= (\mathbf{x}^1,\dots,f(\mathbf{x}^i,\mathbf{x}^j),\dots,g(\mathbf{x}^i,\mathbf{x}^j),\dots,\mathbf{x}^n), \qquad i\neq j \in \{1,\dots,N\} \end{aligned}$$

The map $Q : \mathcal{X}\times\mathcal{X} \to \mathcal{X}\times\mathcal{X}$ will be called *multidimensionally compatible* if it holds

$$\mathbf{x}^i_{jk} = \mathbf{x}^i_{kj}, \qquad i\neq j\neq k\neq i \in \{1,\dots,N\}.$$

Remark 3.2. The companion map Q^c of the multidimensionally compatible map Q, satisfies the following Yang-Baxter equations

$$Q^c_{ij}\circ Q^c_{ik}\circ Q^c_{jk} = Q^c_{jk}\circ Q^c_{ik}\circ Q^c_{ij}, \qquad i\neq j\neq k\neq i \in \{1,\dots,N\}.$$

3.2. The non-abelian companion elastic collision maps are multidimensionally compatible

Here we prove that the non-abelian extension of (28) that is system (1), is multidimensionally compatible. First, we need to extend (1) to multi-dimensions, that can be done by the following identification

$$\left(v^i,\mu^i,v^{i\prime},\mu^{i\prime};v^j,\mu^j,v^{j\prime},\mu^{j\prime}\right) \equiv \left(v^i,\mu^i,v^i_j,\mu^i_j;v^j_i,\mu^j_i,v^j,\mu^j\right), \qquad i\neq j\in\{1,\dots,N\}.$$

In terms of this identification (1) reads

$$v^i_j - v^j = v^j_i - v^i, \tag{31a}$$

$$\mu^i_j + \mu^j = \mu^j_i + \mu^i, \tag{31b}$$

$$\mu^i_j\mu^j = \mu^j_i\mu^i, \tag{31c}$$

$$\mu^i_j v^j + v^i_j\mu^j = \mu^j_i v^i + v^j_i\mu^i, \tag{31d}$$

where $i \neq j \in \{1,\dots,N\}$. This is a linear set of equations with respect to $v^i_j, v^j_i, \mu^i_j, \mu^j_i$. Solving this set of equations for $v^i_j, v^j_i, \mu^i_j, \mu^j_i$, we obtain the defining formulas for the maps Q_{ij}. Indeed, the non-abelian maps Q_{ij} explicitly read

$$Q_{ij} : (v^i,\mu^i;v^j,\mu^j) \mapsto \left(v^i_j,\mu^i_j;v^j_i,\mu^j_i\right), \qquad i\neq j\in\{1,\dots,N\}, \tag{32}$$

where

$$\mu^i_j = K^{i,j}\mu^i\left(K^{i,j}\right)^{-1}, \tag{33}$$

$$v^i_j = \left(\mu^i_j v^j - \mu^j_i v^i - \left(v^i - v^j\right)\mu^i\right)\left(K^{i,j}\right)^{-1}, \tag{34}$$

and the expressions $K^{i,j}$ are defined by the formulas

$$K^{i,j} := \mu^i - \mu^j.$$

We will need the following lemma.

Lemma 3.3. *Consider the maps (32), then the following relations hold*

(1) $K^{i,j}_k + K^{j,k}_i + K^{k,i}_j = 0$ *(additive closure relation);*

(2) $\mu^j_k \left(\mu^i_k\right)^{-1} \mu^k_i = \mu^k_j \left(\mu^i_j\right)^{-1} \mu^j_i$ *(multiplicative closure relation).*

(3) *The expressions*

$$ {}^i\Gamma^{j,k} := K^{i,j}_k K^{i,k} \tag{35}$$

are symmetric with respect to the interchange j to k, and they satisfy

$$ {}^i\Gamma^{j,k} \mu^i \left({}^i\Gamma^{j,k}\right)^{-1} {}^i\Psi^{j,k} + {}^i\Omega^{j,k} = 0 \tag{36}$$

where ${}^i\Psi^{j,k}$ and ${}^i\Omega^{j,k}$ are defined as follows

$$ {}^i\Psi^{j,k} := \left(1 - \mu^j_k \left(\mu^i_k\right)^{-1}\right)\left(v^k - v^i\right) - \left(1 - \mu^k_j \left(\mu^i_j\right)^{-1}\right)\left(v^j - v^i\right), \tag{37}$$

$$ {}^i\Omega^{j,k} := \left(\mu^i_j - \mu^k_j\right)\left(v^j - v^i\right) - \left(\mu^i_k - \mu^j_k\right)\left(v^k - v^i\right), \tag{38}$$

which are clearly antisymmetric with respect to the interchange j to k.

Proof. The proof of the lemma is given in Appendix A. □

Now we are ready to formulate the main Theorem of this Section.

Theorem 3.4. *Maps (32) are multidimensionally compatible.*

Proof. The proof of this theorem is presented in Appendix B. □

3.3. Yang-Baxter maps and the Sylvester equation

Theorem 3.4 states that maps (32) are multidimensionally-compatible. In the following proposition we present the companion maps of (32) which are Yang-Baxter maps due to Proposition 3.1.

Proposition 3.5. *The companion maps Q^c_{ij} of (32) read*

$$ Q^c_{ij} : (v^i, \mu^i; v^j_i, \mu^j_i) \mapsto \left(v^i_j, \mu^i_j; v^j, \mu^j\right), \qquad i \neq j \in \{1, \dots, N\}, \tag{39}$$

where

$$ \begin{aligned} v^i_j &= v^j_i + h^{i,j}, \qquad & v^j &= v^i + h^{i,j}, \\ \mu^i_j &= \mu^j_i + \left(g^{i,j}\right)^{-1}, \qquad & \mu^j &= \mu^i - \left(g^{i,j}\right)^{-1}, \end{aligned} \tag{40}$$

and $h^{i,j}, g^{i,j}$ satisfy the following system of Sylvester equations

$$ \begin{gathered} \mu^i g^{i,j} - g^{i,j} \mu^j_i = 1, \\ \left(\mu^j_i + \left(g^{i,j}\right)^{-1}\right) h^{i,j} + h^{i,j}\left(\mu^i - \left(g^{i,j}\right)^{-1}\right) = u^j_i \left(g^{i,j}\right)^{-1} - \left(g^{i,j}\right)^{-1} u^i. \end{gathered} \tag{41}$$

Proof. In order to find the companion maps Q^c_{ij} we need to solve (31) for $v^i_j, \mu^i_j, v^j, \mu^j$ in terms of $v^i, \mu^i, v^j_i, \mu^j_i$. Omitting the identity solution $v^i_j = v^j_i, \mu^i_j = \mu^j_i, v^j = v^i, \mu^j = \mu^i$, we consider the auxiliary variables $g^{i,j}$, and $h^{i,j}$, defined by

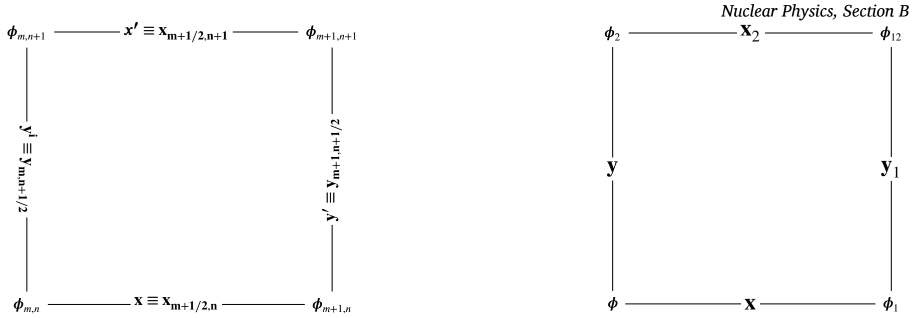
$$ \mu^i_j = \mu^j_i + (g^{i,j})^{-1}, \quad v^i_j = v^j_i + h^{i,j}. \tag{42}$$

Substituting the expressions above into (31a) and (31b), we respectively obtain

$$ \mu^j = \mu^i - (g^{i,j})^{-1}, \quad v^j = v^i + h^{i,j}. \tag{43}$$

Substituting (42) and (43) into (31b) and (31c), they become exactly the system of Sylvester equations (41) and that completes the proof. □

We remark that in a similar manner we can find the inverse of the maps Q^c_{ij}, hence the original maps Q_{ij} are quadrirational. Further recent developments on non-Abelian Yang-Baxter maps can be found in [26–29].



(a) Descriptive notation

(b) Compendious notation

Fig. 2. Variables assigned on vertices and edges of an elementary cell of the $\mathbb{Z}^2$ graph.

4. Non-Abelian elastic collision maps as difference systems

4.1. Reinterpretation of a multidimensionaly compatible map as a system of difference equations on the $\mathbb{Z}^N$ graph

There is a natural association of a map with a difference system defined on the edges of an elementary quadrilateral of the $\mathbb{Z}^2$ graph. Indeed, a map $R : \mathcal{X} \times \mathcal{X} \ni (\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}', \mathbf{y}') \in \mathcal{X} \times \mathcal{X}$, can be considered as a difference system defined on the edges of an elementary quadrilateral of the $\mathbb{Z}^2$ graph by making the following identifications

$$\begin{aligned} \mathbf{x} &\equiv \mathbf{x}_{m+1/2,n}, & \mathbf{y} &\equiv \mathbf{y}_{m,n+1/2} \\ \mathbf{x}' &\equiv \mathbf{x}_{m+1/2,n+1}, & \mathbf{y}' &\equiv \mathbf{y}_{m+1,n+1/2}, \end{aligned} \qquad m, n \in \mathbb{Z}, \tag{44}$$

so that the “primes” have been interpreted as increments on the independent variables. Moreover, we can adopt the compendious notation (see Fig. 2)

$$\begin{aligned} \mathbf{x} &:= \mathbf{x}_{m+1/2,n}, & \mathbf{y} &:= \mathbf{y}_{m,n+1/2}, & \mathbf{x}_1 &:= \mathbf{x}_{m+3/2,n}, & &etc. \\ \mathbf{x}_2 &:= \mathbf{x}_{m+1/2,n+1} \equiv \mathbf{u}, & \mathbf{y}_1 &:= \mathbf{y}_{m+1,n+1/2} \equiv \mathbf{v}, & \mathbf{y}_2 &:= \mathbf{y}_{m,n+3/2}, & &etc. \end{aligned} \qquad m, n \in \mathbb{Z},$$

where with subscripts we denoted discrete shifts on the associated $\mathbb{Z}^2$ graph. We extend our notation to the $\mathbb{Z}^N$ graph as follows

$$\mathbf{x}^i := \mathbf{x}, \quad \mathbf{x}^j := \mathbf{y}, \quad \mathbf{x}^i_j := \mathbf{x}_j, \quad \mathbf{x}^j_i := \mathbf{y}_i, \qquad i \neq j \in \{1, \dots, N\}. \tag{45}$$

Note that in this notation the superscripts represent the associated edges of the $\mathbb{Z}^N$ graph where the variables are assigned, e.g. see Fig. 2. When $\mathbf{x}$ represents an $M \in \mathbb{N}$ component vector, so $\mathbf{x}^i$ stands for the vector $\mathbf{x}$ assigned on the i-th edge of the $\mathbb{Z}^N$ graph, we denote its components as follows

$$\mathbf{x}^i = \left(x^{1,i}, \dots, x^{M,i}\right), \qquad i \in \{1, \dots, N\}, \qquad M \in \mathbb{N}.$$

In this concise notation introduced above, the difference system $\mathbf{x}^i_j = F(\mathbf{x}^i, \mathbf{x}^j)$, is associated with the map $(\mathbf{x}^i, \mathbf{x}^j) \mapsto \left(\mathbf{x}^i_j, \mathbf{x}^j_i\right) = \left(F(\mathbf{x}^i, \mathbf{x}^j), F(\mathbf{x}^j, \mathbf{x}^i)\right)$, where F a rational function. This difference system will be called multidimensional compatible *iff*

$$\mathbf{x}^i_{jk} = \mathbf{x}^i_{kj}, \qquad i \neq j \neq k \neq i \in \{1, \dots, N\}. \tag{46}$$

For the rest of this article and unless else specified, we have

$$\mathbf{x}^i := \mathbf{x} = \left(\mu^i, v^i\right), \qquad \mathbf{x}^j := \mathbf{y} = \left(\mu^j, v^j\right), \qquad i \neq j \in \{1, \dots, N\}.$$

With the reinterpretation considered above, the abelian version of elastic collision map appeared in the discrete integrable systems theory in [1,30,2], see also [31] for their connection with discrete Hirota’s Korteweg de Vries equation.

4.2. Non-Abelian companion elastic collision maps and associated difference systems

With the reinterpretation we presented in the previous Section, we can treat the defining relations of the maps (32) that is (33) and (34), as difference systems that relate the two-component variables defined on the edges of an elementary quad of the $\mathbb{Z}^N$ graph.

Theorem 4.1. *Consider the following Lax system*

$$\Psi_i = L(v^i, \mu^i; \lambda)\Psi, \qquad \Psi_j = L(v^j, \mu^j; \lambda)\Psi, \tag{47}$$

where the Lax matrix L is given by

$$L(v^i,\mu^i;\lambda):=\begin{pmatrix}\mu^i+\lambda & \lambda v^i\\ 0 & \mu^i-\lambda\end{pmatrix}.$$

Let all entries of the Lax matrix be assumed to belong to an associative algebra $\mathcal{A}$, and λ, which is referred to as the spectral parameter, assumed to be an element of the centre of the algebra. The following holds.

(1) The compatibility conditions of the Lax system (47) i.e.

$$L(v^i_j,\mu^i_j;\lambda)L(v^j,\mu^j;\lambda)=L(v^j_i,\mu^j_i;\lambda)L(v^i,\mu^i;\lambda) \tag{48}$$

that are supposed to be valid for every value of λ are equivalent to the following system of difference equations defined on the edges of the $\mathbb{Z}^N$ graph

$$v^i_j-v^j=v^j_i-v^i, \tag{49a}$$
$$\mu^i_j+\mu^j=\mu^j_i+\mu^i, \tag{49b}$$
$$\mu^i_j\mu^j=\mu^j_i\mu^i, \tag{49c}$$
$$\mu^i_jv^j+v^i_j\mu^j=\mu^j_iv^i+v^j_i\mu^i, \tag{49d}$$

where $i\neq j\in\{1,\dots,N\}$;

*(2) they correspond to the following systems of difference equations defined on the vertices (*vertex systems*) of the $\mathbb{Z}^N$ graph*

$$\psi_i-\psi=\phi_i\phi^{-1}, \tag{50a}$$
$$\sigma_i-\sigma=\phi_i^{-1}\phi, \tag{50b}$$
$$\chi_i+\chi=\phi_i(\omega_i-\omega)\phi^{-1}. \tag{50c}$$

(3) The relations (50), serve as hetero-Bäcklund transformations between the following three vertex systems

$$\phi_{ij}\left(\phi_j^{-1}-\phi_i^{-1}\right)=\left(\phi_i-\phi_j\right)\phi^{-1}, \tag{51a}$$
$$\phi_{ij}\phi_j^{-1}(\chi_j+\chi)+(\chi_{ij}+\chi_j)\phi_j\phi^{-1}=\phi_{ij}\phi_1^{-1}(\chi_i+\chi)+(\chi_{ij}+\chi_i)\phi_i\phi^{-1}, \tag{51b}$$

and

$$\phi_{ij}\left(\phi_j^{-1}-\phi_i^{-1}\right)=\left(\phi_i-\phi_j\right)\phi^{-1}, \tag{52a}$$
$$\phi_{ij}(\omega_{ij}-\omega_j)\phi_j^{-1}+\phi_i(\omega_i-\omega)\phi^{-1}=\phi_{ij}(\omega_{ij}-\omega_i)\phi_i^{-1}+\phi_j(\omega_j-\omega)\phi^{-1}, \tag{52b}$$

and

$$(\psi_{ij}-\psi_j)(\psi_j-\psi)=(\psi_{ij}-\psi_i)(\psi_i-\psi), \tag{53a}$$
$$(\psi_{ij}-\psi_j)(\chi_j+\chi)+(\chi_{ij}+\chi_j)(\psi_j-\psi)=(\psi_{ij}-\psi_i)(\chi_i+\chi)+(\chi_{ij}+\chi_i)(\psi_i-\psi). \tag{53b}$$

(4) The difference systems (49) are linear with respect to the variables $v^i_j, v^j_i, \mu^i_j, \mu^j_i$ and provided that the expressions $\mu^i-\mu^j$ are invertible, they can be solved with respect to these variables to obtain

$$\mu^i_j=(\mu^i-\mu^j)\mu^i(\mu^i-\mu^j)^{-1}, \tag{54a}$$
$$v^i_j=(\mu^i_jv^j-\mu^j_iv^i-\left(v^i-v^j\right)\mu^i)(\mu^i-\mu^j)^{-1}. \tag{54b}$$

The system of equations in edge variables (49) is multidimensionally compatible.

(5) The systems of equations in vertex variables (50) are multidimensional compatible.

Proof. (1) The proof follows by direct computation. Indeed, the compatibility conditions (48) of the Lax system (47) explicitly read

$$(\mu^i_j+\lambda)(\mu^j+\lambda)=(\mu^j_i+\lambda)(\mu^i+\lambda), \tag{55a}$$
$$(\mu^i_j-\lambda)(\mu^j-\lambda)=(\mu^j_i-\lambda)(\mu^i-\lambda), \tag{55b}$$
$$(\mu^i_j+\lambda)v^j+v^i_j(\mu^j-\lambda)=(\mu^j_i+\lambda)v^i+v^j_i(\mu^i-\lambda). \tag{55c}$$

Demanding that (55) hold for every λ, (55a) is equivalent to (55b) and together with (55c) we obtain exactly the system of differences equations (49).

(2) The procedure to obtain a system of vertex equations associated with its edge system counterpart, is nowadays referred to as *potentialization* [32–38]. Namely, equations (49a), (49b) and (49c) guarantee the existence of the potential functions χ, ψ, ϕ, respectively

$$v^i = \chi_i + \chi, \qquad v^j = \chi_j + \chi, \tag{56a}$$

$$\mu^i = \psi_i - \psi, \qquad \mu^j = \psi_j - \psi, \tag{56b}$$

$$\mu^i = \phi_i \phi^{-1}, \qquad \mu^j = \phi_j \phi^{-1}, \tag{56c}$$

Then equation (49d) can be written in the conservation form

$$\left(\phi_i^{-1} v^i \phi\right)_j - \phi_i^{-1} v^i \phi = \left(\phi_j^{-1} v^j \phi\right)_i - \phi_j^{-1} v^j \phi. \tag{57}$$

Moreover, (57) guarantees the existence of the potential function ω which is defined by

$$\phi_i^{-1} v^i \phi = \omega_i - \omega, \qquad \phi_j^{-1} v^j \phi = \omega_j - \omega. \tag{58}$$

Furthermore, taking

$$\rho := \phi^{-1},$$

due to (56c) equation (49b) becomes the conservation relation

$$\left(\rho_i \rho^{-1}\right)_j - \rho_i \rho^{-1} = \left(\rho_j \rho^{-1}\right)_i - \rho_j \rho^{-1}, \tag{59}$$

that guarantees the existence of the following potential σ

$$\sigma^i - \sigma = \rho_i \rho^{-1}, \qquad \sigma^j - \sigma = \rho_j \rho^{-1}. \tag{60}$$

From (56) and (57), we eliminate v^i, v^j, μ^i, μ^j and together with (60) we obtain (50).

(3) From the difference systems in vertex variables (50), we use the compatibility conditions $\phi_{ij} = \phi_{ji}$, $\chi_{ij} = \chi_{ji}$, $\psi_{ij} = \psi_{ji}$, $\omega_{ij} = \omega_{ji}$, to either eliminate ψ and ω, obtaining (51); eliminate ψ and χ, obtaining (52); or eliminate ϕ and ω, obtaining (53).

(4) The proof of this fact is essentially the same as the proof of Theorem 3.4.

(5) In order to prove that the vertex systems (50) are multidimensional compatible we have to show that $\forall i \neq j \neq k \in \{1, \ldots, N\}$, $\phi_{ijk} = \phi_{ikj}$, $\chi_{ijk} = \chi_{ikj}$, $\psi_{ijk} = \psi_{ikj}$ and $\omega_{ijk} = \omega_{ikj}$. Let us first prove that $\phi_{ijk} = \phi_{ikj}$. From item (4) of this theorem we have that $\mu^i_{jk} = \mu^i_{kj}$. Substituting the definition of the potential ϕ (56c) to the previous relations we obtain $\phi_{ijk}\phi_{jk}^{-1} = \phi_{ikj}\phi_{kj}^{-1}$. So in order $\phi_{ijk} = \phi_{ikj}$ to hold, it should be that $\phi_{jk} = \phi_{kj}$, but this trivially holds since the potential ϕ exists. Similarly we prove that the remaining potentials are multidimensional consistent. □

Note that, equations (49b), (49c) also appear in the context of discrete differential geometry and serve as the defining relations of the so-called *skew parallelogram nets* [39,11], cf. [40]. In that respect (49) could be realized as deformations of skew parallelogram nets since for $v^i = v^j_j = v^j = v^j_i = c$, where the constant c is an element of the field that the algebra is defined over, we recover (49b), (49c). In [35,41,26,42,28], reductions of (49b), (49c) on certain subspaces of a $\mathbb{Z}_n$-graded algebra over a non-commutative ring were considered together with the associated non-abelian Yang-Baxter maps and the corresponding difference systems in edge and in vertex variables.

4.3. Abelian elastic collisions of point-mass particles as difference equations on the $\mathbb{Z}^N$ graph

Under the identifications (44), (45), the defining relations of the abelian map Q_{ij} (28), define the following difference systems in edge variables

$$v^i_j - v^j = v^j_i - v^i, \qquad m^i \left(v^i_j - v^i\right) = m^j \left(v^j_i - v^j\right), \qquad i \neq j \in \{1, \ldots, N\}. \tag{61}$$

The first equation of (61), guarantees the existence of a potential function χ, such that

$$v^i = \chi_i + \chi, \qquad v^j = \chi_j + \chi.$$

In terms of the potential function χ the second equation of (61), reads

$$\left(m^i - m^j\right)(\chi_{ij} - \chi) - \left(m^i + m^j\right)(\chi_i - \chi_j) = 0. \tag{62}$$

On the other hand, The second equation of (61), guarantees the existence of a potential function ψ, such that

$$v^i = \frac{\psi_i - \psi}{m^i}, \qquad v^j = \frac{\psi_j - \psi}{m^j}.$$

Then, in terms of the potential function ψ the first equation of (61), reads

$$\left(\mu^i - \mu^j\right)(\psi_{ij} - \psi) + \left(\mu^i + \mu^j\right)(\psi_i - \psi_j) = 0, \tag{63}$$

where $\mu^i := 1/m^i$ and $\mu^j := 1/m^j$. The linear difference systems in vertex variables (62) and (63), are related by the substitution

$$\chi_i+\chi=\frac{\psi_i-\psi}{m^i},\qquad \chi_j+\chi=\frac{\psi_j-\psi}{m^j}.$$

5. The closure relations and systems of three-dimensional vertex equations

In Lemma 3.3 we have provided two algebraic relations which are satisfied by the multidimensional compatible maps (32) on any cubic-cell of the Z^N-graph. Clearly, these algebraic relations also hold for (54a), (54b) that serve as the difference systems in edge variables associated with the multidimensional compatible maps (32). These relations serve as closure relations (cf. [43]) since they hold on any cubic-cell of the N-cube lattice where the non-abelian systems (54a), (54b) are defined.

5.1. Three-dimensional systems of vertex equations

In the following Proposition we provide all closure relations associated with the difference systems (54a), (54b), as well as the corresponding systems of three-dimensional vertex equations. As a result we obtain coupled systems of three-dimensional vertex equations.

Proposition 5.1. *A. On any cubic-cell of the N-cube lattice, the solutions of the difference systems (54) satisfy the following closure relations*

(i) $K^{i,j}_k+K^{j,k}_i+K^{k,i}_j=0$, *where* $K^{i,j}:=\mu^i-\mu^j$;

(ii) $S^{i,j}_kS^{j,k}_iS^{k,i}_j=1$, *where* $S^{i,j}:=\mu^j\left(\mu^i\right)^{-1}$;

(iii) $T^{i,j}_k+T^{j,k}_i+T^{k,i}_j=0$, *where* $T^{i,j}:=u^i-u^j$;

(iv) $U^{i,j}_k+U^{j,k}_i+U^{k,i}_j=0$, *where* $U^{i,j}:=\phi_i^{-1}u^i\phi-\phi_j^{-1}u^j\phi$, *and* ϕ *the potential function defined in (56c).*

B. The solutions of the difference systems in vertex variables (51), (52), (53) satisfy the following systems of three-dimensional vertex equations

$$\begin{gathered}\phi_{ij}\phi_{ik}^{-1}\phi_{jk}=\phi_{jk}\phi_{ik}^{-1}\phi_{ij},\\ \left(\text{in terms of } \psi \text{ reads}: (\psi_{ik}-\psi_k)(\psi_{jk}-\psi_k)^{-1}(\psi_{ij}-\psi_i)(\psi_{ik}-\psi_i)^{-1}(\psi_{jk}-\psi_j)(\psi_{ij}-\psi_j)^{-1}=1\right),\\ \left(\phi_{ik}^{-1}\left(\chi_{ik}+\chi_k\right)-\phi_{jk}^{-1}\left(\chi_{jk}+\chi_k\right)\right)\phi_k+\left(\phi_{ij}^{-1}\left(\chi_{ij}+\chi_i\right)-\phi_{ik}^{-1}\left(\chi_{ik}+\chi_i\right)\right)\phi_i\\ +\left(\phi_{jk}^{-1}\left(\chi_{jk}+\chi_j\right)-\phi_{ij}^{-1}\left(\chi_{ij}+\chi_j\right)\right)\phi_j=0,\end{gathered}\qquad(\mathcal{X}_2)$$

$$\begin{gathered}\left(\phi_{ik}-\phi_{jk}\right)\phi_k^{-1}+\left(\phi_{ij}-\phi_{ik}\right)\phi_i^{-1}+\left(\phi_{jk}-\phi_{ij}\right)\phi_j^{-1}=0,\\ \left(\phi_{ik}^{-1}\left(\chi_{ik}+\chi_k\right)-\phi_{jk}^{-1}\left(\chi_{jk}+\chi_k\right)\right)\phi_k+\left(\phi_{ij}^{-1}\left(\chi_{ij}+\chi_i\right)-\phi_{ik}^{-1}\left(\chi_{ik}+\chi_i\right)\right)\phi_i\\ +\left(\phi_{jk}^{-1}\left(\chi_{jk}+\chi_j\right)-\phi_{ij}^{-1}\left(\chi_{ij}+\chi_j\right)\right)\phi_j=0,\end{gathered}\qquad(\mathcal{X}_4)$$

$$\begin{gathered}\left(\phi_{ik}-\phi_{jk}\right)\phi_k^{-1}+\left(\phi_{ij}-\phi_{ik}\right)\phi_i^{-1}+\left(\phi_{jk}-\phi_{ij}\right)\phi_j^{-1}=0,\\ \left(\phi_{ik}\left(\omega_{ik}-\omega_k\right)-\phi_{jk}\left(\omega_{jk}-\omega_k\right)\right)\phi_k^{-1}+\left(\phi_{ij}\left(\omega_{ij}-\omega_i\right)-\phi_{ik}\left(\omega_{ik}-\omega_i\right)\right)\phi_i^{-1}\\ +\left(\phi_{jk}\left(\omega_{jk}-\omega_j\right)-\phi_{ij}\left(\omega_{ij}-\omega_j\right)\right)\phi_j^{-1}=0.\end{gathered}\qquad(\mathcal{X}_4^{\dagger})$$

Proof. The proof of the first two items (i) and (ii) of part A. of the Proposition have been essentially proved in Lemma 3.3.

Let us prove item (iii). Since $T^{i,j}:=v^i-v^j$, we have

$$T^{i,j}_k+T^{j,k}_i+T^{k,i}_j=\underbrace{v^i_k-v^k_i}_{=u^k-u^i}+\underbrace{v^j_i-v^i_j}_{=u^i-u^j}+\underbrace{v^k_j-v^j_k}_{=u^j-u^k}=0,$$

where we have substituted (31a).

(iv) Since $U^{i,j}:=\phi_i^{-1}u^i\phi-\phi_j^{-1}u^j\phi$, we have

$$U^{i,j}_k+U^{j,k}_i+U^{k,i}_j=\underbrace{\phi_{ij}^{-1}\left(u^j_i\phi_i-u^i_j\phi_j\right)}_{=\left(\phi_j^{-1}u^j-\phi_i^{-1}u^i\right)\phi}+\underbrace{\phi_{jk}^{-1}\left(u^k_j\phi_j-u^j_k\phi_k\right)}_{=\left(\phi_k^{-1}u^k-\phi_j^{-1}u^j\right)\phi}+\underbrace{\phi_{ik}^{-1}\left(u^i_k\phi_k-u^k_i\phi_i\right)}_{=\left(\phi_i^{-1}u^i-\phi_k^{-1}u^k\right)\phi}=0,$$

where we have substituted (57).

B. Rewriting the closure relations $(i)-(iv)$ in terms of the potentials (56a), (56b), they become exactly the three-dimensional vertex system $(\mathcal{X}_2)$. On the other hand, rewriting them in terms of the potentials (56a), (56c), they coincide with $(\mathcal{X}_4)$. Finally, expressing the closure relations in terms of potentials (56c), (57), we obtain $(\mathcal{X}_4^{\dagger})$. □

We recall that the first equation of $(\mathcal{X}_2)$, is related to a non-abelian three-dimensional equation that was introduced in [44], that is

$$(\psi_{ik}-\psi_k)(\psi_{jk}-\psi_k)^{-1}(\psi_{ij}-\psi_i)(\psi_{ik}-\psi_i)^{-1}(\psi_{jk}-\psi_j)(\psi_{ij}-\psi_j)^{-1}=1 \tag{64}$$

via the Bäcklund transformation $\psi_i-\psi=\phi_i\phi^{-1}$. The abelian version of (64) coincides with the equation referred to as (χ_2) in [45]. Also, the first equation of the three-dimensional vertex systems $(\mathcal{X}_4)$ and $(\mathcal{X}_4^{\dagger})$, first appeared in [44] and in the Abelian limit coincides with the equation referred to as (χ_4) in [45].

6. A unified approach of discrete analytic functions

In this Section we show that both the linear and the nonlinear theories of discrete analytic functions can be considered as special cases of (5). We start with the following definition.

Definition 6 *(Discrete analytic functions on an embedding of the $\mathbb{Z}^2$ graph [46,47]).* Let $\sigma:\mathbb{Z}^2\to\mathbb{C}$ be an injective embedding of the $\mathbb{Z}^2$ graph to the complex plane $\mathbb{C}$ and let the function $F:\mathbb{C}\to\mathbb{C}$. Then the composition $\chi=F\circ\sigma$ is another embedding of the $\mathbb{Z}^2$ graph to the complex plane $\mathbb{C}$. We say that the function F is *discrete analytic on the embedding σ at an elementary quad that consists of the vertices $\{(m,n),(m+1,n),(m,n+1),(m+1,n+1)\}$ of the graph*, iff the equation

$$(\sigma_i-\sigma_j)(\chi_{ij}-\chi)=(\sigma_{ij}-\sigma)(\chi_i-\chi_j), \tag{65}$$

holds for the quad. We say that the function F is *discrete analytic on the embedding σ*, iff the equation (65) holds for every elementary quad of the graph.

Remark 6.1. Since the definition is “local” i.e. it is restricted to elementary quads, it can be extended to any planar quad graph.

Remark 6.2. Equation (65) is a discrete analogue of $\bar{\partial} f(z)=0$, that in discrete integrable systems theory is referred to as *the discrete Moutard equation* [48,49].

In the literature, *discrete analytic functions* usually are defined with a prescribed embedding σ. In Ferrand’s definition [50–57] the embedding σ is a square embedding $\{m+in\in\mathbb{C}\,|\,(m,n)\in\mathbb{Z}^2\}$. Duffin’s definition [4,58] assumes that the embedding consists of rhombi only. In article [59] Mercat deals with rectangle embeding σ. In this article we assume parallelogram embedding, i.e. for every quad of the graph the following constraint holds

$$\sigma_{ij}+\sigma-\sigma_i-\sigma_j=0. \tag{66}$$

However, we would like to stress that in Mercat approach [46], where essentially discrete analytic transformations are the transformations that preserves the so called *discrete conformal structure*, the role of the embeding σ is secondary.

The important observation is that equation (65) can be rewritten as

$$(\sigma_{ij}-\sigma_j)(\chi_{ij}+\chi_j)+(\sigma_j-\sigma)(\chi_j+\chi)+(\sigma-\sigma_i)(\chi_i+\chi)+(\sigma_i-\sigma_{ij})(\chi_{ij}+\chi_i)=0. \tag{67}$$

If we define the directed integral over the directed edge (a,b) of the graph as

$$\int_{(a,b)}\chi(\sigma)d\sigma:=\frac{\chi(b)+\chi(a)}{2}(\sigma(b)-\sigma(a)),$$

then equation (65) serves as the discrete analogue of the Cauchy integral theorem $\oint f(z)dz=0$ [4], which can be written as

$$\oint_{\Diamond}\chi(\sigma)d\sigma=0.$$

In the formula above $\Diamond$ means that the integration is performed over an elementary quadrilateral of the $\mathbb{Z}^2$ graph.

An important characterisation of the discrete Moutard equation is that it leads to the discrete Laplace equation by the sublattice approach (see [49])

$$\begin{aligned}&(\sigma_{ij}-\sigma)^{-1}(\sigma_i-\sigma_j)(\chi_{ij}-\chi)+(\sigma_{-i-j}-\sigma)^{-1}(\sigma_{-i}-\sigma_{-j})(\chi_{-i-j}-\chi)+\\&+(\sigma_{-ij}-\sigma)^{-1}(\sigma_j-\sigma_{-i})(\chi_{-ij}-\chi)+(\sigma_{i-j}-\sigma)^{-1}(\sigma_{-j}-\sigma_i)(\chi_{i-j}-\chi)=0,\end{aligned} \tag{68}$$

that serves as the discrete analogue of $(\partial_x^2+\partial_y^2)f(x+iy)=0$.

Remark 6.3. The discrete Cauchy integral theorem (67) (or equivalently (65)), is a local property (it is defined on a single quad) and can thus be easily extended to an arbitrary quad-graph. In addition, the transition from (65) to the Laplace type equation (68) can be performed in an arbitrary quad-graph.

Finally, we would like to stress that the formulas (65), (67) and (68) are written in a form that is valid in the non-abelian case.

6.1. Non-abelian unification of the theories of discrete analytic functions

We will show now that both the linear theory (see Definition 6) and the non-linear theory (see [60]) of discrete analytic functions have a unified description.

The system of difference equations (50) under the assumptions:

(1) the values of functions χ and ω commute with the values of any of the functions ϕ, ψ or σ;
(2) the functions χ, ω, ϕ, ψ and σ are embeddings of the $\mathbb{Z}^2$ graph in an associative algebra $\mathcal{A}$ i.e. they are considered as maps $\mathbb{Z}^2 \to \mathcal{A}$;
(3) the values of the functions ϕ, ψ, σ belong to the domain and values of functions χ, ω belong to the codomain of the maps $F_i : \mathcal{A} \to \mathcal{A}$, such that $\chi = F_1 \circ \phi$, $\chi = F_2 \circ \psi$, $\omega = F_3 \circ \sigma$, $\omega = F_4 \circ \phi$, $\omega = F_5 \circ \psi$ and $\omega = F_6 \circ \sigma$[1];

reads

$$\chi_i + \chi = \phi_i \phi^{-1}(\omega_i - \omega), \tag{69a}$$

$$\chi_i + \chi = (\psi_i - \psi)(\omega_i - \omega), \tag{69b}$$

$$(\sigma_i - \sigma)(\chi_i + \chi) = (\omega_i - \omega). \tag{69c}$$

Furthermore, if we assume that if $\omega_i - \omega \neq 0$ then from equations (69a), (69b) and (69c) we infer that

$$\psi_i - \psi = \phi_i \phi^{-1}, \tag{70}$$

$$\sigma_i - \sigma = \phi_i^{-1} \phi, \tag{71}$$

and the compatibility of (70) and (71) leads to the following difference equations with one dependent variable each:

$$\phi_{ij}(\phi_j^{-1} - \phi_i^{-1}) = (\phi_i - \phi_j)\phi^{-1}, \tag{72}$$

$$(\psi_{ij} - \psi_j)(\psi_j - \psi) = (\psi_{ij} - \psi_i)(\psi_i - \psi) \tag{73}$$

$$(\sigma_{ij} - \sigma_j)(\sigma_j - \sigma) = (\sigma_{ij} - \sigma_i)(\sigma_i - \sigma) \tag{74}$$

In addition the functions μ^i and v^i

$$\mu^i := \psi_i - \psi = \phi_i \phi^{-1}, \tag{75a}$$

$$v^i := \chi_i + \chi = \phi_i \phi^{-1}(\omega_i - \omega), \tag{75b}$$

are naturally defined on the edges of the $\mathbb{Z}^2$ graph.

Summarizing, both the linear theory of analytic functions and the non-linear theory arise as special cases of system (69), or more precisely as special cases of the structure $\mathcal{O}(F, \sigma)$, where

(i) the function $\sigma : \mathbb{Z}^2 \to \mathcal{A}$ is an embedding of the $\mathbb{Z}^2$ graph into an algebra $\mathcal{A}$;
(ii) $F : \mathcal{A} \supset \operatorname{Im}\sigma \ni x \mapsto y = F(x) \in \mathbb{C}$, is a function and $\chi := F \circ \sigma$;
(iii) the embeddings σ and χ commute i.e. if $x \in \operatorname{Im}\sigma$ and $y \in \operatorname{Im}\chi$ then $xy = yx$;
(iv) the embeddings satisfy

$$(\sigma_i - \sigma_j)(\chi_{ij} - \chi) = (\sigma_{ij} - \sigma)(\chi_i - \chi_j), \tag{76}$$

that is equivalent to

$$(\sigma_{ij} - \sigma_j)(\chi_{ij} + \chi_j) + (\sigma_j - \sigma)(\chi_j + \chi) + (\sigma - \sigma_i)(\chi_i + \chi) + (\sigma_i - \sigma_{ij})(\chi_{ij} + \chi_i) = 0, \tag{77}$$

that implies

$$\begin{aligned} &(\sigma_{ij} - \sigma)^{-1}(\sigma_i - \sigma_j)(\chi_{ij} - \chi) + (\sigma_{-i-j} - \sigma)^{-1}(\sigma_{-i} - \sigma_{-j})(\chi_{-i-j} - \chi) + \\ &+ (\sigma_{-ij} - \sigma)^{-1}(\sigma_j - \sigma_{-i})(\chi_{-ij} - \chi) + (\sigma_{i-j} - \sigma)^{-1}(\sigma_{-j} - \sigma_i)(\chi_{i-j} - \chi) = 0; \end{aligned} \tag{78}$$

(v) in general no restriction on the embedding σ is imposed.

[1] We recall that $\chi = F_1 \circ \phi$ means that for every $(n_1, n_2) \in \mathbb{Z}^2$ we have $\phi(n_1, n_2) \mapsto F_1 \circ \phi(n_1, n_2) = \chi(n_1, n_2)$ and similarly for the remaining functions.

6.1.1. Linear theory of discrete analytic functions

If one considers $\mathcal{A} = \mathbb{C}$, the system of equations (69c) and (74) coincides with the Definition 6 of discrete analytic functions on parallelogram embedings. Indeed, equation (69c) after elimination ω gives (65), whereas equation (74) in the abelian case can be rewritten as $(\sigma_{ij} - \sigma_i - \sigma_j + \sigma)(\sigma_i - \sigma_j) = 0$, which means that either the faces of the domain are parallelograms or they locally degenerate to a line.

We plan to dedicate a separate article to investigating further consequences of system (69) in the theory of discrete analytic functions.

6.1.2. Non-linear theory of analytic functions

We show now that the non-linear approach of discrete analytic functions theory (see e.g. [5,61–63,60,64]) is a special case of the non-abelian theory.

We follow the steps we have used in Lemma 2.2. We take

$$\mu^i := \psi_i - \psi = \begin{bmatrix} 0 & a^i \\ b^i & 0 \end{bmatrix}, \qquad v^i := \begin{bmatrix} \chi_i + \chi & 0 \\ & \chi_i + \chi \end{bmatrix}.$$

where the entries of the matrices are functions $\mathbb{Z}^2 \to \mathbb{C}$. Equations (49b), (49c) can be rewritten as

$$a^i_j = \frac{a^i - a^j}{b^i - b^j} b^i, \quad b^i_j = \frac{b^i - b^j}{a^i - a^j} a^i.$$

Multiplying the sides of the equations we get

$$(a^i b^i)_j = a^i b^i,$$

so the product of the two variables $(\alpha^i)^2 := a^i b^i$ is a function of the i-th independent variable only. So we can set

$$a^i = \alpha^i y^i, \quad b^i = \frac{\alpha^i}{y^i}.$$

We conclude that μ^i should be of the form

$$\mu^i := \psi_i - \psi = \begin{bmatrix} 0 & \alpha^i y^i \\ \frac{\alpha^i}{y^i} & 0 \end{bmatrix},$$

where α^i are given $\mathbb{C}$ valued functions which depend only on i-th independent variable of the graph, and y^i are $\mathbb{C}$ valued functions that have to be determined. It turns out that the functions y^i must obey

$$y^i_j y^i = y^j_i y^j, \tag{79a}$$

$$\alpha^i (y^i_j - y^i) = \alpha^j (y^j_i - y^j). \tag{79b}$$

Equation (79a) guarantees the existence of potential τ such that

$$y^i = \tau_i \tau,$$

while equation (79b) becomes the discrete modified KdV equation [65]

$$\alpha^i (\tau_{ij}\tau_j - \tau_i \tau) = \alpha^j (\tau_{ij}\tau_i - \tau_j \tau).$$

Then equation (69a) reads

$$(\chi_i + \chi) = \tau_i \tau (\omega_i - \omega).$$

Applying the point transformation

$$\chi \to (-1)^{n_1+n_2} \chi, \qquad y^i \to -(-1)^{n_1+n_2} y^i,$$

we obtain the system of equations

$$(\chi_i - \chi) = \tau_i \tau (\omega_i - \omega),$$
$$\alpha^i (\tau_{ij}\tau_j + \tau_i \tau) = \alpha^j (\tau_{ij}\tau_i + \tau_j \tau),$$

which constitutes the foundations of the alternative non-linear integrable discretization of analytic functions [60].

Declaration of competing interest

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Appendix A. Proof of Lemma 3.3

We prove the three items of Lemma 3.3.

(1) Since $K^{i,j} := \mu^i - \mu^j$, we have

$$K^{i,j}_k + K^{j,k}_i + K^{k,i}_j = \mu^i_k - \mu^j_k + \mu^j_i - \mu^k_i + \mu^k_j - \mu^i_j = 0,$$

where we have substituted the expressions μ^p_q, $p \neq q \in \{i, j, k\}$ from (33).

(2) Applying the relations (33), we verify the second point of the lemma.

(3) We now prove the third point. There is

$$\begin{aligned} {}^i\Gamma^{j,k} = K^{i,j}_k K^{i,k} = \left(\mu^i_k - \mu^j_k\right) K^{i,k} &= \left(K^{i,k}\mu^i \left(K^{i,k}\right)^{-1} - K^{j,k}\mu^j \left(K^{j,k}\right)^{-1}\right) K^{i,k} \\ &= K^{j,k} \left(\underbrace{\left(K^{j,k}\right)^{-1} K^{i,k}\mu^i - \mu^j \left(K^{j,k}\right)^{-1} K^{i,k}}_{{}^i\Delta^{j,k}} \right). \end{aligned}$$

The expressions ${}^i\Delta^{j,k}$ are antisymmetric with respect to the interchange $j \leftrightarrow k$. Indeed, under expansion and recollection of terms ${}^i\Delta^{j,k}$ reads

$${}^i\Delta^{j,k} = \left(\mu^j - \mu^k\right)^{-1} \left(\mu^i\right)^2 + \left(\left(\mu^k\right)^{-1} - \left(\mu^j\right)^{-1}\right)^{-1} - \left(\underbrace{\left(1 - \mu^k \left(\mu^j\right)^{-1}\right)^{-1} - \left(1 - \left(\mu^k\right)^{-1} \mu^j\right)^{-1}}_{E^{j,k}} \right) \mu^i$$

The first two terms in the expression above are clearly antisymmetric under the interchange j to k. The same is true for the term $E^{j,k}$ since we have

$$E^{j,k} = \left(1 - \mu^k \left(\mu^j\right)^{-1}\right)^{-1} - \left(1 - \left(\mu^k\right)^{-1} \mu^j\right)^{-1} = 1 - \left(1 - \mu^j \left(\mu^k\right)^{-1}\right)^{-1} - \left(1 - \left(1 - \left(\mu^j\right)^{-1} \mu^k\right)^{-1}\right) = -E^{k,j},$$

where we have used twice the identity $\left(1 - AB^{-1}\right)^{-1} + \left(1 - BA^{-1}\right)^{-1} = 1$. To recapitulate, since both $K^{j,k}$ and ${}^i\Delta^{j,k}$, are anti-symmetric expressions under the interchange j to k, their product that coincides with ${}^i\Gamma^{j,k}$ (${}^i\Gamma^{j,k} = K^{j,k\,i}\Delta^{j,k}$), is symmetric.

Let us now prove relations (36). We have

$$\begin{aligned} &{}^i\Gamma^{j,k}\mu^i \left({}^i\Gamma^{j,k}\right)^{-1} {}^i\Psi^{j,k} = \\ &\quad {}^i\Gamma^{j,k}\mu^i \left(\underline{\left({}^i\Gamma^{j,k}\right)^{-1}} \left(1 - \mu^j_k \left(\mu^i_k\right)^{-1}\right) \left(v^k - v^i\right) - \underline{\left({}^i\Gamma^{j,k}\right)^{-1}} \left(1 - \mu^k_j \left(\mu^i_j\right)^{-1}\right) \left(v^j - v^i\right) \right) \\ &\quad = \underline{{}^i\Gamma^{j,k}}\mu^i \left(K^{i,k}\right)^{-1} \left(\mu^i_k\right)^{-1} \left(v^k - v^i\right) - \underline{{}^i\Gamma^{k,j}}\mu^i \left(K^{i,j}\right)^{-1} \left(\mu^i_j\right)^{-1} \left(v^j - v^i\right) \\ &\quad = \left(\mu^i_k - \mu^j_k\right)\left(v^k - v^i\right) - \left(\mu^i_j - \mu^k_j\right)\left(v^j - v^i\right) = -{}^i\Omega^{j,k}. \end{aligned}$$

where we have used (35) that results

$${}^i\Gamma^{j,k} = \left(1 - \mu^j_k \left(\mu^i_k\right)^{-1}\right) \mu^i_k K^{i,k} = {}^i\Gamma^{k,j} = \left(1 - \mu^k_j \left(\mu^i_j\right)^{-1}\right) \mu^i_j K^{i,j}, \tag{80}$$

to eliminate the underlined terms of the latter expressions.

Appendix B. Proof of Theorem 3.4

In order to prove that mappings Q_{ij} are multidimensional compatible we have to prove that $v^i_{jk} = v^i_{kj}$, $m^i_{jk} = m^i_{kj}$, $\forall i \neq j \neq k \neq i \in \{1, \ldots, N\}$.

Let us first prove $m^i_{jk} = m^i_{kj}$, or equivalently $\mu^i_{jk} = \mu^i_{kj}$, where $\mu^i := \left(m^i\right)^{-1}$. Shifting (54a) to the k-direction, we have

$$\mu^i_{jk} = \left(\mu^i_k - \mu^j_k\right)\mu^i_k\left(\mu^i_k - \mu^j_k\right)^{-1}. \tag{81}$$

There is

$$\mu^i_k - \mu^j_k = \left(\mu^i - \mu^k\right)\mu^i\left(\mu^i - \mu^k\right)^{-1} - \left(\mu^j - \mu^k\right)\mu^j\left(\mu^j - \mu^k\right)^{-1} = {}^i\Gamma^{j,k}\left(K^{i,k}\right)^{-1},$$

so (81) reads

$$\mu^i_{jk} = {}^i\Gamma^{j,k}\left(K^{i,k}\right)^{-1}\mu^i_k K^{i,k}\left({}^i\Gamma^{j,k}\right)^{-1}$$

Substituting μ^i_k that reads $\mu^i_k = K^{i,k}\mu^i\left(K^{i,k}\right)^{-1}$ into the relations above we obtain the multidimensional compatibility formula

$$\mu^i_{jk} = {}^i\Gamma^{j,k}\mu^i\left({}^i\Gamma^{j,k}\right)^{-1}, \tag{82}$$

that is clearly symmetric under the interchange j to k since $\Gamma^{j,k}$ is symmetric (see Lemma 3.3) under the same interchange. Finaly, it can be shown easily that

$$\mu^i_{jk} = \mu^j_{ik} + K^{i,j}_k. \tag{83}$$

Now we prove that $v^i_{jk} - v^i_{kj} = 0$, $\forall i \neq j \neq k \neq i \in \{1,\dots,N\}$. Shifting (54b) in the $k-th$ direction we have

$$v^i_{jk}K^{i,j}_k = \mu^i_{jk}v^j_k - \mu^j_{ik}v^i_k + \left(v^j_k - v^i_k\right)\mu^i_k.$$

Using (35), (83) and $\mu^i_k = K^{i,k}\mu^i\left(K^{i,k}\right)^{-1}$, the relations above read

$$v^i_{jk}\,{}^i\Gamma^{j,k} = \mu^i_{jk}(v^j_k - v^i_k)K^{i,k} + {}^i\Gamma^{i,j}\left(K^{i,k}\right)^{-1}v^i_k K^{i,k} + \left(v^j_k - v^i_k\right)K^{i,k}\mu^i.$$

Since ${}^i\Gamma^{j,k}$ is symmetric under $j \leftrightarrow k$, we have

$$\left(v^i_{jk} - v^i_{kj}\right){}^i\Gamma^{j,k} = \mu^i_{jk}\,{}^i\Omega^{j,k} + {}^i\Gamma^{j,k}\left(\left(K^{i,k}\right)^{-1}v^i_k K^{i,k} - \left(K^{i,j}\right)^{-1}v^i_j K^{i,j}\right) + {}^i\Omega^{j,k}\mu^i, \tag{84}$$

where ${}^i\Omega^{j,k} := \left(v^j_k - v^i_k\right)K^{i,k} - \left(v^k_j - v^i_j\right)K^{i,j}$. Using (54b) shifted accordingly, we substitute v^j_k, v^i_k, v^k_j and v^i_j into ${}^i\Omega^{j,k}$ and the latter coincides with the expressions (38) of Lemma 3.3. In order to prove the multidimensional compatibility we have to prove that the Rhs of (84) identically vanishes. Indeed, after using (82) the Rhs of (84) reads

$$\begin{aligned}&{}^i\Gamma^{j,k}\mu^i\left({}^i\Gamma^{j,k}\right)^{-1}\left(\underbrace{{}^i\Omega^{j,k} + {}^i\Gamma^{j,k}\left(\mu^i\right)^{-1}\left(\left(K^{i,k}\right)^{-1}v^i_k K^{i,k} - \left(K^{i,j}\right)^{-1}v^i_j K^{i,j}\right)}_{{}^i\Lambda^{j,k}}\right)\\ &\qquad + {}^i\Omega^{j,k}\mu^i.\end{aligned} \tag{85}$$

We use (54b) shifted accordingly and after expansion and recollection of terms, the expression ${}^i\Lambda^{j,k}$ reads

$${}^i\Lambda^{j,k} = \underbrace{\left(\mu^j_k\left(\mu^i_k\right)^{-1}\mu^k_i - \mu^k_j\left(\mu^i_j\right)^{-1}\mu^j_i\right)}_{=0 \text{ due to Lemma 3.3}}v^i + \left(\left(1 - \mu^j_k\left(\mu^i_k\right)^{-1}\right)\left(v^k - v^i\right) - \left(1 - \mu^k_j\left(\mu^i_j\right)^{-1}\right)\left(v^j - v^i\right)\right)\mu^i$$

or (see (37))

$${}^i\Lambda^{j,k} = {}^i\Psi^{j,k}\mu^i. \tag{86}$$

Using (86), the expressions (85), that constitute the Rhs of (84) read

$${}^i\Gamma^{j,k}\mu^i\left({}^i\Gamma^{j,k}\right)^{-1}{}^i\Lambda^{j,k} + {}^i\Omega^{j,k}\mu^i = \underbrace{\left({}^i\Gamma^{j,k}\mu^i\left({}^i\Gamma^{j,k}\right)^{-1}\Psi^{j,k} + \Omega^{j,k}\right)}_{=0 \text{ due to Lemma 3.3}}\mu^i = 0$$

and that completes the proof.

Data availability

No data was used for the research described in the article.

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