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Geometric and Homological Methods in Group Theory: Constructing Small Group Resolutions

by

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Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another University for the award of any other degree.

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Summary

Given two groups K and H for which we have the free crossed resolutions, $B_* \xrightarrow{\epsilon} K$ and $C_* \xrightarrow{\epsilon'} H$ respectively. Our aim is to construct a free crossed resolution, $A_* \xrightarrow{\varepsilon} G$, by way of induction on the degree n, for any semidirect product $G = K \rtimes H$.

First we show how to find a set Z_1 of generators for the free group A_1 and the corresponding unique epimorphism from the free group on those generators to the semidirect product. This gives us the 1-dimensional free crossed resolution $A_1 \xrightarrow{\varepsilon} G$, (see Proposition 4.1).

Next we define a set of generators Z_2 that together with Z_1, constitute a generating set for the free crossed module $A_2 \xrightarrow{\delta_2} A_1$, where δ_2 is crossed module homomorphism. Proposition 4.1 together with this free crossed module $\delta_2 : A_2 \longrightarrow A_1$ define a 2-dimensional free crossed resolution for $A_2 \xrightarrow{\delta_2} A_1 \xrightarrow{\varepsilon} G$ (see Proposition 4.9).

We then define an exact sequence $A_3 \xrightarrow{\delta_3} A_2 \xrightarrow{\delta_2} A_1$, where A_3, is an $(A_1/\delta_2 A_2)$-module on generating set Z_3 with module homomorphism $\delta_3 : A_3 \longrightarrow A_2$ defined on the generators. Proposition 4.11 says that we have a crossed complex of length 3, i.e., $A_3 \xrightarrow{\delta_3} A_2 \xrightarrow{\delta_2} A_1 \xrightarrow{\varepsilon} G$, where $\mathrm{Im}\delta_3 \subseteq \mathrm{Ker}\delta_2$.

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Contents

Introduction

In this work our aim is to investigate the construction of relatively small resolutions of groups, in the context of crossed complexes. Essentially, a crossed complex is like a chain complex of modules, in positive degrees, except that in degrees 1 and 2 there are some non-abelian structure. While a chain complex is a good algebraic structure for calculating **homology** of a space, a crossed complex may also contain information about the first and second **homotopy** groups.

Our ultimate aim would be to generalise, to crossed complexes, a construction of C.T.C. Wall, [13], for the construction of free resolutions in the context of chain complexes. Recall that in the classical context a **free resolution** $\varepsilon : A_* \twoheadrightarrow \mathbb{Z}$ for a group G, [10], is a complex of free (left) $\mathbb{Z}G$-modules

$$\left(\cdots \longrightarrow A_3 \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_0 \longrightarrow 0 \longrightarrow \cdots \right)$$

together with a quasi isomorphism ε to the complex

$$(\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots)$$

where $\mathbb{Z}$ has trivial G-action. **Equivalently** it is a complex of $\mathbb{Z}G$-modules

$$\cdots \longrightarrow A_3 \xrightarrow{\partial_3} A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

which is **free** in degrees ≥ 0 and is **exact**.

There are many methods for finding 'large' resolutions, for example using the notion of **nerve** or **classifying space**, [15]:

Let K be the simplicial nerve of the group G, that is,

$$\begin{aligned} K_0 &= \{*\}, \\ K_1 &= G, \\ K_n = G^n &= \{(g_1, \ldots, g_n) : g_k \in G\}, \end{aligned}$$

with the usual simplicial **face maps**

$$\begin{aligned} d_0(g_1, \ldots, g_n) &= (g_2, \cdots g_n), \\ d_n(g_1, \ldots, g_n) &= (g_1, \cdots g_{n-1}), \\ d_k(g_1, \ldots, g_n) &= (g_1, \ldots, g_k g_{k+1}, \ldots, g_n), \end{aligned}$$

and **degeneracy maps**

$$s_k(g_1, \ldots, g_n) = (g_1, \ldots, g_k, 1, g_{k+1}, \ldots, g_n).$$

If one takes the **geometric realisation** $F = |K|$ of this simplicial set one obtains a **classifying space** for G, that is

- The fundamental group of F is G itself,

$$\pi_1(F) \cong G$$

- The other homotopy groups are trivial,

- It has a universal cover $\widetilde{F}$ that is a **contractible** cell complex

$$\widetilde{F} \xrightarrow{\simeq} *$$

- The complex of cellular chains on $\widetilde{F}$ is a free resolution

$$A = C_*(\widetilde{F}, \mathbb{Z}) \xrightarrow{\simeq} \mathbb{Z}.$$

- The nerve, and this **standard resolution**, are functorial in G.

The only disadvantage of this standard construction is that the number of n-dimensional cells in the nerve, and the number of generators of A_n in the resolution, grow exponentially with n.

On the other hand, for some particular groups and classes of groups there are much smaller resolutions known. Any finite **cyclic** group G has a resolution with just **one** generator in each dimension,

$$\cdots \longrightarrow \mathbb{Z}G \cdot a_3 \xrightarrow{\partial_3} \mathbb{Z}G \cdot a_2 \xrightarrow{\partial_2} \mathbb{Z}G \cdot a_1 \xrightarrow{\partial_1} \mathbb{Z}G \cdot a_0 \xrightarrow{\varepsilon} \mathbb{Z}$$

where $\varepsilon a_o = 1$ and the boundary maps are

$$\partial a_n = \begin{cases} (1-g)a_{n-1} & n \text{ odd} \\ (1+g+g^2+\cdots+g^{k-1})a_{n-1} & n \text{ even} \end{cases}$$

if the group G has order k and $g \in G$ is a generator.

In general however it is not easy to find free resolutions with a small number of generators, that is, it is not easy to find exact sequences of boundary maps, even though we are only dealing with free modules.

One straightforward method to construct new resolutions from old is the following. Suppose the group G is a **cartesian product** $K \times H$. Then one can construct a (small) free resolution for G out of (small) free resolutions for K and for H.

If we consider classifying spaces F_K for K and F_H for H, then

- the product $F_K \times F_H$ is a classifying space for $K \times H$
- the universal cover $\widetilde{F_K \times F_H}$ is contractible
- the cellular chain complex $C_*(\widetilde{F_K \times F_H}) \twoheadrightarrow \mathbb{Z}$ provides a resolution for the cartesian product $K \times H$.

This idea is easy to state in terms of chain complexes. Given two chain complexes B and C which are free resolutions for K and H, then a free resolution A for $K \times H$ may be constructed using the **tensor product** of complexes $B \otimes C$.

In this way, starting from the small free resolutions of finite cyclic groups given above, one may construct a free resolution for a product of cyclic groups, and hence inductively for all finitely generated abelian groups, in which the number of generators grows only **linearly** with the degree.

The obvious question is now whether a similar construction can be used if the group G is not a direct product but only a semidirect product, for example. In particular, we would like to know if there is a **twisted tensor product** of complexes which could be used instead of the tensor product.

A construction of C.T.C. Wall

The work in this thesis is inspired by a construction of C.T.C. Wall for resolutions of group extensions. Consider any (not necessarily split) extension of groups, [13],

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1$$

and suppose we are given resolutions B and C for the groups K and H. Then a resolution A for G may be constructed as a 'twisted' tensor product of B and C.

The idea behind Wall's construction is as follows. The free resolutions B for K and C for H are specified by

- graded sets of generators (X_p) and (Y_q)

- their boundaries

$$\varepsilon(x_0), \varepsilon(y_0) \in \mathbb{Z}, \qquad \partial_p(x_p) \in B_{p-1}, \qquad \partial_q(y_q) \in C_{q-1}.$$

Then a free resolution A for G is constructed, with generators

$$Z_n = \{\, x_p \otimes y_q \;:\; x_p \in X_p,\; y_q \in Y_q,\; p+q=n \,\}$$

and **certain boundary maps**, which may be defined inductively. As usual, it is these boundary maps on A which are hard to define, but Wall, [13], shows that the exactness of the resolutions B and C implies that they exist. He then shows that exactness for these boundary maps on A follows by a spectral sequence argument.

Crossed modules

The notions of crossed modules, and of free crossed complexes, date back many years, to the work of J.H.C. Whitehead, [14], who called them simply 'homotopy systems'. Their use has been developed more recently by a number of people, especially in the work of R. Brown and P. Higgins, and by their students.

A **crossed module** is a pair of groups V, W (not necessarily abelian),

- a group homomorphism $\partial : V \longrightarrow W$,
- and a left action of W on V, written ${}^{w}v$,

satisfying

$$\begin{aligned} \partial({}^{w}v) &= w\,\partial v\,w^{-1}, \\ {}^{\partial v}v' &= v\,v'\,v^{-1}. \end{aligned}$$

The **homotopy groups** of a crossed module are

$$\begin{aligned} \pi_1(\partial : V \longrightarrow W) &= W/\partial V \\ \pi_2(\partial : V \longrightarrow W) &= \ker \partial \end{aligned}$$

An example from algebraic topology (which also has application to presentations of groups) shows the importance of crossed modules. Let $F^{(2)}$ be a pointed connected 2-dimensional cell complex and let $F^{(1)}$ be its 1-skeleton. Then there is a so-called **connecting homomorphism**

$$\partial : \pi_2(F^{(2)}, F^{(1)}) \longrightarrow \pi_1(F^{(1)})$$

in the long exact sequence of homotopy groups of the pair $(F^{(2)}, F^{(1)})$. This is the fundamental example of a **free crossed module**, and in fact any free crossed module can be obtained in this way. One observes that the homotopy groups of this crossed module are

$$\begin{aligned} \pi_2(\partial) &= \pi_2 F^{(2)}, \\ \pi_1(\partial) &= \pi_1 F^{(2)}. \end{aligned}$$

The application to presentations of groups is the following. Let G be a group given by a **presentation**. In other words, $G = \langle Z_1 | Z_2 \rangle$, where

- Z_1 is a set of generators for G: there is an epimorphism
$$\varepsilon : \langle Z_1 \rangle \twoheadrightarrow G,$$
where $\langle \mathbb{Z}_1 \rangle$ is the free group on $\mathbb{Z}_1$;

- Z_2 is a set of relators for G: it comes with an injection
$$\theta_2 : Z_2 \hookrightarrow \langle Z_1 \rangle$$
such that the kernel of ε is generated as a normal subgroup of $\langle Z_1 \rangle$ by the image of θ_2.

Associated to any presentation of a group one has a 2-dimensional cell complex $F^{(1)} \subset F^{(2)}$, given by

$$\bigvee_{Z_1} S^1 \cup_{\theta_2} \bigvee_{Z_2} D^2$$

and associated to this cell complex one has a **free crossed module** with $\pi_1 \cong G$.

In fact the construction can be made completely algebraic: the free crossed module is simply the 'canonical' or 'universal' extension of the injection θ_2,

$$\begin{array}{ccc} \langle\langle Z_2 \rangle\rangle & \xrightarrow{\partial_2} & \langle Z_1 \rangle \\ \cong \downarrow & & \downarrow \cong \\ \pi_2(F^{(2)}, F^{(1)}) & \longrightarrow & \pi_1 F^{(1)} \end{array}$$

As a group, $\langle\langle Z_2 \rangle\rangle$ is generated by all formal conjugates ${}^{w}z$ of elements $z \in Z_2$ by words $w \in \langle \mathbb{Z}_1 \rangle$, modulo some 'obvious' trivial combinations of relations. However it is much more elegant to phrase things in the language of free crossed modules:

Given a group A_1 and a function $\theta_2 : Z_2 \longrightarrow A_1$ we write

$$\partial_2 : \langle\langle Z_2 \rangle\rangle \longrightarrow A_1$$

to denote the **free** crossed C_1-module generated by $Z_2 \xrightarrow{\theta_2} A_1$, where C_1 is the cokernel of ∂_2'.

In the case $A_1 = \langle Z_1 \rangle$, a free group, we said

$$\langle\langle Z_2 \rangle\rangle = \pi_2 \left(\bigvee_{Z_2} D^2 \cup_{\theta_2} \bigvee_{Z_1} S^1 \,,\ \bigvee_{Z_1} S^1 \right)$$

and in general, $\langle\langle Z_2 \rangle\rangle$ is generated by $\{{}^{w_1}z_2\}$, subject to relations

$$mnm^{-1} = {}^{(\partial_2 m)}n, \quad \text{and} \quad \partial_2({}^{w_1}z_2) = a_1 \cdot \theta_2(z_2) \cdot a_1^{-1}.$$

Of course this will not usually be an abelian group.

Presentations of extensions of groups

Returning to the idea of understanding the structure of groups in terms of that of its normal subgroups and quotients, we may consider the following theorem from combinatorial group theory:

Theorem: Consider any (not necessarily split) extension of groups [8]

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1$$

and suppose we are given presentations

$$K \cong \langle X_1 | X_2 \rangle, \qquad H \cong \langle Y_1 | Y_2 \rangle$$

Then the group G has a presentation

$$G \cong \langle Z_1 | Z_2 \rangle$$

where, as sets,

$$Z_1 \cong (X_1 \times \{*\}) \cup (\{*\} \times Y_1)$$

$$Z_2 \cong (X_2 \times \{*\}) \cup (X_1 \times Y_1) \cup (\{*\} \times Y_2)$$

Note that what we have not made clear in the above statement is how the function $\theta_2 : Z_2 \longrightarrow \langle Z_1 \rangle$ is constructed. Let us translate this theorem, using the correspondence between presentations and free crossed modules that we described above, (see Proposition 4.9):

Theorem: Consider any split extension of groups,

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1$$

and suppose we are given free crossed modules

$$B_2 \xrightarrow{\delta} B_1, \qquad C_2 \xrightarrow{\delta'} C_1$$

with $\pi_1(\delta) \cong K$, $\pi_1(\delta') \cong H$.

Then there exists a free crossed module

$$A_2 \xrightarrow{\partial} A_1, \qquad \pi_1(\partial) \cong G$$

with generators: $x_1 \otimes y_0, \quad x_0 \otimes y_1, \quad x_2 \otimes y_0, \quad x_1 \otimes y_1, \quad x_0 \otimes y_2,$

where $x_0, y_0 = *$, x_p a generator in B_p, y_q generator in C_q.

Once again, the hard part of proving this theorem is defining the boundary map ∂.

The work of Ellis–Kholodna

Another inspiration for this thesis was the work of G. Ellis and I. Kholodna, [7], who proposed an extension of the above idea to dimension 3. Unfortunately one of their general results contains an error, which we will discuss later.

Ellis–Kholodna introduced the following concept. A **3-presentation** of a group G consists of:

- a 2-presentation $\langle Z_1 | Z_2 \rangle$,
- and hence a free crossed module

$$\partial_2 : \langle\langle Z_2 \rangle\rangle \longrightarrow \langle Z_1 \rangle$$

- together with a set Z_3 and an injective function

$$\theta_3 : Z_3 \hookrightarrow \langle\langle Z_2 \rangle\rangle$$

whose image generates $\ker(\partial_2)$ as a $\mathbb{Z}G$-module. These elements of this image, are sometimes called '**relations between relations**' or '**homotopical syzygies**'.

A 3-presentation may be represented as an exact sequence

$$\langle Z_3 \rangle_{\mathbb{Z}G} \xrightarrow{\partial_3} \langle\langle Z_2 \rangle\rangle \xrightarrow{\partial_2} \langle Z_1 \rangle \longrightarrow G \longrightarrow 1,$$

where $\langle \mathbb{Z}_3 \rangle_{\mathbb{Z}G}$ is the free $\mathbb{Z}G$-module, on generators: $x_3 \otimes y_0$, $x_2 \otimes y_1$, $x_1 \otimes y_2$, $x_0 \otimes y_3$, where $x_0, y_0 = *$, x_p is a generator in B_p, and y_q is a generator in C_q.

The theorem that Ellis and Kholodna claimed to have proved can be expressed as follows:

Theorem (3-presentations of extensions of groups)

Consider any split extension of groups

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1$$

and suppose we are given 3-presentations

$$\langle X_3 \rangle_{\mathbb{Z}K} \xrightarrow{\delta_3} \langle\langle X_2 \rangle\rangle \xrightarrow{\delta_2} \langle X_1 \rangle \longrightarrow K \longrightarrow 1$$
$$\langle Y_3 \rangle_{\mathbb{Z}H} \xrightarrow{\delta_3'} \langle\langle Y_2 \rangle\rangle \xrightarrow{\delta_2'} \langle Y_1 \rangle \longrightarrow H \longrightarrow 1$$

Then the group G has a presentation

$$\langle Z_3\rangle_{\mathbb{Z}G} \xrightarrow{\partial_3} \langle\langle Z_2\rangle\rangle \xrightarrow{\partial_2} \langle Z_1\rangle \longrightarrow G \longrightarrow 1$$

where the sets of generators are given by

$$Z_n = \{ x_p \otimes y_q \ : \ x_p \in X_p, \ y_q \in Y_q, \ p+q=n \}$$

Crossed complexes

One can think of the non-abelian exact sequence used to represent a 3-presentation above as the tail end of a crossed complex. Explicitly, a **crossed complex** is a diagram of group homomorphisms

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_4 \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1$$

in which $\partial_2 : C_2 \longrightarrow C_1$ is a crossed module and, for $n \geq 3$,

- C_n is a $\mathbb{Z}G$-module, where $G = \pi_1(\partial_2)$,
- ∂_n respects the G actions, and $\partial_{n-1}\partial_n$ is trivial.

In particular, $C_{\geq 3}$ is just a classical chain complex of $\mathbb{Z}G$-modules, and so our work in this thesis of generalising Wall's proof from chain complexes to crossed complexes has to be concentrated mainly in degrees 1, 2 and 3. It is easy to see how to extend the basic **definitions** from chain complexes to crossed complexes:

- A crossed complex C is **free** if ∂_2 is a free crossed module and C_n is a free $\mathbb{Z}G$-module for $n \geq 3$.

- A free crossed complex C which is exact (except at C_1) is termed a **free crossed resolution** of the group $G = \pi_1(\partial_2)$.

- An n-**presentation** of a group G, for $0 \leq n \leq \infty$ is an n-dimensional connected cell complex with $\pi_1 = G$ and π_i trivial for $1 < i < n$,

$$F^{(n)} = \bigvee_{Z_1} S^1 \cup_{\theta_2} \bigvee_{Z_2} D^2 \cup_{\theta_3} \bigvee_{Z_3} D^3 \cup_{\theta_4} \cdots \cup_{\theta_n} \bigvee_{Z_n} D^n.$$

- An n-presentation gives a **free crossed resolution of length** n

$$C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \longrightarrow G \longrightarrow 1,$$

  where $C_k = \pi_k(F^{(k)}, F^{(k-1)})$, with generating set Z_k.

For clarity we may write an n-presentation simply as

$$\langle Z_1 | Z_2 | Z_3 | \ldots | Z_n \rangle.$$

Then we may make the following general **conjecture**:

Consider any (not necessarily split) extension of groups

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1$$

and suppose we are given n-presentations of K and H,

$$\langle X_1|X_2|X_3|\dots|X_n\rangle, \qquad \langle Y_1|Y_2|Y_3|\dots|Y_n\rangle.$$

Then the group G has an n-presentation

$$\langle Z_1|Z_2|Z_3|\dots|Z_n\rangle$$

where, if we write $X_0 = Y_0 = \{*\}$,

$$Z_k \;\cong\; \{x_p \otimes y_q \;:\; x_p \in X_p,\; y_q \in Y_q,\; p+q=k\} \;\cong\; \bigcup_{p+q=k} X_p \times Y_q$$

for each $k = 1, \dots, n$.

Once more, the hard part of proving this conjecture is defining funtions θ_k on the generating sets Z_n which, on the free crossed complex of length n, will define an exact complex.

Chapter 1

Group Theory

This chapter will establish the group theory necessary for understanding the structures used throughout this thesis, but most especially in chapter 4, where the main result is developed.

The first section contains definitions and notation, for understanding semidirect products and then we discuss group extensions.

In the final section we will define modules, and free modules, as well as the notion of tensor product $A \otimes B$, where A is a right module and B is a left module. We shall also give some properties of these structures.

1.1 Transversals and Semidirect Products

A subgroup K of a group G is called **normal** and denoted $K \lhd G$ if $gKg^{-1} = K$ for all $g \in G$. Given that K is a normal subgroup of a group G, then every $g \in G$ defines an automorphism α_g of K

$$\alpha_g : K \longrightarrow K, \qquad k \mapsto gkg^{-1},$$

and this in turn defines a homomorphism

$$\alpha : G \longrightarrow Aut(K), \qquad g \mapsto \alpha_g. \tag{1.1}$$

If K is a subgroup of G, we may choose a **right transversal** H for K in G consisting of one element from each right coset of K in G, so that every element $g \in G$ can be written uniquely as $g = kh$ for $k \in K$, $h \in H$. If K is in fact a normal subgroup then left transversals and right transversals are the same, $G = KH = HK$. Then the two unique ways of expressing elements of G are related by

$$g = kh = hk', \qquad k = \alpha_h(k') = hk'h^{-1}.$$

If K is a normal subgroup, consider the canonical epimorphism to the quotient group

$$q : G \longrightarrow G/K, \qquad g \mapsto Kg,$$

in this case the normal subgroup K is the kernel of this epimorphism.

We can define a function

$$j : G/K \longrightarrow G, \qquad qj = \mathrm{id}_{G/K},$$

that is a **splitting** or a **cross section** of the quotient map.

Cross sections and transversals of normal subgroups are just two ways of saying the same thing. Both of them give a particular choice of elements $h \in G$, one for each coset of K in G, so that all cosets are represented exactly once by cosets of the form Kh.

In terms of the cross sections j, we see that every element g of G can be written as

$$g = kh = hk', \qquad h = j(Kg), \quad k = gh^{-1}, \quad k' = h^{-1}g.$$

We can assume we always choose the cross section in the obvious way for the identity coset:

$$j(K \cdot 1) = 1.$$

That is, we assume the transversal H contains the identity element of G.

A **complement** for a normal subgroup K of a group G is a transversal H which is actually a subgroup of G. It is not always possible to find a complement. If a complement exists then G is called a **semidirect product** of H and K, which we write as

$$G \cong K \rtimes H.$$

In terms of cross sections, G is a semidirect product if there exists a function j with $q \circ j$ the identity, such that j is a homomorphism of groups. This is the same as the condition that the image of j is a subgroup of G. For the image of a homomorphism is always a group, and conversely if the image of the function j is a subgroup of G then

$$j(Kg) \cdot j(Kg') = j(Kg'')$$

for some g'', and applying the homomorphism q gives $Kg'' = Kg \cdot Kg'$, so j is a homomorphism also.

We will see further examples of cross sections, semidirect products, and their generalisations, below.

1.2 Group Extensions

A sequence of groups and group homomorphisms

$$1 \longrightarrow K \xrightarrow{i} G \xrightarrow{p} H \longrightarrow 1, \tag{1.2}$$

is called **exact** if the homomorphism i is injective, p is surjective and

$$\mathrm{Im}(i) = \mathrm{Ker}(p).$$

This last condition is the same as saying that $p(i(k)) = 1$ for all k in K and also that every element g in the kernel of p may be represented as $i(k)$ for some element

k in K. So in this situation, the sequence (1.2) is called an **extension of K by H**, then

- The image $i(K) \cong K$ is a normal subgroup of G. This is because it is also the kernel of the homomorphism p.
- The quotient group $G/i(K)$, of G modulo the subgroup $i(K)$, contains the (right) cosets of $\mathrm{Im}(i)$ and it is isomorphic to the group H by the isomorphism theorem,

$$G/i(K) = G/\ker(p) \cong p(G) = H.$$

- Since p is a surjection, $p^{-1}(h)$, for any $h \in H$, is one of the (right) cosets of $\mathrm{Im}(i)$, so we can choose a mapping $j : H \longrightarrow G$ such that $j(1) = 1$ and $p \circ j = \mathrm{id}_H$. The map j selects a representative of each coset. We shall denote this set of coset representatives by $j(H)$, and call it a **cross section** of G.

When $K \xrightarrow{i} G \xrightarrow{p} H$ is a group extension of K by H and $j(H)$ is a cross section of G, then every element g of G can be written in the form,

$$g = i(k)j(h), \tag{1.3}$$

for some unique $k \in K$ and $h \in H$. To prove that this representation exists and is unique, we apply the homomorphism p to the equation (1.3) and we get

$$p(g) = p(i(k)j(h)) = p(i(k))p(j(h)) = 1 \cdot h = h$$

and so h is uniquely determined. Now solving (1.3) is the same as solving

$$g \cdot j(h)^{-1} = i(k) \tag{1.4}$$

This has a solution because $p(g) = h = p(j(h))$ and so the left hand side of equation (1.4) is an element of $\ker(p) = i(K)$. The solution is unique because i is injective.

Thus all elements in G can be represented uniquely by a product of the type $i(k)j(h)$.

We now consider three other possible products in G,

$$j(h)i(k), \qquad j(h)j(h'), \qquad i(k)j(h)i(k')j(h').$$

(i) applying p to the first product, $j(h)i(k)$ gives,

$$p(j(h)i(k)) = p(j(h))p(i(k)) = h.$$

However, by (1.3), every element of G can be written uniquely as a product of an element of $\mathrm{Im}(i)$ and an element of $j(H)$, so that

$$j(h)i(k) = i({}^{h}k)j(h), \tag{1.5}$$

for some unique ${}^{h}k \in K$ which is determined by

$$i({}^{h}k) = j(h) \cdot i(k) \cdot j(h)^{-1}.$$

Therefore we get a mapping $\alpha : H \times K \longrightarrow K$ defined by $(hk) \mapsto {}^h k$ which gives the **action** of the set H on the group K relative to a cross section $\jmath(H)$ of G

(ii) applying p to the second product, $\jmath(h)\jmath(h')$ yields,

$$p(\jmath(h)\jmath(h')) = p(\jmath(h))p(\jmath(h')) = hh'.$$

Then (1.3) says that

$$\jmath(h)\jmath(h') = i(\{h, h'\})\jmath(hh') \tag{1.6}$$

for some unique $\{h, h'\} \in K$ which is determined by

$$i(\{h, h'\}) = \jmath(h)\jmath(h') \cdot \jmath(hh')^{-1}.$$

Therefore we get a mapping $c_2 : (h, h') \mapsto \{h, h'\}$ of $H \times H$ into K which we will call the **cocycle** of H relative to $\jmath(H)$. It is clear from the definition that the cocycle is trivial, $c_2(h, h') = 1$, if and only if the map $\jmath : H \longrightarrow G$ is a homomorphism of groups.

(iii) finally, applying p to $i(k)\jmath(h)i(k')\jmath(h')$ we see that (1.5) and (1.6) yield,

$$\begin{aligned} i(k)\jmath(h)i(k')\jmath(h') &= i(k)i({}^h k')\jmath(h)\jmath(h') \\ &= i(k)i({}^h k')i(\{hh'\})\jmath(hh') \\ &= i(k{}^h k'\{h, h'\})\jmath(hh') \end{aligned} \tag{1.7}$$

So that the set action α and the cocycle c_2 are enough to put all products in G in the form of (1.3), conversely (1.3) and (1.7) construct G from K and H.

When we have a group extension G, of K by H, then the set action α and the cocycle c_2 have the following properties. For all $h, h' \in H$ and $k \in K$, with $\alpha_h : k \mapsto {}^h k$ an automorphism of K, then

$$ {}^1 k = k; \tag{1.8}$$

$$ \{h,1\} = 1 = \{1,h\}; \tag{1.9}$$

$$ {}^h({}^{h'}k)\{h,h'\} = \{h,h'\}{}^{hh'}k; \tag{1.10}$$

$$ \{h,h'\}\{hh',h''\} = {}^h\{h',h''\}\{h,h'h''\}. \tag{1.11}$$

Observe, since i is an injective homomorphism, (1.10) follows from:

$$ i({}^h({}^{h'}k)\{h,h'\}) = i({}^h({}^{h'}k))i(\{h,h'\})) $$

$$ = j(h)i({}^{h'}k)j(h)^{-1}j(h)j(h')j(hh')^{-1} $$

$$ = (j(h)(j(h')i(k)j(h')^{-1})j(h)^{-1})j(h)j(h')j(hh')^{-1} $$

$$ = j(h)j(h')i(k)j(hh')^{-1} $$

$$ = j(h)j(h')(j(hh')^{-1}j(hh'))i(k)j(hh')^{-1} $$

$$ = i(\{h,h'\})i({}^{hh'}k) = i(\{h,h'\}{}^{hh'}k). $$

Conversely, let K and H be groups and $\alpha : H \longrightarrow \text{Aut}(K)$ be a mapping, and $c_2 : H \times H \longrightarrow K$ be a mapping, if they satisfy conditions (1.8)-(1.11) above, then $G_{(\alpha,c_2)} = K \times H$ together with the following multiplication:

$$(k,h)(k',h') = (k^h k'\{h,h'\}, hh') \tag{1.12}$$

is a group extension of K by H:

- Associativity:

$$\begin{aligned}
(k,h)\left[(k',h')(k'',h'')\right] &= (k,h)\left(k'\,{}^{h'}k''\{h',h''\}, h'h''\right) \\
&= \left(k\,{}^{h}\left(k'\,{}^{h'}k''\{h',h''\}\right)\{h,h'h''\}, h(h'h'')\right) \\
&= \left(k\,{}^{h}k'\,{}^{h}\left({}^{h'}k''\right)\,{}^{h}\{h',h''\}\{h,h'h''\}, h(h'h'')\right) \\
&= \left(k\,{}^{h}k'\,{}^{h}\left({}^{h'}k''\right)\{h,h'\}\{hh',h''\}, h(h'h'')\right) \text{ by (1.11)} \\
&= \left(k\,{}^{h}k'\{h,h'\}\,{}^{hh'}k''\{hh',h''\}, (hh')h''\right) \text{ by (1.10)} \\
&= \left(k\,{}^{h}k'\{h,h'\}, hh'\right)(k'',h'') \\
&= \left[(k,h)(k',h')\right](k'',h''),
\end{aligned}$$

- Identity element: $(k,h)(1,1) = \left(k\,{}^{h}1\{h,1\}, h\cdot 1\right) = (k\cdot 1\cdot 1, h) = (k,h)$.

Similarly $(1,1)(k,h) = (1^1 k\{1,h\}, 1\cdot h) = (k,h)$

- Inverse element: first (k',h') is a left inverse for (k,h) if $(k',h')(k,h) = (1,1)$ where $k' = \left({}^{h'}k\{h',h\}\right)^{-1}$ and $h' = h^{-1}$, which shows that every element has a left inverse. Let (k'', h'') be a left inverse for (k', h'),

$$\begin{aligned}(k'', h'') &= (k'', h'')\,[(k', h')\,(k, h)] \\ &= [(k'', h'')\,(k', h')]\,(k, h) \\ &= (k, h)\end{aligned}$$

so (k', h') is infact an actual inverse for (k, h).

It is the case that any group extension G of K by H is equivalent to $G_{(\alpha,c_2)}$ for some α and c_2.

Some types of extension are particularly simple:

The **centre** of a group G is the subset

$$Z(G) = \{g \in G : gx = xg \text{ for all } x \in G\},$$

of G. It is a subgroup of G. An extension is called **central** if $i(K)$ is contained in the centre $Z(G)$ of G. By definition, the action α is trivial (given by ${}^h k = k$ for all $h \in H$ and all $k \in K$) for all central extensions, but the cocycle c_2 may not be trivial.

The **semidirect product** $K \rtimes H$ of the previous section is an extension of K by H,

$$1 \longrightarrow K \xrightarrow{i} K \rtimes H \xrightarrow{p} H \longrightarrow 1,$$

with α given by the conjugation action. By definition, the cocycle is trivial for these types of extensions where the map j can be chosen to be the homomorphism $j(h) = (1, h)$. These types of extensions are also called **split** extensions. We can always identify a semidirect product G with the set of ordered pairs (k, h) and the group structure given by

$$(k, h) \cdot (k', h') = (k \cdot \alpha_h(k'),\ h \cdot h').$$

An extension is both split and central if and only if both the action and the cocycle are trivial, that is, if the extension is just the direct product of groups, $K \times H$.

1.2.1 Examples of Group Extensions

In appendix 5 we give some examples of the action, α, and cocycle, c_2, which correspond to particular group extensions. Here we have shown three particular group extensions:

- The cyclic group of order six is an extension of the cyclic group of order three by the cyclic group of order two,

$$1 \longrightarrow C_3 \longrightarrow C_6 \longrightarrow C_2 \longrightarrow 1$$

Since the extension is abelian it is central. There are several possible choices for the cross section j, and one of them gives a homomorphism, so the extension can also be seen to be split. Of course, the cyclic group of order six is isomorphic to the direct product of the groups of orders two and three,

$$C_6 \cong C_2 \times C_3$$

- The symmetric group of degree three is a split extension of the group of order 3 by the group of order two

$$1 \longrightarrow C_3 \longrightarrow S_3 \longrightarrow C_2 \longrightarrow 1$$

  This extension is not central: the cocycle is trivial but the action is not.

The basic form of all examples is the same: G is a group with a normal subgroup K, and H is the quotient. The essential question is, given the availability of nice algorithms for working with (or simply "nice properties of") two groups K and H, are these passed on to all of the possible extensions G, of K by H?

Here we give a couple of generalisations:

1. For all groups K, H the direct product $K \times H$ is a split central extension (that is, an extension with trivial action and trivial cocycle) given by

$$1 \longrightarrow K \xrightarrow{i} K \times H \xrightarrow{p} H \longrightarrow 1$$

with $i(k) = (k, 1)$, $p(k, h) = h$ and $j(h) = (1, h)$.

As a special case, recall that if m, n are coprime integers then the Chinese Remainder Theorem gives an isomorphism

$$C_m \times C_n \cong C_{mn}, \qquad (x^r, y^t) \mapsto c^{rbn+tam}$$

where x, y, c are the generators of the groups and a, b are integers such that

$$am + bn = 1$$

and so

$$rbn + tam \equiv \begin{cases} r \mod m, \\ t \mod n. \end{cases}$$

Therefore, for cyclic groups of coprime order $K = C_m$ and $H = C_n$ we can write down an isomorphism of two split central extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C_m & \xrightarrow{i} & C_m \times C_n & \xrightarrow{p} & C_n & \longrightarrow & 1 \\ \downarrow{=} & & \downarrow{=} & & \downarrow{\cong} & & \downarrow{=} & & \downarrow{=} \\ 1 & \longrightarrow & C_m & \xrightarrow{i'} & C_{mn} & \xrightarrow{p'} & C_n & \longrightarrow & 1 \end{array}$$

and we get

$$i'(x) = c^{bn}, \qquad p'(c) = y, \qquad j'(y) = c^{am}.$$

2. There is a split extension of the cyclic group of order n by the group of order 2 which gives the dihedral group,

$$1 \longrightarrow C_n \xrightarrow{i} D_{2n} \xrightarrow{p} C_2 \longrightarrow 1$$

with $i(x) = a$, $p(a^r b^t) = y^t$ $(t = 0, 1)$, cross section $j(y) = b$, and

$$i({}^y x) \quad = \quad bab^{-1} \quad = \quad bab \quad = \quad a^{n-1} \quad = \quad i(x^{n-1})$$

so the action is given by ${}^y x = x^{n-1}$.

1.3 Modules

In what follows let G be a group written multiplicatively and R be a ring with an identity element $1 \neq 0$.

Definition 1.1. *A* **left** R-**module** M *is an additive abelian group* M *together with a scalar multiplication i.e., a map,* $R \times M \longrightarrow M$*, defined by* $(r, m) \mapsto rm$*, with properties:*

(1) $(r_1 + r_2)m = r_1 m + r_2 m$ *for all* $r_1, r_2 \in R$ *and* $m \in M$*;*

(2) $r(m_1 + m_2) = rm_1 + rm_2$ *for all* $r \in R$ *and* $m_1, m_2 \in M$*;*

(3) $(r_1 r_2)m = r_1(r_2 m)$ *for all* $r_1, r_2 \in R$ *and* $m \in M$*;*

(4) $1m = m$ *for all* $m \in M$*.*

There is a similar statement for a **right** R-**module** *where the scalar multiplication is given by the map* $M \times R \longrightarrow M$*,* $(m, r) \mapsto mr$ *and corresponding properties.*

1.3.1 Examples of Modules

1. Any ring R can be considered as a left or a right R-module.

2. Given a subring S of a ring R, then for any $s \in S$ and $r \in R$ we have $sr \in R$ so that R together with addition makes R into a left (as well as right) S-module. The distributive, associative and unit laws form the four conditions of the definition.

3. If $R = \mathbb{Z}$ is the ring of integers, then an R-module M, is really just an abelian group.

4. Suppose R is a ring and G is a group, then let RG be the following set,

$$\left\{ \sum_{g \in G} r_g g : r_g \in R, r_g \neq 0 \text{ for a finite number of } g \in G \right\}.$$

   If we define the following addition and multiplication on RG,

$$\sum_{g \in G} r_g g + \sum_{g \in G} r'_g g = \sum_{g \in G} (r_g + r'_g) g,$$
$$(\sum_{g \in G} r_g g)(\sum_{g' \in G} r_{g'} g') = \sum_{g,g' \in G} (r_g r'_{g'}) g g'$$

   then RG is called the **group ring** of the group G over the ring R. If we now identify $r \in R$ with $r1 \in RG$ then we see that R is a subring of RG. Thus RG becomes a left (or right) R-module with

$$r \cdot (\sum_{g \in G} r_g g) = (\sum_{g \in G} (r r_g) g)$$

5. If K is a subgroup of the group G then RK is a subring of RG so that RG is also a left (or right) RK-module.

1.3.2 Submodules

Definition 1.2. *Let M be an R-module and S a subset of M, then we call S a* **submodule** *of M if:*

(1) $0 \in S$;

(2) $s_1, s_2 \in S \Rightarrow s_1 + s_2 \in S$;

(3) $s \in S \Rightarrow -s \in S$;

These three conditions state that S is a subgroup of the abelian group M.

(4) $s \in S$ *and* $r \in R \Rightarrow rs \in S$.

This last condition states that S is closed under scalar multiplication.

From this definition we see that a 'submodule' of a left R-module is itself a left R-module. We can replace definition 1.2 by the following statement:

Result 1.3. *A subset S of an R-module is a submodule if and only if (a) $S \neq \emptyset$; (b) $s_1, s_2 \in S$ and $r_1, r_2 \in R \Longrightarrow r_1 s_1 + r_2 s_2 \in S$.*

Proof. Certainly if $0 \in S$ then $S \neq \emptyset$. If $S \neq \emptyset$, then there exists a $s \in S$, so that $-s \in S$ by 1.2(3) and $s + -s \in S$ by 1.2(2) i.e., $0 \in S$. Condition 1.2(3) is redundant. First note if $s_1 + s_2 = s_2$, then $s_1 = 0$, since $s_1 + s_2 - s_2 = s_2 - s_2$. In R, $0 + 0 = 0$, so for any $s \in S$ $(0+0)s = 0s$, so $0s + 0s = 0s$, therefore $0s = 0$. Next $0 = 0s = (1 + (-1))s = s + (-1)s$, so $(-1)s = -s$ therefore 1.2(3) follows from 1.2(4) if $r = -1$.

If $r_1 s_1 \in S$ and $r_2 s_2 \in S$ by 1.2(4) and $r_1 s_1 + r_2 s_2 \in S$ by 1.2(2). Conversely, given $s \in S$, $r \in R$ then $rs = rs + 0 = rs + 0s \in S$ by 1.3(b), and given $s_1, s_2 \in S$, then $s_1 + s_2 = 1s_1 + 1s_2 \in S$ by 1.3(b). □

1.3.3 Free modules

Free modules can be defined by their universal property or with an explicit construction.

Definition 1.4. *Let F be a left R-module and X be a set. Then F is called a* **free left R-module** *on the* **basis** *X if X is regarded as a subset of F and all homomorphisms of modules from F are determined by their values on the elements of X. That is, if there is a map $\iota : X \longrightarrow F$ such that for any left R-module A and any function $f : X \longrightarrow A$, there exists a unique R-homomorphism $g : F \longrightarrow A$ such that $f = g \circ \iota$*

Let $F = \bigoplus_{x \in X} R_x$ with $\{R_x\}_{x \in X}$ a family of left R-modules and $R_x = Rx$ for each $x \in X$. Associate $r \in R$ with $rx \in R_x$ then every element of F can be uniquely written as a finite sum $\sum_{i=1}^{n} r_i x_i$, $r_i \in R$, $x_i \in X$. Let $\iota : X \longrightarrow F$ be the map that sends x to $1 \cdot x$. So that any R-module A and map $f : X \longrightarrow A$ we have the unique homomorphism $g : F \longrightarrow A$ of the definition and we see that it is well-defined and unique

$$g(\sum r_i x_i) = \sum r_i f(x_i).$$

We see that $g\iota = f$ and we have proved the following theorem:

Theorem 1.5. *Let X be a set, then there exists a free left R-module F with X as basis.*

If X is the empty set then the free module is the zero module.

Now let G be a group and let R be the group ring

$$\mathbb{Z}G = \left\{ \sum_{g \in G} r_g g : r_g \in \mathbb{Z}, r_g \neq 0 \text{ for a finite number of } g \in G \right\}$$

For every free left R-module F with X as basis there is a homomorphism of R-modules, called the **augmentation** map,

$$\epsilon : F \longrightarrow \mathbb{Z}$$

Here $\mathbb{Z}$ is the **trivial $\mathbb{Z}G$-module**, that is, for all $g \in G$ and $a \in \mathbb{Z}$ we have $g \cdot a = a$. The augmentation map is the only homomorphism of R-modules which sends each element x of the basis to the element $1 \in \mathbb{Z}$. That is,

$$\text{If} \quad b = \sum_{x \in X} \sum_{g \in G} n_{x,g} g \cdot x \in F \quad \text{then} \quad \epsilon(b) = \sum_{x \in X} \sum_{g \in G} n_{x,g} \in \mathbb{Z}.$$

1.3.4 Tensor Product

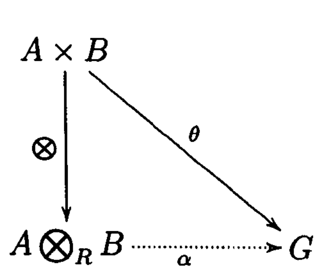
Definition 1.6. *Suppose R is a ring and G is any abelian group. Given a right R-module A and a left R-module B then a function $\theta : A \times B \longrightarrow G$ is called an* R-**bihomomorphism** *or an* R-**bilinear map** *if it satisfies the following conditions,*

(i) $\theta(a, b + b') = \theta(a, b) + \theta(a, b')$;

(ii) $\theta(a + a', b) = \theta(a, b) + \theta(a', b)$;

(iii) $\theta(ar, b) = \theta(a, rb)$.

Definition 1.7. *A* **tensor product of** *A* **by** *B* **over** *R is an abelian group $A \bigotimes_R B$ consisting of a right R-module A and a left R-module B together with a R-bihomomorphism $\bigotimes : A \times B \longrightarrow A \bigotimes_R B$ satisfying the following universal property: – for any abelian group G and R- bihomomorphism $A \times B \longrightarrow G$ there exists a unique homomorphism $\alpha : A \otimes_R B \longrightarrow G$ of abelian groups that makes the following diagram commute*



As for free modules above, the universal property proves that the tensor product is **unique**, because for any two different tensor products the universal property would automatically give a pair of (unique) isomorphisms between them. In order to prove the tensor product **exists** there is an explicit construction which we can give:

Consider the free $\mathbb{Z}$-module $Z(A,B)$ which is freely generated by the set $A \times B$, where A is a right R-module and B is a left R-module. Let $D(A,B)$ be the submodule of $Z(A,B)$ generated by the following elements

1. $(a_1 + a_2, b) - (a_1, b) - (a_2, b)$,

2. $(a, b_1 + b_2) - (a, b_1) - (a, b_2)$,

3. $(ar, b) - (a, rb)$,

for all $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$ and $r \in R$. Then we have constructed the tensor product as the quotient $\mathbb{Z}$-module

$$A \bigotimes_R B \ := \ Z(A,B)/D(A,B).$$

We will write $a \otimes b$ for the element $(a,b) + D(A,B)$ in $Z(A,B)/D(A,B)$.

We remark also that if A is at the same time a left S-module and a right R-module, and B is a left R module, then the tensor product $A \otimes_R B$ is still a left

S-module, with

$$s \cdot (a \otimes b) \quad = \quad (sa) \otimes b$$

On the other hand, if A and B are just abelian groups, then the tensor product over $R = \mathbb{Z}$ gives an abelian group $A \otimes B$.

Chapter 2

Chain Complexes and Resolutions

In this chapter we define chain complexes and their homology groups. We also introduce the notion of exactness of a sequence of groups and homomorphisms of groups.

We also discuss a construction by C.T.C. Wall [13], where he gives a resolution for group extensions with the use of R-modules. This is a major motivation for the work undertaken in this thesis.

2.1 Chain Complexes and Resolutions for Groups

The established theory of chain complexes, [10], was highly useful in the development of a comparison theorem for crossed complexes.

2.1.1 Chain complexes and homology

A **chain complex** X of R-modules is a sequence of R-modules and R-module homomorphisms

$$X: \quad \cdots \longrightarrow X_{n+1} \xrightarrow{\partial_{n+1}} X_n \xrightarrow{\partial_n} X_{n-1} \longrightarrow \cdots \xrightarrow{\partial_2} X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\partial_0} 0$$

such that for each n the composite $\partial_{n-1}\partial_n = 0$, in other words, the kernel of each ∂ contains the image of the previous one.

We call an element c of the submodule $C_n(X) = \ker \partial_n$, of the R-module X_n, an n**-cycle** of X and an element c of the submodule $\partial_{n+1}X_{n+1}$, of the R-module X_n, an n**-boundary** of X, then

$$H_n(X) = C_n(X)/\partial_{n+1}X_{n+1},$$

describes the homology modules as 'cycles mod boundaries', this allows us to write the coset of c in $H_n(X)$, as $\{c\} = c + \partial_{n+1}X_{n+1}$. Two elements, c, $c' \in X_n$, are in the same coset if and only if $c - c' \in \partial_{n+1}X_{n+1}$. We say they are **homologous** and write $c \sim c'$.

The nth **homology group** of a chain complex X is the quotient

$$H_n(X) = \ker(\partial_n)/\partial_{n+1}(X_{n+1})$$

A chain complex is **exact** if all of its homology groups are zero. This is the same as saying that the kernel of each ∂ equals the image of the previous one.

A chain complex is **acyclic** if all of the homology groups H_n for $n > 0$ are zero. This is the same as saying that for $n \geq 1$ the kernel of each ∂_n equals the image of the previous one.

Given two complexes X and X', then a **chain homomorphism** $f : X \longrightarrow X'$, is a sequence of R-module homomorphisms $f_n : X_n \longrightarrow X'_n$, one for each n in the sequence, such that $f_{n-1}\partial_n = \partial'_n f_n$, this condition states we have a **commutative diagram** of R-modules and R-module homomorphisms,

$$\begin{array}{lccccccc} X & \cdots \longrightarrow & X_{n+1} & \xrightarrow{\partial_{n+1}} & X_n & \xrightarrow{\partial_n} & X_{n-1} & \longrightarrow \cdots \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & \\ X' & \cdots \longrightarrow & X'_{n+1} & \xrightarrow[\partial'_{n+1}]{} & X'_n & \xrightarrow[\partial'_n]{} & X'_{n-1} & \longrightarrow \cdots . \end{array} \tag{2.1}$$

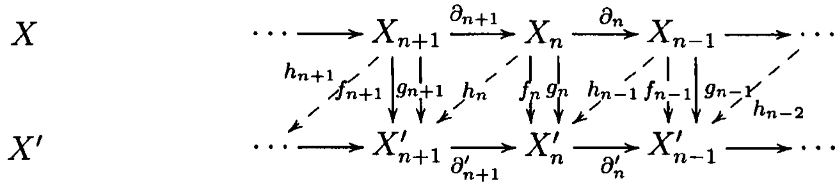
We can define a function

$$\begin{array}{rccc} & H_n(f) = f_* : H_n(X) & \longrightarrow & H_n(X'), \\ \text{by} & c + \partial_{n+1}X_{n+1} & \mapsto & f_n c + \partial'_{n+1}X'_{n+1}, \end{array}$$

it can be shown to be a homomorphism.

Given two chain homomorphisms $f, g : X \longrightarrow X'$, then a **chain homotopy** h, between these chain homomorphisms, is a sequence of R-module homomorphisms

$h_n : X_n \dashrightarrow X'_{n+1}$, for each $n \in \mathbb{Z}$,



such that

$$h_{n-1}\partial_n + \partial'_{n+1}h_n = f_n - g_n, \tag{2.2}$$

we say that f and g are **homotopic**, and write $f \simeq g$.

If $h : f \simeq g : X \longrightarrow X'$, then $H_n(f) = H(g) : H_n(X) \longrightarrow H_n(X')$ for all $n \in \mathbb{Z}$. Consider $c \in \ker \partial$, then $\partial_n c = 0$ and by (2.2), $f_n c - g_n c = \partial'_{n+1} h_n c$. Then $f_n c$ and $g_n c$ are homologous.

Given a chain homomorphism $f : X \longrightarrow X'$, if there exists a chain homomorphism $g : X' \longrightarrow X$ such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_{X'}$, then we call f a **chain equivalence**.

Given that $f : X \longrightarrow X'$ is a chain equivalence, then the induced map $H_n(f) : H_n(X) \longrightarrow H_n(X')$ is an isomorphism for each n. If we have chain homotopies $h : f \simeq g : X \longrightarrow X'$ and $h' : f' \simeq g' : X' \longrightarrow X''$ then we have the composite chain homotopy,

$$f'h + h'g : f'f \simeq g'g : X \longrightarrow X''.$$

'Subcomplexes' and 'quotient complexes' have properties like those of submodules and quotient modules. A **subcomplex** Y of a complex X, is a family of submodules Y_n of the module X_n, one for each n, such that $\partial Y_n \subset Y_{n-1}$, for all n. So that Y itself is a complex with boundary induced by $\partial = \partial_X$, and the injection $j : Y \longrightarrow X$ is a chain homomorphism. If $Y \subset X$, the **quotient complex** X/Y is the family $(X/Y)_n = X_n/Y_n$ of quotient modules with boundary $\partial' : X_n/Y_n \longrightarrow X_{n-1}/Y_{n-1}$ induced by ∂_X. The projection is a chain homomorphism $X \longrightarrow X/Y$, and the short sequence $Y_n \rightarrowtail X_n \twoheadrightarrow (X/Y)_n$ of modules is exact for each n. If $f : X \longrightarrow X'$ is a chain homomorphism, then $\ker f = \{\ker f_n\}$ is a subcomplex of X, $\mathrm{Im} f = \{f_n X_n\}$ a subcomplex of X', while $X'/\mathrm{Im} f$ is the 'cokernel' of f and $X/\ker f$ the 'coimage'. A pair of chain transformations $X \xrightarrow{f} X' \xrightarrow{g} X''$ is **exact** at X' if $\mathrm{Im} f = \ker g$; that is, if each sequence $X_n \longrightarrow X'_n \longrightarrow X''_n$ of modules is exact at X'_n. For any chain homomorphism $f : X \longrightarrow X'$, then

$$0 \longrightarrow \ker f \longrightarrow X \xrightarrow{f} X' \longrightarrow \mathrm{Coker} f \longrightarrow 0$$

is an exact sequence of complexes.

Contracting Homotopy

A chain complex is **positive**, (or **non-negative**), if $X_n = 0$ for $n < 0$, and it's homology will also be *positive*. There is a similar statement for a **negative** complex when $n > 0$.

Any module M may be considered as **trivial** chain complex with $M_n = 0$ for $n \neq 0$ and $M_0 = M$, then a **complex over** M is a positive complex X together with a trivial complex M and a chain homomorphism $\epsilon : X \longrightarrow M$, which is just a module homomorphism $\epsilon_0 : X_0 \longrightarrow M$ such that $\epsilon_0 \partial_1 = 0$

$$\begin{array}{lccccccc}
X & \cdots \longrightarrow & X_2 & \xrightarrow{\partial_2} & X_1 & \xrightarrow{\partial_1} & X_0 & \\
 & & \downarrow 0 & \overset{h_1}{\nwarrow} & \downarrow 0 & \overset{h_0}{\nwarrow} & f_0 \uparrow \; \downarrow \epsilon_0 & \\
M & \cdots \longrightarrow & 0 & \xrightarrow[0]{} & 0 & \xrightarrow[0]{} & M. &
\end{array}$$

In this case a **contracting homotopy** for ϵ is a chain homomorphism $f : M \longrightarrow X$, with $\epsilon_0 f_0 = \mathrm{id}_M$, together with a homotopy $h : \mathrm{id}_X \simeq f\epsilon$. So that a contracting homotopy is a module homomorphism $f_0 : M \longrightarrow X_0$ together with a homotopy $h_n : X_n \longrightarrow X_{n+1}$, for $n \geqslant 0$, such that

$$\epsilon_0 f_0 = 1_M, \qquad \partial_1 h_0 + f_0 \epsilon_0 = 1_{X_0}, \qquad \partial_{n+1} h_n + h_{n-1} \partial_n = \mathrm{id}_{X_n} \qquad n > 0.$$

Equivalently, extend the chain complex as shown below, then $h : \mathrm{id} \simeq 0$ of the identity and zero homomorphisms of the extended complex to itself.

$$\begin{array}{lcccccccc}
X & \cdots \longrightarrow & X_2 & \xrightarrow{\partial_2} & X_1 & \xrightarrow{\partial_1} & X_0 & \xrightarrow{\partial_0 = \epsilon_0} & X_{-1} = M. \\
 & & & \underset{h_1}{\nwarrow} & & \underset{h_0}{\nwarrow} & & \underset{h_{-1} = f_0}{\nwarrow} &
\end{array}$$

If we have that $\epsilon : X \longrightarrow M$ has a contracting homotopy, then it's homology groups are $H_0(X) \cong M$ and $H_n(X) = 0$ for $n > 0$.

A chain map $f : X \longrightarrow Y$ between two chain complexes is a sequence of R-module homomorphisms $f_n : X_n \longrightarrow Y_n$ which commutes with the ∂ maps,

$$f_{n-1}\partial_n^X = \partial_n^Y f_n$$

for all n.

A chain map f gives well-defined homomorphisms between the homology groups,

$$f_* : H_n(X) \longrightarrow H_n(Y).$$

for all n.

2.1.2 Resolutions for groups

Definition 2.1. *A* **resolution** *for a group G, or a resolution by $\mathbb{Z}G$-modules of the trivial module, is an exact complex of $\mathbb{Z}G$-modules*

$$\mathbf{S}: \qquad \cdots \longrightarrow X_n \xrightarrow{\partial_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where $\mathbb{Z}$ is the trivial module, that is, $\mathbb{Z}G$ acts trivially on it.

S *is called a* **free** *resolution if X_i is free for all i.*

The **augmentation map** $\epsilon : X_0 \longrightarrow \mathbb{Z}$ can be regarded as a chain map from the complex X to the complex which has $\mathbb{Z}$ in degree 0 and zero in all other degrees,

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & X_n & \xrightarrow{\partial_n} & X_{n-1} & \longrightarrow \cdots \longrightarrow & X_1 & \xrightarrow{\partial_1} & X_0 & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \epsilon} & & \downarrow \\
\cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

An alternative definition is that this is a **resolution** for the group G if this chain map induces isomorphisms in homology,

$$\epsilon_* : H_n(X) \cong 0 \ \ (n > 0), \qquad \epsilon_* : H_0(X) \cong \mathbb{Z}.$$

2.1.3 Example: resolutions for the finite cyclic groups

In this section we will give an explicit free resolution for the cyclic group C_m of order m,

$$C_m \ = \ \langle x | x^m = 1 \rangle$$

As usual we consider the trivial $\mathbb{Z}C_m$-module $\mathbb{Z}$, i.e. $x \cdot a = a$ for all $a \in \mathbb{Z}$, and the augmentation map $\epsilon : P \longrightarrow \mathbb{Z}$, on any free $\mathbb{Z}C_m$-module P with basis B, which is given by

$$\epsilon \left(\sum_{p \in B} \sum_{i=0}^{m-1} r_{p,i} x^i \cdot p \} \right) \ = \ \sum_{p \in B} \sum_{i=0}^{m-1} n_{p,i}.$$

Then the resolution is constructed as follows.

- Consider the two elements

$$N_x = \sum_{i=0}^{m-1} x^i \quad \text{and} \quad L_x = 1 - x \quad \text{in} \quad \mathbb{Z}C_m,$$

- consider the sequence of modules and homomorphisms

$$\cdots \longrightarrow P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where

- $P_n = \mathbb{Z}C_m \cdot p_n$ is a free (left) $\mathbb{Z}C_m$-module on one generator p_n for each $n \geq 0$.
- $\partial_n(p_n) = L_x \cdot p_{n-1}$ if n is odd,
- $\partial_n(p_n) = N_x \cdot p_{n-1}$ if n is even,
- and $\epsilon(x^i \cdot p_0) = 1$ as usual.

- We see that $\partial_n \partial_{n+1} = 0$,

$$N_x L_x = L_x N_x = x^m - 1 = 0 \in \mathbb{Z}C_m,$$

and also $\epsilon \partial_1 = 0$,

$$\epsilon L_x = \epsilon(1 - x) = 1 - 1 = 0 \in \mathbb{Z}C_m,$$

thus we have a chain complex P of free $\mathbb{Z}C_m$-modules ending with the trivial $\mathbb{Z}C_m$-module $\mathbb{Z}$.

- In order to show that P is a resolution, we need to show that any element of the kernel of a boundary map ∂_n can be lifted, and expressed as an element in the image of ∂_{n+1}. That is:

Consider a general element $b \in P_n$ and $\partial_n(b) = 0$. Then $b = \partial_{n+1}(b')$ for some $b' \in P_{n+1}$:

– Observe that when n is odd and $b = (\sum r_i x^i\}) \cdot p_n$ we see that

$$\partial_n \left(\left(\sum r_i x^i \} \right) \cdot p_n \} \right) = \left(\sum r_i x^i \} \right) \cdot L_x \cdot p_{n-1} = 0$$

precisely when

$$\left(\sum r_i x^i \}) \cdot (1 - x) \right) = 0$$

$$r_0 + \left(\sum (r_i - r_{i-1}\}) \, x^i \} \right) - r_{m-1} = 0$$

which implies $r_{i-1} = r_i$, since $x^i \neq 1 \in C_m$ for any $i \in \{0, 1, ..., m-1\}$

and

$$\begin{aligned} \ker \partial_n &= \left\{ \left(\sum r_0 x^i \right) \cdot p_n | r_0 \in \mathbb{Z} \right\} \\ &= \left\{ r_0 \cdot \left(\sum x^i \right) \cdot p_n | r_0 \in \mathbb{Z} \right\} \\ &= \{ \partial_{n+1} \left(r_0 \cdot p_{n+1} \right) | r_0 \in \mathbb{Z} \} \\ &\subseteq \operatorname{Im}(\partial_{n+1}). \end{aligned}$$

– Whereas when n is even we have

$$\begin{aligned} \partial_n \left(\left(\sum r_i x^i \right) \cdot p_n \right) &= \left(\sum r_i x^i \right) \cdot \left(\sum x^j \right) \cdot p_{n-1} \\ &= \left(\sum_{k=i+j} \left(\sum_i r_i \right) x^k \right) \cdot p_{n-1} = 0 \end{aligned}$$

which implies $\sum_i r_i = 0$. We have the same thing for the kernel of the augmentation map:

$$\epsilon \left(\left(\sum r_i x^i \right) \cdot p_n \right) = \sum r_i = 0$$

So in both cases we can assume $r_0 = -r_1 - \cdots - r_{m-1}$ for a general element of the kernel. Therefore we can write

$$\begin{aligned}
\sum r_i x^i &= -r_1 - r_2 - \cdots - r_{m-1} \\
&\quad +r_1 x + r_2 x^2 + \cdots + r_{m-1} x^{m-1} \\
&= (-r_1 - r_2(1+x) - \cdots - r_{m-1}(1 + x + \cdots + x^{m-2})) \\
&\quad \cdot(1-x) \\
&= \left(-\sum r_i \left(\sum_{j=0}^{i-1} x^j\right)\right) \cdot (1-x)
\end{aligned}$$

and so

$$\begin{aligned}
\ker(\partial_n) &= \left\{\left(-\sum r_i \left(\sum x^j\right)\right) \cdot (1-x) \cdot p_n | r_i \in \mathbb{Z}\right\} \\
&= \left\{\partial_{n+1}\left(\left(-\sum r_i \left(\sum x^j\right)\right) \cdot p_{n+1}\right) | r_i \in \mathbb{Z}\right\} \\
&\subseteq \operatorname{Im}(\partial_{n+1}).
\end{aligned}$$

and $\ker(\epsilon) \subseteq \operatorname{Im}(\partial_1)$ similarly.

Thus we have the free resolution of the trivial $\mathbb{Z}C_m$- module $\mathbb{Z}$.

2.2 A construction by C.T.C. Wall

In 1960 C.T.C. Wall [13] constructed a free resolution (A, δ) for a group extension G,

$$\cdots \xrightarrow{\delta_{n+1}} A_n \xrightarrow{\delta_n} \cdots \longrightarrow A_2 \xrightarrow{\delta_2} A_1 \xrightarrow{\delta_1} A_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

He achieved his goal by creating a direct sum of chain complexes derived from the resolutions for the subgroups K and H of G.

Given the following group extension

$$K \xrightarrow{i} G \xrightarrow{p} H,$$

where K is a group written multiplicatively we form the integral group ring

$$\mathbb{Z}K = \left\{ \sum_{k \in K} \lambda_k k : \lambda_k \in \mathbb{Z}, \lambda_k = 0 \text{ for almost all } k \in K \right\}$$

of K and view the additive abelian group of integers $\mathbb{Z}$ as a trivial $\mathbb{Z}K$-module on a single generator.

Suppose that we start off with an exact $(\text{Im}(\partial_{p+1}) = \text{Ker}(\partial_p))$ chain complex, $(\partial_p \partial_{p+1} = 0)$

$$\mathbf{B} : \cdots \longrightarrow B_{p+1} \xrightarrow{\partial_{p+1}} B_p \xrightarrow{\partial_p} B_{p-1} \longrightarrow \cdots \longrightarrow B_2 \xrightarrow{\partial_2} B_1 \xrightarrow{\partial_1} B_0$$

together with a chain map,

$$\epsilon : \mathbf{B} \longrightarrow \mathbb{Z}$$

inducing isomorphisms in homology, and that **B** consists of free $\mathbb{Z}K$-modules

$$B_p = \left\{ \sum_{i=1}^{\alpha_p} (\sum_{k \in K} \lambda_{k,i} k) b_{p,i} : \lambda_{k,i} \in \mathbb{Z} \right\}$$

with generating set $\{b_{p,i}\}$, $1 \leq i \leq \alpha_p$, in dimension p. Since the modules are free, the boundary maps and the chain transformation ϵ are defined by their values on the generators as follows

$$\partial_p \left(\sum_i (\sum_{k \in K} \lambda_{k,i} k) \cdot b_{p,i} \right) = \sum_i \left(\sum_{k \in K} \lambda_{k,i} k \right) \cdot \partial_p(b_{p,i}),$$

$$\epsilon \left(\sum_i (\sum_{k \in K} \lambda_{k,i} k) \cdot b_{0,i} \right) = \sum \lambda_{k,i},$$

where $1 \leq i \leq \alpha_p$, $p \geq 1$.

The sequence **B**,

$$\mathbf{B} : \cdots \longrightarrow B_{p+1} \xrightarrow{\partial_{p+1}} B_p \xrightarrow{\partial_p} \cdots \longrightarrow B_2 \xrightarrow{\partial_2} B_1 \xrightarrow{\partial_1} B_0 \xrightarrow{\epsilon} \mathbb{Z} \qquad (2.3)$$

is then a free resolution for the group K.

Suppose also we are given the following exact sequence,

$$\mathbf{C} : \cdots \longrightarrow C_{q+1} \xrightarrow{\partial'_{q+1}} C_q \xrightarrow{\partial'_q} \cdots \longrightarrow C_2 \xrightarrow{\partial'_2} C_1 \xrightarrow{\partial'_1} C_0 \xrightarrow{\epsilon'} \mathbb{Z} \qquad (2.4)$$

which is a free resolution for the group H with each

$$C_q = \left\{ \sum_{j=1}^{\alpha'_q} (\sum_{h \in H} \mu_{h,j} h) \cdot c_{q,j} : \mu_{h,j} \in \mathbb{Z} \right\}$$

a free $\mathbb{Z}H$–module with $\{c_{q,j}\}$, $1 \leq j \leq \alpha'_q$, as a generating set and boundary maps as follows,

$$\partial'_q\left(\sum_j\left(\sum_{h\in H}\mu_{h,j}h\right)\cdot c_{q,j}\right)=\sum_j\left(\sum_{h\in H}\mu_h h\right)\cdot\partial'_q(c_{q,j})$$
$$\epsilon'\left(\sum_j\left(\sum_{h\in H}\mu_{h,j}h\right)\cdot c_{0,j}\right)=\sum\mu_{h,j},$$

where $1 \leq j \leq \alpha'_q$, $q \geq 1$.

From the resolutions **B** and **C**, we want to form a resolution **A**, for the group extension G, with its known injection $K \xrightarrow{i} G$ and surjection $G \xrightarrow{p} H$. This will be achieved in two stages.

- First stage, apply a tensor product to the resolution **B** and call it **D** to give the following chain complex

$$\mathbf{D}:\cdots\longrightarrow(\mathbb{Z}G\otimes_{\mathbb{Z}K}B_p)\longrightarrow\cdots\longrightarrow(\mathbb{Z}G\otimes_{\mathbb{Z}K}B_0)\longrightarrow(\mathbb{Z}G\otimes_{\mathbb{Z}K}\mathbb{Z})\cong\mathbb{Z}H, \tag{2.5}$$

  of free left $\mathbb{Z}G$–modules $(\mathbb{Z}G \otimes_{\mathbb{Z}K} B_p)$, $p \geq 0$ with $\{(1 \bigotimes b_{p,i})\}$ as a set of generators in dimension p, and induced boundary $d_0 : (\mathbb{Z}G \otimes_{\mathbb{Z}K} B_{p+1}) \longrightarrow (\mathbb{Z}G \otimes_{\mathbb{Z}K} B_p)$, for $p \geq 1$ as follows,

$$d_0 = \ 1_G \bigotimes \partial_p$$

  where ∂_p is the boundary on **B** and 1_G is the identity map on $\mathbb{Z}G$.

- For each $q \geq 0$ we now take the direct sum of α'_q copies of the complex $\mathbf{D}$, one for each generator of C_q, and call this new complex $\mathbf{D}_q$,

$$\mathbf{D}_q \cdots \longrightarrow \bigoplus_{j=1}^{\alpha'_q}(\mathbb{Z}G \otimes_{\mathbb{Z}K} B_{p+1}) \longrightarrow \bigoplus_{j=1}^{\alpha'_q}(\mathbb{Z}G \otimes_{\mathbb{Z}K} B_p) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{j=1}^{\alpha'_q}(\mathbb{Z}G \bigotimes_{\mathbb{Z}K} B_0) \longrightarrow \bigoplus_{j=1}^{\alpha'_q} \mathbb{Z}G \otimes_{\mathbb{Z}K} \mathbb{Z}.$$

Now we observe that this is a complex of free $\mathbb{Z}G$-modules, and also of free $\mathbb{Z}H$-modules, and that the last module in this complex may be written as

$$\bigoplus_{j=1}^{\alpha'_q} \mathbb{Z}G \otimes_{\mathbb{Z}K} \mathbb{Z} \;\cong\; \bigoplus_{j=1}^{\alpha'_q} \mathbb{Z}H \;\cong\; C_q$$

- Second stage, lay the $\mathbf{D}_q$'s out in columns. However, before that we will just do some relabelling, let

$$\bigoplus_{j=1}^{\alpha'_q} \mathbb{Z}G \otimes_{\mathbb{Z}K} B_p = A_{p,q}$$

so that $\mathbf{D}_q$ becomes

$$\cdots \longrightarrow A_{p,q} \longrightarrow A_{p-1,q} \cdots \longrightarrow A_{1,q} \longrightarrow A_{0,q} \longrightarrow C_q.$$

The subsequent array of free modules with the columns $\mathbf{D}_q$ that has been constructed from the resolutions $\mathbf{B}$ and $\mathbf{C}$ is shown in Figure 2.1 below. We have also the maps

$$d_0 : A_{p,q} \longrightarrow A_{p-1,q}$$

and the parts (i) and (iii) of the following Proposition below hold.

- We then construct inductively maps

$$d_k : A_{p,q} \longrightarrow A_{p+k-1,q-k},$$

by Wall's method, such that part (ii) of the Proposition 2.2 below also holds, using the exactness of the complex **B**.

- The Proposition then says that this data is always sufficient to give a resolution of the group extension. Wall's proof of this Proposition is by a spectral sequence argument, but we give a bare-hands proof here since we would like to generalise it to crossed complexes for which no spectral sequence machinery is available.

Proposition 2.2. *Given a bigraded family $\{A_{p,q}\}_{p,q\geq 0}$ of R-modules, for some ring R, together with R-homomorphisms $d_k : A_{p,q} \longrightarrow A_{p+k-1,q-k}$ for $0 \leq k \leq q$ and $p+k > 0$, such that:*

(i) *for each q, $(A_{*,q}, d_0)$ is an acyclic chain complex;*

(ii) $\sum_{i=0}^{k} d_i d_{k-i} = 0$*;*

(iii) *if $C_q = H_0(A_{*,q}, d_0)$ and ∂'_q is the induced R-homomorphism $C_q \longrightarrow C_{q-1}$ which is well-defined by* (ii)*, then (C_*, ∂'_*) is also an acyclic chain complex.*

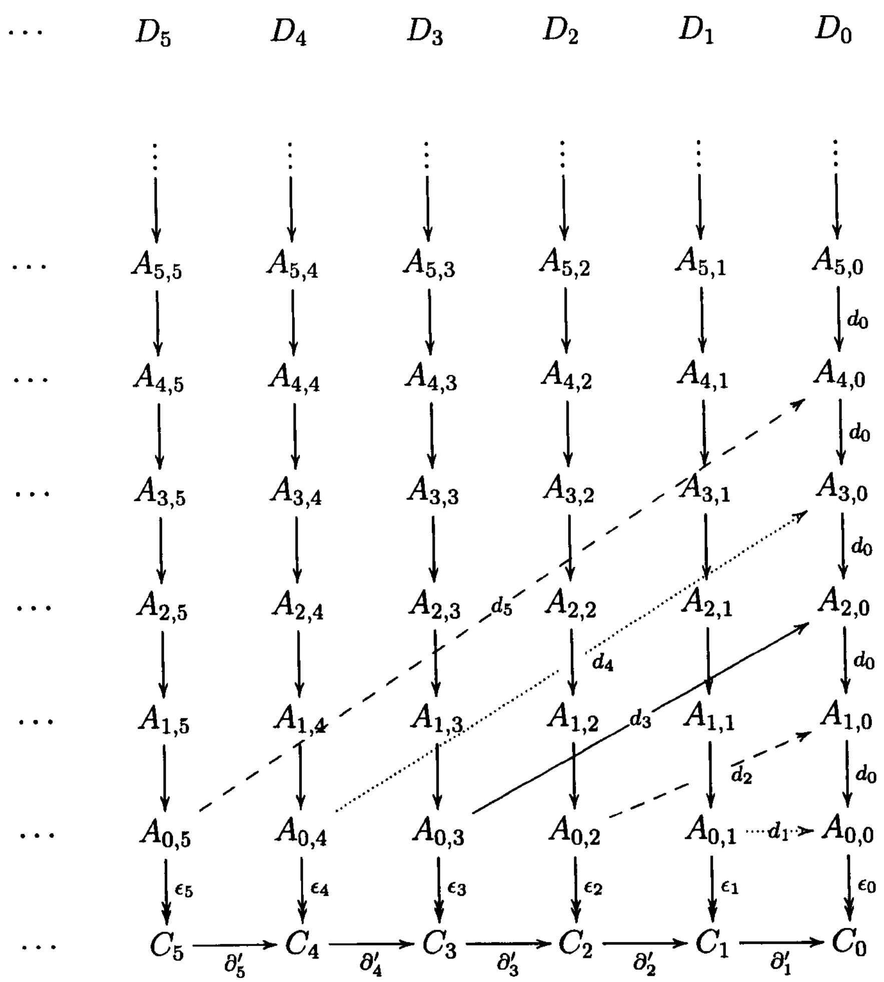


Figure 2.1: Vertical complexes are $\left(\bigoplus_{\alpha'_q} \mathbb{Z}G \otimes_{\mathbb{Z}K} \mathbf{B},\ d_0\right)$

Let $A_n = A_{0,n} \bigoplus A_{1,n-1} \bigoplus \dots \bigoplus A_{n,0}$, *for each* $n \geq 0$, $\delta_n = \sum d_k : A_n \longrightarrow A_{n-1}$. *Then* (A_n, δ_n) *is an acyclic chain complex.*

Proof. Let $(x_0, \dots, x_{n-1}) \in \mathrm{Ker}(\delta_{n-1})$ (where $x_0 \in A_{0,n-1}, \dots, x_{n-1} \in A_{n-1,0}$) so that by the definition of δ_{n-1} we have the following equations:

(1) $d_1x_0 + d_0x_1 = 0$

(2) $d_2x_0 + d_1x_1 + d_0x_2 = 0$

$\vdots$

$(n-2)$ $d_{n-2}x_0 + d_{n-3}x_1 + d_{n-4}x_2 + \dots + d_0x_{n-2} = 0$

$(n-1)$ $d_{n-1}x_0 + d_{n-2}x_1 + d_{n-3}x_2 + \dots + d_1x_{n-2} + d_0x_{n-1} = 0.$

To show that $(x_0, \dots, x_{n-1}) \in \mathrm{Im}(\delta_n)$ we need to find $(a_0, \dots, a_n) \in A_n$, (i.e., $a_0 \in A_{0,n}, \dots, a_n \in A_{n,0}$) such that the following equations hold:

(1^*) $d_1a_0 + d_0a_1 = x_0$

(2^*) $d_2a_0 + d_1a_1 + d_0a_2 = x_1$

$\vdots$

$(j-1^*)$ $d_{j-1}a_0 + d_{j-2}a_1 + \dots + d_0a_{j-1} = x_{j-2}$

(j^*) $d_ja_0 + d_{j-1}a_1 + \dots + d_1a_{j-1} + d_0a_j = x_{j-1}$

$\vdots$

(n^*) $d_n a_0 + d_{n-1} a_1 + \cdots + d_1 a_{n-1} + d_0 a_n = x_{n-1}$.

Rearranging these gives:

$(1')$ $d_0 a_1 = x_0 - d_1 a_0$

$(2')$ $d_0 a_2 = x_1 - d_2 a_0 - d_1 a_1$

$\vdots$

$(j-1')$ $d_0 a_{j-1} = x_{j-2} - d_{j-1} a_0 - d_{j-2} a_1 - \cdots - d_1 a_{j-2}$

(j') $d_0 a_j = x_{j-1} - d_j a_0 - d_{j-1} a_1 - \cdots - d_1 a_{j-1}$

$\vdots$

(n') $d_0 a_n = x_{n-1} - d_n a_0 - d_{n-1} a_1 - \cdots - d_1 a_{n-1}$.

To find $a_0 \in A_{0,n}$, and $a_1 \in A_{1,n-1}$, observe that $d_0 x_1 \in d_0 A_{1,n-2}$, so $[d_0 x_1] = [0] \in C_{n-2}$, and by (1) above, $d_1 x_0 = -d_0 x_1 \in d_0 A_{1,n-2}$. So we have $[d_1 x_0] = [0] = \partial'_{n-1}[x_0]$ which shows $[x_0] \in \mathrm{Im}(\partial'_n)$. So there exists an $a_0 \in A_{0,n}$ such that $\partial'_n[a_0] = [d_1 a_0] = [x_0] \in C_{n-1}$, i.e. $x_0 - d_1 a_0 \in \mathrm{Ker}(\partial'_{n-1}) = d_0 A_{0,n-1}$ and there exists an $a_1 \in A_{1,n-1}$ which satisfies $(1')$ above.

Suppose, by induction, $a_2 \in A_{2,n-2}, \ldots, a_{j-1} \in A_{j-1,n-j+1}$, exist which satisfy equations $(2')$, …, $(j-1')$ above, by part (ii) of the proposition we get:

$(1'')$ $d_0 d_j a_0 + d_1 d_{j-1} a_0 + d_2 d_{j-2} a_0 + \cdots + d_{j-1} d_1 a_0 = 0$

(2″) $d_0 d_{j-1} a_1 + d_1 d_{j-2} a_1 + d_2 d_{j-3} a_1 + \cdots + d_{j-1} d_0 a_1 = 0$

$\vdots$

(j'') $d_0 d_1 a_{j-1} + d_1 d_0 a_{j-1} = 0$

By applying d_0 to the right hand side of equation (j'), we get

$$d_0 x_{j-1} - d_0 d_j a_0 - d_0 d_{j-1} a_1 - \cdots - d_0 d_1 a_{j-1},$$

now substitute using equations (1″), (2″),...,(j''), to get

$$\begin{aligned} d_0 x_{j-1} &+ (d_1 d_{j-1} a_0 + d_2 d_{j-2} a_0 + \cdots + d_{j-1} d_1 a_0) \\ &+ (d_1 d_{j-2} a_1 + d_2 d_{j-3} a_1 + \cdots + d_{j-1} d_0 a_1) \\ &\vdots \\ &+ (d_1 d_0 a_{j-1}), \end{aligned}$$

rearranging this gives

$$\begin{aligned} d_0 x_{j-1} &+ d_1 (d_{j-1} a_0 + d_{j-2} a_1 + \cdots + d_0 a_{j-1}) \\ &+ d_2 (d_{j-2} a_0 + d_{j-3} a_1 + \cdots + d_0 a_{j-2}) \\ &\vdots \\ &+ d_{j-1} (d_1 a_0 + d_0 a_1) \end{aligned}$$

now substitutions using equations (1*),...,($j-1$*), gives

$$d_0 x_{j-1} + d_1 x_{j-2} + d_2 x_{j-3} + \cdots + d_{j-1} x_0$$

which equals zero by equation $(j-1)$. So, by part (i) of the proposition, there exists an $a_j \in A_{j,n-j}$ that satisfies equation (j'), hence given an element in $\text{Ker}(\delta_{n-1})$ it has been shown that there is an element is in the $\text{Im}(\delta_n)$ as required, therefore (A_n, δ_n) is an acyclic chain complex. □

Chapter 3

Crossed Complexes and Crossed Resolutions

This chapter introduces the remaining tools necessary for constructing free crossed resolutions for group extensions.

We shall give the definitions of crossed modules, crossed complexes and crossed resolutions of groups. We will explore the meaning of free with respect to a crossed resolution and give a crossed resolution version of the comparison theorem, [12].

Finally we state the tensor product of crossed complexes.

3.1 Crossed Complexes and Resolutions

Given two groups G and H, and a homomorphism $\partial : G \longrightarrow H$ where H acts on G, defined by $(h, g) \mapsto {}^h g$. Then ∂ is called the boundary homomorphism if it satisfies the following axioms:

$$\text{CM1.} \quad \partial({}^h g) = h\partial(g)h^{-1}, \quad \text{and} \quad \text{CM2.} \quad gg'g^{-1} = {}^{\partial g}g',$$

for all $g, g' \in G$ and $h \in H$. In this case $\partial : G \longrightarrow H$ is called a **crossed module.** Sometimes referred to as a **crossed H-module.**

Some basic examples:

(1) A conjugation crossed module: $\partial : K \hookrightarrow G$, where ∂ is the inclusion map of a normal subgroup K of the group G, and the action is conjugation ${}^g k = gkg^{-1}$

(2) An automorphism crossed module: $\alpha : G \longrightarrow Aut(G)$, where $\alpha(g) = \alpha_g$ is an inner automorphism of G, and the action is given by conjugation, ${}^{\alpha_g}g' = gg'g^{-1}$.

(3) A central extension crossed module: $\partial : G \twoheadrightarrow H$, where ∂ is a projection with kernel contained in the center of G, and the action ${}^h g = \bar{h}g\bar{h}^{-1}$, where $\bar{h} \in (\partial^{-1}h)$.

Let $\partial : G \longrightarrow H$ be a crossed module, then there are some very important consequences of the axioms for a crossed modules:

Remarks 3.1.

- *The kernel of ∂ lies in the center of G, and so in particular it is abelian: by CM2. for $g, g' \in G$ with $\partial g = 1$, we have $gg'g^{-1} = {}^{\partial g}g' = g'$, which shows that $g \in \mathrm{Ker}\partial$ commutes with all $g' \in G$ as stated;*

- *The image of ∂ is a normal subgroup of H: by CM1. the conjugate of the image of $g \in G$ is still contained in the image of ∂;*

- *The group G is abelian if and only if the image of ∂ acts on it trivially: by CM2. $gg'g^{-1} = {}^{\partial g}g'$, we need ∂g to be trivial for all $g \in G$ if G is to be abelian.*

3.1.1 Crossed Complexes

A crossed complex, [5], is basically a chain complex of abelian groups, except that there is some slightly non-abelian information in low degrees. This non-abelian information is given by a crossed module at the bottom of the crossed complex.

In detail, a **crossed complex C** consists of a group C_1, together with groups C_n for $n \geq 2$ with C_1-actions, and homomorphisms:

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \cdots \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \longrightarrow C_0 = \{*\},$$

in which

1. each map ∂_n respects the action of C_1,

2. the composite $\partial_n \partial_{n-1}$ is trivial,

3. $\partial_2 : C_2 \longrightarrow C_1$ is a crossed module

4. for $n \geq 3$, C_n is a $\mathbb{Z}G$-module, where $G \cong C_1/\partial_2 C_2$.

Sometimes one considers crossed complexes of **groupoids** with a set of base-points C_0. Since we are only considering groups we have defined

$$C_0 = \{*\}.$$

Example: the crossed complex of a filtered space

The **fundamental crossed complex** πX of a pointed filtered space

$$\{*\} = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X = \bigcup X_n$$

is the crossed complex given by the connecting homomorphisms between relative homotopy groups, with the action of the fundamental group:

$$\cdots \longrightarrow \pi_{n+1}(X_{n+1}, X_n) \xrightarrow{\partial_{n+1}} \pi_n(X_n, X_{n-1}) \xrightarrow{\partial_n} \pi_{n-1}(X_{n-1}, X_{n-2}) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_2(X_2, X_1) \xrightarrow{\partial_2} \pi_1(X_1) \longrightarrow \{*\}.$$

The main examples of filtered spaces that we meet are CW complexes (or "cell complexes"), which are filtered by their n-dimensional sketeta, $n \geq 0$. Simplicial sets are also examples since their geometric realisations are CW complexes. The fundamental crossed complex of a CW complex or of a simplicial set is in fact a **free** crossed complex, the freeness of the complex is discussed later [5].

3.1.2 Homology and Crossed Resolutions

We have seen that for a crossed complex **C**

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \cdots \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \longrightarrow C_0 = \{*\},$$

the image of ∂_2 is normal in C_1, so we can define the first homology group

$$H_1(\mathbf{C}) = C_1/\partial_2 C_2.$$

Note that this group is not always an abelian group.

The group C_2 in a crossed complex is not always abelian, but we have seen that the kernel of ∂_2 is abelian group, so we can define

$$H_2(\mathbf{C}) = \ker(\partial_2)/\partial_3 C_3.$$

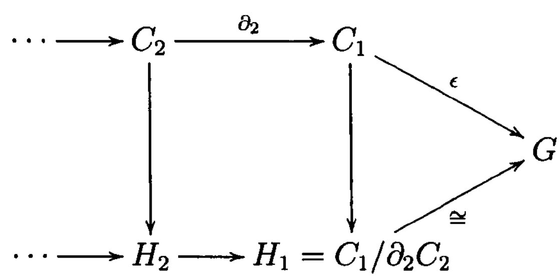
For $n \geq 3$ the all the groups involved are abelian so it is obvious that the homology can be defined in exactly the same way as for chain complexes,

$$H_n(\mathbf{C}) = \ker(\partial_n)/\partial_{n+1} C_{n+1}.$$

We can therefore make the following definition:

A crossed complex $(\mathbf{C}, \partial)$ is a **crossed resolution** of a group G if

- there is an augmentation $\epsilon : C_1 \longrightarrow G$, i.e. $\epsilon\partial_2$ is trivial, which induces an isomorphism



$$H_1(C) \cong G$$

- the homology is trivial for all $n \geq 2$,

$$H_n(C) = 1, \qquad n \geq 2$$

A crossed complex is a **free crossed resolution** of a group if it is a resolution which is also free, but the definition of **free** is quite complicated, and will be explained in detail in section [2].

3.2 Free Crossed Resolutions of Groups

In this work it is very important to understand the notions of free crossed modules and of free crossed complexes of groups. These were introduced by J.H.C. Whitehead, [14] (who called them **homotopy systems**) and were later developed and applied by authors such as Baues, Brown, Ellis and Huebschmann, [1], [5], [7], [9].

In many ways free crossed complexes are similar to free groups, and we present in this section the three basic ideas that are essential for understanding them:

- The data required to specify a free crossed complex, that is, what is meant by a collection of **generators** in this context.
- The **universal** property of a free crossed complex, which will also guarantee the **uniqueness** of the free crossed complex generated by the data.
- The **construction** of free crossed complexes, as explicit sets of elements that can be built up from the generators.

3.2.1 Generating data

A **free crossed complex** C is given by

1. a set X_1 whose elements are the generators of a **free group** C_1.
2. a set X_2 and a function $\theta_2 : X_2 \longrightarrow C_1$, such that the elements of X_2 are the two-dimensional generators of a **free** C_1**-crossed module**

$$\partial_2 : C_2 \longrightarrow C_1$$

3. a set X_3 and a function $\theta_3 : X_3 \longrightarrow C_2$ such that the composition

$$X_3 \xrightarrow{\theta_3} C_2 \xrightarrow{\partial_2} C_1$$

is trivial so one may write $\theta_3 : X_3 \longrightarrow \ker(\partial_2) \hookrightarrow C_2$.

The elements of X_3 are the generators of a **free left G-module** C_3, where $G = C_1/\partial_2 C_2$.

4. for each $n \geq 4$, similarly, a set X_n and a function $\theta_n : X_n \longrightarrow C_{n-1}$ such that the composition

$$X_n \xrightarrow{\theta_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2}$$

is trivial, so one may write $\theta_n : X_n \longrightarrow \ker(\partial_{n-1}) \hookrightarrow C_{n-1}$.

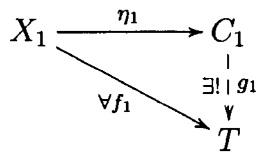
The elements of X_n are the generators of a **free left G-module** C_n.

3.2.2 Freeness and universal properties

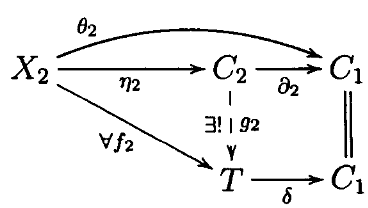
The different uses of the word **free** above are made more precise if we explain more explicitly the different universal properties the free crossed complex has in each dimension.

1. The inclusion of generators $\eta_1 : X_1 \longrightarrow C_1$ in dimension 1 has the usual universal property of free groups: if we are given any group T, then any function $f_1 : X_1 \longrightarrow T$ will extend to a unique group homomorphism g_1 from

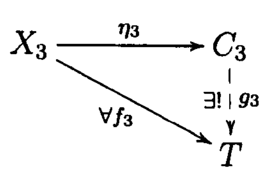
C_1 to T,



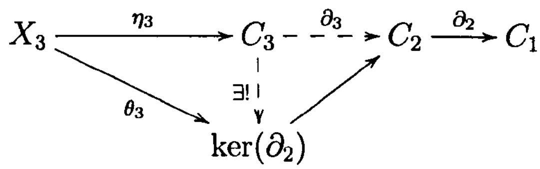
2. The inclusion of generators $\eta_2 : X_2 \longrightarrow C_2$ in dimension 2 has the following universal property for crossed modules: the condition $\theta_2 = \partial_2\eta_2$ holds and if we are given any crossed C_1-module $\delta : T \longrightarrow C_1$, then any function $f_2 : X_2 \longrightarrow T$ satisfying the condition $\theta_2 = \delta f_2$ will extend to a unique crossed C_1-module homomorphism (g_2, id_{C_1}) from $\partial_2 : C_2 \longrightarrow C_1$ to $\delta : T \longrightarrow C_1$



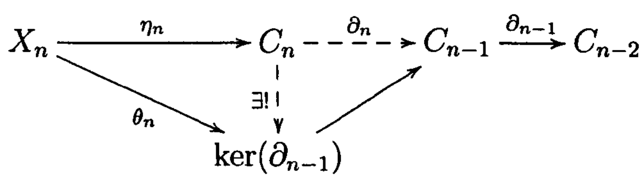
3. The inclusion of generators $\eta_3 : X_3 \longrightarrow C_3$ in dimension 3 has the universal property that, if we are given any left $\mathbb{Z}G$-module T, then any function $f_3 : X_3 \longrightarrow T$ will extend to a unique C_1-module homomorphism g_3 from C_3 to T.



In particular we can take T to be the kernel of $\partial_2 : C_2 \longrightarrow C_1$, since the axioms for a crossed module imply that $\partial_2 C_2$ acts trivially on $\ker(\partial_2)$. Therefore the function θ_3 from X_3 to $\ker(\partial_2)$ has a unique extension to a homomorphism from C_3, giving a boundary map $\partial_3 : C_3 \longrightarrow \ker(\partial_2) \hookrightarrow C_2$.



4. In exactly the same way, the inclusion of generators $\eta_n : X_n \longrightarrow C_n$ in dimensions $n \geq 4$ has the usual universal property for maps from X_n to left $C_1/\partial_2(C_2)$-modules, and in particular the function θ_n has a unique extension to a homomorphism from C_n, giving a boundary map $\partial_n : C_n \longrightarrow \ker(\partial_{n-1}) \hookrightarrow C_{n-1}$.



3.2.3 Constructions

Using the universal properties above it follows that, up to isomorphism, free groups (or crossed modules, or crossed complexes, or any algebraic structure) are completely determined by the generating sets (and the functions θ_n). That is, there is essentially only one way to generate the free structure once its generators are given. To be completely explicit, actual constructions can be given which say exactly what elements the free structure contains. Although, up to isomorphism, everything is completely determined, these explicit constructions may involve some arbitary choices of notation.

1. The free group C_1 on the generating set X_1 may be constructed as the set of all words

$$x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}, \qquad n \geq 0,\ \varepsilon_i = \pm 1,\ x_i \in X_1,$$

which are **reduced** in the sense that no letter x is ever followed or preceeded by x^{-1} because we can always write $x^{\varepsilon} x^{-\varepsilon} = 1$, the empty word. Multiplication is given by concatenation (followed by reduction, if it is necessary), and inverses of words are given by reversing the words and changing the signs of each exponent ε.

For convenience we often denote the (left) conjugation action of C_1 on itself by

$$^{c}d \;=\; c \cdot d \cdot c^{-1} \qquad c, d \in C_1.$$

2. The free crossed C_1-module $\partial_2 : C_2 \longrightarrow C_1$, on X_2 and $\theta_2 : X_2 \longrightarrow C_1$, may be constructed as follows.

   - Let C_2' be the free group on the set $C_1 \times X_2$. For convenience a generator (c, x) in $C_1 \times X_2$ is usually written as ^{c}x, or just as x in the case $c = 1$. Then the elements of the free group C_2' are given by the reduced words

$$^{c_1}x_1^{\varepsilon_1} \cdot {}^{c_2}x_2^{\varepsilon_2} \dots {}^{c_n}x_n^{\varepsilon_n}, \qquad n \geq 0,\ \varepsilon_i = \pm 1,\ c_i \in C_1,\ x_i \in X_2,$$

- Let $C_2 = C_2'/N$ where N is the Peiffer commutator subgroup, that is, the normal subgroup of C_2' generated by all words of the form

$$[{}^{c_1}x_1, {}^{c_2}x_2]' \;:=\; {}^{c_1}x_1 \cdot {}^{c_2}x_2 \cdot {}^{c_1}x_1^{-1} \cdot {}^{c_1\theta_2(x_1)c_1^{-1}c_2}x_2^{-1}$$

for all generators ${}^{c_1}x_1$, ${}^{c_2}x_2$ of C_2'.

- Let C_1 act on C_2 by the rule

$${}^{c}\left({}^{c_1}x_1^{\varepsilon_1} \cdot {}^{c_2}x_2^{\varepsilon_2} \ldots {}^{c_n}x_n^{\varepsilon_n}\right) \;=\; {}^{cc_1}x_1^{\varepsilon_1} \cdot {}^{cc_2}x_2^{\varepsilon_2} \ldots {}^{cc_n}x_n^{\varepsilon_n}$$

This C_1-action is defined first on C_2', but it is easy to check that it sends Peiffer commutators to Peiffer commutators

$${}^{c}[{}^{c_1}x_1, {}^{c_2}x_2]' = [{}^{cc_1}x_1, {}^{cc_2}x_2]'$$

so the action is well defined on the quotient $C_2 = C_2'/N$ also.

- Let $\partial_2 : C_2 \longrightarrow C_1$ be the homomorphism defined as the extension of $\theta_2 : X_2 \longrightarrow C_1$,

$$\partial_2\left({}^{c_1}x_1^{\varepsilon_1} \cdot {}^{c_2}x_2^{\varepsilon_2} \ldots {}^{c_n}x_n^{\varepsilon_n}\right) \;=\; {}^{c_1}\theta_2 x_1^{\varepsilon_1} \cdot {}^{c_2}\theta_2 x_2^{\varepsilon_2} \ldots {}^{c_n}\theta_2 x_n^{\varepsilon_n}$$

Again, this homomorphism is defined first on C_2', and clearly the first of the two crossed module axioms, $\partial_2({}^{c}a) = {}^{c}\partial_2(a)$, holds in C_1. It is easy to check that the Peiffer commutators lie in the kernel

$$\partial_2[{}^{c_1}x_1, {}^{c_2}x_2]' = 1$$

so the homomorphism is well defined on the quotient $C_2 = C_2'/N$ also. The vanishing of the Peiffer commutators implies now that the second of the two crossed module axioms, $aba^{-1} = {}^{\partial_2(a)}b$, holds in C_2.

3. For all $n \geq 3$, the free $C_1/\partial_2 C_2$-module C_n on the generating set X_n may be constructed, using classical notation, as the direct sum of 1-generator $\mathbb{Z}[C_1/\partial_2 C_2]$-modules

$$\bigoplus_{x \in X_n} \mathbb{Z}[C_1/\partial_2 C_2] \cdot x.$$

with the left module structure $c \cdot \sum \lambda_x x = \sum (c \cdot \lambda_x x)$ and the abelian group structure $\sum \lambda_x x + \sum \mu_x x = \sum (\lambda_x + \mu_x) x$.

In lower dimensions we have to use multiplicative and generally nonabelian notation rather than additive notation. When it is is convenient we will often use multiplicative notation in higher dimensions as well, even though the structure is abelian. We will then not use directly the group ring structure of $\mathbb{Z}[C_1/\partial_2 C_2]$ but instead write $(c \pm c')x$ as ${}^c x \cdot {}^{c'} x^{\pm 1}$.

Thus, we construct C_n as the group with generators ${}^{c_1} x_n$ for all $c_1 \in C_1$ and $x_n \in X_n$, and write ${}^1 x_n$ simply as x_n, subject to the relations

$$ {}^{\partial_2 c_2} x_n = x_n, \qquad {}^{c_1} x_n \cdot {}^{c_1'} x_n' = {}^{c_1'} x_n' \cdot {}^{c_1} x_n $$

for all $c_2 \in C_2$, $x_n, x_n' \in X_n$ and $c_1, c_1' \in C_1$. Using this notation the C_1-action on C_n is defined by the same rule as for the C_1-action on C_2 given above.

3.3 Lifts of Group Actions

The following lemma is a crossed complex version of the classical comparison theorem in homological algebra that, given a free complex and an exact complex, and any homomorphism between their zeroth homology groups, there exists a homomorphism of complexes which induces the given map in homology.

Lemma 3.2. *Consider the following diagram*

$$\begin{array}{ccccccccccccccc} \cdots \xrightarrow{\partial_{n+2}} & B_{n+1} & \xrightarrow{\partial_{n+1}} & B_n & \xrightarrow{\partial_n} \cdots \longrightarrow & B_3 & \xrightarrow{\partial_3} & B_2 & \xrightarrow{\partial_2} & B_1 & \xrightarrow{\epsilon} & G & \longrightarrow & 0 \\ & & & & & & & & & & & \downarrow{\scriptstyle \alpha} & & \\ \cdots \xrightarrow{\partial'_{n+2}} & C_{n+1} & \xrightarrow{\partial'_{n+1}} & C_n & \xrightarrow{\partial'_n} \cdots \longrightarrow & C_3 & \xrightarrow{\partial'_3} & C_2 & \xrightarrow{\partial'_2} & C_1 & \xrightarrow{\epsilon'} & H & \longrightarrow & 0 \end{array} \qquad (3.1)$$

where the top row is a free crossed resolution of the group G, the bottom row is a crossed resolution of the group H, and $\alpha : G \longrightarrow H$ is a group homomorphism, then there exist maps $\alpha_* : B_* \longrightarrow C_*$ lifting α to a homomorphism of crossed complexes.

Proof. We suppose that the free crossed complex of the group G has a set of generators X_n in each degree $n \geqslant 1$, so that a set of generators of the group G is given by the elements $\epsilon(x_1)$ for $x_1 \in X_1$. Similarly, the group H has a set of generators Y_n in each degree $n \geqslant 1$, with the set of generators $\epsilon'(y_1)$ for each $y_1 \in Y_1$.

We need to prove existence of $\alpha_* : B_* \longrightarrow C_*$ such that $\partial'_{n+1}\alpha_{n+1} = \alpha_n\partial_{n+1}$ for all $n \geqslant 1$ and $\epsilon'\alpha_1 = \alpha\epsilon$ by induction on n, i.e. that the following diagram commutes

$$\begin{array}{ccccccccccccccc}
\cdots \xrightarrow{\partial_{n+2}} & B_{n+1} & \xrightarrow{\partial_{n+1}} & B_n & \xrightarrow{\partial_n} \cdots \longrightarrow & B_3 & \xrightarrow{\partial_3} & B_2 & \xrightarrow{\partial_2} & B_1 & \xrightarrow{\epsilon} & G & \longrightarrow & 0 \\
 & \downarrow{\scriptstyle \alpha_{n+1}} & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha} & & \\
\cdots \xrightarrow{\partial'_{n+2}} & C_{n+1} & \xrightarrow{\partial'_{n+1}} & C_n & \xrightarrow{\partial'_n} \cdots \longrightarrow & C_3 & \xrightarrow{\partial'_3} & C_2 & \xrightarrow{\partial'_2} & C_1 & \xrightarrow{\epsilon'} & H & \longrightarrow & 0
\end{array} \qquad (3.2)$$

For each of the generators $x_1 \in X_1$, we have $\alpha\epsilon(x_1) \in H(= \mathrm{Im}\epsilon')$ since ϵ' is an epimorphism. We can choose a section $\jmath : H \longrightarrow C_1$ such that $\jmath(1_H) = 1_{C_1}$ and $\epsilon'\jmath = \mathrm{id}_H$. Now we define a function α_1 on the generators to be $\alpha_1(x_1) = \jmath(\alpha\epsilon(x_1))$. Since B_1 is free on these generators, this function extends uniquely to a homomorphism $\alpha_1 : B_1 \longrightarrow C_1$ of groups. So that $\epsilon'\alpha_1 = \alpha\epsilon$ and we see that the first square commutes,

$$\begin{array}{ccccc}
X_1 & \xrightarrow{\eta_1} & B_1 & \xrightarrow{\epsilon} & G \\
 & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha} \\
 & & C_1 & \underset{\jmath}{\overset{\epsilon'}{\rightleftarrows}} & H
\end{array}$$

and then freeness yields the required unique crossed module homomorphism.

Now we need to define α_2, since $\partial_2 : B_2 \longrightarrow B_1$ is a free crossed B_1-module we define α_2 on the generators $x_2 \in X_2$. For each $x_2 \in X_2$ we have that

$$(\epsilon'\alpha_1\partial_2)({}^{x_1}x_2) = \epsilon'((\alpha_1\partial_2)({}^{x_1}x_2)) = (\alpha\epsilon\partial_2)({}^{x_1}x_2) = 1_H$$

So we can define $\alpha_2({}^{x_1}x_2)$ to be an element in C_2 that maps to $(\alpha_1\partial_2)({}^{x_1}x_2)$ under

∂'_2, i.e. $\partial'_2\alpha_2 = \alpha_1\partial_2$. Which makes the following diagram commutative,

$$\begin{array}{ccccccc}
X_2 & \xrightarrow{\eta_2} & B_2 & \xrightarrow{\partial_2} & B_1 & \xrightarrow{\epsilon} & G \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha} \\
 & & C_2 & \xrightarrow[\partial'_2]{} & C_1 & \xrightarrow[\epsilon']{} & H
\end{array}$$

Now suppose that $n \geqslant 2$ and that we have already constructed the homomorphisms $\alpha_i : B_i \longrightarrow C_i$ for $1 \leqslant i \leqslant n$, so we have

$$\epsilon'\alpha_1 = \alpha\epsilon, \quad \text{and} \quad \partial'_{i+1}\alpha_{i+1} = \alpha_i\partial_{i+1}, \quad \text{for} \quad 2 \leqslant i \leqslant n.$$

$$\begin{array}{ccccccc}
X_{n+1} & \xrightarrow{\eta_{n+1}} & B_{n+1} & \xrightarrow{\partial_{n+1}} & B_n & \xrightarrow{\partial_n} & B_{n-1} \\
 & \searrow & \downarrow{\scriptstyle \alpha_{n+1}} & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha_{n-1}} \\
 & & C_{n+1} & \xrightarrow[\partial'_{n+1}]{} & C_n & \xrightarrow[\partial'_n]{} & C_{n-1}
\end{array}$$

Since B_{n+1} is a free crossed $\mathbb{Z}G$-module, the sequence $C_{n+1} \xrightarrow{\partial'_{n+1}} C_n \xrightarrow{\partial'_n} C_{n-1}$ is exact and the module homomorphism $\alpha_n\partial_{n+1} : B_{n+1} \longrightarrow C_n$ is such that

$$\partial'_n(\alpha_n\partial_{n+1}) = (\partial'_n\alpha_n)\partial_{n+1} = (\alpha_{n-1}\partial_n)\partial_{n+1} = \alpha_{n-1}(\partial_n\partial_{n+1}) = 0,$$

and it follows that there exists a homomorphism $\alpha_{n+1} : B_{n+1} \longrightarrow C_{n+1}$ such that $\partial'_{n+1}\alpha_{n+1} = \alpha_n\partial_{n+1}$. □

There is a further property of the lifts defined by this comparison Lemma that we have not proven here: if $\alpha_*, \beta_* : B_* \longrightarrow C_*$ are *two* lifts of α then they are 'homotopic' as crossed complex homomorphisms, [4]. Some important examples of this are given next.

3.3.1 Some Properties of Lifts

Now suppose $\alpha : H \times K \longrightarrow K$ is a group action, and let a and b be two elements of the group H. Then we have various homomorphisms of K, denoted $\alpha_a(x) = \alpha(a, x)$.

Since α is a group action, these homomorphisms are related. For example,

$$\alpha_a \circ \alpha_b = \alpha_{ab} : K \longrightarrow K,$$

$$(\alpha_a)^{-1} = \alpha_{a^{-1}} : K \longrightarrow K,$$

because

$$(\alpha_a \circ \alpha_b)(k) = {}^a({}^b k) = {}^{ab}k = \alpha_{ab}(k),$$

$$(\alpha_a \circ \alpha_{a^{-1}})(k) = {}^a({}^{a^{-1}} k) = {}^{aa^{-1}}k = \alpha_1(k) = k.$$

Now if B is a free crossed resolution of the group K then the comparison lemma says that these group homomorphisms can be lifted to crossed complex homomorphisms

$$\alpha(a) : B \longrightarrow B$$

$$\alpha(b) : B \longrightarrow B$$

$$\alpha(ab) : B \longrightarrow B$$

$$\alpha(a^{-1}) : B \longrightarrow B.$$

In general however neither the equality $\alpha(a) \circ \alpha(b) = \alpha(ab)$ nor $(\alpha(a))^{-1} = \alpha(a^{-1})$ will be true. Lifts of actions exist for each element of H, but we cannot expect composites of the lifts to agree with the lifts of composites. We cannot expect the lift of an inverse be the same as the inverse of a lift, and we cannot even expect lifts to be invertible.

Example 3.3. *Let*

$$H = \langle h | h^2 \rangle,$$

$$K = \langle k | k^3 \rangle,$$

with the action

$$\alpha(h, k) = {}^h k = k^2.$$

Because h has order 2, so does the homomorphism $\alpha_h : K \longrightarrow K$. The inverse of α_h is just α_h itself. Now a free crossed resolution B of K may be constructed starting with

$$B_1 = \langle x_1 \rangle$$

and the augmentation map $\epsilon : B_1 \longrightarrow K$ given by $\epsilon(x_1) = k$ and in general

$$\epsilon({x_1}^n) = k^n = k^{n \bmod 3}.$$

To find a lift $\alpha(h) : B \longrightarrow B$ of the action homomorphism $\alpha_h : K \longrightarrow K$, one must first find a lift $\alpha_1(h) : B_1 \longrightarrow B_1$. Let

$$(\alpha_1(h))(x_1) = {x_1}^2$$

Then

$$\epsilon \circ \alpha_1(h) : x_1 \mapsto \epsilon({x_1}^2) = k^2$$

$$\alpha_h \circ \epsilon : x_1 \mapsto \alpha_h(k) = k^2$$

as required. But $\alpha_1(h) : B_1 \longrightarrow B_1$ *is not surjective, so the lift is not invertible.*

Lemma 3.4. *Suppose* $\alpha : H \times K \longrightarrow K$ *is a group action, and let* $h \in H$*. If* B *is a free crossed resolution of* K *then there exists a (not necessarily unique) function*

$$\nu_1(h) : B_1 \longrightarrow B_2$$

such that for each element $b \in B_1$ *the following equation holds,*

$$(\alpha_1(h) \circ \alpha_1(h^{-1}))(b) = \partial_2\left((\nu_1(h))(b)\right) \cdot b$$

Proof. The images of the elements

$$b, \qquad (\alpha_1(h) \circ \alpha_1(h^{-1}))(b)$$

under the augmentation map $\epsilon : B_1 \longrightarrow K$ are equal, since

$$\begin{aligned} \epsilon \circ \alpha_1(h) \circ \alpha_1(h^{-1}) &= \alpha_h \circ \epsilon \circ \alpha_1(h^{-1}) \\ &= \alpha_h \circ \alpha_{h^{-1}} \circ \epsilon \\ &= \epsilon. \end{aligned}$$

Therefore

$$(\alpha_1(h) \circ \alpha_1(h^{-1}))(b) \cdot b^{-1}$$

is in the kernel of ϵ and so it is in the image of ∂_2. That is, there exists an element of B_2 that we can call $(\nu_1(h))(b)$ such that

$$(\alpha_1(h) \circ \alpha_1(h^{-1}))(b) \cdot b^{-1} = \partial_2\left((\nu_1(h))(b)\right)$$

as required. □

3.4 Tensor Products of Crossed Complexes

Crossed complexes have a geometrically-motivated tensor product, introduced in [3], using an equivalence with the category of (cubical) ω-groupoids. The tensor product $C \otimes D$ has a presentation by generators $c_m \otimes d_n$ in degree $m+n$ for each $c_m \in C_m$ and $d_n \in D_n$, $m, n \geq 0$, and certain bilinearity and boundary relations:

$$c_m \otimes (d_n \cdot d'_n) = \begin{cases} (c_0 \otimes d_n) \cdot (c_0 \otimes d'_n) & \text{if } m = 0 \text{ or } n \geq 2, \\ (c_m \otimes d_1) \cdot {}^{*\otimes d_1}(c_m \otimes d'_1) & \text{if } m \geq 1 \text{ and } n = 1, \end{cases} \tag{3.3}$$

$$(c_m \cdot c'_m) \otimes d_n = \begin{cases} (c_m \otimes d_0) \cdot (c'_m \otimes d_0) & \text{if } m \geq 2 \text{ or } n = 0, \\ {}^{c_1 \otimes *}(c'_1 \otimes d_n) \cdot (c_1 \otimes d_n) & \text{if } m = 1 \text{ and } n \geq 1, \end{cases} \tag{3.4}$$

$$^{a\otimes *}(c \otimes d) \;=\; {}^{a}c \otimes d \qquad \text{if } a \in C_1,\ b = * \text{ and } m \geq 2, \qquad (3.5)$$

$$^{*\otimes b}(c \otimes d) \;=\; c \otimes {}^{b}d \qquad \text{if } a = *,\ b \in D_1 \text{ and } n \geq 2, \qquad (3.6)$$

$$\delta(c_1 \otimes d_1) \;=\; (c_1 \otimes *) \cdot (* \otimes d_1) \cdot (c_1 \otimes *)^{-1} \cdot (* \otimes d_1)^{-1} \qquad (3.7)$$

$$\delta(c_m \otimes *) \;=\; \partial_m c_m \otimes * \qquad \text{if } m \geq 2, \qquad (3.8)$$

$$\delta(* \otimes d_n) \;=\; * \otimes \partial_n d_n \qquad \text{if } n \geq 2, \qquad (3.9)$$

$$\delta(c_1 \otimes d_n) \;=\; {}^{c_1 \otimes *}(* \otimes d_n) \cdot (* \otimes d_n)^{(-1)} \cdot (c_1 \otimes \partial_n d_n)^{-1} \quad \text{if } n \geq 2, \qquad (3.10)$$

$$\delta(c_m \otimes d_1) \;=\; \partial_m c_m \otimes d_1 \cdot ({}^{*\otimes d_1}(c_m \otimes *) \cdot (c_m \otimes *)^{-1})^{(-1)^m} \quad \text{if } m \geq 2, \qquad (3.11)$$

$$\delta(c_m \otimes d_n) \;=\; \partial c_m \otimes d_n \cdot (c_m \otimes \partial d_n)^{(-1)^m} \qquad \text{if } m, n \geq 2 \qquad (3.12)$$

The following result may be found in the work of H. J. Baues and R. Brown, [1, 2]:

- If C and D are free crossed complexes, then so is the tensor product $C \otimes D$.

Generators for the tensor product $C \otimes D$ may be denoted $c_m \otimes d_n \in (C \otimes D)_{m+n}$ where c_m, d_n are generators of C, D respectively. If c_m, d_n are not generators, then $c_m \otimes d_n$ is to be interpreted according to (3.3)–(3.6) above. The boundary maps in the free crossed complex $C \otimes D$ are given by (3.7)–(3.12).

Theorem 3.5. *Suppose that B and C are free crossed resolutions of groups K and H respectively. Then the tensor product $B \otimes C$ is a free crossed resolution of $K \times H$.*

This means that if B has sets of generators X_n in each dimension $n \geq 1$ and C has sets of generators Y_n, then a free crossed resolution for the direct product group $K \times H$ is given by $B \otimes C$ which, as we saw above, has generators $x_p \otimes y_q$ in dimension $n = p+q$ for each p-dimensional generator $x_p \in X_p$ and each q-dimensional generator $y_q \in Y_q$. As well as giving the generators we need to say how the boundary maps are defined on the generators, and by the definition of the tensor product above we have

$$\partial(x_m \otimes y_n) = \begin{cases} x_1 y_1 {x_1}^{-1} {y_1}^{-1} & \text{if } p, q = 1, \\ \partial^{\mathrm{I}}_m(x_m \otimes y_n) \cdot \partial^{\mathrm{II}}_n(x_m \otimes y_n)^{(-1)^m} & \text{otherwise.} \end{cases} \tag{3.13}$$

Here the operators $\partial^{\mathrm{I}}_k(c \otimes d)$ and $\partial^{\mathrm{II}}_k(c \otimes d)$ are defined for $k \geq 2$ by $\partial c \otimes d$ and $c \otimes \partial d$ respectively, for $k = 1$ by ${}^{c\otimes *}(* \otimes d) \cdot (* \otimes d)^{-1}$ and ${}^{*\otimes d}(c \otimes *) \cdot (c \otimes *)^{-1}$ respectively, and vanish if $k = 0$.

It is our eventual aim to find some sort of twisted tensor product of crossed complexes, generalising the methods of Wall for chain complexes, which provides a free crossed resolution for any group extension, not just the direct product. We state the following conjecture which generalises the above result for direct products $G = K \times H$, and at the same time is a generalisation of Wall's results from chain complexes to crossed complexes.

Conjecture 3.6. *Suppose that B and C are free crossed resolutions of groups K*

and H respectively, and that B has sets of generators X_n in each dimension $n \geq 1$ and C has sets of generators Y_n.

Let G be a group defined as an extension of K by H. Then G has a free crossed resolution A with sets of generators

$$Z_n = \bigcup_{n=p+q} X_p \times Y_q$$

in each dimension $n \geq 1$.

We will sometimes use the notation $x_p \tilde{\otimes} y_q$ for a generator $z_n = (x_p, y_q) \in X_p \times Y_q$.

To prove this conjecture we need to specify expresssions for the boundaries $\partial(z_n)$, or prove that such expressions exist, for each $z_n = (x_p, y_q) \in X_p \times Y_q$, such that the crossed complex A is indeed a resolution, that is, A is exact in dimensions ≥ 2 and $H_1 A \cong G$. In order to specify the boundary maps of A we will have to put together the other information we are given: the boundary maps of B and C, together with the action and cocycle data which determines the group G.

We can also state the corresponding conjectures when we are only considering n-presentations rather than free crossed resolutions:

Theorem 3.7. *Suppose we are given n-presentations, with sets of generators X_k and Y_k for $1 \leq k \leq n$, of groups K and H respectively. Then any group G defined* *as an extension of* K *by* H *has a* n*-presentation with sets of generators*

$$Z_k = \bigcup_{k=p+q} X_p \times Y_q$$

in each dimension $1 \leq k \leq n$.

The first case to consider is $n = 2$, that is, $(p, q) = (0, 2)$, $(1, 1)$ or $(2, 0)$. This is a classical result on finding presentations (that is, generators and relations) for groups if we have been given presentations of a normal subgroup and the quotient.

The case $n = 3$ was considered by Ellis and Kholodna, [7], who gave a partial answer when G is a split extension. For large values of n the situation is abelian and should be similar to that of Wall's construction.

Chapter 4

Crossed Resolutions

4.1 Crossed Resolutions for Semidirect Products

Suppose K and H are groups and let $G = K \rtimes H$ be their semidirect product, defined by the action $\alpha : H \longrightarrow \mathrm{Aut}(K)$ where $h \mapsto \alpha(h) = \alpha_h : K \longrightarrow K$ such that $\alpha_h(k) = {}^h k$.

Given the free crossed resolutions $B_* \xrightarrow{\epsilon} K$,

$$\cdots \xrightarrow{\partial_{n+2}} B_{n+1} \xrightarrow{\partial_{n+1}} B_n \xrightarrow{\partial_n} \cdots \longrightarrow B_3 \xrightarrow{\partial_3} B_2 \xrightarrow{\partial_2} B_1 \xrightarrow{\epsilon} K \longrightarrow 1 \tag{4.1}$$

of the group K, and $C_* \xrightarrow{\epsilon'} H$,

$$\cdots \xrightarrow{\partial'_{n+2}} C_{n+1} \xrightarrow{\partial'_{n+1}} C_n \xrightarrow{\partial'_n} \cdots \longrightarrow C_3 \xrightarrow{\partial'_3} C_2 \xrightarrow{\partial'_2} C_1 \xrightarrow{\epsilon'} H \longrightarrow 1 \tag{4.2}$$

of the group H, we want to construct a free crossed resolution $A_* \xrightarrow{\varepsilon} G$

$$\cdots \longrightarrow A_n \xrightarrow{\delta_n} A_{n-1} \xrightarrow{\delta_{n-1}} \cdots \longrightarrow A_3 \xrightarrow{\delta_3} A_2 \xrightarrow{\delta_2} A_1 \xrightarrow{\varepsilon} G \longrightarrow 1, \qquad (4.3)$$

for their semidirect product G. The following diagram shows the free crossed resolutions of the groups K and H, and the short exact sequence for the group extension G.

$$\begin{array}{c} \cdots \xrightarrow{\partial_{n+2}} B_{n+1} \xrightarrow{\partial_{n+1}} B_n \xrightarrow{\partial_n} \cdots \longrightarrow B_3 \xrightarrow{\partial_3} B_2 \xrightarrow{\partial_2} B_1 \xrightarrow{\epsilon} K \longrightarrow 1 \\ \iota \downarrow \uparrow \varrho \\ G \\ \rho \downarrow \uparrow \jmath \\ \cdots \xrightarrow{\partial'_{n+2}} C_{n+1} \xrightarrow{\partial'_{n+1}} C_n \xrightarrow{\partial'_n} \cdots \longrightarrow C_3 \xrightarrow{\partial'_3} C_2 \xrightarrow{\partial'_2} C_1 \xrightarrow{\epsilon'} H \longrightarrow 1 \end{array} \qquad (4.4)$$

For $n \geqslant 1$, let X_n denote the set of generators in degree n of the groups B_n, and let Y_n denote the set of generators in degree n of groups C_n.

We choose a cross section, $\jmath : H \longrightarrow G$, for the group extension G, such that $\rho\jmath = \mathrm{id}_H$. We assume $\jmath(1_H) = 1_G$, and recall that $\jmath$ can be taken to be a homomorphism if and only if the group extension is in fact a semidirect product. Similarly, we can choose a projection ϱ with $\varrho\iota = \mathrm{id}_K$.

Given the free crossed resolutions $B_* \xrightarrow{\epsilon} K$, $C_* \xrightarrow{\epsilon'} H$, then the free groups B_1, C_1, generated by the sets X_1, Y_1 respectively, determine the sets $X = \{\epsilon(x)\}_{x \in X_1}$, $Y = \{\epsilon'(y)\}_{y \in Y_1}$, which are generating sets for K, H respectively.

A crossed complex version of the comparison theorem tells us that the action lifts to the crossed complex homomorphisms, $\alpha_*(\epsilon'(c))$, such that the diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & B_2 & \xrightarrow{\partial_2} & B_1 & \xrightarrow{\epsilon} & K & \longrightarrow 1 \\
 & \downarrow{\scriptstyle \alpha_2(\epsilon'(c))} & & \downarrow{\scriptstyle \alpha_1(\epsilon'(c))} & & \downarrow{\scriptstyle \alpha_{\epsilon'(c)}} & \\
\cdots \longrightarrow & B_2 & \xrightarrow[\partial_2]{} & B_1 & \xrightarrow[\epsilon]{} & K & \longrightarrow 1
\end{array}$$

commutes, i.e. $\partial_n \circ \alpha_n(\epsilon'(c)) = \alpha_{n-1}(\epsilon'(c)) \circ \partial_n$ and $\epsilon \circ \alpha_1(\epsilon'(c)) = \alpha_{\epsilon'(c)} \circ \epsilon$.

Our aim is to construct a free crossed resolution,

$$\cdots \longrightarrow A_n \xrightarrow{\delta_n} A_{n-1} \xrightarrow{\delta_{n-1}} \cdots \longrightarrow A_2 \xrightarrow{\delta_2} A_1 \xrightarrow{\varepsilon} G \longrightarrow 1,$$

inductively on the degree $n = 1, 2, 3, \ldots$.

4.1.1 Degree 1

The following Proposition shows how to find a set A_1 of generators for the extension G, and the corresponding epimorphism $A_1 \longrightarrow G$.

Proposition 4.1. *Let $A_1 = \langle Z_1 \rangle$ be the free group with the set of generators*

$$Z_1 \;:=\; \{*\} \times Y_1 \;\cup\; X_1 \times \{*\}.$$

Then there exists a unique homomorphism $\varepsilon : A_1 \longrightarrow G$ such that the following diagram commutes

$$\begin{array}{ccccc}
C_1 & \xleftarrow{\rho_1} & A_1 & \xleftarrow{\iota_1} & B_1 \\
\downarrow{\scriptstyle \epsilon'} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \epsilon} \\
H & \xleftarrow{\rho} & G & \xleftarrow{\iota} & K
\end{array}$$

where ι_1 *and* ρ_1 *are the homomorphisms of free groups defined by*

$$\iota_1(x_1) = (x_1, *), \qquad \rho_1(x_1, *) = 1, \qquad \rho_1(*, y_1) = y_1$$

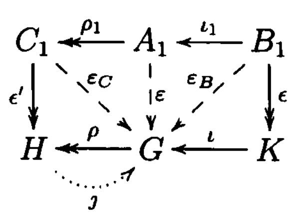
on the generators $x_1 \in X_1$ *of* B_1 *and* $(*, y_1), (x_1, *) \in Z_1$ *of* A_1.

The homomorphism $\varepsilon : A_1 \longrightarrow G$ *is an epimorphism.*

Proof. To show existence of $\varepsilon : A_1 \longrightarrow G$ we will first define $\varepsilon_B : B_1 \longrightarrow G$ and $\varepsilon_C : C_1 \longrightarrow G$ and then we define

$$\varepsilon(x_1, *) = \varepsilon_B(x_1), \qquad \varepsilon(*, y_1) = \varepsilon_C(y_1).$$

The following diagram describes the situation, although it is not exactly a commutative diagram since the equation $\varepsilon_C \rho_1 = \varepsilon$ is clearly not expected to hold.



We point out that the first row of the diagram is not exact.

We first set $\varepsilon_B(x_1) = \iota\epsilon(x_1)$, so that

$$\varepsilon\iota_1(x_1) = \varepsilon(x_1, *) = \varepsilon_B(x_1) = \iota\epsilon(x_1).$$

Next we set $\varepsilon_C(y_1) = \jmath\epsilon'(y_1)$ for all generators $y_1 \in Y_1$, by the universal property of free groups we have a homomorphism ε_C on all of the group C_1. We can also see that

$$\rho(\varepsilon_C(y_1)) = \rho(\jmath(\epsilon'(y_1) = \epsilon'(y_1)$$

on each of the generators $y_1 \in Y_1$ and so on the group C_1 we also have the equation

$$\rho\varepsilon_C = \epsilon'.$$

We also see that $\rho\varepsilon = \epsilon'\rho_1$ on the free group A_1 because it holds for all generators:

$$\rho\varepsilon(x_1, *) = \rho\iota\epsilon(x_1) = 1 = \epsilon'\rho_1(x_1, *),$$

$$\rho\varepsilon(*, y_1) = \rho\epsilon_C\rho_1(*, y_1) = \rho\jmath\epsilon'\rho_1(*, y_1) = \epsilon'\rho_1(*, y_1).$$

Therefore we have defined the homomorphism ε so that the following diagram commutes

$$\begin{array}{ccccc} C_1 & \xleftarrow{\rho_1} & A_1 & \xleftarrow{\iota_1} & B_1 \\ \downarrow{\scriptstyle \epsilon'} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \epsilon} \\ H & \xleftarrow{\rho} & G & \xleftarrow{\iota} & K \end{array}$$

Now we have to show that ε is an epimorphism. Let $g \in G$, so that $\rho(g) \in H$. Since ϵ' and ρ_1 are epimorphisms, $\epsilon'\rho_1$ is an epimorphism, so that there exists $a \in A_1$ such that $\rho(g) = (\epsilon'\rho_1)(a)$. Observe,

$$\rho(g^{-1}\varepsilon(a)) = \rho(g^{-1})\rho(\varepsilon(a)) = \rho(g^{-1})(\epsilon'\rho_1)(a) = 1,$$

so that $g^{-1}\varepsilon(a) \in \mathrm{Ker}\rho$. Thus $g^{-1}\varepsilon(a) \in \mathrm{Im}\iota$, so there exists $k \in K$ such that $g^{-1}\varepsilon(a) = \iota(k)$. Since ϵ is epimorphism, there exists $b \in B_1$ such that $\epsilon(b) = k$. Now

$$g^{-1}\varepsilon(a) = \iota(k) = \iota(\epsilon(b)) = (\iota\epsilon)(b) = (\varepsilon\iota_1)(b) = \varepsilon(\iota_1(b)).$$

Thus $g = \varepsilon(a\iota_1(b^{-1}))$, and so ε is epimorphism. $\square$

Proposition 4.1 can be viewed as a definition for the 1-dimensional resolution for G.

As remarked above, the sequence of free groups

$$C_1 \overset{\rho_1}{\twoheadleftarrow} A_1 \overset{\iota_1}{\leftarrowtail} B_1$$

is not exact.

Lemma 4.2. *In this situation,* $\mathrm{Ker}\rho_1$ *is the normal subgroup of* A_1 *generated by* $\mathrm{Im}\iota_1$.

Proof. Since $\rho_1\iota_1(x_1) = 1$ we see that the kernel of ρ_1 contains the image of ι_1, and since the kernel is normal it also contains the normal subgroup of A_1 generated by the image of ι_1. It remains to prove the opposite inclusion, that any element of the kernel of ρ_1 can be written as a product of conjugates of elements of $\mathrm{Im}\iota_1$.

Suppose $a \in \mathrm{Ker}\rho_1$. Since A_1 is the free product of the images of ι_1 and $\jmath_1$, we can write a as a reduced word

$$a = \iota_1 b_1 \cdot \jmath_1 c_1 \cdot \iota_1 b_2 \cdot \jmath_1 c_2 \cdots \iota_1 b_n \cdot \jmath_1 c_n$$

with b_1 and c_n possibly trivial. If $n \leq 2$ the result is clear, since $1 = \rho_1 a = c_1 \cdot c_2$ so $a = \iota_1 b_1 \cdot \jmath_1 c_1 \cdot \iota_2 b_2 \cdot \jmath_2 c_1^{-1}$ which is in the normal subgroup generated by $\mathrm{Im}\iota_1$ as required.

If not, we proceed by induction on n. In fact, if we write

$$a = a' \cdot \jmath_1 c \cdot \iota_1 b \cdot \jmath_1 c',$$

then

$$1 = \rho_1 a = \rho_1 a' \cdot \rho_1 \jmath_1 c \cdot \rho_1 \iota_1 b \cdot \rho_1 \jmath_1 c' = \rho_1 a' \cdot c \cdot c'$$

and so c is the inverse of $c' \cdot \rho_1 a'$ and

$$a = a' \cdot \jmath_1 \rho_1 a'^{-1} \cdot \jmath_1 c'^{-1} \cdot \iota_1 b \cdot \jmath_1 c'.$$

Now by induction $a' \cdot \jmath_1 \rho_1 a'^{-1}$ (a word of length smaller than the length of a) is in the normal subgroup generated by $\mathrm{Im}\iota_1$, and so is a. □

4.1.2 Degree 2

Now we define the 2-dimensional free crossed resolution $A_* \xrightarrow{\varepsilon} G$,

$$A_2 \xrightarrow{\delta_2} A_1 \xrightarrow{\varepsilon} G \tag{4.5}$$

but first a lemma that will be useful later.

Lemma 4.3. *Suppose $\delta_2 : A_2 \longrightarrow A_1$ is a free crossed module defined by generating sets Z_1, Z_2 and a function $\theta_2 : Z_2 \longrightarrow A_1$ which gives the boundaries of the generators. If $\mathrm{Im}\theta_2 \subseteq K$ where K is any normal subgroup of A_1, then $\mathrm{Im}\delta_2 \subseteq K$ also.*

Proof. Any element $a \in A_2$ can be written $a = {}^{a_1}z_1^{\epsilon_1}{}^{a_2}z_2^{\epsilon_2}\cdots{}^{a_n}z_n^{\epsilon_n}$, where $a_i \in A_1$, $z_i \in Z_2$, $\epsilon_i = \pm 1$. Then

$$\begin{aligned}\delta_2(a) &= \delta_2({}^{a_1}z_1^{\epsilon_1}{}^{a_2}z_2^{\epsilon_2}\cdots{}^{a_n}z_n^{\epsilon_n}) \\ &= (a_1\theta_2 z_1 a_1^{-1})^{\epsilon_1}(a_2\theta_2 z_2 a_2^{-1})^{\epsilon_2}\cdots(a_n\theta_2 z_n a_n^{-1})^{\epsilon_n}.\end{aligned}$$

If $\theta_2 z_i \in K \trianglelefteq A_1$, then conjugates $a_i\theta_2 z_i a_i^{-1} \in K$, and so are inverses and products of these. Hence $\mathrm{Im}\delta_2 \subseteq K$. □

It follows, from Lemma 4.3, that if the boundaries of the generators, $z_i \in Z_2$ of A_2, are contained in a normal subgroup K, then the boundaries of any element a of A_2 are also contained in K.

The following definition generalises the formulas (3.7), (3.8), (3.9) that we saw for the tensor product of crossed complexes in dimension 2.

Definition 4.4. *Suppose that $\delta_2 : A_2 \longrightarrow A_1$ is a crossed module and that B_1, C_1 and A_1 are free groups with generating sets X_1, Y_1 and $\{*\} \times Y_1 \cup X_1 \times \{*\}$ respectively.*

Suppose also that for each generator y_1 of C_1 there is a given group homomorphism $\alpha_1(\epsilon'(y_1)) : B_1 \longrightarrow B_1$.

1. Define

$$* \otimes_\alpha (-) : C_1 \longrightarrow A_1, \qquad (-) \otimes_\alpha * : B_1 \longrightarrow A_1.$$

to be the unique group homomorphisms specified on the generators by

$$* \otimes_\alpha y_1 = (*, y_1), \qquad x_1 \otimes_\alpha * = (x_1, *).$$

We also use the notation $\jmath_1(y_1) = * \otimes_\alpha y_1$ *and* $\iota_1(x_1) = x_1 \otimes_\alpha *$,

$$\jmath_1 : C_1 \longrightarrow A_1, \qquad \iota_1 : B_1 \longrightarrow A_1.$$

2. *Define a function*

$$[\,,\,]_\alpha : B_1 \times C_1 \longrightarrow A_1$$

inductively by

$$\begin{aligned}
[b_1, c_1]_\alpha &= 1 \qquad \textit{if either } b_1 \textit{ or } c_1 \textit{ is } 1\\
[x_1, y_1]_\alpha &= (x_1 \otimes_\alpha *)(* \otimes_\alpha y_1)(\alpha_1(\epsilon'(y_1))(x_1) \otimes_\alpha *)^{-1}(* \otimes_\alpha y_1)^{-1}\\
[b_1 x_1, y_1]_\alpha &= [b_1, y_1]_\alpha \cdot \iota_1(\alpha_1(\epsilon'(y_1))(b_1)) \cdot [x_1, y_1]_\alpha \cdot \iota_1(\alpha_1(\epsilon'(y_1))(b_1))^{-1}\\
[x_1, y_1 c_1]_\alpha &= \jmath_1(y_1) \cdot [c_1, x_1]_\alpha \cdot \jmath_1(y_1)^{-1} \cdot [y_1, \alpha_1(\epsilon'(c_1))(x_1)]_\alpha
\end{aligned}$$

Here $x_1 \in X_1$ *and* $y_1 \in Y_1$ *are generators and* b_1 *and* c_1 *are general elements of* B_1 *and* C_1 *respectively. Note that the term* $\alpha_1(\epsilon'(y_1))(x_1)$ *may be a general element of* B_1*; it need not be a generator.*

3. *For any function*

$$- \otimes_\alpha - : X_1 \times Y_1 \longrightarrow A_2$$

we define the extension

$$- \otimes_\alpha - : B_1 \times C_1 \longrightarrow A_2$$

to be the function given inductively by

$$\begin{aligned} b_1 \otimes_\alpha c_1 &= 1 \qquad \text{if } b_1 = 1 \text{ or } c_1 = 1 \\ (b_1 x_1) \otimes_\alpha y_1 &= b_1 \otimes_\alpha y_1 \cdot {}^{\alpha_1(\epsilon'(y_1))(b_1)\otimes_\alpha *}(x_1 \otimes_\alpha y_1) \\ x_1 \otimes_\alpha (y_1 c_1) &= {}^{*\otimes_\alpha y_1}(x_1 \otimes_\alpha c_1) \cdot (\alpha_1(\epsilon'(c_1))(x_1) \otimes_\alpha y_1). \end{aligned}$$

Here $x_1 \in X_1$ and $y_1 \in Y_1$ are generators and b_1 and c_1 are general elements of B_1 and C_1 respectively. Note that the term $\alpha_1(\epsilon'(y_1))(x_1)$ may be a general element of B_1; it need not be a generator.

Definition 4.5. *Suppose that*

$$B_2 \xrightarrow{\partial_2} B_1, \qquad \text{and} \qquad C_2 \xrightarrow{\partial_2'} C_1$$

are free crossed modules with sets of generators X_1, X_2 and Y_1, Y_2 respectively. Suppose also that for each generator y_1 of C_1 there is a given group homomorphism $\alpha_1(\epsilon'(y_1)) : B_1 \longrightarrow B_1$.

Let $X_0 = Y_0 = \{*\}$ and let

$$\begin{aligned} Z_1 &= X_1 \times Y_0 \cup X_0 \times Y_1, \\ Z_2 &= X_2 \times Y_0 \cup X_1 \times Y_1 \cup X_0 \times Y_2. \end{aligned}$$

Then we define

$$A_2 \xrightarrow{\delta_2} A_1$$

to be the free crossed module with sets of generators Z_1, Z_2 *with boundary maps defined on the generators by*

$$\begin{aligned}
\delta_2(* \otimes_\alpha y_2) &= * \otimes_\alpha \partial_2' y_2 = \jmath_1 \partial_2' y_2 \\
\delta_2(x_1 \otimes_\alpha y_1) &= [x_1, y_1]_\alpha \\
\delta_2(x_2 \otimes_\alpha *) &= \partial_2 x_2 \otimes_\alpha * = \iota_1 \partial_2 x_2.
\end{aligned}$$

Here a generator $(x_p, y_q) \in X_p \times Y_q$ *is denoted by* $x_p \otimes_\alpha y_q \in A_{p+q}$ *as usual.*

We can define $\jmath_2 : C_2 \longrightarrow A_2$,

$$\jmath_2({}^{c_1}c) = * \otimes_\alpha ({}^{c_1}c) = {}^{*\otimes_\alpha c_1}(* \otimes_\alpha c),$$

and $\iota_2 : B_2 \longrightarrow A_2$,

$$\iota_2({}^{b_1}b) = ({}^{b_1}b) \otimes_\alpha * = {}^{b_1 \otimes_\alpha *}(b \otimes_\alpha *),$$

to be the unique extensions to crossed module homomorphisms of the injective functions $y_2 \mapsto * \otimes_\alpha y_2$ and $x_2 \mapsto x_2 \otimes_\alpha *$.

Lemma 4.6. *The homomorphism* $\jmath_2 : C_2 \longrightarrow A_2$ *defined above preserves the crossed module action, i.e.,*

$$\delta_2 \jmath_2(c) = \jmath_1 \partial_2'(c),$$

for all $c \in C_2$.

Proof. Let $c' \in C_2$, $c \in C_1$, then

$$\begin{aligned} \jmath_1\partial_2'({}^c c') &= \jmath_1(c\partial_2'(c')(c)^{-1}) \\ &= * \otimes_\alpha (c\partial_2'(c')(c)^{-1}) \\ &= * \otimes_\alpha (\partial_2'({}^c c')) \\ &= * \otimes_\alpha \delta_2(({}^c c')) \\ &= \delta_2(\jmath_2({}^c c')) \end{aligned}$$

□

The following lemma will be needed to prove exactness of the 2-dimensional free crossed resolution that has to be constructed:

Lemma 4.7. *For any elements $b \in B_1$ and $y \in Y_1$ there exist elements $a', a'' \in A_2$ such that the following equations hold in the free group A_1,*

$$(j_1 y)(\iota_1 b)(j_1 y)^{-1} = (\delta_2 a')(\iota_1 b') \tag{4.6}$$

$$(j_1 y)^{-1}(\iota_1 b)(j_1 y) = (\delta_2 a'')(\iota_1 b'') \tag{4.7}$$

where the elements $b', b'' \in B_1$ are given by

$$b' = (\alpha_1(\epsilon'(y)))(b)$$

$$b'' = (\alpha_1(\epsilon'(y^{-1})))(b)$$

Proof. Since b is an element of a free group $B_1 = \langle X_1 \rangle$ we have

$$b = {x_1}^{\pm 1} \cdot {x_2}^{\pm 1} \cdot \dots \cdot {x_k}^{\pm 1} \cdot \dots \cdot {x_r}^{\pm 1},$$

for some $x_k \in X_1$, for $1 \leq k \leq r$. Now the left hand sides of the equations (4.6) and (4.7) can be written as

$$(j_1 y)(\iota_1 b)(j_1 y)^{-1} = ((j_1 y)(\iota_1 x_1)(j_1 y)^{-1})^{\pm 1} \cdot \dots \cdot ((j_1 y)(\iota_1 x_r)(j_1 y)^{-1})^{\pm 1}.$$

$$(j_1 y)^{-1}(\iota_1 b)(j_1 y) = ((j_1 y)^{-1}(\iota_1 x_1)(j_1 y))^{\pm 1} \cdot \dots \cdot ((j_1 y)^{-1}(\iota_1 x_r)(j_1 y))^{\pm 1}.$$

For equation (4.7):

This equation is true because of the form of the definition of $\delta_2(x \otimes_\alpha y)$ for $x \in X_1$ and $y \in Y_1$.

First consider those values of k for which the power is $+1$, and let

$$b'_k = (\alpha_1 \epsilon' y)(x_k),$$

$$a'_k = x_k \otimes_\alpha y.$$

For these values of k we then have equations

$$\begin{aligned}(\delta_2 a'_k)(\iota_1 b'_k) &= \delta_2(x_k \otimes_\alpha y)\ \iota_1((\alpha_1 \epsilon' y)(x_k)) \\ &= (j_1 y)(\iota_1 x_k)(j_1 y)^{-1} \iota_1((\alpha_1 \epsilon' y)(x_k))^{-1} \iota_1((\alpha_1 \epsilon' y)(x_k)) \\ &= (j_1 y)(\iota_1 x_k)(j_1 y)^{-1}\end{aligned}$$

by definition of $\delta_2(x_k \otimes_\alpha y)$.

Now consider the values of k for which the power is -1, and let

$$b'_k = (\alpha_1 \epsilon' y)(x_k)^{-1},$$

$$a'_k = {}^{(\iota_1 b'_k)}(x_k \otimes_\alpha y)^{-1}.$$

For these values of k we then have equations

$$\begin{aligned}
(\delta_2 a'_k)(\iota_1 b'_k) &= \delta_2({}^{(\iota_1 b'_k)}(x_k \otimes_\alpha y))^{-1} \cdot (\iota_1 b'_k) \\
&= (\iota_1 b'_k)\delta_2(x_k \otimes_\alpha y)^{-1}(\iota_1 b'_k)^{-1} \cdot (\iota_1 b'_k) \\
&= (\iota_1 b'_k)\delta_2(x_k \otimes_\alpha y)^{-1} \\
&= \iota_1((\alpha_1 \epsilon' y)(x_k))^{-1} \left((j_1 y)(\iota_1 x_k)(j_1 y)^{-1}\iota_1((\alpha_1 \epsilon' y)(x_k))^{-1}\right)^{-1} \\
&= \left((j_1 y)(\iota_1 x_k)(j_1 y)^{-1}\right)^{-1}
\end{aligned}$$

by definition of $\delta_2(x_k \otimes_\alpha y)$.

Putting these together for all values of $k = 1, 2, \ldots, r$ gives

$$\begin{aligned}
(j_1 y)(\iota_1 b)(j_1 y)^{-1} = ((j_1 y)(\iota_1 x_1)(j_1 y)^{-1})^{\pm 1} \cdots ((j_1 y)(\iota_1 x_r)(j_1 y)^{-1})^{\pm 1}. \\
= \ (\delta_2 a'_1)(\iota_1 b'_1) \ \cdots \ (\delta_2 a'_r)(\iota_1 b'_r).
\end{aligned}$$

This is nearly the equation we wanted to prove, except for the order of the elements. The terms on the right hand side must now be rearranged, by adding actions to the

a'_k, to get the required expression $(\delta_2 a')(\iota_1 b')$. That is, if we set

$$b' = b'_1 b'_2 b'_3 \cdots b'_r,$$

$$a' = a'_1 \cdot {}^{\iota_1 b'_1}a'_2 \cdot {}^{\iota_1(b'_1 b'_2)}a'_3 \;\cdots\; {}^{\iota_1(b'_1 \cdots b'_{k-1})}a'_k \;\cdots\; {}^{\iota_1(b'_1 \cdots b'_{r-1})}a'_r,$$

then we have

$$\delta_2 a' = \delta_2(a'_1) \cdot \iota_1(b'_1)\delta_2(a'_2)\iota_1(b'_1)^{-1} \cdot \iota_1(b'_1 b'_2)\delta_2(a'_3)\iota_1(b'_1 b'_2)^{-1} \cdots$$
$$\cdots \iota_1(b'_1 \cdots b'_{r-1})\delta_2(a'_r)\iota_1(b'_1 \cdots b'_{r-1})^{-1},$$

and since ι_1 is a homomorphism most of these terms cancel. We therefore have

$$\begin{aligned}(\delta_2 a')(\iota_1 b') &= (\delta_2 a'_1)(\iota_1 b'_1)(\delta_2 a'_2)(\iota_1 b'_2) \;\cdots\; (\delta_2 a'_r)(\iota_1 b'_r) \\ &= (j_1 y)(\iota_1 b)(j_1 y)^{-1}\end{aligned}$$

where

$$\begin{aligned}b' &= (\alpha_1(\epsilon'(y_1)))(x_1^{\pm 1}) \cdots (\alpha_1(\epsilon'(y_1)))(x_r^{\pm 1}) \\ &= (\alpha_1(\epsilon'(y_1)))(b)\end{aligned}$$

as required.

For equation (4.6): As before we write $x \in B_1$ in terms of the generators

$$b = {x_1}^{\pm 1} \cdot {x_2}^{\pm 1} \cdot \cdots \cdot {x_k}^{\pm 1} \cdot \cdots \cdot {x_r}^{\pm 1}.$$

Now for each k we consider the elements $b_k, b''_k \in B_1$ defined as follows

$$b''_k = (\alpha_1(\epsilon'(y^{-1})))(x_k^{\pm 1}),$$

$$b_k = b''^{-1}_k = (\alpha_1(\epsilon'(y^{-1})))(x_k^{\mp 1}),$$

and we can use equation (4.7) above to see that there exist elements $a'_k \in A_2$ with

$$\begin{aligned}(j_1 y)^{-1}(\iota_1 b_k)(j_1 y) &= \delta_2(a'_k) \cdot \iota_1(\alpha_1(\epsilon'(y)))(b_k) \\ &= \delta_2(a'_k) \cdot \iota_1(\alpha_1(\epsilon'(y)) \circ \alpha_1(\epsilon'(y^{-1})))(x_k^{\mp 1}).\end{aligned}$$

From Lemma 3.4 we can write this as

$$(j_1 y)^{-1}(\iota_1 b_k)(j_1 y) = \delta_2(a'_k \cdot \iota_2(\nu_1(\epsilon' y)(x_k^{\mp 1}))) \cdot \iota_1(x_k^{\mp 1}).$$

Let

$$a''_k = {}^{(j_1 y \cdot \iota_1 b_k)^{-1}}\left(a'_k \cdot \iota_2(\nu_1(\epsilon' y)(x_k^{\mp 1}))\right).$$

Then

$$\begin{aligned}(\delta_2 a''_k)(\iota_1 b''_k) &= (j_1 y \cdot \iota_1 b_k)^{-1} \cdot \left((j_1 y)(\iota_1 b_k)(j_1 y)^{-1}(\iota_1 x_k^{\pm 1})\right) \cdot (j_1 y \cdot \iota_1 b_k) \cdot (\iota_1 b''_k) \\ &= (j_1 y)^{-1}(\iota_1 x_k^{\pm 1}) \cdot (j_1 y).\end{aligned}$$

Exactly as we did before for the proof of equation (4.7), we must now multiply these equations for $k = 1, \ldots, r$ and rearrange the terms. That is, if we set

$$b'' = b''_1 b''_2 b''_3 \cdots b''_r = (\alpha_1(\epsilon'(y^{-1})))(b),$$

$$a'' = a''_1 \cdot {}^{\iota_1 b''_1}a''_2 \cdot {}^{\iota_1(b''_1 b''_2)}a''_3 \cdots {}^{\iota_1(b''_1 \cdots b''_{k-1})}a''_k \cdots {}^{\iota_1(b''_1 \cdots b''_{r-1})}a''_r,$$

then we have

$$(\delta_2 a'')(\iota_1 b'') = (\delta_2 a''_1)(\iota_1 b''_1)(\delta_2 a''_2)(\iota_1 b''_2) \cdots (\delta_2 a''_r)(\iota_1 b''_r)$$
$$= (j_1 y)^{-1}(\iota_1 b)(j_1 y)$$

as required. □

The following Lemma is very useful later in proving exactness of our free 2-dimensional resolution.

Lemma 4.8. *Suppose $\delta_2 : A_2 \longrightarrow A_1$ is a crossed module and $\varepsilon : A_1 \longrightarrow G$ is a group homomorphism such that the composite $\varepsilon\delta_2 : A_2 \longrightarrow G$ is the trivial homomorphism. Then*

$$a_1 \cdot \delta_2(a_2) \cdot a'_1 \in \mathrm{Im}\delta_2 \Leftrightarrow a_1 \cdot a'_1 \in \mathrm{Im}\delta_2$$
$$a_1 \cdot \delta_2(a_2) \cdot a'_1 \in \mathrm{Ker}\delta_2 \Leftrightarrow a_1 \cdot a'_1 \in \mathrm{Ker}\delta_2$$

for all elements $a_1, a'_1 \in A_1$ and $a_2 \in A_2$.

Proof. These results follow from the identity $a_1 \cdot \delta_2(a_2) \cdot a'_1 = \delta_2({}^{a_1}a_2) \cdot a_1 \cdot a'_1$ □

Putting together the above results we can now show that, using the 2-dimensional free resolutions B and C of K and H we have constructed a free 2-dimensional resolution of their semidirect product G.

Proposition 4.9. *Suppose that $G = K \rtimes H$ is a semidirect product with action α of H on K, and that we are given*

- *free crossed resolutions of length 2*

$$B_2 \xrightarrow{\partial_2} B_1 \xrightarrow{\epsilon} K, \quad \text{and} \quad C_2 \xrightarrow{\partial_2'} C_1 \xrightarrow{\epsilon'} H$$

for the groups K and H, with generating sets X_p and Y_q for $p, q = 1, 2$.

- *for each $h \in H$, a lift of $K \xrightarrow{\alpha_h} K$ to an endomorphism $B_1 \xrightarrow{\alpha(h)} B_1$, so that*

$$\epsilon \circ (\alpha(h)) = \alpha_h \circ \epsilon : B_1 \longrightarrow K.$$

Then the homomorphism ε of Proposition 4.1 and the free crossed module δ_2 of definition 4.5 define a free crossed resolution of length 2 for the group G,

$$A_2 \xrightarrow{\delta_2} A_1 \xrightarrow{\varepsilon} G.$$

Proof. We have already proved in Proposition 4.1 that ε is an epimorphism. It remains to show that the kernel of ε is equal to the image of δ_2. The proof has two parts:

1. We show that the image of δ_2 is contained in the kernel of ε. Since the kernel is normal we know from lemma 4.3 that it is enough to prove that the image of each

generator is contained in the kernel. We can see that

$$\begin{aligned}
\varepsilon(\delta_2(* \otimes_\alpha y_2)) &= \varepsilon(* \otimes_\alpha \partial_2' y_2) = \jmath\,(\epsilon'(\partial_2' y_2)) = 1 \\
\varepsilon(\delta_2(x_1 \otimes_\alpha y_1)) &= \varepsilon\left(\jmath_1 y_1\, \iota_1 x_1 (\jmath_1 y_1)^{-1} (\iota_1(\alpha(\epsilon' y_1))(x_1))^{-1}\right) \\
&= \jmath\epsilon' y_1\, \iota\epsilon x_1 (\jmath\epsilon' y_1)^{-1} (\iota\epsilon(\alpha(\epsilon' y_1))(x_1))^{-1} \\
&= \iota\,(\alpha_{\epsilon' y_1}(\epsilon x_1))\, \iota(\epsilon(\alpha(\epsilon' y_1))(x_1))^{-1} = 1 \\
\varepsilon(\delta_2(x_2 \otimes_\alpha *)) &= \varepsilon(\partial_2 x_2 \otimes_\alpha *) = \epsilon(\partial_2 x_2) = 1.
\end{aligned}$$

By lemma 4.3 we see that δ_2 sends every element to the kernel of ε.

2. We need to show that the kernel of ε is contained in the image of δ_2. This will be achieved in two steps which rely on the exactness of the 2-dimensional resolutions first for the quotient group H and then for the normal subgroup K.

For the first step we show that it is enough to show the result for elements in the kernel of ρ. That is, we show that for any element a in the kernel of ε there is an element a' that is also in the kernel of ρ, satisfying the condition that if a' is in the image of δ_2 then so is a.

If $a \in A_1$ such that $\varepsilon(a) = 1$, then

$$\epsilon' \rho_1 a = \rho \varepsilon a = 1,$$

that is, $\rho_1 a$ is in the kernel of ϵ'. By the exactness of the free 2-dimensional resolution for H we can therefore find an element $c_2 \in C_2$ such that

$$\partial_2' c_2 = \rho_1 a.$$

Now consider

$$a' = a \cdot \delta_2 \jmath_2 c_2^{-1} = a \cdot \jmath_1 \partial_2' c_2^{-1} = a \cdot \jmath_1 \rho_1 a^{-1} \in A_1.$$

Then Lemma 4.8 says proving the result for a is equivalent to proving the result for a', and we note that

$$\rho_1 a' = \rho_1 a \cdot \rho_1 \jmath_1 \rho_1 a^{-1} = \rho_1 a \cdot \rho_1 a^{-1} = 1.$$

For the second step we use the fact that an element in the kernel of ρ is in the normal subgroup of A_1 generated by the image of B_1, by Lemma 4.2. If this element was in the image of B_1, i.e. $\iota_1(B_1)$, we get the result by the exactness of the resolution for K. If not, we use induction on the number of conjugations by elements of C_1 needed to write it. Then Lemma 4.7 will give us the inductive step.

We are given $a \in \mathrm{Ker}\varepsilon$ and we may assume $a \in \mathrm{Ker}\rho_1$, which by Lemma 4.2 means that

$$a = \jmath_1 c_1 \iota_1 b_1 \jmath_1 c_1^{-1} \cdots \jmath_1 c_r \iota_1 b_r \jmath_1 c_r^{-1}.$$

with

$$c_j = y_{j,1}^{\pm 1} y_{j,2}^{\pm 1} \cdots y_{j,m_j}^{\pm 1}$$

$$b_i = x_{i,1}^{\pm 1} x_{i,2}^{\pm 1} \cdots x_{i,n_i}^{\pm 1}$$

for $y_j \in Y_1$, $x_i \in X_1$.

Now we prove that $\varepsilon(a) = 1$ implies that $a \in \mathrm{Im}\delta_2$ by induction on

$$K = \max_{1 \leqslant j \leqslant r}(m_j).$$

For $K = 0$, all $c_j = 1$ and we have $a = \iota_1 b_1 \cdots \iota_1 b_r = \iota_1 b$, so that

$$\varepsilon\iota_1 b = \iota\epsilon b = 1 \;\Rightarrow\; \epsilon b = 1 \;\Rightarrow\; b = \partial_2 b' \quad \text{for some } \; b' \in B_2,$$

so that $a = \iota_1 b = \iota_1 \partial_2 b' = \delta_2 \iota_2 b'$.

For $K = 1$, the element a is a product of elements of the form

$$\jmath_1 y_{j,1}^{\pm 1} \cdot \iota_1 b_j \cdot \jmath_1 y_{j,1}^{\mp 1}$$

which by Lemma 4.7 can be written as

$$\delta_2(a'_j)\iota(b'_j)$$

and by Lemma 4.8 proving the result for the product of these elements is equivalent to proving it for the product $\iota_1 b'_1 \cdots \iota_1 b'_r = \iota_1 b'$ as we did in the case $K = 0$.

For general K, the element a is a product of expressions of the form

$$\jmath_1 c'_j \cdot \jmath_1 y_{j,m_j}^{\pm 1} \cdot \iota_1 b_j \cdot \jmath_1 y_{j,m_j}^{\mp 1} \cdot \jmath_1 {c'_j}^{-1}$$

where c'_j is a reduced word of length $\leq K - 1$. By Lemma 4.7 these can be written as

$$\jmath_1 c'_j \cdot \delta_2(a'_j) \cdot \iota(b'_j) \cdot \jmath_1 {c'_j}^{-1}$$

and by Lemma 4.8 proving the result for the product of these expressions is equivalent to proving it for the product of the expressions

$$\jmath_1 c'_j \cdot \iota(b'_j) \cdot \jmath_1 c'^{-1}_j.$$

The result holds for the product of these expression by the inductive hypothesis. □

4.1.3 Degree 3

Suppose that $\delta_3 : A_3 \longrightarrow A_2$ is a morphism of $A_1/\delta_2 A_2$-modules and that C_1, B_1 and A_1 are free groups on generating sets X_1, Y_1, and $X_0 \times Y_1 \cup X_1 \times Y_0$ respectively, while $C_2 \longrightarrow C_1$, $B_2 \longrightarrow B_1$ and $A_2 \longrightarrow A_1$ are free crossed modules

Definition 4.10. *Suppose that*

$$B_3 \xrightarrow{\partial_3} B_2 \xrightarrow{\partial_2} B_1, \qquad \text{and} \quad C_3 \xrightarrow{\partial'_3} C_2 \xrightarrow{\partial'_2} C_1$$

are exact sequences, and that B_3, C_3 are free left $(B_1/\partial_2 B_2)$- $(C_1/\partial'_2 C_2)$-modules on generating sets X_3 and Y_3 respectively. Suppose also that for each generator y_1 of C_1 there is a given group homomorphism $\alpha_2(y_1) : B_2 \longrightarrow B_2$.

Define

$$Z_3 = X_3 \times Y_0 \cup X_2 \times Y_1 \cup X_1 \times Y_2 \cup X_0 \times Y_3$$

where $X_0 = \{*\}$, $Y_0 = \{*\}$, then we can define a sequence

$$A_3 \xrightarrow{\delta_3} A_2 \xrightarrow{\delta_2} A_1$$

where A_3 *is a free* $A_1/\delta_2 A_2$*-module on generating set* Z_3*, with module homomorphism* δ_3 *defined on the generators by*

$$\begin{aligned}
\delta_3(* \otimes_\alpha y_3) &= \jmath_2 \partial'_3 y_3 \\
\delta_3(x_1 \otimes_\alpha y_2) &= {}^{\iota_1 x_1}(\jmath_2 y_2)(\jmath_2 y_2)^{-1}(x_1 \otimes_\alpha \partial'_2 y_2)\iota_2(\kappa(y_2)(x_1)) \\
\delta_3(x_2 \otimes_\alpha y_1) &= {}^{\jmath_1 y_1}(\iota_2 x_2)(\iota_2 \alpha_2(y_1) x_2)^{-1}(\partial_2 x_2 \otimes_\alpha y_1)^{-1} \\
\delta_3(x_3 \otimes_\alpha *) &= \iota_2 \partial_3 x_3,
\end{aligned}$$

where the boundary of the element $\kappa(y_2)(x_1) \in B_2$ *is* $\tilde{\alpha}(y_2)(x_1) \cdot x_1^{-1}$ *(see below).*

Here a generator $(x_p, y_q) \in X_p \times Y_q$ *is denoted by* $x_p \otimes_\alpha y_q \in A_{p+q}$ *as usual.*

Proposition 4.11. *Given the following sequence*

$$A_3 \xrightarrow{\delta_3} A_2 \xrightarrow{\delta_2} A_1$$

(as defined above), then $\mathrm{Im}\delta_3 \subseteq \mathrm{Ker}\delta_2$*.*

Proof.

$$\begin{aligned}
\delta_2\delta_3(* \otimes_\alpha y_3) &= \delta_2(\jmath_2 \partial'_3 y_3) \\
&= \jmath_1 \partial'_2 \partial'_3 y_3 = 1
\end{aligned}$$

$$\begin{aligned}
\delta_2\delta_3(x_1 \otimes_\alpha y_2) &= \delta_2({}^{\iota_1 x_1}(\jmath_2 y_2)(\jmath_2 y_2)^{-1}(x_1 \otimes_\alpha \partial'_2 y_2)\iota_2(\kappa(y_2)(x_1)) \\
&= \delta_2({}^{\iota_1 x_1}(\jmath_2 y_2))\delta_2(\jmath_2 y_2)^{-1}\delta_2(x_1 \otimes_\alpha \partial'_2 y_2)\iota_2(\kappa(y_2)(x_1))
\end{aligned}$$

Now we explain where $\tilde{\alpha}$ and κ came from in the definition of $\delta_3(x_1 \otimes_\alpha y_2)$ above.

Note that $\delta_2(x_1 \otimes_\alpha \partial' y_2) = [x_1, \partial' y_2]_\alpha = \jmath_1(\partial_2' y_2)\iota_1 x_1 \jmath_1(\partial_2' y_2)^{-1}\iota_1(\tilde{\alpha}(y_2)(x_1))^{-1}$ where $\tilde{\alpha}(y_2)$ is the composite of the lifts of the $\alpha_{\epsilon' y}$ for each generator y in $\partial_2'(y_2)$. The lift of the composite, on the other hand, is the identity: it is the lift of $\alpha_{\epsilon' \partial_2' y_2} = 1$. Therefore there is a function $\kappa(y_2) : B_1 \longrightarrow B_2$ whose boundary is $\tilde{\alpha}(y_2)(x_1) \cdot x_1^{-1}$, and

$$\delta_2\delta_3(x_1 \otimes_\alpha y_2) = \iota_1 x_1 \jmath_1(\partial_2' y_2)\iota_1 x_1^{-1} \jmath_1(\partial_2' y_2)^{-1}[\partial' y_2, x_1]_\alpha \iota_1((\tilde{\alpha}(y_2)(x_1)x_1^{-1}) = 1.$$

Now consider

$$\delta_2\delta_3(x_2 \otimes_\alpha y_1) = \delta_2({}^{\jmath_1 y_1}(\iota_2 x_2)(\iota_2\alpha_2(\epsilon'(y_1))(x_2))^{-1}(\partial_2 x_2 \otimes_\alpha y_1)^{-1})$$

Recall that $\alpha_2(\epsilon'(y_1))(x_2)$ is not necessarily a generator of B_2, but

$$\delta_2\delta_3(x_2 \otimes_\alpha y_1) = \jmath_1 y_1 \iota_1 \partial_2 x_2 \jmath_1 y_1^{-1} \iota_1 \partial_2(\alpha_2(\epsilon' y_1)(x_2))^{-1}[\partial_2 x_2, y_1]_\alpha^{-1} = 1$$

Finally

$$\begin{aligned}\delta_2\delta_3(x_3 \otimes_\alpha *) &= \delta_2(* \otimes_\alpha \partial_3 x_3) = \delta_2(\iota_2 \partial_3 x_3) \\ &= * \otimes_\alpha \partial_2\partial_3 x_3 = \iota_1 \partial_2 \partial_3 y_3 = 1\end{aligned}$$

□

So we have shown that the $\text{Im}\delta_3 = \text{Ker}\delta_2$ all that is left to prove is exactness.

Conjecture 4.12. *The sequence in proposition 4.11 is exact*

$$A_3 \xrightarrow{\delta_3} A_2 \xrightarrow{\delta_2} A_1.$$

Assuming conjecture 4.12 to be true, then this together with Wall's construction (section 2.2), for $A_n \xrightarrow{\delta_n} A_{n-1}$ for $n \geq 4$, yields the required free crossed resolution for the semidirect product G, of the group K be the group H.

In a paper written by G. Ellis and I. Kholodna [7], which was inspirational in that they proposed a free crossed resolution to dimension 3, we found a slight error: the relation $\delta_2\delta_3$ given there does not hold (see appendix 5 for some details). It became apparent that the map $\kappa(y_2)$ defined above was exactly what was missing from their construction, and corresponds in the classical chain complex setting to the map d_2 of Wall. The proof that our candidate is for the correct definition of δ_3 does in fact define a resolution is complicated and will be left for future work.

Chapter 5

Conclusion

In this thesis we have studied the possibility of extending of results of Wall and Ellis–Kholodna, applying the theory of crossed complexes to obtain free crossed resolutions for semidirect products. There are several clear objectives for further work, some closer to realisation than others:

- Find a proof of our conjecture that the complex

$$A_3 \longrightarrow A_2 \longrightarrow A_1 \longrightarrow G$$

  we defined above in chapter 4 is in fact exact.

- Identify where (if at all) we have used the fact that the extension G is split, and generalise (if necessary) our proof to the case of general group extensions.

- Relate our work to that of Brown *et al.*, who explains an algorithm for finding resolutions using the notion of universal covers of crossed complexes. The idea is that the universal cover of a resolution is contractible, and explicit contracting homotopies provide information on how to form resolutions of extensions. In particular, they have found a dimension 4 crossed resolution of the symmetric group S_3, which is a semidirect product of C_3 and C_2, with $k+1$ generators in degree $k \leq 4$.

- Develop a theory of twisted tensor products of crossed complexes in order to obtain a result of the form 'a free crossed resolution for a group extension is obtained from the twisted tensor product of free crossed resolutions of the normal subgroup and quotient group'.

- Recalling that the category of crossed complexes is equivalent to the categories of ∞-groupoids, or simplicial T-complexes, or cubical ω-groupoids, investigate whether the formulas we obtain have easier geometric or algebraic interpretation in these other settings.

- Investigate the possibility, in some special cases, of obtaining analogues of spectral sequence or homological perturbation theory arguments for crossed complexes. It is well known that in general this will not be possible.

Appendix

Details of the examples from section 1.2.1

Here we give some examples of the action, α, and cocycle, c_2, which correspond to particular group extensions.

1. The cyclic group of order four is an extension of the cyclic group of order two by the cyclic group of order two,

$$1 \longrightarrow C_2 \longrightarrow C_4 \longrightarrow C_2 \longrightarrow 1$$

   Since the extension is abelian it is of course central, but it is not a semidirect product because the transversal of C_2 in C_4 cannot be chosen to be a subgroup of C_4.

2. The Klein four group is an extension of the cyclic group of order two by the

cyclic group of order two,

$$1 \longrightarrow C_2 \longrightarrow V \longrightarrow C_2 \longrightarrow 1$$

Since the extension is abelian it is central, and it is also split since the cross section j can be chosen to be a homomorphism. Of course, the Klein four group is isomorphic to the direct product of the group of order two with itself,

$$V \cong C_2 \times C_2$$

3. The cyclic group of order 8 is a central extension of the cyclic group of order 2 by the cyclic group of order 4

$$1 \longrightarrow C_2 \longrightarrow C_8 \longrightarrow C_4 \longrightarrow 1$$

and it is also a central extension of the cyclic group of order 4 by the cyclic group of order 2

$$1 \longrightarrow C_4 \longrightarrow C_8 \longrightarrow C_2 \longrightarrow 1$$

The cross sections cannot be chosen to be homomorphisms in either case so these extensions do not split.

4. The direct product $C_4 \times C_2$ is, of course, a split, central extension of the cyclic group of order 4 by the group of order 2,

$$1 \longrightarrow C_4 \longrightarrow C_4 \times C_2 \longrightarrow C_2 \longrightarrow 1$$

5. The dihedral group of order 8 is also a split extension of the cyclic group of order 4 by the group of order 2,

$$1 \longrightarrow C_4 \longrightarrow D_8 \longrightarrow C_2 \longrightarrow 1$$

and we can write $D_8 \cong C_4 \rtimes C_2$. The action is not trivial, and the extension is not central.

6. The quaternion group of order 8 is also an extension of the cyclic group of order 4 by the group of order 2,

$$1 \longrightarrow C_4 \longrightarrow Q_8 \longrightarrow C_2 \longrightarrow 1$$

This extension is neither split nor central.

Ellis and Kholodna - Proposition 3

Graham Ellis and Irina Kholodna, [7], state that given 3-presentations, $\{x|a := x^r|b := xaa^{-1}\}$ and $\{y|a' := y^s|b' := ybb^{-1}\}$ for the cyclic groups, K and H, then we can construct a '3-presentation' for a semidirect product, $G = K \rtimes H$, where K is a normal subgroup of G, and H its quotient group. Given a semidirect product $K \rtimes H$, then we also have an action, $\alpha : H \longrightarrow \mathrm{Aut}(K)$, of H on K, such that $\alpha(h)(k) = {}^h k$, where $h \in H$ and $k \in K$.

Consider the 3-presentations for K and H listed above, we can associate with them the following 'free crossed complex resolutions', B and C,

$$B: \qquad M(\underline{s}') \xrightarrow{\partial_3'} C(\underline{r}') \xrightarrow{\partial_2'} F(\underline{x}') \xrightarrow{\epsilon'} K.$$

$$C: \qquad M(\underline{s}) \xrightarrow{\partial_3} C(\underline{r}) \xrightarrow{\partial_2} F(\underline{x}) \xrightarrow{\epsilon} H,$$

where $\underline{r}$ generates the kernel of ϵ and $\underline{s}$ generates the kernel of ∂_2. Also, $F(\underline{x})$ is the free group generated by $\underline{x}$, $C(\underline{r}) \xrightarrow{\partial_2} F(\underline{x})$ is a 'free crossed module' and $M(\underline{s})$ is a 'free' $F(\underline{x})$-module.

Now observe that for each $h \in H$ we can construct a non-unique commutative diagram

$$\begin{array}{ccc} C(\underline{r}') & \xrightarrow{\alpha_2(h)} & C(\underline{r}') \\ \downarrow{\scriptstyle \partial_2'} & & \downarrow{\scriptstyle \partial_2'} \\ F(\underline{x}') & \xrightarrow{\alpha_1(h)} & F(\underline{x}') \end{array}$$

where if $y \in F(\underline{x})$ and $\epsilon(y^s) = 1_H$, $y \in H$, $u' \in F(\underline{x}')$ and $w \in C(\underline{r}')$. Note that $\alpha_1(h)$ and $\alpha_2(h)$ are homomorphisms, $\alpha_2(h)$ must preserve the action of $F(\underline{x}')$ on $C(\underline{r}')$. Let $\alpha_1(h)(u')$ be denoted by ${}^{\alpha(x)}u$ and let $\alpha_2(h)(u')$ be denoted by ${}^{\alpha(x)}w$. Now Proposition 3, of [7], says:

Proposition 1. **[Ellis-Kholodna]** *Given that we have free crossed complex resolutions, C and B, for the groups H and K, (with associated 3-presentations, $\{\, \underline{x} \mid \underline{r} \mid \underline{s} \,\}$, and $\{\, \underline{x}' \mid \underline{r}' \mid \underline{s}' \,\}$),*

$$C: \qquad M(\underline{s}) \xrightarrow{\partial_3} C(\underline{r}) \xrightarrow{\partial_2} F(\underline{x}) \xrightarrow{\epsilon} H,$$

$$B: \qquad M(\underline{s}') \xrightarrow{\partial_3'} C(\underline{r}') \xrightarrow{\partial_2'} F(\underline{x}') \xrightarrow{\epsilon'} K,$$

then there exist a free crossed resolution for the semidirect product, $G = K \rtimes_\alpha H$,

$$A: \qquad M(S) \xrightarrow{\delta_3} C(R) \xrightarrow{\delta_2} F(X) \xrightarrow{\epsilon} G,$$

which has an associated 3-presentation $\{\ X \mid R \mid S$ where $X = \underline{x} \cup \underline{x}'$, $R = \underline{r} \cup \underline{x} \underset{\alpha}{\times} \underline{x}' \cup \underline{r}'$ and $S = \underline{s} \cup \underline{x} \underset{\alpha}{\times} \underline{r}' \cup \underline{x}' \underset{\alpha}{\times} \underline{r} \cup \underline{s}'$.

The boundary maps δ_3, δ_2 are determined by:

$\delta_3(s) \mapsto \partial_3(s)$	$\delta_2(r) \mapsto \partial_2(r)$
$\delta_3(r, x') \mapsto {}^{x'}r r^{-1} c(r, x')$	$\delta_2(x, x') \mapsto x x' x^{-1} ({}^{\alpha(x)}x')^{-1}$
$\delta_3(x, r') \mapsto {}^{x}r' ({}^{\alpha(x)}r')^{-1} c(x, r')^{-1}$	$\delta_2(r') \mapsto \partial_2'(r')$
$\delta_3(s') \mapsto \partial_3'(s')$	

and the function, $c(-,-) : F(\underline{x}) \times F(\underline{x}')C(\underline{r} \cup \underline{x}' \times \underline{x} \cup \underline{r}')$ *is defined by,*

$$\begin{aligned} c(x,x') &= xx'x^{-1}({}^{\alpha(x)}x')^{-1} \\ c(1,v') = c(u,1) &= 1 \\ c(u,v_1'v_2') &= c(u,v_1')^{({}^{\alpha(u)}v_1')}c(u,v_2') \\ c(u_1u_2,v') &= {}^{u_1}c(u_2,v')c(u_1,{}^{\alpha(u_2)}v') \end{aligned}$$

for all $x \in \underline{x}$, $x' \in \underline{x}'$, $u, u_1, u_2 \in F(\underline{x})$, $v', v_1', v_2' \in F(\underline{x}')$. ■

Example 2. *Take the semidirect product* $S_3 = C_3' \rtimes_\alpha C_2$, *with the following 3-presentation for the cyclic groups of orders* 3 *and* 2 *respectively,* $C_3' = \{\ x \mid a' \mid p'\ \}$ *and* $C_2 = \{\ y \mid b \mid q\ \}$. *Then the action of* C_2 *on* C_3', *in* S_3, *is given by* ${}^yx = x^2$ *and we have the following free crossed resolutions for* C_2 *and* C_3', *with their respective boundary maps,*

$$C: \qquad M(q) \xrightarrow{\partial_3} C(b) \xrightarrow{\partial_2} F(y) \xrightarrow{\epsilon} C_2,$$

$$\partial_2(b) = y^2 \qquad\qquad \epsilon(y^2) = 1$$

$$\epsilon(y) = y.$$

$$\partial_3(q) = {}^ybb^{-1} \qquad \begin{aligned} \partial_2({}^ybb^{-1}) &= y\partial_2(b)y^{-1}\partial_2(b)^{-1} \\ &= yy^2y^{-1}y^{-2} \\ &= 1 \end{aligned} \qquad ,$$

$$B' : \qquad M(p') \xrightarrow{\partial'_3} C(a') \xrightarrow{\partial'_2} F(x') \xrightarrow{\epsilon'} C'_3,$$

$$\partial'_2(a') = x^3 \qquad\qquad \epsilon'(x^3) = 1$$

$$\epsilon'(x) = x.$$

$$\partial'_3(p') = {}^{x}a'a'^{-1} \qquad \partial'_2({}^{x}a'a'^{-1}) = x\partial'_2(a')x^{-1}\partial'_2(a')^{-1}$$

$$= xx^3x^{-1}x^{-3}$$

$$= 1$$

Then a free crossed resolution for the semidirect product, S_3,

$$A : \qquad M(S) \xrightarrow{\delta_3} C(R) \xrightarrow{\delta_2} F(X) \xrightarrow{\varepsilon} G,$$

which has an associated 3-presentation $\{\ X \mid R \mid S\ \}$ where $X = \{x, y\}$, $R = \{a, b, (y, x)\}$ and $S = \{p, q, (b, x), (y, a)\}$.

The boundary maps δ_3, δ_2 are determined by:

$$\delta_3(p) = \partial'_3(p) = {}^{x}aa^{-1} \qquad \delta_2({}^{x}aa^{-1}) = \partial'_2({}^{x}aa^{-1}) = 1$$

$$\delta_3(q) = \partial_3(q) = {}^{y}bb^{-1} \qquad \delta_2({}^{y}bb^{-1}) = \partial_2({}^{y}bb^{-1}) = 1$$

$$\delta_3(b,x) = {}^{x}bb^{-1}c(b,x) \qquad \delta_2({}^{x}bb^{-1}c(b,x)) = \delta_2({}^{x}bb^{-1}c(b,x))$$

$$= x\delta_2(b)x^{-1}\delta_2(b)^{-1}\delta_2(c(b,x))$$

$$= xy^2x^{-1}y^{-2}\delta_2(c(yy,x))$$

$$= xy^2x^{-1}y^{-2}\delta_2({}^{y}c(y,x)c(y,{}^{\alpha(y)}x))$$

$$= xy^2x^{-1}y^{-2}\delta_2({}^{y}c(y,x)c(y,xx))$$

$$= xy^2x^{-1}y^{-2}\delta_2({}^{y}c(y,x)c(y,x)^{\alpha(y)x}c(y,x))$$

$$= xy^2x^{-1}y^{-2}\delta_2({}^{y}c(y,x)c(y,x)^{x^2}c(y,x))$$

$$= xy^2x^{-1}y^{-2}y(yxy^{-1}({}^{\alpha(y)}x)^{-1})y^{-1}(yxy^{-1}({}^{\alpha(y)}x)^{-1})$$

$$x^2(yxy^{-1}({}^{\alpha(y)}x)^{-1})x^{-2}$$

$$= xy^2x^{-1}y^{-2}yyxy^{-1}x^{-2}y^{-1}yxy^{-1}x^{-2}x^2yxy^{-1}x^{-2}x^{-2}$$

$$= x^{-3}$$

$$\neq 1$$

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