ON COHOMOLOGIES AND REPRESENTATIONS OF GROUPS WITH NORMAL ENGEL SUBGROUPS

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Abstract. Let λ, U be representations of a group G with a normal Engel subgroup N. The paper studies triviality conditions for the cohomology group $\mathcal{H}^1(G, \lambda, U)$ when λ and U are sectionally spectrally disjoint and examines some decompositions of the extension $\mathfrak{e}(\lambda, U, \xi)$ of λ by U associated with non-trivial (λ, U)-cocycles ξ.

1 Introduction and preliminaries

This paper is a continuation of papers [6] and [7], where we studied cohomology and extensions of representations of nilpotent groups, and briefly considered some triviality conditions for cohomologies of groups with normal Engel subgroups.

In this paper we present further results on cohomologies and representations of this important class of groups which includes, in particular, all nilpotent and many solvable groups.

Throughout the paper G is a *connected locally compact group* with a *connected closed normal subgroup* N. Let $G^{[1]}$ be the minimal closed subgroup of G containing all commutators $ghg^{-1}h^{-1}$, $g \in G$, $h \in N$. Each $h \in G$ defines a map ad_h from G into $G^{[1]}$: $\mathrm{ad}_h(g) = ghg^{-1}h^{-1}$ for $g \in G$. Recall that h is an *Engel element* if,

$$\text{for each } g \in G, \text{ there is } n_g \in \mathbb{N} \text{ such that } \mathrm{ad}_h^{n_g}(g) = e. \tag{1.1}$$

A group G is an *Engel group*, if it consists of Engel elements. Nilpotent groups are Engel groups, while solvable groups are not always Engel groups.

For Hilbert spaces L and $\mathfrak{H}$, $B(\mathfrak{H},L)$ is the space of all bounded operators from $\mathfrak{H}$ to L ($B(\mathfrak{H}) = B(\mathfrak{H}, \mathfrak{H})$ and $B(L) = B(L, L)$). All representations on Hilbert spaces we consider in this paper are weakly continuous. Let λ, U be representations of G on L and $\mathfrak{H}$, respectively. A weakly continuous map ξ: $G \to B(\mathfrak{H}, L)$ is

$$\text{a } (\lambda, U)\text{-}\textit{cocycle} \text{ if } \xi(gh) = \lambda(g)\xi(h) + \xi(g)U(h) \text{ for } g, h \in G; \tag{1.2}$$

$$\text{a } (\lambda, U)\text{-}\textit{coboundary} \text{ if } \xi(g) = \lambda(g)T - TU(g) \text{ for all } g \in G \tag{1.3}$$

and some $T \in B(\mathfrak{H}, L)$. Let $\mathcal{Z}(G, \lambda, U)$ be the set of all (λ, U)-cocycles and $\mathcal{B}(G, \lambda, U)$ the set of all (λ, U)-coboundaries (instead of 1-cycles, 1-boundaries and 1-cohomology, we write cycles, boundaries and cohomology). Then $\mathcal{H}^1(G, \lambda, U) = \mathcal{Z}(G, \lambda, U)/\mathcal{B}(G, \lambda, U)$ is the *cohomology group of G related to the representations λ, U*. If each (λ, U)-cocycle is a (λ, U)-coboundary then $\mathcal{H}^1(G, \lambda, U) = 0$. If $\dim L = 1$ and the representation $\iota(g) \equiv \mathbf{1}_L$ is the trivial representation on L, then $\mathcal{H}^1(G, \iota, U)$ is the cohomology group of U (see [1], [2], [3], [4]).

For any map μ from G to $B(L)$, $B(\mathfrak{H})$, or $B(\mathfrak{H}, L)$, μ^N denotes its restriction to N.

In Section 2 we develop a spectral criterion of triviality of the group $\mathcal{H}^1(G, \lambda, U)$ for pairs (G, N) (it was established in [6] for the case of nilpotent G and considered briefly in [7]). In Theorem 2.8

we show that if the representations λ and U are *spectrally, or sectionally spectrally disjoint* (see Definition 2.6) at some Engel elements of N then $\mathcal{H}^1(G,\lambda,U)=0$.

The importance of the notion of the sectional spectral disjointness is demonstrated in Theorem 3.9. Suppose that π is a representation of G on a Hilbert space X and its restriction π^N to an Engel subgroup N has an invariant subspace L:

$$\pi = \begin{pmatrix} \lambda & \xi \\ \rho & U \end{pmatrix} \text{ and } \pi^N = \begin{pmatrix} \lambda^N & \xi^N \\ 0 & U^N \end{pmatrix}$$

with respect to the decomposition $X = L \oplus L^\perp$. It is shown that if $\dim L < \infty$ and each character in $\mathrm{sign}(\lambda^N)$ (see Definition 3.4) is sectionally spectrally disjoint with the representation U^N, then L is also π-invariant, i.e., $\rho = 0$.

One of the motivations for the investigation of (λ,U)-cocycles ξ is the study of the structure of the *extensions* of representations λ by U performed by ξ: the representations

$$\mathfrak{e}(g) = \mathfrak{e}(\lambda,U,\xi)(g) = \begin{pmatrix} \lambda(g) & \xi(g) \\ 0 & U(g) \end{pmatrix} \text{ for } g \in G, \text{ on } X = L \oplus \mathfrak{H}. \tag{1.4}$$

It is well known that L has an $\mathfrak{e}$-invariant complement H (L *splits* $\mathfrak{e}$) i.e., $X = L \dotplus H$, if and only if ξ is a (λ,U)-coboundary for some $T \in B(\mathfrak{H},L)$ in which case $H = \{x - Tx\colon x \in \mathfrak{H}\}$.

Section 4 deals with the structure and the decomposition of $\mathfrak{e}(\lambda,U,\xi)$ when the (λ,U)-cocycle ξ is not a boundary. In particular, in Theorem 4.2 we consider some conditions on ξ that guarantee the *approximate splitting* (see Definition 4.1) of $\mathfrak{e}(\lambda,U,\xi)$ by L. In Corollary 4.4 we obtain a complete decomposition of finite-dimensional extensions $\mathfrak{e}(\lambda,U,\xi)$ into non-decomposable components.

2 Spectrally disjoint representations and group cohomology

In this section we develop a spectral criterion of triviality of $\mathcal{H}^1(G,\lambda,U)$ for a group G with a normal subgroup N. In Theorem 2.8 we show that if λ and U are sectionally spectrally disjoint at some Engel elements of N then $\mathcal{H}^1(G,\lambda,U)=0$.

We say that operators A and B are *spectrally disjoint* (see [6]) if

$$\mathrm{Sp}(A) \cap \mathrm{Sp}(B) = \varnothing.$$

We say that maps $\mu\colon G \to B(L)$ and $U\colon G \to B(\mathfrak{H})$ are *spectrally disjoint* at $h \in G$, if the operators $U(h)$ and $\mu(h)$ are spectrally disjoint.

The following well known result of Rosenblum (see, for example, Corollary 0.13 [9]) will be often used below.

Lemma 2.1 *Let operators $A \in B(L)$ and $B \in B(\mathfrak{H})$ be spectrally disjoint. If $AY = YB$ for some $Y \in B(\mathfrak{H},L)$, or $YA = BY$ for some $Y \in B(L,\mathfrak{H})$ then $Y = 0$.*

Let $U\colon G \to B(\mathfrak{H})$ be a representation of G on $\mathfrak{H}$. As N is a normal subgroup,

$$h_g = ghg^{-1} \in N, \text{ so that } gh = h_g g, \tag{2.1}$$

and, clearly, $\mathrm{Sp}(U(h_g)) = \mathrm{Sp}(U(h))$. From this and from Lemma 2.1 we get

Corollary 2.2 *Let a map* $\mu\colon G \to B(L)$ *and a representation* $U\colon G \to B(\mathfrak{H})$ *be spectrally disjoint at* $h \in G$. *Then, for each* $g \in G$,

$$\mu(h)Y = YU(h_g) \text{ for } Y \in B(\mathfrak{H}, L), \text{ implies } Y = 0;$$
$$Z\mu(h) = U(h_g)Z \text{ for } Z \in B(L, \mathfrak{H}), \text{ implies } Z = 0.$$

Let λ and U be representations of G on L and $\mathfrak{H}$. Sometimes we consider the case when λ or U has the upper triangular form. For example, let $\mathfrak{H} = \oplus_{i=1}^{k} \mathfrak{H}_i$, $k \le \infty$, and $P_{\mathfrak{H}_i}$ be the projections on $\mathfrak{H}_i$. Set $U_{ij}(g) = P_{\mathfrak{H}_i} U(g)|_{\mathfrak{H}_j}$ and $U_i(g) = U_{ii}(g)$. If $U_{ij} = 0$ for all $i > j$, i.e.,

$$U = \begin{pmatrix} U_1 & U_{12} & U_{13} & \cdots \\ 0 & U_2 & U_{23} & \cdots \\ 0 & 0 & U_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{ then we write } U = \backslash U_i]_{i=1}^{k}.$$

Each U_i is a representation of G on $\mathfrak{H}_i$. Every map $\xi\colon G \to B(\mathfrak{H}, L)$ has the form

$$\xi = \{\xi_i\}_{i=1}^{k}, \text{ where } \xi_i\colon G \to B(\mathfrak{H}_i, L),\ \xi_i(g) = \xi(g)|_{\mathfrak{H}_i}. \tag{2.2}$$

Lemma 2.3 *Let* $U = \backslash U_i]_{i=1}^{k}$ *and let* λ *be spectrally disjoint with each* U_i *at some* $g^i \in G$. *If* $\lambda(g)R = RU(g)$ *for some* $R \in B(\mathfrak{H}, L)$ *and all* $g \in G$, *then* $R = 0$.

Proof. We have $\mathfrak{H} = \oplus_{i=1}^{k}\mathfrak{H}_i$. As in (2.2), $R = \{R_i\}_{i=1}^{k}$, where $R_i \in B(\mathfrak{H}_i, L)$. Since $\lambda(g)R = RU(g)$ for $g \in G$, we have $\lambda(g^1)R_1 = R_1U_1(g^1)$. As $\lambda(g^1)$ is spectrally disjoint with $U_1(g^1)$, it follows from Lemma 2.1 that $R_1 = 0$. Hence $\lambda(g^2)R_2 = R_2U_2(g^2)$. As before, $R_2 = 0$. Continuing this process, we get $R = \{R_i\}_{i=1}^{k} = 0$. ■

Recall that λ^N, U^N and ξ^N denote the restrictions of λ,U,ξ to the normal subgroup N.

Proposition 2.4 *Let* $U = \backslash U_i]_{i=1}^{k}$, $k \le \infty$, *let* λ *be spectrally disjoint with each* U_i *at some* $h^i \in N$ *and let* ξ *be a* (λ, U)*-cocycle.*

If ξ^N *is a* (λ^N, U^N)*-coboundary* ($\xi(h) = \lambda(h)T - TU(h)$ *for all* $h \in N$ *and some* $T \in B(\mathfrak{H}, L)$), *then* ξ *is a* (λ, U)*-coboundary with the same unique representing operator* T:

$$\xi(g) = \lambda(g)T - TU(g) \text{ for } g \in G.$$

Proof. Set $\eta(g) = \xi(g) - (\lambda(g)T - TU(g))$ for $g \in G$. Then η is also a (λ, U)-cocycle of G and $\eta(h) = 0$ for $h \in N$. Hence, as $h_{g^{-1}} \in N$ for all $h \in N$, $g \in G$, we have from (1.2)

$$\eta(gh_{g^{-1}}) = \lambda(g)\eta(h_{g^{-1}}) + \eta(g)U(h_{g^{-1}}) = \eta(g)U(h_{g^{-1}}),$$
$$\eta(gh_{g^{-1}}) \overset{(2.1)}{=} \eta(hg) = \lambda(h)\eta(g) + \eta(h)U(g) = \lambda(h)\eta(g).$$

Hence

$$\eta(g)U(h_{g^{-1}}) = \lambda(h)\eta(g) \text{ for } g \in G \text{ and } h \in N. \tag{2.3}$$

As in (2.2), $\eta = \{\eta_i\}_{i=1}^{k}$, where $\eta_i\colon G \to B(\mathfrak{H}_i, L)$, from (2.3) we have

$$\eta_1(g)U_1(h^1_{g^{-1}}) = \lambda(h^1)\eta_1(g) \text{ for } g \in G.$$

Since λ and U_1 are spectrally disjoint at $h^1 \in N$, we get from Corollary 2.2 that $\eta_1 = 0$.

Hence we have from (2.3) that $\eta_2(g)U_2(h_{g^{-1}}) = \lambda(h)\eta_2(g)$ for all $g \in G$ and $h \in N$. Since λ and U_2 are spectrally disjoint at $h^2 \in N$, it follows as above that $\eta_2 = 0$. Continuing this process, we get $\eta = 0$. So $\xi(g) = \lambda(g)T - TU(g)$ for all $g \in G$.

Let also $\xi(g) = \lambda(g)S - SU(g)$ for all $g \in G$ and some $S \in B(\mathfrak{H}, L)$. Set $R = T - S$. Then $\lambda(g)R = RU(g)$ for $g \in G$. It follows from Lemma 2.3 that $R = 0$. So $T = S$. ■

Proposition 2.4 yields the following result.

Corollary 2.5 *Let $U = \backslash U_i]_{i=1}^k$, $k \leq \infty$, and let λ be spectrally disjoint with each U_i at some $h^i \in N$. If $\mathcal{H}^1(N, \lambda^N, U^N) = 0$ then $\mathcal{H}^1(G, \lambda, U) = 0$.*

We call a continuous map $\chi\colon G \to \mathbb{C}$ *a character* of G if $\chi(gh) = \chi(g)\chi(h)$ for $g, h \in G$. It is unitary, if

$$|\chi(g)| = 1 \text{ for } g \in G, \text{ i.e., } \chi(g^{-1}) = \overline{\chi(g)}. \tag{2.4}$$

Following [6], we call the spectral condition used in the statement of Corollary 2.5 by *sectional spectral disjointness*. Let us define it precisely adding a restriction on the number k of summands in the direct sum of subspaces.

Definition 2.6 *We say that a representation λ of G (respectively, a character χ on G) is* ***sectionally*** *spectrally disjoint with a representation U of G at some $\{g^i\}_{i=1}^k$ in G, if $U = \backslash U_i]_{i=1}^k$, $k < \infty$, and λ (respectively, χ) is spectrally disjoint with each diagonal U_i at g^i, i.e.,*

$$\mathrm{Sp}(\lambda(g^i)) \cap \ \mathrm{Sp}(U_i(g^i)) = \varnothing \quad (\textit{respectively, } \chi(g^i) \notin \ \mathrm{Sp}(U_i(g^i))).$$

Recall (see (1.1)) that $h \in G$ is an Engel element if, for each $g \in G$, there is $n_g \in \mathbb{N}$ with $\mathrm{ad}_h^{n_g}(g) = e$. The following result was proved in Corollary 2.9 [6].

Corollary 2.7 *If λ is spectrally disjoint with U at an Engel element of G, then $\mathcal{H}^1(G, \lambda, U) = 0$.*

Using Corollaries 2.5 and 2.7, we obtain the main result of this section.

Theorem 2.8 *Let λ be sectionally spectrally disjoint with U at some elements $\{h^i\}_{i=1}^k$ in N, i.e., $U = \backslash U_i]_{i=1}^k$, $k < \infty$, and λ is spectrally disjoint with U_i at $h^i \in N$.*

(i) *If $\mathcal{H}^1(N, \lambda^N, U_i^N) = 0$ for all i, then $\mathcal{H}^1(G, \lambda, U) = 0$.*

(ii) *If all $\{h^i\}_{i=1}^k$ are Engel elements of N, then $\mathcal{H}^1(G, \lambda, U) = 0$.*

(iii) *If N is an Engel group then $\mathcal{H}^1(G, \lambda, U) = 0$.*

Proof. (i) As λ^N is spectrally disjoint with each diagonal U_i^N at $h^i \in N$, the condition $\mathcal{H}^1(N, \lambda^N, U_i^N) = 0$ implies $\mathcal{H}^1(G, \lambda, U_i) = 0$ for each i by Corollary 2.5 (set $k = 1$ there). It was shown in Lemma 2.3 [6] that if $k < \infty$ and $\mathcal{H}^1(G, \lambda, U_i) = 0$ for all i, then $\mathcal{H}^1(G, \lambda, U) = 0$.

(ii) As λ^N is spectrally disjoint with each diagonal U_i^N at an Engel element $h^i \in N$, it follows from Corollary 2.7 that $\mathcal{H}^1(N, \lambda^N, U_i^N) = 0$ for all i. Hence $\mathcal{H}^1(G, \lambda, U) = 0$ by (i).

(iii) follows from (ii). ■

3 Elementary representations and spectral continuity

3.1 Spectral continuity and invariant subspaces

Let G be a connected group with a normal connected subgroup N. Let π be a representation of G on a Hilbert space X and let L be a subspace invariant for π^N. Set $\mathfrak{H} = L^{\perp}$. Then

$$X = L \oplus \mathfrak{H},\ \pi(g) = \begin{pmatrix} \lambda(g) & \xi(g) \\ \rho(g) & U(g) \end{pmatrix} \text{ and } \pi(h) = \begin{pmatrix} \lambda(h) & \xi(h) \\ 0 & U(h) \end{pmatrix} \tag{3.1}$$

for $g \in G$, $h \in N$. So the restrictions λ^N and U^N of the maps λ and U to N are representations of N. We will introduce now the notion of spectral continuity of maps and show that the spectral continuity of λ^N and its spectral disjointness with U^N are sufficient for the subspace L to be also invariant for π.

Definition 3.1 *A map μ: $N \to B(L)$ is* ***spectrally continuous*** *at $h \in N$ if, for each neighbourhood V of $\mathrm{Sp}(\mu(h))$ in $\mathbb{C}$, there is a neighbourhood W_V of h in N such that $\mathrm{Sp}(\mu(h')) \subseteq V$ for $h' \in W_V$.*

In other words, μ is spectrally continuous if the multi-valued map $h \mapsto \mathrm{Sp}(\mu(h))$ from N into $\mathbb{C}$ is upper semicontinuous on N (see [8]). If μ is a norm-continuous representation (in particular, if $\dim L < \infty$) then μ is spectral continuous (see [5], p. 53, Problem 86).

We need now the following result.

Lemma 3.2 *Let maps λ: $G \to B(L)$ and U: $G \to B(\mathfrak{H})$ be spectrally disjoint at some $h \in N$. If the map λ^N is spectrally continuous at h, then $\lambda(h_g)$ and $U(h)$ are spectrally disjoint for all g in some neighborhood of e in G.*

Proof. As $\mathrm{Sp}(\lambda(h)) \cap \mathrm{Sp}(U(h)) = \varnothing$ and λ^N is spectrally continuous at h, there is a neighbourhood V of h in N such that $\mathrm{Sp}(\lambda(h')) \cap \mathrm{Sp}(U(h)) = \varnothing$ for all $h' \in V$. As N is a normal subgroup, there is a neighbourhood W of e in G such that $h_g \in V$ for all $g \in W$. Thus $\mathrm{Sp}(\lambda(h_g)) \cap \mathrm{Sp}(U(h)) = \varnothing$ for $g \in W$. ∎

If λ is a representation, Lemma 3.2 holds even without spectral continuity of λ^N at h.

As π is a representation, $\pi(g)\pi(h_{g^{-1}}) = \pi(gh_{g^{-1}}) = \pi(hg) = \pi(h)\pi(g)$, whence

$$\rho(g)\lambda(h_{g^{-1}}) = U(h)\rho(g) \text{ for all } h \in N,\ g \in G. \tag{3.2}$$

Theorem 3.3 *Let λ^N in (3.1) be sectionally spectrally disjoint with $U^N = \backslash U_i]_{i=1}^k$, $k < \infty$, at $\{h^i\}_{i=1}^k$. If λ^N is spectrally continuous at all h^i then $\rho = 0$, so that L is π-invariant.*

Proof. Since λ^N is sectionally spectrally disjoint with U^N at $\{h^i\}_{i=1}^k$, the operators $\lambda^N(h^i)$ are spectrally disjoint with $U_i(h^i)$, for $i = 1, ..., k$. As λ^N is spectrally continuous at h^i, the operators $\lambda(h^i_{g^{-1}})$ and $U_i(h^i)$ are spectrally disjoint for all g^{-1} in some neighborhood W_i of e in G by Lemma 3.2. Set $W = \cap_{i=1}^k W_i$ and $V = W \cap W^{-1}$. Then $V^{-1} = V$ and $\lambda(h^i_{g^{-1}})$ and $U_i(h^i)$ are spectrally disjoint for all $g \in V$ and $i = 1, ..., k$. We will show firstly that $\rho(g) = 0$ for all $g \in V$.

By Definition 2.6, $\mathfrak{H} = \oplus_{i=1}^k \mathfrak{H}_i$, where all subspaces $\oplus_{i=1}^j \mathfrak{H}_i$, $1 \le j \le k$, are invariant for U^N. With respect to this decomposition, one can write the components of operators $U(g)$, $\xi(g)$, $\rho(g)$ in a matrix form:

$$U(g) = (U_{ij}(g))_{i,j=1}^k, \xi(g) = (\xi_1(g), ..., \xi_k(g)), \rho(g) = (\rho_1(g), ..., \rho_k(g))^T,$$
$$U_{ij}(g) = 0 \text{ if } i > j, \text{ and we write } U_i \text{ for } U_{ii}. \tag{3.3}$$

Starting with $i = k$ and moving to $i = 1$, we will prove that

$$\rho_i(g) = 0 \text{ for all } g \in V \text{ and } 1 \le i \le k. \tag{3.4}$$

For $i = k$, we have from (3.2) and (3.3) that

$$\rho_k(g)\lambda(h_{g^{-1}}^k) = U_k(h^k)\rho_k(g) \text{ for all } g \in G.$$

It follows from the argument at the beginning of the proof that $\lambda(h_{g^{-1}}^k)$ and $U_k(h^k)$ are spectrally disjoint for $g \in V$. So, by Lemma 2.1, $\rho_k(g) = 0$ for $g \in V$.

Let $j < k$ and assume that (3.4) is already proved for all $i > j$. From this and from (3.3) it follows that $U_{ij}(h^j) = 0$ and $\rho_i(g) = 0$ for $i > j$ and $g \in V$. Applying now (3.2) to $g \in V$, $h = h^j$ gives

$$\rho_j(g)\lambda(h_{g^{-1}}^j) = U_j(h^j)\rho_j(g) \text{ for } g \in V.$$

By the argument at the beginning of the proof, $\lambda(h_{g^{-1}}^j)$ and $U_j(h^j)$ are spectrally disjoint for all $g \in V$. Hence $\rho_j(g) = 0$ for all $g \in V$ by Lemma 2.1. Thus (3.4) holds, so that $\rho(g) = 0$ for $g \in V$.

Let now

$$K = \{g \in G : \rho(g) = 0 \text{ and } \rho(g^{-1}) = 0\}.$$

Since $V^{-1} = V$, K contains V by (3.4). As π is a representation,

$$\rho(g_1g_2) = \rho(g_1)\lambda(g_2) + U(g_1)\rho(g_2) \text{ for } g_1, g_2 \in K.$$

So it follows that $g_1g_2 \in K$. As $K^{-1} = K$, K is a subgroup of G. It is open because $V \subseteq K$ is a neighborhood of e. Since K is obviously closed and G is connected, $K = G$. ■

3.2 Elementary representations and invariant subspaces

Denote by N^* the set of all continuous characters on N. The following notions were introduced in [6].

Definition 3.4 *A weakly continuous representation μ of N on L is called*

(i) *a χ-**representation** for some $\chi \in N^*$, if*

$$\mathrm{Sp}(\mu(h)) = \{\chi(h)\} \textit{ for all } h \in N.$$

(ii) ***elementary** if, for some $k < \infty$,*

$$L = \oplus_{i=1}^k L_i \textit{ and } \mu = \backslash\mu_i]_{i=1}^k \textit{ with respect to this decomposition,} \tag{3.5}$$

where each μ_i is a χ_i-representation of N on L_i for some $\chi_i \in N^*$ (some χ_i may repeat).

Denote by $\mathrm{sign}(\mu)$ *the list of all distinct characters in the set* $\{\chi_1, ..., \chi_k\}$.

It was shown in [6] that the list of the characters $\mathrm{sign}(\mu)$ does not depend on the choice of triangularization of L in (3.5).

Lemma 3.5 *Elementary representations of N are spectrally continuous at all $h \in N$.*

Proof. If μ is an elementary representation then

$$\mathrm{Sp}(\mu(h)) = \cup_{i=1}^{k} \mathrm{Sp}(\mu_i(h)) = \cup_{i=1}^{k} \chi_i(h),$$

where $\chi_i \in N^*$. As all characters χ_i are continuous, μ is spectrally continuous. ■

Clearly, if $\chi \neq \omega$ in N^* then χ is spectrally disjoint with any ω-representation. Note also that a χ-representation of N is sectionally spectrally disjoint with U^N if and only if χ is sectionally spectrally disjoint with U^N.

We preserve the notations and the assumption introduced in (3.1) that L is an invariant subspace for the restriction π^N of a representation π to N. We now refine Theorem 3.3 in the case when $\lambda^N = \pi^N|_L$ is an elementary representation.

Theorem 3.6 *Let λ^N in* (3.1) *be elementary with respect to a decomposition*

$$L = \oplus_{i=1}^{k} L_i, \tag{3.6}$$

i.e., $\lambda^N = \backslash\lambda_i^N]_{i=1}^k$ and each λ_i^N is a χ_i-representation of N on L_i for some $\chi_i \in N^*$.

If all $\{\chi_i\}_{i=1}^k$ are distinct and sectionally spectrally disjoint with U^N, then L is π-invariant and the representation $\lambda = \pi|_L$ has an upper triangular form with respect to the decomposition (3.6), *that is, $\lambda = \backslash\lambda_i]_{i=1}^k$.*

Proof. Set $X_1 = L_2 \oplus ... \oplus L_k \oplus \mathfrak{H}$. With respect to the decomposition $X = L_1 \dotplus X_1$,

$$\pi = \begin{pmatrix} \lambda_1 & \xi_1 \\ \rho_1 & \pi_1 \end{pmatrix}, \ \pi^N = \begin{pmatrix} \lambda_1^N & \xi_1^N \\ 0 & \pi_1^N \end{pmatrix}, \ \text{where } \pi_1^N = \begin{pmatrix} \lambda_2^N & * & * & * \\ 0 & \ddots & * & * \\ 0 & 0 & \lambda_k^N & * \\ 0 & 0 & 0 & U^N \end{pmatrix}.$$

As all $\{\chi_i\}_{i=1}^k$ are distinct, λ_1^N is spectrally disjoint with all λ_i^N, $2 \le i \le k$. As χ_1 is sectionally spectrally disjoint with U^N, we have that λ_1^N is sectionally spectrally disjoint with π_1^N. As λ_1^N is spectrally continuous on N by Lemma 3.5, we get $\rho_1 = 0$ by Theorem 3.3, so that L_1 is π-invariant.

Consider now the representation π_1 on X_1. Repeating the above process, we get that L_2 is invariant for π_1. So $L_1 \oplus L_2$ is π-invariant. Continuing this process, we conclude the proof. ■

Corollary 3.7 *Let π be a representation of G on X. Let the representation π^N be elementary with respect to a decomposition*

$$X = \oplus_{i=1}^{k} X_i,$$

i.e. $\pi^N = \backslash\pi_i^N]_{i=1}^k$ and each π_i^N is an χ_i-representation on X_i for some $\chi_i \in N^*$. If characters $\{\chi_i\}_{i=1}^k$ are distinct, then π has the upper triangular form with respect to this decomposition of X.

In the rest of this subsection we assume that N is a *connected Engel group*. The result below was obtained in [6] (Corollary 2.18 and the discussion after Definition 2.15).

Proposition 3.8 *Let μ be a representation of a connected Engel group N on L.*

(i) *If μ is elementary then L uniquely decomposes into a direct sum of μ-invariant subspaces:*

$$L = \sum_{\chi \in \mathrm{sign}(\mu)} L_\chi, \text{ each } L_\chi \text{ is } \mu\text{-invariant and } \mu|_{L_\chi} \text{ is a } \chi\text{-representation.} \tag{3.7}$$

(ii) *If $\dim L < \infty$ then μ is elementary and* (3.7) *holds.*

Let π be a representation of G on X and L be a π^N-invariant subspace. Suppose that $\lambda^N = \pi^N|_L$ is an elementary representation (in particular, $\dim L < \infty$ by Proposition 3.8). Then, by Proposition 3.8, decomposition (3.7) of L holds with $\mu = \lambda^N$. We obtain now the following refinement of Theorems 3.3 and 3.6.

Theorem 3.9 *Suppose that each $\chi \in \mathrm{sign}(\lambda^N)$ is sectionally spectrally disjoint with U^N in* (3.1). *Then*

(i) *L and all subspaces L_χ, $\chi \in \mathrm{sign}(\lambda^N)$, in* (3.7) *are π-invariant*;

(ii) *there is a π-invariant subspace H such that $X = L \dot{+} H$, where $\pi|_H$ is similar to U*;

(iii) *for each $\chi \in \mathrm{sign}(\lambda^N)$, $\chi(h) = \chi(h_g) = \chi(ghg^{-1})$ for $g \in G$, $h \in N$.*

Proof. (i) All characters in $\mathrm{sign}(\lambda^N)$ are distinct (see Definition 3.4). Taking each $\chi \in \mathrm{sign}(\lambda^N)$ as χ_1 in the proof of Theorem 3.6, we obtain that all subspaces L_χ in (3.7) are π-invariant, which proves (i).

(ii) As all $\chi \in \mathrm{sign}(\lambda^N)$ are sectionally spectrally disjoint with U^N, all $\lambda^N|_{L_\chi}$ are sectionally spectrally disjoint with U^N. Since N is an Engel group, $\mathcal{H}^1(G, \lambda|_{L_\chi}, U) = 0$ by Theorem 2.8(iii). As λ is the direct sum of the representations $\lambda|_{L_\chi}$, $\chi \in \mathrm{sign}(\lambda^N)$, we have $\mathcal{H}^1(G, \lambda, U) = 0$.

By (i), $\pi = \begin{pmatrix} \lambda & \xi \\ 0 & U \end{pmatrix}$, where ξ is a (λ, U)-cocycle. By the above, ξ is a (λ, U)-coboundary. Hence $\xi(g) = \lambda(g)T - TU(g)$ for some $T \in B(\mathfrak{H}, L)$. Then $H = \{-Tx + x\colon x \in \mathfrak{H}\}$ is π-invariant, $X = L \dot{+} H$ and $\pi|_H$ is similar to U.

(iii) For $\chi \in \mathrm{sign}(\lambda^N)$, set $\lambda_\chi = \lambda|_{L_\chi}$. Then $\chi(h) = \mathrm{Sp}(\lambda_\chi(h))$, $h \in N$. As $h_g = ghg^{-1} \in N$ and L_χ is π-invariant,

$$\chi(ghg^{-1}) = \mathrm{Sp}(\lambda_\chi(ghg^{-1})) = \mathrm{Sp}(\lambda_\chi(g)\lambda_\chi(h)\lambda_\chi^{-1}(g)) = \mathrm{Sp}(\lambda_\chi(h)) = \chi(h)$$

which completes the proof. ■

Proposition 3.8 and Theorem 3.9 yield

Corollary 3.10 *Let N be an Engel group and π be a representation of G on X. Suppose that the representation $\pi^N = \backslash\pi_i^N]_{i=1}^k$ is elementary with respect to some decomposition of X (this is the case, for example, if $\dim X < \infty$).*

Then X uniquely decomposes into a direct sum of π-invariant subspaces:

$$X = \sum_{\chi \in \mathrm{sign}(\pi^N)} X_\chi, \text{ each } X_\chi \text{ is } \pi\text{-invariant and } \pi^N|_{L_\chi} \text{ is a } \chi\text{-representation.}$$

We illustrate Corollary 3.10 with the following example.

Example 3.11 Consider the connected groups

$$G = \left\{ g(a,b,x) = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} : \ a,b \in \mathbb{R}_+, \ x \in \mathbb{R} \right\} \text{ and } N = \{g(a,a,x) \in G\}.$$

Then G is solvable and N is a normal commutative subgroup of G. As $\omega|_{G^{[1]}} = 1$ for each character ω on G, we have $\omega(g) = \omega_{z,u}(g(a,b,x)) = a^z b^u$ for some $z, u \in \mathbb{C}$. Set $\delta = z + u$. Then

$$\chi_\delta(g(a,a,x)) = \omega^N_{z,u}(g(a,a,x)) = a^\delta.$$

Let π be a representation of G on X, $\dim X = n$. As G is solvable, π has upper triangular form with characters $\{\omega_{z_i,u_i}\}_{i=1}^n$ on the diagonal by the Lie theorem. Set $\delta_i = z_i + u_i$ and $\chi_i = \chi_{\delta_i}$. Then $\chi_i = \omega^N_{z_i,u_i}$ are characters on N and $\operatorname{sign}(\pi^N) = \{\chi_i\}_{i=1}^m$, where $m \le n$ and all χ_i are distinct. By Corollary 3.10,

$$X = \sum_{i=1}^m X_{\chi_i}, \text{ each } X_{\chi_i} \text{ is } \pi\text{-invariant and all } \pi^N|_{X_{\chi_i}} \text{ are } \chi_i\text{-representations of } N.$$

(i) Let $X = \mathbb{C}^2$. The representation $\pi(g) = g$ of G on X has two characters $\omega_{1,0}$, $\omega_{0,1}$, so that $\delta_1 = \delta_2 = 1$ and $\operatorname{sign}(\pi^N) = \{\chi\}$, where $\chi(g(a,a,x)) = \chi_{\delta_1}(g(a,a,x)) = a$. So X does not decompose into a sum of invariant subspaces.

(ii) The representation π: $g(a,b,x) \to \begin{pmatrix} a^2 & m(a^2 - b^3) \\ 0 & b^3 \end{pmatrix}$ for some $m \in \mathbb{C}$, on $X = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ is elementary with characters $\omega_{2,0}$, $\omega_{0,3}$. As $\delta_1 = 2$, $\delta_2 = 3$, we have $\operatorname{sign}(\pi^N) = \{\chi_1, \chi_2\}$, where

$$\chi_1(g(a,a,x)) = \chi_{\delta_1}(g(a,a,x)) = a^2 \text{ and } \chi_2(g(a,a,x)) = \chi_{\delta_2}(g(a,a,x)) = a^3.$$

So X decomposes into the direct sum of π-invariant subspaces $X_{\chi_1} = \mathbb{C}e_1$ and $X_{\chi_2} = \mathbb{C}(-me_1 \oplus e_2)$.

For $g = g(a,a,x) \in N$, we have

$$\pi^N(g)e_1 = a^2 e_1 = \chi_1(g)e_1 \text{ and } \pi^N(g)(-me_1 \oplus e_2) = a^3(-me_1 \oplus e_2) = \chi_2(g)(-me_1 \oplus e_2).$$

So $\pi^N|_{X_{\chi_i}}$, $i = 1, 2$, are χ_i-representations.

4 Decomposition of extensions of representations

For representations λ on L and U on $\mathfrak{H}$, and for a (λ, U)-cocycle ξ, we consider the representation

$$\mathfrak{e} = \mathfrak{e}(\lambda, U, \xi) = \begin{pmatrix} \lambda & \xi \\ 0 & U \end{pmatrix} \text{ on } X = L \oplus \mathfrak{H} \tag{4.1}$$

called the extension of λ by U performed by ξ (see (1.4)).

We say that L *splits* $\mathfrak{e}$, if L has an $\mathfrak{e}$-invariant complement H, i.e., $X = L \dot{+} H$ and $\mathfrak{e}(g)H = H$ for all g. It is well known that L splits $\mathfrak{e}$ if and only if ξ is a (λ, U)-coboundary: $\xi(g) = \lambda(g)T - TU(g)$ for some $T \in B(H, L)$; in this case $H = \{x - Tx \colon x \in H\}$.

In this section we study various types of decomposition of $\mathfrak{e}(\lambda, U, \xi)$.

Definition 4.1 *Let $\mathfrak{e} = \mathfrak{e}(\lambda, U, \xi)$. We say that L* ***approximately splits*** $\mathfrak{e}$ *if there are $\mathfrak{e}$-invariant subspaces $(Y_i, Z_i)_{i=1}^{\infty}$ in X such that*

$$X = Y_i \dotplus Z_i,\ Y_{i+1} \subsetneqq Y_i,\ Z_i \subsetneqq Z_{i+1} \text{ for all } i, \text{ and } L = \cap_i Y_i. \tag{4.2}$$

Clearly, if $Y_i = L$ for some i, then L splits $\mathfrak{e}$.

Let $\mathfrak{H} = \oplus_{i=1}^{\infty} \mathfrak{H}_i$ and let $U = \oplus_{i=1}^{\infty} U_i$, where $U_i = U|_{\mathfrak{H}_i}$, with respect to this decomposition. Then (see (2.2)) each (λ, U)-cocycle ξ has the form $\xi = \{\xi_i\}_{i=1}^{\infty}$, where $\xi_i = \xi|_{\mathfrak{H}_i}\colon G \to B(\mathfrak{H}_i, L)$ are (λ, U_i)-cocycles.

Theorem 4.2 (i) *Let $\xi = \{\xi_i\}_{i=1}^{\infty}$ be a (λ, U)-cocycle. If all ξ_i are (λ, U_i)-coboundaries, then L approximately splits $\mathfrak{e}(\lambda, U, \xi)$.*

(ii) *If $\mathcal{H}^1(G, \lambda, U_i) = 0$ for all $i \in \mathbb{N}$, then L approximately splits $\mathfrak{e}(\lambda, U, \xi)$ for each (λ, U)-cocycle ξ.*

(iii) *Let N be an Engel group and let, for each $i \in \mathbb{N}$, λ be sectionally spectrally disjoint with U_i at some elements of N. Then L approximately splits $\mathfrak{e}(\lambda, U, \xi)$ for each (λ, U)-cocycle ξ.*

Proof. (i) If all $\xi_i \in B(\mathfrak{H}_i, L)$ are (λ, U_i)-coboundaries, $\xi_i = \lambda T_i - T_i U_i$ for $i \in \mathbb{N}$ and some operators $T_i \in B(\mathfrak{H}_i, L)$. Hence the subspace $\widehat{\mathfrak{H}_i} = \{-T_i x + x\colon x \in \mathfrak{H}_i\}$ of $L \oplus \mathfrak{H}_i$, $i \in \mathbb{N}$, is $\mathfrak{e}$-invariant, as

$$\mathfrak{e}(g)(-T_i x \oplus x) = (-\lambda(g) T_i x + \xi_i(g) x) \oplus U_i(g) x = -T_i U_i(g) x \oplus U_i(g) x \in \widehat{\mathfrak{H}_i}.$$

Set $Y_i = L \oplus \left(\oplus_{k=i+1}^{\infty} \mathfrak{H}_i\right)$ and $Z_i = \sum_{k=1}^{i} \widehat{\mathfrak{H}_k}$. They are $\mathfrak{e}$-invariant and (4.2) holds. So L approximately splits $\mathfrak{e}$.

Part (ii) follows from (i).

(iii) As λ is sectionally spectrally disjoint with each U_i, $i \in \mathbb{N}$, at some elements in N and as N is an Engel group, we have from Theorem 2.8(iii) that $\mathcal{H}^1(G, \lambda, U_i) = 0$ for all $i \in \mathbb{N}$. Hence (iii) follows from (ii). ■

Let N be an Engel group, let $X = L \oplus \mathfrak{H}$ and $\mathfrak{e} = \mathfrak{e}(\lambda, U, \xi)$. Suppose that

$$\lambda^N \text{ is elementary and } \mathfrak{H} = \sum_{j=1}^{m} \mathfrak{H}_j, \text{ where } \mathfrak{H}_i \text{ are } U\text{-invariant subspaces,} \tag{4.3}$$

$m < \infty$. Then, by (3.7) and Corollary 3.10,

$$L = \sum_{\chi \in \operatorname{sign}(\lambda^N)} L_\chi, \text{ each } L_\chi \text{ is } \lambda\text{-invariant and } \lambda^N|_{L_\chi} \text{ is an } \chi\text{-representation.} \tag{4.4}$$

For $j \in [1, m]$, let

$$U_j = U|_{\mathfrak{H}_j},\ F_j = \{\chi \in \operatorname{sign}(\lambda^N)\colon \chi \text{ is } \textbf{not} \text{ sectionally spectrally disjoint with } U_j^N\}. \tag{4.5}$$

Set $\lambda_\chi = \lambda|_{L_\chi}$ and, for $F \subseteq \operatorname{sign}(\lambda^N)$, set

$$L_F = \sum_{\chi \in F} L_\chi,\ \lambda_F = \sum_{\chi \in F} \lambda_\chi \text{ and } F^c = \operatorname{sign}(\lambda^N) \setminus F.$$

Then all $\chi \in F_j^c$ are sectionally spectrally disjoint with U_j^N.

Theorem 4.3 (i) *For each* $j \in [1,m]$, *there is an operator* $T_j \in B(\mathfrak{H}_j, L_{F_j^c})$ *such that*

$$X_j = L_{F_j} \dotplus \widehat{\mathfrak{H}}_j \text{ is an } \mathfrak{e}\text{-invariant subspace and } \mathfrak{e}|_{X_j} = \begin{pmatrix} \lambda_{F_j} & \eta_j \\ 0 & \sigma_j \end{pmatrix},$$

where $\widehat{\mathfrak{H}}_j = \{-T_j x + x\colon\ x \in \mathfrak{H}_j\}$ *and* σ_j *is similar to* U_j: $\sigma_j \widehat{T}_j = \widehat{T}_j U_j$ *and* $\widehat{T}_j = -T_j + \mathbf{1}_{\mathfrak{H}_i}$ *is invertible.*

(ii) *Let* $F_j \cap F_k = \varnothing$ *if* $j \neq k$ *in* $[1,m]$. *Set* $\Phi = \cup_{j\in[1,m]} F_j$. *Then*

$$X = L_{\Phi^c} \dotplus X_1 \dotplus \ldots \dotplus X_m \text{ is the direct sum of } \mathfrak{e}\text{-invariant subspaces.}$$

Proof. (i) As all L_χ are λ-invariant, each (λ, U)-cocycle ξ has the form $\xi = (\xi_{\chi j})$, $\chi \in \operatorname{sign}(\lambda^N)$, $j \in [1,m]$, where $\xi_{\chi j}$ is a (λ_χ, U_j)-cocycle. If $\chi \in F_j^c$ then λ_χ^N is sectionally spectrally disjoint with U_j^N. As N is an Engel group, $\mathcal{H}^1(G, \lambda_\chi, U_j) = 0$ by Theorem 2.8. So $\xi_{\chi j}$ is a coboundary:

$$\xi_{\chi j} = \lambda_\chi S_{\chi j} - S_{\chi j} U_j \text{ for some operator } S_{\chi j} \in B(\mathfrak{H}_j, L_\chi). \tag{4.6}$$

Set $T_j = \sum_{\chi \in F_j^c} S_{\chi j}$ and $\widehat{\mathfrak{H}}_j = \{-T_j x + x\colon\ x \in \mathfrak{H}_j\}$. Then $T_j \in B(\mathfrak{H}_j, L_{F_j^c})$ and

$$\sum_{\chi \in F_j^c} \xi_{\chi j}|_{\mathfrak{H}_j} \overset{(4.6)}{=} \sum_{\chi \in F_j^c} (\lambda_\chi S_{\chi j} - S_{\chi j} U_j)|_{\mathfrak{H}_j} = (\lambda T_j - T_j U_j)|_{\mathfrak{H}_j} \in B(\mathfrak{H}_j, F_j^c). \tag{4.7}$$

The subspace $X_j = L_{F_j} \dotplus \widehat{\mathfrak{H}}_j$ is $\mathfrak{e}$-invariant, since L_{F_j} is $\mathfrak{e}$-invariant and, by (4.1), for $x \in \mathfrak{H}_j$,

$$\begin{aligned}
\mathfrak{e}(g)(-T_j x + x) &= (-\lambda(g) T_j x + \xi(g)x) + U_j(g)x \\
&= \left(-\lambda(g) T_j x + \sum_{\chi \in F_j} \xi_{\chi j}(g)x + \sum_{\chi \in F_j^c} \xi_{\chi j}(g)x\right) + U_j(g)x \\
&\overset{(4.7)}{=} \sum_{\chi \in F_j} \xi_{\chi j}(g)x + (-\lambda(g) T_j x + (\lambda(g) T_j - T_j U_j(g))x) + U_j(g)x \\
&= \sum_{\chi \in F_j} \xi_{\chi j}(g)x + (-T_j U_j(g)x + U_j(g)x) \in L_{F_j} \dotplus \widehat{\mathfrak{H}}_j = X_j.
\end{aligned}$$

Then $\widehat{T}_j := -T_j + \mathbf{1}_{\mathfrak{H}_i} \in B(\mathfrak{H}_j, \widehat{\mathfrak{H}}_j)$ is invertible and, by the above formula, $\mathfrak{e}|_{X_j} = \begin{pmatrix} \lambda_{F_j} & \eta_j \\ 0 & \sigma_j \end{pmatrix}$, where

$$\sigma_j(g) \widehat{T}_j x = \sigma_j(g)(-T_j x + x) = -T_j U_j(g)x + U_j(g)x = \widehat{T}_j U_j(g)x.$$

(ii) As $\mathfrak{H}_j \cap \mathfrak{H}_k = \{0\}$ for $j \neq k$, we have $\widehat{\mathfrak{H}}_j \cap \widehat{\mathfrak{H}}_k = \{0\}$. As also $F_j \cap F_k = \varnothing$ for $j \neq k$, we have $X_j \cap X_k = \{0\}$. Since $L = L_{\Phi^c} \dotplus \sum_{j=1}^m L_{F_j}$, the proof follows from (4.3) and (i). ■

Let us now consider the finite-dimensional case: $\dim X < \infty$.

As N is an Engel group, λ^N and U^N are elementary representations by Proposition 3.8. So, by Theorem 3.9, decomposition (4.4) of L holds and there is a finite set $\operatorname{sign}(U^N)$ of characters ω in N^* such that

$$\mathfrak{H} = \sum_{\omega \in \operatorname{sign}(U^N)} \mathfrak{H}_\omega, \text{ each } \mathfrak{H}_\omega \text{ is } U\text{-invariant and } U^N|_{\mathfrak{H}_\omega} \text{ is an } \omega\text{-representation.} \tag{4.8}$$

The representations λ_χ^N and U_ω^N are **not** spectrally disjoint only if $\chi = \omega$. Hence (see (4.5)), for each $\omega \in \text{sign}(U^N)$, either $F_\omega = \varnothing$, or $F_\omega = \{\chi\} \subseteq \text{sign}(\lambda^N)$ if $\omega = \chi$. Set

$$\Omega = \text{ sign}(\lambda^N) \cap \text{ sign}(U^N),\ \Omega_\lambda = \text{ sign}(\lambda^N) \diagdown \Omega \text{ and } \Omega_U = \text{ sign}(U^N) \diagdown \Omega.$$

The representation $\mathfrak{e}^N$ is elementary with $\text{sign}(\mathfrak{e}^N) = \text{sign}(\lambda^N) \cup \text{ sign}(U^N) = \Omega_\lambda \cup \Omega \cup \Omega_U$. So Theorem 4.3 gives the following result (cf. Corollary 3.10).

Corollary 4.4 *Let* $\mathfrak{e} = \mathfrak{e}(\lambda, U, \xi)$, $X = L \oplus \mathfrak{H}$ *and* $\dim X < \infty$. *Then there are operators* $T_\chi \in B(\mathfrak{H}_\chi, L)$ *such that*

$$X = \sum_{\chi \in \Omega_\lambda} L_\chi \dotplus \sum_{\chi \in \Omega} X_\chi \dotplus \sum_{\chi \in \Omega_U} \widehat{\mathfrak{H}}_\chi \text{ is the direct sum of } \mathfrak{e}\text{-invariant subspaces,}$$

where $\widehat{\mathfrak{H}}_\chi = \{-T_\chi x + x \colon x \in \mathfrak{H}_\chi\}$ *for* $\chi \in \text{sign}(U^N)$ *and* $X_\chi = L_\chi \dotplus \widehat{\mathfrak{H}}_\chi$, $\chi \in \Omega$.

For $\chi \in \Omega_U$, *the representation* $\mathfrak{e}|_{\widehat{\mathfrak{H}}_\chi}$ *is similar to* $U|_{\mathfrak{H}_\chi}$: $\mathfrak{e}\widehat{T}_\chi|_{\mathfrak{H}_\chi} = \widehat{T}_\chi U|_{\mathfrak{H}_\chi}$, *where*

$$\widehat{T}_\chi = -T_\chi + \mathbf{1}_{\mathfrak{H}_\chi} \in B(\mathfrak{H}_\chi, \widehat{\mathfrak{H}}_\chi) \text{ is invertible.}$$

For $\chi \in \Omega$, $X_\chi = L_\chi \dotplus \widehat{\mathfrak{H}}_\chi$ *and* $\mathfrak{e}|_{X_\chi} = \begin{pmatrix} \lambda|_{L_\chi} & \eta_\chi \\ 0 & \sigma_\chi \end{pmatrix}$, *where* σ_χ *is similar to* $U|_{\mathfrak{H}_\chi}$: $\mathfrak{e}\widehat{T}_\chi|_{\mathfrak{H}_\chi} = \widehat{T}_\chi U|_{\mathfrak{H}_\chi}$. *Moreover,* $\mathfrak{e}^N|_{X_\chi}$ *is a* χ*-representation.*

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References

[1] P. Delorme, 1-*cohomologie des representations unitaires des groupes de Lie semisimples et resolubles*, Bull. Soc. Math. France, 105 (1977), 281-336.

[2] A. Guichardet, *Sur la cohomologie des groupes topologiques*, II, Bull. Soc. Math., 96 (1972), 305-332.

[3] A. Guichardet, *Cohomology of topological groups and positive definite functions*, J. Multivariate Anal., 3 (1973), 249-261.

[4] A. Guichardet, *Cohomologie des groupes topologiques et des algebres de Lie*, Cedic/Fernand Nathan, Paris, 1980.

[5] P.R. Halmos, *A Hilbert space problem book*, D. van Nostrand Co. Inc., Princeton, Toronto, London, 1967.

[6] E. Kissin, V.S. Shulman, *Non-unitary representations of nilpotent groups*, I: *cohomologies, extensions and neutral cocycles*, J. Func. Anal., 269 (2015), 2564-2610.

[7] E. Kissin, V.S. Shulman, *Cohomology of groups with normal Engel subgroups*, Proceedings of 10th International Science and Technology Conference, Vologda, 2019, 4-8 (ISBN 978-5-87851-872-7) (INFOS-2019). DOI.10.131140/RG.2.2.25937.38243.

[8] K. Kuratowski, *Topology*, 1–2 , Acad. Press, London, 1966–1968.

[9] R. Radjavi, P. Rosenthal, *Invariant subspaces*, Springer-Verlag, Berlin, 1973.

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