THEORIES OF THE UNIFORM POSITIVE COLUMN

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ABSTRACT

The uniform longitudinal electric field commonly found in positive columns is discussed in relation to the "quasi-neutral" state of the plasma. The types of motion of the positive ions and the electrons are contrasted and it is shown that although the former have a mean energy close to that of the gas atoms, the small mass of the latter restricts energy interchange and allows the electron temperature to attain high values. The process of ambipolar diffusion arising from differences of mobility of the positive and negative ions is examined and the factors determining the cross-sectional variation of space potential and ion density are studied. Consideration is given to the effects of the ion sheath adjacent to the tube walls. The self-adjustment of the electron temperature to allow equal rates of production and loss of ions is noted and various mechanisms of ionisation are reviewed. The effects of cumulative ionisation processes and resonance radiation absorption are exemplified by a discussion of the low pressure mercury vapour discharge. The division of energy between the various processes is related to the longitudinal field and current. The interrelationships between the equations determining the discharge parameters are studied and broad methods of solution are indicated with a particular example. It is concluded that in most respects present theories of the uniform positive column provide a fairly adequate qualitative explanation of the phenomena.

/The quantitative

The quantitative aspect is not entirely satisfactory due to lack of data on elementary processes, to mathematical manipulative difficulties and to limitations of experimental techniques. It is anticipated that more basic data will become available with the passage of time and mechanical methods of solution may be applied to the more intractable groups of equations. No doubt techniques such as the use of high frequency will facilitate experimental investigation.

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THEORIES OF THE UNIFORM POSITIVE COLUMN

1. Scope

This dissertation is concerned mainly with the uniform positive column in discharges in monatomic gases at sufficiently low pressures and currents for ionised atoms to constitute only a small proportion of the total population of atoms, and for volume recombination to be insignificant. The conditions considered include only those in which the electrons, gas atoms, excited atoms and ions are remote from thermal equilibrium. The subnormal discharge is not discussed in detail and the treatment of the effects of non-uniform gas temperature is largely excluded. The topics of the constricted discharge and the behaviour of molecular gases and gases tending to form heavy negative ions are omitted.

2. Introduction

The experimental study of the positive column involves the collection of data concerning its observable characteristics and their dependence upon the various independently variable parameters. Theory is concerned with the interpretation of this information in terms of basic properties of quasi-elementary particles and photons and their manifestations in statistical phenomena such as diffusion, drift motion and random motion. In the past there have been

/attempts

attempts to use measurements on the positive column as a source of information on these basic properties. The present tendency is to draw upon the results of special experiments made under simplified conditions in order to elucidate the often complex phenomena of the discharge in general, and the positive column in particular.

The general problem of the positive column is to arrive at an insight into the various elementary processes taking place, the factors which influence their relative importance and the manner in which the divers mechanisms cooperate to yield a coherent macroscopic phenomenon. In this scheme, physics is required to contribute hypotheses of the component processes with sufficient definition and clarity to allow their quantitative formulation in equations relating individual parameters of the discharge. If sufficient relationships have been established the task becomes the mathematical one of manipulating the equations to yield statements concerning those characteristics of the positive column which are open to experimental measurement. This treatment can be restricted so that specific observed phenomena of a limited type can be shown to be consistent with particular hypothetical mechanisms. Much of the work which has been done on positive columns falls within this category. The more general problem involves deriving all the details of the macroscopic phenomena in terms of the assumed elementary processes. Owing to the complex network of interdependence of some of the factors, this solution presents difficulties.

Agreement between the observed and calculated outward characteristics of the positive column may be taken as justification of the physical assumptions made with regard to the inner mechanisms. There still remains a further aspect requiring attention. The formal mathematical linking of the premises and conclusions can, from the physical point of view, be usefully illuminated by consideration of special cases in which the mutual interplay of selected component factors comes into prominence. Such examples can yield an imaginative insight into the large scale phenomena which forms the counterpart of the physical concepts of the elementary processes.

3. The Equality of Positive and Negative Ion Densities

The existence of the distinctive region now known as the positive column was recognised from the early days of discharge investigations. Its simple properties, such as approximately constant longitudinal electric field, frequently uniform luminosity and ability to act as a medium of indefinite extension intermediate between cathode and anode phenomena, each with definite dimensional restrictions, soon became familiar.

An assumption made in nearly all theories of the positive column is that the main core of this part of the discharge is a "plasma" containing equal numbers of positive and negative ions per unit volume.

/This assertion

This assertion is made on the basis of Poisson's equation, in which the second differential of the space potential becomes zero for a uniform homogeneous field, thus determining that there shall be no net space charge. This assumption of the "quasi-neutrality" of the discharge is unsatisfactory in two respects. Firstly, it is undesirable that a detailed theoretical structure should be built upon empirical observations of the existence of a uniform longitudinal field within the discharge, without further examination of the underlying causes of this potential distribution. Secondly, the above conclusion is usually based on argument from uniform field assumptions in which the observed radial component of electric field in the positive column is ignored. Strictly speaking, this component may only be omitted for a filamentary axial element of the discharge.

The first objection is partially met by the theories of Morse[73], Schumann[84] and more especially Rogowski[80]. Using equations expressing Poisson's law, conductivity relationships and ionic conservation, it has been possible to set up a differential equation governing the longitudinal distribution of electric field.

$$\frac{d}{dz}\left(X\frac{dX}{dz}\right) = \left\{\frac{4\pi}{\mu_e \mu_p} j_z + \frac{X}{\mu_e}\frac{dX}{dz}\right\}\left\{\mu_e M X - \frac{u}{X}\right\} \qquad 1$$

where X = longitudinal field
z = longitudinal coordinate
e = electronic charge
μ_e = electron mobility
μ_p = positive ion mobility
j_z = longitudinal current density
M = number of ions generated per electron per volt
u = number of ions lost to the wall/cc/sec per electron/cc

/It is

It is readily seen that a uniform longitudinal field ($\frac{dX}{dz}=0$, $\frac{d^2X}{dz^2}=0$) is a possible solution of this equation, although the conditions under which it is the necessary solution are less obviously defined.

This solution requires that the final factor on the right hand side of the equation shall be zero. Hence the value of X may be deduced. A physical interpretation of these mathematical considerations arises from the observation that by making this factor zero, its two constituent terms, representing the number of ions generated by collision per unit volume of the discharge per second and the number of ions lost per unit volume per second by diffusion to the wall, are being equated. This implies that in the uniform field regions of the discharge no net gain or loss of ions by longitudinal transfer takes place. This could have been postulated ab initio from the evidence that the uniform positive column can be given indefinite longitudinal extension without altering its properties, and without altering the associated electrode phenomena. The latter approach has the advantage of simplicity and clarity, but lacks the generality of the argument from the general differential equation wherein the positive column conditions are set into relationship with the whole field system in the discharge. Moreover, equation 1 yields the desired result from considerations of basic processes, whereas the argument from the ion balance relationship requires the use of a deduction from the empirically observed properties of the positive column.

/The conclusion

The conclusion that $dx/dz = 0$ is a possible solution of equation 1 is not dependent on the form of the functions used to signify the rate of ionisation and rate of loss of ions to the wall. Variation of the form of these functions will merely alter the particular value of field required to satisfy the equation when $dx/dz = 0$. The mathematical indication of the possibility of existence of one or more values of x representing solutions of this kind does not, of course, necessitate the physical existence of mechanisms able to yield the required rates of ionisation and loss, and it may be expected that instances will occur in which the discharge cannot be uniform.

As an illustration of the fact that the possible solutions of equation 1 are not exhausted by the constant field result, it may be noted that oscillatory solutions representing striated discharges can be derived from it.[73] Such discharges are often visibly manifest, but certain apparently uniform columns have been shown to contain striations in rapid motion.[3, 98]

The second objection to the hypothesis of quasi-neutrality arises from the neglect to consider the radial field variation and its implications on Poisson's equation. This matter has been dealt with by Tonks and Langmuir[100] who have assumed the equality of the positive and negative ion densities, calculated the resulting radial field distribution and its contribution to the Poisson term and given examples in which this is negligible over the central major part of the tube

/cross-section.

cross-section. Much work on the positive column has been concerned with the discharge in which these conditions prevail, and the tube has been regarded as being filled with plasma (in which charge densities of opposite sign are equal) surrounded by a thin boundary sheath in which net space charge exists. Under low current conditions the sheath expands, eventually to fill the tube. It is then no longer possible to use the hypothesis of quasi-neutrality and a special analysis is required for this "subnormal" discharge.[22, 24]

Since field distribution and net space charge density are fundamentally related, the experimental evidence in support of the assumption that the positive and negative ion densities are equal consists of observations of uniform longitudinal field and agreements between calculated and observed radial variations of space potential.

4. The Motion of Electrons

Townsend[101] has shown that in electric fields typical of the positive column, the electron motion consists of a large random component, upon which is superimposed a small drift velocity. For the purposes of positive column analysis it is usual to assume that the electrons move sufficiently independently of the atoms for each type of particle to be characterised by its own separate Maxwellian distribution of velocities. It has been shown[15] that the average fractional energy lost by an electron in elastic collision with an atom is $2m_e/m_\alpha$ where m_e and m_α are the masses of the electron

/and atom

and atom respectively. The smallness of this quantity indicates the difficulty of energy transfer between the kinds of particles and affords some explanation of how it is that the average energy of the electrons can differ markedly from that of the atoms. The energy distribution within the electron cloud is the result of the interplay between the constant drain of energy by elastic collision with atoms, the selective loss processes involving excitation, ionisation and diffusion to the wall, and the replenishment of energy from the electric field. If a Maxwellian distribution is to exist in the presence of such disturbances, there must be a very efficient mechanism for interchange of energy between electrons. It has been suggested that this might be provided by Coulomb forces or by oscillatory processes, but this question appears to be not yet satisfactorily settled. Most investigators have been conscious of the possible departures from the Maxwell distribution law but, lacking a better expression, this function has been widely used. In several instances limited experimental justification has been obtained, but in some cases there has been adverse evidence. Frequently, the methods used to detect departure from the Maxwellian distribution have been insensitive. The introduction of the Druyvestyn expression[18,20] would take account of the effect of elastic collisions but would not represent the disturbances due to the other losses. This formula seems to have been little used in this application.

Another concept which has been extensively used is that of electron temperature, which is defined by analogy with the heat motion of atoms. In so far as this analogy is strictly pursued the use of the electron temperature concept implies the acceptance of a Maxwellian distribution. Alternatively, it is possible in some circumstances to use a quantity, T_e (electron temperature), defined by the equation $\frac{3}{2}kT_e = \overline{E_e}$ in a purely algebraic manner as an equivalent expression for the mean energy of the electrons $\overline{E_e}$, without any implication with regard to the distribution law. (The factor k is Boltzmann's constant). Owing to this difference in interpretation it may not be always immediately evident to what extent the use of the quantity T_e in a given theoretical treatment renders the whole analysis dependent on the assumption of Maxwellian distribution.

5. Cross-Sectional Variation of Discharge Parameters

If the gas pressure and current density in a positive column are sufficiently low to ensure that volume recombination of ions is negligible, the diffusion of ions to the walls of the vessel assumes a major role. This is the condition studied by Schottky[83] and by Tonks and Langmuir[100]. The analyses expounded by them are concerned, amongst other things, with the problem of radial distribution of potential and ion density.

5.1. Schottky's Theory of Radial Variation of Discharge Parameters

This theory is mainly concerned with the discharge in a circular cylindrical vessel but briefly mentions the case of plain parallel walls. It is based upon the assumption that local regions of the positive column receive or donate no net quantity of ions as a result of longitudinal transfer. Thus there must be an equality between the rate of loss of ions by outward flow and the rate of production of new ions. This relationship must apply to any right cylindrical volume element coaxial with the outer vessel, and is valid for ions of either sign. Equal quantities of charge must be imparted to the new ions of each sign and consequently the radial currents of ions of opposite polarity must be equal in magnitude and may be set equal to the so-called "ambipolar diffusion current". This constitutes the hypothesis of ambipolarity. The result can also be obtained by postulating that the tube wall is non-conducting and that therefore it cannot continue indefinitely to receive an excess of charge of one sign. This might seem to be a much weaker argument, involving an additional stipulation which is not required in the first approach, but it serves to cast some light on what may be a necessary requirement for the existence of a uniform longitudinal field.

/The balance

The balance between outward flow of ions and production of new ions may be expressed by the equation

$$\frac{\partial}{\partial r}\left(r j_A\right) - erq = 0 \qquad 2$$

where: r = radial coordinate

j_A = ambipolar current density

q = number of ion pairs generated per second per unit volume.

The positive and negative ion densities are assumed equal to each other and are designated by the quantity n_i. Schottky's justification of this important assumption perhaps seems a little inadequate. The equation can be solved when suitable expressions in terms of n_i are substituted for j_A and q.

The ambipolar current is regarded as arising from a combination of chaotic motion and drift motion under the influence of radial field. The very high mobility of electrons results in the initial rapid accumulation of negative charge on the wall until a sufficiently large retarding potential is built up to ensure exact balance between the rates of arrival of ions and electrons. The radial current densities for the two types of ions may be written down as the sum of ordinary diffusion and drift current terms, and, remembering the ambipolar and quasi-neutral conditions, it may be readily shown that

$$j_A = -e D_A \frac{\partial n_i}{\partial r} \qquad 3$$

/Here

Here D_A represents the ambipolar diffusion coefficient which Schottky defines as $(V_{Te} + V_{Tp})/\{(1/\mu_p) + (1/\mu_e)\}$.
V_{Te} is the volt equivalent of the electron temperature, given by

$$eV_{Te} = kT_e$$

V_{Tp} is correspondingly given by $eV_{Tp} = kT_p$
T_p is the positive ion temperature, which is approximately equal to the gas temperature.

It is assumed that there is no tendency to formation of heavy negative ions such as occurs with electronegative gases, for example oxygen. The mobility of the electrons greatly exceeds that of the positive ions and the ambipolar diffusion coefficient becomes approximately equal to $\mu_p V_{Te}$. If heavy negative ions are formed entirely different conditions apply owing to the close relationship then existing between the atomic and negative ion temperatures and the much reduced tendency for a negative wall charge to be built up. This case will be excluded from subsequent discussion.

Schottky takes the rate of production of ions as proportional to the electron density. Thus $q = \nu n_e$ where ν is a constant. An attempt is made to justify this by considering the energy supplied for the formation of ions in relation to the total power input to the tube, as expressed by the product of current and electric field.

/On this basis

On this basis the equation

$$q = x \frac{\mu_p + \mu_e}{V_I} \left(\frac{\partial V}{\partial z}\right)^2 n_e = \nu n_e \qquad \text{—— 4}$$

is set up.

x is the fraction of the input power which goes to the production of ions.

V_I is the ionisation potential of the ions.

V is the space potential.

The proportionality of q and n_e can only be proved if the constancy of ν is established. It is not self evident that the factor $x(\partial V/\partial z)^2$ will be independent of n_e, so this manipulation does not serve as adequate justification for the assumption. Subsequent investigators have considered the elementary collisions which can yield ions.[55] They have concluded that the rate of ionisation can be taken as proportional to the electron density providing that the latter is so low that an inappreciable number of ionisations occur by two-stage processes.

Within these limitations, the expressions for ambipolar current and rate of ionisation may be set in equation 2 yielding the following differential equation.

$$\frac{\partial}{\partial r}\left(r \frac{\partial n_e}{\partial r}\right) + \frac{\nu}{D_A} r n_e = 0 \qquad \text{—— 5}$$

/The solution

The solution of this equation is a zero order Bessel function.

$$n_e = n_{ea} J_0\left(r\sqrt{\frac{\nu}{D_A}}\right) \qquad 6$$

where n_{ea} is the electron density at the tube axis. The original argument then proceeded by putting $n_e = 0$ when $r = r_w$ (the tube radius). The first zero of this Bessel function occurs when the parameter $r\sqrt{\frac{\nu}{D_A}}$ = 2.405 and it was assumed that this applied for $r = r_w$.

$$r_w\sqrt{\frac{\nu}{D_A}} = 2.405 \qquad 7$$

In this way, a quantitative relationship was set up between the ambipolar diffusion coefficient, the rate of ionisation and the tube radius. This expresses the balance between the production and loss of ions, and influences the electron temperature through D_A.

By returning to the original equation expressing the radial flow of electrons as the sum of a diffusion term and a field drift term, and substituting the ambipolar diffusion current $-eD_A\frac{\partial n_e}{\partial r}$ the ion concentration distribution can be related to the radial potential gradient. On integration the following potential distribution is obtained.

$$V - V_a = V_A \log(n_e / n_{ea}) \qquad 8$$

Where V_a is the potential at the axis

$$V_A = (V_{Te}\mu_e + V_{Tp}\mu_p)/(\mu_p + \mu_e) \doteq V_{Te}$$

/This leads

This leads to the major difficulty of Schottky's theory, namely that the insertion of $n_e = 0$ into equation 8 yields an infinite wall potential. Schottky himself recognised this defect, which he attributed to the failure of the original assumption of quasi-neutrality near the wall. Nevertheless, the remainder of his analysis proceeds on the assumption that the above treatment provides an acceptable approximation to conditions in the major part of the tube. As a commentary it may be noted that this difficulty does not invalidate the rigorous application of equation 6 to those parts of the discharge where a true plasma exists, but it renders inappropriate the use without further justification of the Bessel function in the form

$$n_e = n_{ea} J_0\left(2.405 \frac{r}{r_w}\right) \qquad 9$$

even as an approximation. Further consideration of the conditions in the region of the wall is required before the associated relationship 7 between the rate of ionisation, tube radius and rate of loss of ions can be accepted as a useful approximation.

In spite of these difficulties, and in many instances in the light of further theoretical and experimental investigation, the formulae in question have been frequently used for approximate solutions. The nature of the assumptions made in these cases can be best seen by

/expressing

expressing the accurate law for the plasma region in the form

$$n_e = n_{ea} J_0 (2.405\, r/r_0) \qquad (10)$$

where r_0 is a modified tube radius bearing a relationship to the actual tube radius which is so far unknown. This equation must certainly fail for values of r approaching r_0, and will not be applicable when r exceeds r_0. In anticipating reasonable accuracy in the plasma region when using the unmodified formula, the assumption is being made that $r_w \doteq r_0$. Methods of estimation of r_0 in relation to r_w have been proposed [34, 100] and will be dealt with in sections 5.2, 5.7, 5.8 and 5.9.

In summary, omitting for the present the subject of longitudinal conditions, Schottky's achievement was to establish the broad features of the radial ion density and potential distributions in the plasma and to obtain an approximate relationship between electron temperature, ion mobility, tube radius and rate of ionisation. This analysis is limited in accuracy by uncertainties arising from lack of knowledge of conditions in the space charge region adjacent to the wall. It is limited in scope with regard to pressure by the requirement that conventional mobility laws must apply, necessitating mean free paths much less than the tube diameter. It is limited with regard to the current by the consideration of strict proportionality between electron density and rate of ionisation, which excludes high currents yielding two stage ionising processes, and also by the restriction of the space charge sheath near the wall, which excludes low currents for which the sheath occupies a large part

/of the tube.

of the tube. It does not in itself offer any indication of its failure to comply at low currents or of its radial extent of application. Neither does the original theory estimate the accuracy of which it is capable. The theory is not dependent on the electron energy distribution function in the sense that the behaviour of individual energy groups of electrons is not discussed and appropriate mean quantities may be used for diffusion, electron temperature and rate of production of ions.

5.2. <u>Extensions of Schottky's Theory</u>

Attempts to overcome the difficulties associated with wall potential have been made. J.S. Townsend[105] and F.L. Jones[50] assumed that the Bessel function would be applicable at least as far as $n_e = \frac{n_{ea}}{2}$ and associated a radius r_h with this condition. This allows further algebraic manipulation but does not yield any information about the influence of the actual tube radius on the discharge properties. Any assumption such as, for example, that of proportionality between r_h and the tube radius is not justified without further discussion.

Funk and Seeliger introduced a quite different wall condition by equating the ambipolar wall current given by the Schottky solution of Bessel form (equation 6) with the random electron current reaching the wall calculated from kinetic theory. Thus for unit

/area of

area of the wall

$$-D_A\left(\frac{\partial n_e}{\partial r}\right)_{r=r_w} = D_A\left(n_{ea}\sqrt{\frac{\nu}{D_A}}\right) J_1\left(r\sqrt{\frac{\nu}{D_A}}\right)_{r=r_w} = \frac{n_{ew}\bar{v}_e}{4} \qquad 11$$

where $\bar{v}_e$ is the mean random velocity of the electrons

n_{ew} is the electron density at the wall.

This leads to the following transcendental equation for ψ_w

$$\frac{J_0(\psi_w)}{J_1(\psi_w)} = \frac{4 D_A \psi_w}{\bar{v}_e \, r_w} \qquad 12$$

where $\psi_w = r_w\sqrt{\frac{\nu}{D_A}}$

This equation takes the place of the condition $\psi_w = 2.405$.

An approximate solution of 12 is

$$\psi_w = \frac{2.405}{1 + \frac{4D_A}{\bar{v}_e \, r_w}} = \frac{2.405}{1 + \frac{\mu_p\sqrt{T_e}}{7\times 10^9 r_w}} \qquad 13$$

Funk and Seeliger substituted typical values of the parameters and in their examples ψ_w departed from 2.405 by only a fraction of one per cent, which tends to justify Schottky's original assumption. They elaborated their equation in the form

$$\psi_w = \frac{2.405}{1 + \frac{\mu_p\sqrt{T_e}}{7.10^9 r_w(1-F)}} \qquad 14$$

to allow for the possibility of reflection of electrons at the walls by the factor F. Their results suggested that ψ_w is very insensitive to the variation of F from 0 to a quantity approaching close to unity (e.g. 0.999).

/Boltzmann's Law

Boltzmann's Law may be used to give values of the wall potential based on the electron density at the wall calculated from $J_0(\psi_w)$. These potentials are finite and the obvious limitations of the original theory are thereby removed.

Nevertheless, the treatment is not completely satisfactory. Attention is concentrated on the arbitrary nature of the boundary condition that Schottky used (i.e. $n_w = 0$) but very little is said regarding the other cause of failure of the Schottky solution in the region of the wall, namely the inapplicability of the assumption of quasi-neutrality. The existence of net positive space charge near the wall invalidates the use of the first and zero order Bessel functions for the electron density and ambipolar current at the wall. The net effect of the calculations of Funk and Seeliger is therefore to replace a completely arbitrary boundary condition by a requirement based on sound principles, but estimated from formulae which are not rigorously applicable.

It is of interest to examine the possibilities of making the Funk and Seeliger solution more acceptable by imposing limitations on the sheath thickness for which it may be used. Thus the wall current might be susceptible to calculation from the plasma solution if the sheath were sufficiently thin to allow substantially no increment of ambipolar current by ionisation within the sheath. The wall current could then be equated to the ambipolar current at the plasma edge.

/This would require

This would require the insertion into the Bessel function of the radius of the plasma boundary, which is neither known at this stage nor is suitable for the purposes of equation 12. The substitution of the actual tube radius in its place yields an approximate result which is formally identical with Funk and Seeliger's solution and might be regarded as having no clear advantages in accuracy over the conventional Schottky solution. It is suggested that it is better in principle to obtain the boundary condition in the form of an approximate estimate of the value of a parameter required to satisfy certain evident requirements, rather than as a completely arbitrary assumption. In this sense, the theory of Funk and Seeliger can judicate on the validity of the Schottky wall condition in terms of correctness of order of magnitude, and it is evident that the result is favourable. The former analysis contains residual inaccuracies due to sheath space charge and cannot be regarded as providing an accurate scale factor for the plasma solution. Nevertheless, the boundary requirement 13 is probably superior to 7.

In the same paper, Funk and Seeliger also derived a modification of the Schottky solution which takes account of radial temperature variation. The latter is assumed to obey a parabolic law and corresponding variations of D_A and V are obtained. The final solution for ion density is obtained in the form of a Bessel function plus additional terms constituting a power series in r. It is shown that the derived wall potential is finite and diminishes as the temperature becomes less uniform.

/5.3.

5.3. Tonks' and Langmuir's Theory of Radial Variation of Potential and Ion Density

Tonks and Langmuir approach the problem of cross-sectional potential distribution and ion distribution from a very different standpoint. They regard the electron cloud as possessing large random motion upon which is superimposed a relatively small drift current. The longitudinal field is considered as negligible. The electrons are regarded as having sufficient freedom of motion for the Boltzmann density distribution to apply thus.

$$n_e = n_{ea}\, \varepsilon^{-\frac{eV}{kT_e}} \qquad \text{—— 15}$$

In the Schottky theory this is a derived relationship and at first sight it may seem a little arbitrary that it should be used as an a priori assumption. Nevertheless, it is a classical kinetic theory result for the distribution of particles in a dynamic system involving a field of force.[46]

Tonks and Langmuir consider planar, cylindrical and spherical systems, which they are able to describe in one set of equations involving a parameter which takes different values according to the chosen geometry. Their treatment is more general than that of Schottky in several respects, the earlier analysis being confined to the fairly high pressure, proportional ionisation, planar and cylindrical cases in which many collisions occur in the course of transit of an ion to the wall.

/An expression

An expression for the positive ion density is set up on the basis of the radial ion velocity and current density at the point under consideration. The positive ions are considered to acquire their velocities by acceleration due to the outward field, the law of mobility being governed by pressure conditions. At low pressures, the motion will be of the "free fall" kind. At higher pressures, many collisions will occur and the ion movement will consist of random motion with a superimposed drift. The radial ion current density is obtained by integration of the rate of production of ions over the space between the axis and the point in question. Two alternative laws of generation are studied, namely uniform constant ionisation and ionisation proportional to electron density. In the latter case the integral is evaluated using the Boltzmann distribution of electrons. The result of these operations is to establish an integral equation for n_p in terms of the axial electron density, the electron temperature and the space potential.

It is now possible to equate the expressions for the positive ion and electron densities for the analysis of the plasma, but in the studies of the sheath Poisson's equation is used in the form of the "complete plasma-sheath equation".

$$\nabla^2 V = -4\pi e(n_p - n_e) \qquad 16$$

/The departure

The departure from the rigid limitations of the equation $n_p = n_e$ allows a more satisfactory discussion of the wall conditions than is possible on Schottky's theory. This is of significance over the whole plasma because the solution of the equations for this central part of the discharge is incomplete without the use of proper boundary conditions.

In order to generalise the theory, the potential V is everywhere replaced by the dimensionless quantity η defined as $(-eV/kT_e)$. The potential V is an intrinsically negative quantity, being taken with reference to the central potential maximum denoted by $V_a = 0$. The position coordinate r is replaced by the dimensionless quantity s defined as αr, where α assumes different values in the cases of different pressures or ion generation mechanisms. These transformations of the plasma equation are chosen so as to allow the potential distribution to be derived in the general form of dimensionless differential equations involving η and s. The solutions of these equations are obtained by various methods, some of which involve extensive numerical computation. They involve functions of η and s in the forms of power series, exponential expressions and Bessel functions. Whereas a single computation of the numerical relationship between η and s can be made, the results being incorporated in tables or graphs for further use, separate calculations would be necessary for V and r for each chosen values of α and T_e. This demonstrates the convenience of the general solutions, which are independent of distinguishing parameters such as T_e, μ_p etc., but may be readily converted to particular forms if required, by replacing η and s by the quantities in terms of which they were defined.

For example, the solution corresponding to the cylindrical case in which the ion mean free path is short compared with the tube diameter, and the rate of formation of ions is proportional to the electron density, is

$$\mathcal{E}^{-\eta} = J_0(s) \text{———} 17$$

This may be converted to the particular form

$$\mathcal{E}^{-\frac{eV}{kT_e}} = J_0(\alpha r) \text{———} 18$$

where $\alpha = (\nu / y k T_e)^{\frac{1}{2}}$ ——— 19

$y = 0.895\, \lambda_\alpha (k T_\alpha m_p)^{-\frac{1}{2}} = \mu_p / e$

λ_α = mean atomic free path

T_α = gas temperature.

Boltzmann's law indicates that $\mathcal{E}^{-\frac{eV}{kT_e}} = \frac{n_e}{n_{ea}}$

Hence the solution becomes

$$n_e = n_{ea} J_0(\alpha r) = n_{ea} J_0\left(r\sqrt{\frac{\nu}{y k T_e}}\right) = n_{ea} J_0\left(r\sqrt{\frac{\nu}{D_A}}\right) - 20$$

which is identical with Schottky's solution (equation 6).

The particular solutions obtained by Tonks and Langmuir contain the unknown parameter T_e whose determination requires the insertion of a known pair of values of r and V. This is the same boundary problem as was encountered by Schottky. Calculation from the general

/solutions

solutions shows that for some value of s denoted by s_0, $d\eta/ds$ or η tends towards an infinite value. This represents a limit at which the plasma equation is certainly not applicable. The Poisson term $\nabla^2 V$ will become appreciable at some value of s which may be denoted s_ϕ. This will be less than s_0 and the plasma solution will be incorrect for values of s exceeding s_ϕ.

Let η_0 be the value of η corresponding to s_0. The numerical values of η_0 and s_0 are directly calculable from the general solution. As a first approximation, Tonks and Langmuir assume that η_0 and s_0 correspond to conditions at the edge of the plasma. Presuming that the sheath thickness is small compared with the tube diameter they write $s_0 = \alpha r_w$ and $\eta_0 = eV_0/kT_e$. The first of these expressions may be used to convert the particular plasma solutions into a form involving the tube radius. The second expression yields a value for the potential V_0 at the sheath edge. The general form of Tonk's and Langmuir's potential distribution equations in which η is expressed as a function of s/s_0 can be thus changed to a form in which η is related to r/r_w. This transformation is analogous to the modification of Schottky's Bessel solution from the form $J_0\left(r\sqrt{\frac{\nu}{D_A}}\right)$ to the form $J_0\left(2.405\,\frac{r}{r_w}\right)$ and involves the similar advantage of convenient interpretation and disadvantage of loss of strict accuracy. Some further deductions based on these approximations will be considered before proceeding to the refinement of the equations.

5.4. Tonks' and Langmuir's Calculation of Radial Current

Tonks and Langmuir also investigated the value of the outward flowing ion current using the method of spatial integration of the production rate of ions described in the preceding section. Attention was concentrated on the magnitude j_{r0} of the ion current density at the position s_0, which was expressed in general dimensionless form h_0 analogous to η_0. Values of h_0 were calculated for each of the different pressure cases and laws of ion generation, and simple relationships were set out, enabling specific values of j_{r0} to be obtained for particular values of the independent parameters r_w, n_{ea}, ν and Q. (Q is the rate of production of ion pairs per unit volume, where this is constant and uniform over the cross-section). These relationships were expressed in two different forms, an example of which may be quoted, for the case of ion generation proportional to electron density and mean free path small compared with tube radius.

$$j_{r0} = h_0 e r_w n_{ea} \nu \qquad (21)$$

$$j_{r0} = s_0^2 h_0 e n_{ea} y k T_e / r_w \qquad (22)$$

The second form is derived from the first by expressing ν in terms of α (using the definition of α) and writing α in the form s/r, making the further assumption that $s = s_0$ when $r = r_w$. The authors of

/this theory

this theory perhaps tend towards a rather too easy presumption of exact equivalence of the alternative forms of the equations for j_∞. In fact, the relationship between the expressions is similar to that between the two forms of potential distribution equation expressed by Schottky, namely that the first form preserves independence of inexact assumptions about the wall conditions, whilst the second introduces the useful but approximate identification of s_0 with αr_w. It is desirable in this and other aspects of the theory to maintain a sharp distinction between results which are independent of specific assumptions about boundary conditions and those which are not, in order to facilitate subsequent refinements which may arise as the result of more detailed study of the wall conditions.

5.5. Experimental Investigation of Radial Variation of Ion Density and Potential.

The first important experimental investigation of the radial distribution of ion density and potential was carried out by Killian[55], using movable probes in a tube containing mercury vapour. The random electron current was obtained for different probe positions from conventional plots of the probe voltage against the logarithm of probe current, and the space potential was estimated from the position of the breakaway of the curve from the linear form. Killian was content to illustrate his results simply in the form of diagrams of random electron current versus radius and space potential versus radius. In the latter case the measured space potential was

/compared

compared with the quantity $(kT_e \log j_{ec})/e$ computed from experimental values, j_{ec} being the random electron current density. It was shown that there was an approximately constant difference between these quantities over the cross - section of the tube, with the exception of minor deviations near the wall at low vapour pressure. This was put forward as confirmation that the electron density and potential are related by the Boltzmann equation. Further examination shows that the assumption is here being made that j_{ec} is proportional to the electron density, which is only true if the electron temperature is constant over the whole tube. This must, of course, be the case if the Boltzmann equation is to be meaningful. Another consequence expected from the Boltzmann distribution is that the probe current should be constant at a given probe voltage with respect to the plasma potential at the axis irrespective of the distance of the probe from the axis, except when the probe becomes positive with respect to the immediately surrounding space. This was verified experimentally.

Tonks and Langmuir made use of these data for checking their plasma potential distribution equations. They obtained values of η from the measured potentials and electron temperatures, and expressed the radial distances as fractions of the tube radius. The experimental points were plotted in terms of these parameters and compared with the calculated curves, which were apparently obtained without correction for sheath thickness.

/For saturated

For saturated mercury vapour at 1.4°C, the correspondence with the calculations based on long free paths with proportional ionisation is moderately satisfactory. At 38.6°C the best agreement is obtained with calculations based on the assumption of a short mean free path, these results being identical with the Schottky solution. At both of these vapour pressures, discrepancies up to 10% in the value of s/s_0 or 20% in the value of η occur. These were not regarded as significant owing to the well known difficulties of probe measurements. The original papers do not draw attention to the fact that the results were asymmetrical with respect to the tube axis. Doubtless this is the result of disturbance due to the probe support structure. Although the limited success of the experiment in spite of this difficulty must be regarded as a substantial achievement, it cannot be said that ideal conditions were approached. The number of experimentally determined points indicated on the published diagrams is scarcely sufficient for any further conclusion to be drawn as to whether the observed discrepancies are of random or systematic nature.

Tonks and Langmuir also made an estimate of the ionic mean free path λ_p in relation to the tube diameter for the two vapour pressures and showed that the observed transition from the long free path solution to the short free path solution corresponds to a change in the ratio λ_p/r_w from 10 to 0.37. The latter figure is perhaps a little higher than might be expected but in view of uncertainty regarding the value of λ_p , the general agreement with theoretical considerations is satisfactory.

/Experiments

Experiments of the type carried out by Killian could be extended to cover other gases, pressures and current densities. The potential distribution theories which have been discussed are based on the assumption of constant ionisation or ionisation proportional to electron density and experimental verification of the scope of these assumptions is desirable. The acceptability of the Maxwellian energy distributinn function may differ under these various conditions. Generally, these assumptions have been examined from standpoints other than radial potential distribution theory. For the proposed experiments to be useful in indicating departures from the assumed laws of ion generation, it would be necessary to use Tonks' and Langmuir's method of presentation rather than Killian's. The comparison of observed space potential with the quantity $(kT_e \log j_{ec})/e$ serves only to demonstrate the Boltzmann distribution of electrons, which may be equally applicable for various ion generation mechanisms.

5.6. Experimental Investigation of the Wall Current

Most measurements of wall current have been incidental to other investigations and will be dealt with separately.

Tonks and Langmuir give details of an experiment carried out with mercury vapour in the same tube used by Killian. The check consists of calculating the value of n_{ea} from the wall current equation using experimentally determined values of j_{ro} and T_e, and comparing this with the value obtained directly from a conventional voltage, log current probe plot. This procedure involves the usual approximation regarding the wall conditions.

/The results

The results quoted are limited in number. Agreement within 10% between the n_{ea} values is obtained, but this cannot be extended to positive ions. It is not clear whether the observed discrepancies for electrons stem from the experimental technique or inadequacies of the theory. The method of measurement of wall current was not fully described and is open to suspicion.

This experiment covers an insufficient range of mercury vapour pressure to allow confirmation of the theory in the short mean free path regime. It is unfortunate that of the three results quoted, one refers to a bulb temperature of 0°C, at which the long mean free path regime is definitely established, and two refer to a bulb temperature of 15.5°C at which there is some doubt as to the suitability of either of the simplifying assumptions concerning mean free path.

Druyvestyn and Warmoltz[21] carried out measurements of wall current in discharge tubes containing mixtures of sodium vapour and inert gases. They used Schottky's equation relating this current to the ambipolar diffusion coefficient, that is, to the electron temperature and positive ion mobility. The product of the gas pressure and the ion mobility estimated on this basis was calculated. This tended to increase at the higher pressures. Similar behaviour of the electron mean free path was observed and it was suggested that these phenomena might be explained by the presence of a group of slow electrons, i.e. by a non-Maxwellian distribution.

Extensions of this type of experiment to a wider range of pressures, tube currents, tube diameters and gas fillings appear to be desirable.

/5.7.

5.7. Tonks' and Langmuir's Study of Sheath Conditions

The approach to this problem is made by way of the preliminary calculation of the values η_ϕ, s_ϕ of η and s which represent the limit of the accurate applicability of the plasma solution, the value of the wall potential and the mean ion velocity on entering the sheath. With this information it then becomes possible to transform the differential equation for the sheath condition 16 into a form yielding an approximate solution for the thickness. This allows s_0 and s_ϕ to be related to the tube radius, leading to an improved form of the plasma solution.

5.7.1. Limit of Validity of the Plasma Equation

The plasma solution discussed in previous sections has been calculated on the assumption that when equation 16 is expressed in terms of η and s, the term involving $\nabla^2\eta$ is negligible in comparison with either of the other terms, that is the difference between the ion and the electron density is small compared with either taken separately. This may be regarded as invalid when the difference exceeds a chosen fraction ϕ of the separate quantities. The corresponding value of η is represented by η_ϕ and is given by the equation

$$A\left(\frac{d^2\eta}{ds^2}\right)_{\eta=\eta_\phi} = \phi\,\varepsilon^{-\eta_\phi} \qquad \underline{23}$$

where A is a coefficient originating from the details of the

/modification

modification of equation 16. Here the sheath thickness is assumed to be small so that $\nabla^2\eta$ can be written $d^2\eta/ds^2$. To evaluate η_ϕ it was put equal to $\eta_0-\delta\eta_\phi$ and the plasma solution in the vicinity of the sheath was expressed in terms of $\delta\eta$, which signifies the quantity $\eta_0-\eta$. A Taylor expansion was used to obtain $d^2\eta/ds^2$ in terms of $\delta\eta$ and $(d^2\eta/ds^2)_{s=s_0}$. This, together with equation 23, yields $\delta\eta_\phi$ in the form

$$\left(\delta\eta_\phi\right)^3 = A\,\varepsilon^{\eta_0}\Big/\phi\left(\frac{d^2s}{d\eta^2}\right)_{s=s_0} \quad \text{——— } \underline{24}$$

This involves the use of data derived from the plasma solution, namely the values of η_0 and $(d^2s/d\eta^2)_{s=s_0}$. A second use of the Taylor expansion allows δs_ϕ to be obtained.

Tonks and Langmuir use the special case of low pressure and proportional ionisation to exemplify their solution but fail to note that the above treatment is inapplicable for high pressures for which η_0 becomes infinite. Their further development consists of substitution of the value of the coefficient A, from which α is eliminated by using the approximate boundary condition $\alpha = s_0/r_w$ and n_{ea} is eliminated by using the low pressure wall current equation analogous to equation 22. Thus

$$A = h_0 s_0^3 (2kT_e)^{3/2} / 8\pi e r_w^2 m_p^{\frac{1}{2}} j_{r0} \quad \text{——— } \underline{25}$$

Hence

$$(\delta\eta_\phi)^3 = \left(\frac{k^{3/2}}{2\sqrt{2}\,\pi e\phi}\right)\left(\frac{1}{r_w^2 m_p^{\frac{1}{2}}}\right)\left\{\frac{h_0 s_0^3 \varepsilon^{\eta_0}}{(d^2s/d\eta^2)_{s=s_0}}\right\}\left(\frac{T_e^{3/2}}{j_{r0}}\right) \quad \text{——— } \underline{26}$$

/in which

in which the presence of the factors T_e and j_{r0} may be noted. At this point the treatment was restricted to a special case. Experimental data were inserted for T_e and j_{r0} and it was shown that under the conditions studied by Langmuir and Mott Smith (section 8) the values of $\delta\eta_\phi$ and δs_ϕ were small. In this instance the plasma solution is applicable up to a small distance from s_0.

This restriction on the generality of the result is rather disappointing. The difficulties of obtaining a result of more general form will be discussed subsequently. (Section 5.9.).

5.7.2. Calculation of Wall Potential

Tonks and Langmuir equated the wall current j_{rw} obtained from spatial integration of the production rate of ions to the value obtained from kinetic theory assuming a Boltzmann distribution of electrons and a wall potential signified by η_w. Insofar as the Boltzmann distribution is unaffected by the space charge in the sheath, the latter value of the wall current should be independent of the approximation of the tube radius to the plasma radius. The estimate of j_{rw} does involve this inaccuracy by virtue of the fact that it must be obtained from the second form of wall current equation, (Section 5.4.), taking j_{rw} to be equal to j_{r0}. The calculated wall potential is therefore only approximate, but it does not involve the gross errors due to extending the hypothesis of quasi-neutrality into the sheath. The reflection coefficient F of the wall for electrons was taken into account in the kinetic theory estimate of the wall current.

/The result

The result obtained for η_w was

$$\eta_w = \log\left\{(m_p/m_e)^{\frac{1}{2}}/2\pi^{\frac{1}{2}} h_o s_o\right\} + \log(1-F) \quad \text{--- } 27$$

This quantity is calculable for a specified gas and is independent of other parameters such as T_e and arc current, insofar as it is permissible to take F constant.

5.7.3. Calculation of Mean Ion Velocity at the Sheath Edge

The mean ion velocity $\overline{v_{rpo}}$ at the position s_o is calculated by equating j_{ro} and $e n_{po} \overline{v_{rpo}}$, where n_{po} is the positive ion density at s_o, and is taken to be equal to n_{eo}. The velocity may be expressed in terms of the quantity η_{po} defined by

$$\eta_{po} = \frac{\frac{1}{2} m_p \left(\overline{v_{rpo}}\right)^2}{kT_e} \quad \text{--- } 28$$

The values thus calculated are influenced by the inadequacies of the plasma equation in the region s_ϕ to s_o, but an approximate value of $\overline{v_{rpo}}$ will here suffice since the only case considered is that in which the sheath thickness is small. Such approximations will then merely represent second order errors in the final result.

The expression for η_{po} depends on the law of ion generation. In the example given by Tonks and Langmuir, proportional ionisation is considered and the discussion is restricted to the long free path regime. The high pressure case is not discussed in detail. It demands quite different analysis owing to the infinite value of η_o and to the fact that for a certain range of pressures the mean free path for ions may be small compared with the plasma radius but large compared with the sheath thickness.

/For low

For low pressures, the quantity $h_o^2 s_o^2 \varepsilon^{2\eta_o}$ is obtained for η_{po}, being a numerical constant independent of the particular parameters of the discharge.

5.7.4. Calculation of Positive Ion Density in the Sheath

The sheath was assumed to be so thin that no appreciable generation of ions occurred within it. In this case the ion current density is constant throughout the sheath and the ion density must be inversely proportional to the ion velocity, which is itself dependent on the space potential. In this way the ion density at a point in the sheath may be expressed in terms of the ion current and the parameter η. Since the ions pass through the sheath without an appreciable number of collisions, their velocity may be taken as the sum of the mean velocity at the sheath edge and the extra velocity gained by falling through the potential drop in the sheath. The quantity $\eta_{p\phi}$ corresponding to the mean velocity $\overline{v_{rp\phi}}$ at the sheath edge is taken to be $h_o^2 s_o^2 \varepsilon^{2\eta_\phi}$ by analogy with $\overline{v_{rpo}}$ and the mean velocity within the sheath $\overline{v_{rps}}$ is associated with the quantity η_{ps} where

$$\eta_{ps} = \eta_{p\phi} + \eta - \eta_\phi \qquad \text{—— } 29$$

$$\eta_{p\phi} = \frac{\frac{1}{2} m_p \left(\overline{v_{rp\phi}}\right)^2}{kT_e} \qquad \text{—— } 30$$

$$\overline{v_{rps}} = \left(2kT_e \eta_{ps} / m_p\right)^{\frac{1}{2}} \qquad \text{—— } 31$$

/As pointed

As pointed out by Langmuir and Tonks, this simplifies the actual processes of ion acceleration and involves certain inconsistencies which, however, were not regarded as being of much importance.

The combination of the ion current equation with the quantity η_{ps} representing the mean ion velocity, yields the positive ion density in the form $n_{ea} s_o h_o \eta_{ps}^{-\frac{1}{2}}$, and this may be incorporated in the complete plasma sheath equation 16 as follows.

$$A\, d^2\eta / ds^2 + \varepsilon^{-\eta} - s_o h_o \eta_{ps}^{-\frac{1}{2}} = 0 \qquad \text{———} \quad \underline{32}$$

5.7.5. Solution of the Complete Plasma Sheath Equation

The equation is transformed by replacing the variable ds by the dimensionless quantity $d\xi$ chosen so as to render the coefficients free of the quantities T_e, m_p, j_{ro} and r_w. This procedure is analogous to that for the plasma equation. Manipulation and approximation yield the solution

$$\xi = (\eta_{ps}^{\frac{1}{2}} + 2.88)(\eta_{ps}^{\frac{1}{2}} - 1.44)^{\frac{1}{2}} - 1.23 \log(\phi\, \delta\eta_\phi) - 0.1 \qquad \text{———} \quad \underline{33}$$

where ξ is measured from s_o. The sheath thickness as represented by ξ_w can be obtained by substitution of values of ϕ, $\delta\eta_\phi$ and η_{ps} at the wall, the latter being calculable from η_w. Reversion to the original variable yields $(r_w - r_o)/r_w$, the approximate fraction of the tube radius occupied by the sheath.

$$\left(\frac{r_w - r_o}{r_w}\right) = \left(\frac{2}{3s_o}\right)\left(\frac{A}{s_o h_o}\right)^{\frac{1}{2}} \xi_w \qquad \text{———} \quad \underline{34}$$

/5.8.

5.8. Comparison of Calculations of Sheath Thickness by Funk and Seeliger and by Tonks and Langmuir

The results of these investigators are to some extent complementary. Tonks and Langmuir deal with the case of low pressure in which the ion mean free paths are long compared with the plasma dimensions and sheath thickness. The higher pressure condition is considered by the other investigators.

Both methods estimate the wall potential by equating the wall current calculated from Boltzmann's Law to that obtained from the plasma solution. The Funk and Seeliger modification of the plasma solution is accomplished with much greater simplicity than Tonks and Langmuir are able to achieve in their estimate of the wall correction. This simplification is obtained at the expense of recognition of any departure from quasi-neutrality, in which respect the latter analysis is superior.

5.9. Application of Data on Sheath Thickness to Correction of Plasma Potential Solution

Tonks and Langmuir limit their discussion of sheath thickness to a single specific example, namely that of mercury at certain low pressures in a cylindrical vessel of fixed dimensions at given currents. They do not enter into any detail regarding the general method of use of the correction.

The sheath thickness affects the scale of the radial distances in the plasma solution. In the approximate solution a given value of η is associated with a certain value of s and hence with a point at a distance r from the axis given by $r = s r_w / s_0$. In the solution

/corrected

corrected for the sheath thickness r is now given by

$$r = \frac{s}{s_0} r_w \left\{ 1 - \left(\frac{2}{3s_0}\right)\left(\frac{A}{s_0 h_0}\right)^{\frac{1}{2}} \xi_w \right\} \quad \text{——— } \underline{35}$$

Here it is convenient to consider the parameters involved in the correction term. The use of an approximate value of A will involve only second order errors so we may use a form of equation 25, in which j_{ro} is replaced by a factor involving n_{ea} by means of the wall current equation. Accordingly

$$A = \frac{ks_0^2}{4\pi e^2}\left(\frac{T_e}{n_{ea}\, r_w^2}\right) = \frac{ks_0^2}{4\pi e^2}\, t \quad \text{——— } \underline{36}$$

where $$t = \frac{T_e}{n_{ea}\, r_w^2}$$

The expression 33 for ξ_w consists of the sum of a constant term, a function of $\delta\eta_\phi$ and a function of η_{ps}. The choice of a suitable arbitrary value of ϕ allows $\delta\eta_\phi$ to be obtained from equation 24 as the product of a known numerical factor and $t^{\frac{1}{3}}$. The function of η_{ps} is determined by equation 29 in terms of η_0, η_w and $\delta\eta_\phi$ and may therefore be written as the function $\mathcal{F}_1(t, \eta_w)$.

Substitution of these quantities into the correction term yields

$$r = \frac{s}{s_0} r_w \left[1 - C_1 t^{\frac{1}{2}} \left\{ \mathcal{F}_1(t, \eta_w) - C_2 \log t - C_3 \right\} \right] \quad \text{——— } \underline{37}$$

where the constants C_1, C_2, C_3, η_w can be given numerical values for a specific case. Thus the radial scale of the plasma solution is modified by the sheath to an extent governed by the parameter t.

6. The Plasma Balance Equation

Tonks and Langmuir attach considerable physical significance to the result of the combination of their equations defining s and α. In order to obtain a suitable dimensionless form of the differential equation governing the radial potential distribution, s was defined as equal to αr and α was given specific values for the different cases of ion generation, geometry and pressure. Thus for example, in the circumstance in which the geometry is cylindrical, the ion generation is proportional to the electron density and the mean free path of the ions is small compared with the tube dimensions, we have

$$s = \alpha r \qquad\text{———— } 38$$

$$\alpha = (\nu / y k T_e)^{\frac{1}{2}} \qquad\text{———— } 39$$

$$y = \mu p / e$$

If the approximation that $r = r_w$ when $s = s_o$ is adopted the "plasma balance equation" is obtained.

$$s_o = (\nu / y k T_e)^{\frac{1}{2}} r_w \qquad\text{———— } \underline{40}$$

Similar relationships can be obtained for the other cases.

It was perceived that the plasma balance equation played an important role in determining the electron temperature, which must take up a value allowing the radial loss of electrons exactly to balance the production of new ions. The analogous equation in the Schottky theory is $r_w(\nu / D_a)^{\frac{1}{2}} = 2.405$. Here the ion production rate as signified by ν can be readily seen to be related to the loss rate as manifest

/in the

in the ambipolar diffusion coefficient. The electron temperature assumes a less conspecuous role, being incorporated in the quantity D_A.

Insofar as the quantities ν and y remain constant, the above approximate form of the plasma balance equation indicates that the electron temperature should be independent of the discharge current and of the specific nature of the collision processes. This simple situation is not realised owing to the dependence of the ionisation rate on the mean electron energy and hence on T_e. The general outcome of taking this into account is that ν and T_e become determined by a pair of simultaneous equations of which one is the plasma balance equation. The second involves a computation of the average rate of ionisation from considerations of the relevant collision cross-sections and the mean electron energy. (Section 8). Providing that no processes of ionisation through an intermediate stage are active, that is, if the ionisation rate is truly proportional to the electron density, this second equation will relate ν and T_e without the inclusion of any terms dependent on current (Equation 46). Hence the value of ν and T_e determined by these simultaneous equations or by their graphical representation illustrated in Fig. 1 will still be independent of arc current. This result is of great convenience in the general analysis of the discharge as it allows some of the parameters to be solved independently of the complex of interlinked simultaneous equations. Departure from proportional ion generation is accompanied by variation of the electron temperature with current and great complication in analysis ensues from the additional coupling of equations.

/A convenient

A convenient version of the above method of solution of T_e has been described by Von Engel[26, 27], who used the Schottky form of plasma balance equation with the familiar approximate boundary condition and combined it with an expression for the rate of formation of ions derived from general principles to be described in Section 8. This procedure is not essentially different from that described by Tonks and Langmuir but the formulae were manipulated so as to reveal a universal relationship between the quantities (V_{Te}/V_I) and $(C_4 p r_w)$ which is applicable to all gases and tube sizes.

Here
$$C_4 = \left(\beta / \lambda_{en} \mu_{pn} V_I^{\frac{1}{2}}\right)^{\frac{1}{2}} \qquad 41$$

V_I = ionisation potential

λ_{en} = electron mean free path at a normal temperature and pressure (i.e. 0°C and 1 mm. Hg.)

μ_{pn} = positive ion mobility at a normal temperature and pressure (i.e. 0°C and 1 mm. Hg.)

β = slope of curve of probability of ionisation in a single collision versus electron energy.

The convenience of this treatment is that the numerical part of the calculation can be embodied in a single curve or functional relationship, instead of in a network of curves as used by the earlier workers. The similarity concepts can be applied and the interplay of the separate factors can be much more readily envisaged.

7. Modification of the Plasma Balance Equation to Allow for Sheath Thickness

It is possible to set up a more accurate form of the plasma balance equation to take into account the sheath thickness, in the low pressure case for which the latter has been calculated. Thus, in place of $s_0 = \alpha r_w$ one may write

$$s_0 = \alpha r_w \left[1 - C_1 t^{\frac{1}{2}} \left\{ \mathcal{F}_1(t, \eta_w) - C_2 \log t - C_3 \right\} \right] \qquad 42$$

and the plasma balance equation

$$s_0 = \nu (m_p/2kT_e)^{\frac{1}{2}} r_w \qquad 43$$

(which is analogous to equation 40) is changed to the form

$$s_0 = \nu (m_p/2kT_e)^{\frac{1}{2}} r_w \left[1 - C_1 t^{\frac{1}{2}} \left\{ \mathcal{F}_1(t, \eta_w) - C_2 \log t - C_3 \right\} \right] \qquad 44$$

By analogy with the previous procedure, this may be taken in conjunction with the ion generation equation. The appearance of the quantity t (including the factor n_{ea}) now prevents the direct solution of ν and T_e. If a value of n_{ea} is assumed then ν may be calculated for a range of combinations of T_e and r_w values, and graphical solution of the simultaneous equations may be obtained as before. Such solutions may be repeated for other chosen values of n_{ea}.

It is evident that the resultant solutions for ν and T_e will now depend on the arc current (through n_{ea}), in contrast to the situation obtained when the wall sheath thickness was neglected. These considerations lead to the expectation that even in the absence of

/two-stage

two-stage ionisation processes electron temperature will vary with tube current over a certain range of values of t.

The application of a correction to the value of r_ω in the Von Engel form of solution leads essentially to the same result. Thus the function $C_4 p r_\omega$ is replaced by

$$C_4 p r_\omega \left[1 - C_1 t^2 \left\{ \mathcal{F}_1(t, \eta_\omega) - C_2 \log t - C_3 \right\} \right]$$

This modification considerably detracts from the ease of interpretation of the universal relationship which now takes the form

$$\frac{V_{Te}}{V_I} = \mathcal{F}_2 \left\{ (C_4 p r_\omega), t, \eta_\omega \right\} \qquad \text{—— 45}$$

This equation may be illustrated by a family of curves, one for each combination of values of the parameters t and η_ω. These replace the original single curve.

8. Rate of Generation of Ions

On the basis of data obtained by Compton[16], Van Voorhis[107] and Jones[51], Killian[55] obtained an estimate of the rate of generation of ions, assuming that in any collision between an electron and an atom in the ground state, the probability of ionisation is proportional to the energy possessed by the electron in excess of that required for ionisation. A mean value was calculated by assuming a Maxwellian distribution of energies and integrating the rate of ionisation over all energy values. The probability of ionisation in a single collision was taken to be $\beta(V_e - V_I)$ where V_e is the electron energy in volts and V_I is the ionisation potential. Although this is not true for high values of V_e it was considered to be applicable up to $3V_I$ and it was shown that the integral with an infinite upper limit was only

/slightly

slightly different from the integral with $3V_I$ as the upper limit. The data of the forementioned investigators is not in full agreement regarding the value of β, and Killian made separate calculations based on $\beta = 0.20\,p/T_\alpha$ and $\beta = 0.28\,p/T_\alpha$. His final expression for the number of ions produced per second per electron is, with slight rearrangement,

$$\nu = \left(\frac{8e}{\pi m_e}\right)^{\frac{1}{2}} \beta V_{Te}^{\frac{3}{2}} \left(2 + \frac{V_I}{V_{Te}}\right) \varepsilon^{-\frac{V_I}{V_{Te}}} = G \mathcal{F}_3\left(\frac{V_I}{V_{Te}}\right) \qquad 46$$

where m_e = electron mass

p = gas pressure

T_α = gas temperature

$G = \beta V_I^{\frac{3}{2}}$

$$\mathcal{F}_3\left(\frac{V_I}{V_{Te}}\right) = \left(\frac{V_{Te}}{V_I}\right)^{\frac{3}{2}} \left(2 + \frac{V_I}{V_{Te}}\right) \varepsilon^{-\frac{V_I}{V_{Te}}} \left(8e/\pi m_e\right)^{\frac{1}{2}}$$

The values of ν obtained in this way were compared with those derived from wall current measurements. Since it may be assumed that the ions produced in unit length of the positive column must be collected by unit length of the wall, one can write $\nu = 2\pi r_w j_{pw}/eN_e$, where j_{pw} is the wall ion current density and N_e is the total number of electrons per unit length of tube.

j_{pw} can be measured by means of a probe conforming to the wall contour and N_e can be obtained from the integral $2\pi\int_0^{r_w} n_e\, r\, dr$ where n_e is the electron density at distance r from the axis. Thus ν can be calculated wholly from experimental data.

/A further

A further set of values of ν for comparison were obtained from Tonks' and Langmuir's plasma balance equation in the low pressure form $\nu = 703.1\ T_e^{\frac{1}{2}}/r_\omega$ (Here, specific values of s_o and m_p have been introduced into equation 43).

The results of these computations are expressed in Table 1.

Table 1

Bulb Temperature °C	1.4	18.6	38.6
Hg vapour pressure (mm. x 10^{-3})	0.205	1.05	5.4
ν from wall current	41,000	25,800	13,800
ν from ion generation equation			
i. $\beta = 0.20\ p/T_\alpha$	14,600	16,900	12,600
ii. $\beta = 0.28\ p/T_\alpha$	20,400	23,600	17,600
ν from plasma balance equation			
i. low pressure	44,200	37,500	32,000
ii. high pressure			137,000

Data for a current of 5 amps in a tube of 6.2 cms. diameter

It will be seen that there is fair agreement between the values of ν obtained from the wall current measurement and from the ion generation equation at the higher mercury vapour pressures, bearing in mind the considerable doubt about the correct value of β. At low vapour pressure the ion generation equation underestimates the rate of ionisation.

/Killian

Killian attributes this to a departure from the Maxwellian distribution at low pressure, possibly due to the disproportionate number of high energy secondary electrons resulting from ionisation. This arises owing to the increase in ionisation probability with increase in primary electron energy. Although there is little reason to doubt this distribution of energy amongst the secondary electrons, the explanation as a whole is inadequate since the general distribution of energies results as the product of various mechanisms, all of which must be considered together if a complete analysis is to be achieved.

Killian dismisses an alternative explanation involving cumulative ionising mechanisms as being unlikely at this low pressure. This will be discussed in Section 17.

It is notable that where the ion generation equation fails at low pressure, the agreement between ν calculated from the wall current and from the plasma balance equation is best. There seems little doubt that the discrepancies which arise in this connection at vapour pressures of 1.05×10^{-3} and 5.4×10^{-3} mm. of mercury are due to the use of the low pressure form of the plasma balance equation. Tonks' and Langmuir's use of Killian's data to verify the potential distribution equations (Section 5.5.) strongly suggests that the transition from the long mean free path to the short mean free path conditions occurs within the range 2.05×10^{-4} to 5.4×10^{-3} mm. Hg. A value of ν calculated from the high pressure form of the plasma balance equation

/has been

has been added to the table for comparison. The formula used is

$$\nu = 0.895 \left(\frac{s_0}{r_w}\right)^2 kT_e \lambda_\alpha / (kT_\alpha m_\alpha)^{\frac{1}{2}}$$

The quantities have been assigned values as follows:

T_e = 19,900 (Killian's experimental value)

T_α = 323 °K

λ_α = 0.355 cms. (Jeans[46] p.135, 183).

The rate of production of ions calculated in this way considerably exceeds the values obtained by the other methods. Further comment on this will be made in Section 17.

Tonks and Langmuir[100] carried out further calculations based on Killian's data. They used a slightly different definition of β but, when reduced to Killian's form, their value is $0.31\, p/T_\alpha$. They concluded that the ionisation probability was proportional more nearly to the excess electron velocity than to the excess energy. In this way the factor $\left(2 + \frac{V_I}{V_{Te}}\right)$ was converted to $\left(\frac{3}{2} + \frac{V_I}{V_{Te}}\right)$ in the expression for ν, thus reducing the rate of ionisation by approximately 10% for electron temperatures of about 30,000°K in mercury. A further detail was added in regard to the vapour pressure, which was corrected for thermal effusion, although no indication is given of the method adopted. (The experimental arrangement involved the control of the vapour pressure by cooling a local region of the tube, using a bath at a lower temperature than the remainder of the discharge envelope). After these modifications the value of ν for a bulb temperature of 1.4°C came out as 2.1×10^4 by ion generation equation or 4.5×10^4 by plasma balance equation. This was regarded as satisfactory agreement in view of the uncertainty in the value of β, and the sensitiveness of the ion generation equation to

/small

small errors in T_e. It may be noted that there is a small discrepancy in the values of T_e quoted by Killian and by Tonks and Langmuir, which are 38,000°K and 38,800°K respectively. This largely accounts for the slight differences in values of ν calculated from the plasma balance equation.

Langmuir and Mott-Smith[62] used Killian's tube for further measurements of ν over a range of currents 0.5 - 8.0 amps at a fixed temperature of 15.5°C. They found that the disagreement between the estimates based on the ion generation equation and the plasma balance equation increased at the higher currents. For this reason they were less inclined towards Killian's view that cumulative processes are insignificant in this range of conditions. This attitude is supported by the fact that the electron temperature was observed to vary appreciably over the range of current, which is contrary to the theoretical expectation based on single stage proportional ionisation, assuming the sheath to be thin.

It is unfortunate that these measurements were made at 15.5°C, at which temperature the mean free paths are likely to be of intermediate value, so that doubt arises about the applicability of the low pressure version of the plasma balance equation.

The disturbance of the plasma balance equation by the sheath thickness correction is not discussed in detail by any of the above workers.

9. The Longitudinal Current

As a consequence of the equality of the positive and negative charge densities in the plasma, and of the low mobility of the positive ions, practically the whole of the arc current may be regarded as being carried by the drift motion of the electrons in the longitudinal field. The longitudinal component of electron drift velocity is usually assumed constant over the cross-section of the tube. The total current may be obtained by integration of the velocity times the electron density over the cross-section. For this purpose the appropriate form of potential distribution may be obtained from Tonks' and Langmuir's results and the electron density follows from Boltzmann's equation. For example, assuming a sheath of negligible thickness, the case of high pressure yields

$$i_z = 1.36\, \overline{v_{ez}}\, n_{ea}\, r_w^2\, e \qquad \text{---} \quad \underline{47}$$

Here i_z is the arc current and $\overline{v_{ez}}$ is the mean longitudinal drift velocity of the electrons, which will depend upon the distribution law. Assuming the Maxwellian function gives the "mobility equation"[19],

$$i_z = 0.723\, \frac{e^2 X_z \lambda_e n_{ea} r_w^2}{(m_e k T_e)^{\frac{1}{2}}} \qquad \text{---} \quad \underline{48}$$

where X_z is the longitudinal field.

The analogous equation for low pressure is

$$i_z = \frac{8 \times 10^{-12}\, \lambda_{en}\, r_w^2\, X_z\, n_{ea}}{p\, (V_{Te})^{\frac{1}{2}}} \qquad \text{---} \quad \underline{49}$$

/The introduction

The introduction of corrections to allow for sheath thickness involves great complexity. The radial scale of the plasma electron density distribution must be modified and in principle it is also necessary to take into account the electrons in the sheath. This involves using the complete sheath solution which becomes very cumbersome.

10. Longitudinal Field

This subject will initially be discussed without reference to sheath corrections.

The "mobility equations" 48 and 49 are open to experimental verification by direct measurement of the various parameters. Thus X_z, n_{ea} and T_e may be measured by means of probes. The tube radius can be assumed equal to r_w. The electron mean free path can be obtained as a function of T_e from Townsend's or Ramsauer's data.

As an example of more limited checks, the variation of X_z with changes of p has been studied experimentally by Klarfeld[57] and Groos[35]. Although molecular gases give fairly simple results, the inert gases show complicated behaviour, the electric field strength passing through one or several minima. This can be shown to be related through the mobility equation to the peculiarities of the Ramsauer effect.

One of the quantities i_z or X_z in the "mobility equations" may be regarded as an independent parameter subject in practical conditions to external choice. A further equation is required for the solution of the other quantities.

Schottky[83] obtained an expression for the longitudinal field by combining a result from his radial ambipolar theory with a calculation

/of the rate

of the rate of production of ions. The equations have already been discussed in Section 5.1. They are

$$\nu = x \frac{\mu_p + \mu_e}{V_I} X_z^2 \qquad 4$$

$$r_w \sqrt{\frac{\nu}{D_A}} = 2.405 \qquad 7$$

Eliminating ν gives

$$X_z = \frac{2.4}{r_w} \sqrt{\frac{V_I}{x}} \sqrt{V_{Tp} + V_{Te}} \sqrt{\frac{\mu_p \mu_e}{(\mu_p + \mu_e)^2}} \qquad 50$$

The form of this equation suggests the particular conclusion that the longitudinal field is inversely proportional to the tube radius. Schottky quoted some experimental data of Claude[13] in support of this view. Nevertheless, this conclusion must be challenged on the ground that V_{Tp} and V_{Te} (the ion and electron temperatures) and x (the "efficiency" of ionisation i.e. the fraction of the input power used in ionisation) are liable to be functions of r_w. The significance of the Claude result is more probably that the quantity $(V_{Tp} + V_{Te})/x$ happens to remain fortuitously constant in the experimental conditions covered. The validity of Schottky's general method of analysis, as distinct from his particular conclusion, can be established from evidence of this kind, only in conjunction with supporting data on the particle temperatures and the quantity x. It must be noted that Schottky was aware of the need of more information on these points.

/Mention

Mention has already been made of the discussions of the constancy of the longitudinal field due to Morse[73], Schumann[84] and Rogowski[80]. These authors derive expressions for this field based on the application of Poissons equation, the mobility equation and an equation expressing the balance of production and loss of ions. Morse used Schottky's wall current formula

$$j_{rw} = i_z \frac{2.405}{r_w} \left(\frac{\chi}{V_I}\right)^{\frac{1}{2}} D_A^{\frac{1}{2}} \qquad \text{———} \quad \underline{51}$$

to obtain the result

$$X_z = 2.4 \left(\chi V_I D_A\right)^{\frac{1}{2}} \varepsilon \qquad \text{———} \quad \underline{52}$$

Schottky's own field expression was written by Morse in the form

$$X_z = \frac{2.4}{r_w} \left(V_I D_A / \chi\right)^{\frac{1}{2}} \qquad \text{———} \quad \underline{53}$$

From a comparison of these results he concluded that $\chi = 1/r_w \varepsilon$
Substitution of this value into either formula yields

$$X_z = 2.4 \left(V_I D_A \varepsilon / r_w\right)^{\frac{1}{2}} \qquad \text{———} \quad \underline{54}$$

This apparently eliminates the uncertainty associated with the presence of x in the original Schottky formula. The validity of this procedure is undermined by various algebraic defects some of which are evident in this brief exposition. Detailed comparison of Schottky's original wall current and field expressions with the forms used by Morse raises the question whether χ (undefined by Morse except as "a constant") is intended to be identical with x. If this is so, then Morse's

/expressions

expressions need additional factors $(\mu_p + \mu_e)$, which cannot be accommodated in the preceding treatment. Various discrepancies invalidate the specific results, but general interest attaches to the emphasis placed on the part played in the determination of X_z by the power balance consideration embodied in the factor. x .

A relationship between longitudinal field, gas pressure and tube radius is derived by Llewellyn Jones[50], using similarity principles. Starting from Schottky's differential equation 5 relating ion density and radial coordinate, an expression is written down for the rate of ionisation per electron in terms of Townsend's ionisation coefficient ζ . In this way the equation

$$\frac{\nu}{D_A} = \zeta \frac{\mu_e^2 X_z}{D_e \mu_p} \qquad \text{———} \quad 55$$

is obtained. The problem of boundary conditions is evaded[105] by writing $r = r_h$ for $n_e = \frac{n_{ea}}{2}$, r_h being proportional to the tube radius only when the sheath is thin. Taking $J_0\left(r_w \sqrt{\frac{\nu}{D_A}}\right) = \frac{1}{2}$ gives

$$r_h \sqrt{\frac{\nu}{D_A}} = 1.52 \qquad \text{———} \quad 56$$

Combining 55 and 56 and arranging the parameters in combinations obeying the similarity laws yields

$$\frac{\zeta}{p} = \frac{C_5}{r_h^2 p^2}\left(\frac{X_z}{p}\right)^{\frac{1}{2}} \qquad \text{———} \quad 57$$

Ll. Jones obtained ζ/p as a function of X_z/p by taking the probability of ionisation in an elementary collision P_I as proportional to the fractional electron energy excess over that required for ionisation

$(V_e - V_I)/V_e$. The unsatisfactory knowledge of the function P_I at that time is illustrated by the fact that the function here assumed differs from those of either Killian or Tonks and Langmuir. The general procedure of integration over all electron energies is substantially that followed by these workers except that provision is made to accommodate some departure from Maxwell's law at energies less than V_I , by insertion of an additional constant. The expression obtained for ζ/p involves X_z/p and V_{Te} . The latter is regarded as being determined by the balance between electron energy gained by motion in the field and lost by elastic collision. Thus over a certain range of values V_{Te} is taken proportional to X_z/p . Hence the relationship 57 is reduced to the form

$$C_6 r_h^2 p^2 = \left(\frac{X_z}{p}\right)^{\frac{1}{2}} \varepsilon^{\left(\frac{C_7 p}{X_z}\right)} \qquad \text{———} \quad 58$$

Seeliger has obtained this relationship by a different method[85].

The expressions here used for ζ and V_{Te} involve considerations of elastic loss of energy by electrons, whereas Tonks and Langmuir have shown how to obtain V_{Te} and the related quantity γ without introducing any such factors (Section 6). This apparent paradox is resolved when it is noted that the present treatment introduces the longitudinal field, which is a parameter which was absent from the earlier discussion. It is again evident that the calculation of the field demands consideration of the power dissipation , as was concluded from Schottky's work.

/The experimental

The experimental evidence mentioned in Llewellyn Jones' original paper is limited to a study of the relationship between X_z and p for fixed tube radius. Good agreement with theory is obtained for helium from a few mm. up to about 40 mms. pressure. This agreement requires the interpretation that at the higher pressures the ionisation mechanism proceeds via a metastable intermediate state. The theory can be readily adapted for this circumstance by substituting V_M in place of V_I in the formulae. Agreement for neon is unsatisfactory either for direct or indirect ionisation. It is not clear how far the discrepancies can be attributed to variation of sheath thickness, departure from Maxwell's Law, or inadequacy of the assumption that the total input power is converted to elastic loss. The first point is not discussed in the paper, but is probably unimportant at the pressures involved. Values of the constant introduced to accommodate non-Maxwellian distribution were obtained, and suggest a marked deviation from this law for both gases. It is difficult to estimate the success of the special constant in accounting for these deviations. Probably the departure of the experimental neon results from those predicted is mainly attributable to the reduced importance of the energy losses in elastic collisions at the lower pressures.

An expression of the general form of equation 58 is also obtained by Von Engel, as a development of his universal relationship between V_{Te}/V_I and $C_4 p r_w$ (Section 6).

11. Energy Balance

In the foregoing approaches to the calculation of the longitudinal field, it will be seen that energy considerations enter in some form. The Morse and Schottky calculations use the factor x which represents the proportion of input energy which is used for ionisation. Llewellyn Jones assumes that nearly all the input power is absorbed in elastic collisions, in order to obtain a relationship between the mean electron energy and X_z/p.

A more general expression which serves to determine the field equates the power input and the sum of the powers absorbed in the various mechanisms of energy dissipation.

Tonks and Langmuir[100] noted the various categories of energy dissipation thus

i. Wall losses (including the kinetic energy of ions and electrons striking the walls, and the heat of recombination) W_w watts per unit length.

ii. Loss of energy by radiation. W_ρ watts per unit length.

iii. Heat production by elastic collisions and inelastic collisions not leading to ionisation or radiation. W_v watts per unit length.

The power balance may thus be expressed by the equation

$$i_z X_z = W_w + W_\rho + W_v \qquad \text{—— } \underline{59}$$

/Only

Only the first item was considered in much detail. This was taken as the product of the wall current into the sum of the ionisation potential, the voltage equivalent of the electron temperature and the mean potential fall for the ions. These terms are respectively V_I, $T_e/11{,}600$ and $(\eta_w - 0.3)T_e/11{,}600$, where the quantity 0.3 is entered to allow for the mean point of origin of the ions. Thus

$$W_w = 2\pi r_w j_w \left[V_I + \frac{(\eta_w + 0.7)T_e}{11{,}600} \right] \qquad \text{---} \ 60$$

Only single stage proportional ionisation will be considered at present. As shown in previous sections, the wall current is proportional to the axial electron density and to ν, which together with T_e is subject to independent solution from the plasma balance and ion generation equations. The wall potential is dependent only on the gas and pressure regime. Accordingly, the wall losses are expressible as the product of the axial ion density and a coefficient calculable from basic data.

Killian[55] obtained values of the wall loss for a low pressure mercury vapour discharge, making experimental measurements of the above mentioned factors. He also estimated the rate of energy loss by elastic collisions between electrons and atoms, using Comptons expression for the average fractional energy lost at a single collision $2m_e/m_\alpha$. For this purpose an experimental value of mean free path was obtained from the equation for longitudinal current 48 which

/incorporates

incorporates Langevin's mobility equation

$$\mu_e = 0.75 e \lambda_e / m_e \overline{v_e} \qquad \text{—— } 61$$

where $\overline{v_e}$ is the mean electron agitation velocity $\left(\frac{8kT_e}{\pi m_e}\right)^{\frac{1}{2}}$.

The total loss due to this mechanism ("volume loss") is, of course, proportional to the axial electron density.

The balance of the power consumed, after allowing for volume loss and wall loss, was considered to be radiated. Assuming the rate of emission of photons to be equal to the rate of excitation of atoms, the average probability of excitation at a given collision was calculated. Results obtained in this way were compared with data from independent sources. Hertz[42] calculated an average probability of excitation for 5-6 volt electrons from Sponer's results[97], giving a value of 0.03. Killian found that it was necessary to use values from 0.063 to 0.006 in order to account for the power supposedly radiated. This agreement to an order of magnitude is perhaps all that can be expected in view of the indirect nature of the estimates.

Druyvestyn and Warmoltz[21] studied the energy balance in discharges in sodium vapour, with or without the addition of rare gases. A comparison between their method of calculating the energy lost by the positive ions in transit to the wall and that of Killian is of interest. The former authors expressed their power term in the form $2\pi r_w j_w (V_w - V_\phi)$ where j_w is the wall current density and V_w, V_ϕ are the potentials at the wall and sheath edge respectively. The assumption was made that the ions have negligible energy on

/entering

entering the sheath. It was noted that a certain amount of energy may be gained by the positive ions in the plasma but this was thought to be likely to introduce not more than 10% error. It may be observed that such errors may be much more important at low pressures, for which the ions can enter the sheath with substantially all of the energy gained in free fall in the plasma. Druyvestyn and Penning[20] later suggested that the values of V_w and V_ϕ might be calculated by the Tonks and Langmuir method, but this is specifically limited to the low pressure regime.

On the other hand, Killian replaced V_ϕ by the quantity

$$\bar{V} = \left(2\pi \int_0^{r_w} n_e V r \, dr\right) / N_e$$

which is the mean potential of the point of origin of the ions weighted according to the relative number of ions produced at different positions. In the low pressure case, a negligible number of ion collisions will take place in transit to the wall and the power imparted to the wall will be correctly represented by $2\pi r_w j_w (V_w - \bar{V})$. In the high pressure case, ion collisions will occur en route to the wall, resulting in a certain amount of heat production in the gas. Killian's formula treats this energy as if it were part of the wall loss. Although this is an artificial convention, it appears to be convenient and preferable to the approach used by Druyvestyn and Warmoltz, which implicitly assumes this energy to be zero, or at least, makes it necessary for it to be separately evaluated.

/The substance

The substance of the experimental work of Druyvestyn and Warmoltz was the investigation of the distribution of the power in the discharge, using a direct measurement of the input power, a calibrated thermopile radiation measurement, and a calculation of the wall energy based on the probe measurement of wall current and potential, electron density and temperature. Any residual power was assumed to represent volume loss. In pure sodium vapour they found the volume loss to be negligible over the range of currents used. The addition of a few cms. of inert gas raised the volume losses markedly, especially in helium.

A theoretical expression for the volume loss was obtained by assuming the fractional loss of energy at each electron collision to be $2m_e/m_a$ and integrating this over all electron energies using the Maxwell distribution function. A further integration was then required for the varying electron density over the tube cross-section. In principle a different expression for volume loss should be used for the cases of sodium vapour with and without inert gas, owing to the different laws of electron density distribution in the different pressure regimes. In fact, only the high pressure case was dealt with in this detail since evidently the volume loss will be proportional to pressure and it is clear that this loss will be negligible at the low pure metal vapour pressures. They obtained

$$W_v = 1.7 \times 10^{-14} \frac{n_{ea} V_{Te}^{3/2} p r_w^2}{A} \qquad 62$$

where A is the atomic weight of the inert gas.

/Probably

Probably the most difficult discharge tube energy calculations are those relating to radiation. Some of the complexities will be discussed in subsequent sections but for the purpose of completing the general review of the energy balance a simple treatment is sufficient. Here it may be assumed that the radiated power at a given wavelength is equal to the product of the appropriate atomic excitation energy and the total number of exciting collisions per second. An expression for the latter may be written down by analogy with the procedure adopted for the rate of ionisation. It is necessary to assume some relationship between the probability of excitation in a single impact and the energy of the electron involved. This must be integrated over the whole range of electron energies assuming some particular distribution function, and over the tube cross-section assuming the appropriate law of radial variation of electron density.

Druyvestyn and Warmoltz took the usual Maxwellian distribution and assumed that the excitation probability for a single collision was zero for electron energies below the photon energy and constant for electron energies exceeding this value. For the high pressure electron density distribution the following expression was obtained

$$W_\rho = \frac{5.5\, n_{ea}\, \mathcal{E}}{\lambda_e}\, r_\omega^2 \left(\frac{eV_{Te}}{3\pi m_e}\right)^2 \left\{1 + \frac{3}{2}\frac{V_R}{V_{Te}}\right\} \varepsilon^{-\frac{3}{2}\frac{V_R}{V_{Te}}}\, eV_R \qquad \text{—— } 63$$

where $\mathcal{E}$ is an excitation constant

V_R is the excitation potential.

/for circumstances

For circumstances in which radiation takes place at two or more wavelengths, all excited from the ground state, similar terms may be calculated for each wavelength. An attempt to verify the equation was made by comparison of the measured values of W_p with calculated values.

12. Further Consideration of the Longitudinal Field

Each of the separate terms in the right hand side of the power balance equation 59 involves the linear factor n_{ea}, within the limitations prescribed in Section 11. Use of the mobility equation 48 allows the left hand also to be expressed with this factor. Accordingly, the longitudinal current and electron density can be eliminated from the power balance equation. The principal parameters remaining are X_z, λ_e, T_e and ν. The other quantities such as tube radius, ionisation potential and atomic mass are effectively fixed constants for a given discharge and may be given appropriate numerical values. The electron mean free path λ_e can be expressed as a function of T_e by the use of data such as those of Townsend. The simultaneous solution of the plasma balance and ion generation equations yields T_e and ν when the sheath is thin. Hence the value of X_z is directly determined by the power balance equation, and within the limitations of the assumptions so far made, is independent of the magnitude of the longitudinal current.

The approximate constancy of the longitudinal field irrespective or variation in the current within rather wide limits is one of the most familiar experimental facts concerning the positive column and

/has been

has been verified by numerous workers.[23, 25, 35, 37, 47, 49, 57, 65, 67]

13. Additional Consequences of Finite Wall Sheath Thickness

It has already been shown that ν and T_e become functions of n_{ea} when the sheath thickness is not negligible. This results in a somewhat modified ion distribution in the plasma and a completely modified distribution in the sheath region. Accordingly, any quantities which are obtained in subsequent theory by integration of a product or function of ion density over the cross-section require modification when the sheath is taken into account. Such quantities are the longitudinal current and the various terms representing the different kinds of power dissipation. The correction of these terms would not be as simple as the correction of the plasma balance equation as it was possible in the latter to utilise the data of Tonks' and Langmuir's analysis as a modification of the tube radius in the form of a single quantity ξ_w. In the present problem this same correction must be applied to the plasma contribution to the cross-sectional integral, but in addition it is necessary to evaluate that part of the integral covering the sheath, which involves the use of the sheath solution for ion density in full detail.

Fortunately in the power balance equation when ionisation is strictly proportional to electron density, similar integrals involving electron density appear in all of the terms and cancel. Thus the effect of the finite sheath thickness is that the electric field becomes dependent on n_{ea} only through the relationship of the latter with ν and T_e. It is therefore unnecessary to invoke non-proportional

/ionisation

ionisation to account for inconstancy of the longitudinal field in circumstances in which the sheath occupies an appreciable part of the tube.

14. Calculation of Electron Density

It has been shown that if the sheath thickness is negligible and the ionisation is proportional to the electron density, the electron temperature and the longitudinal field may be solved independently of the value of tube current. In this case the electron density is calculable directly from the mobility equation for a specified current, using the determined values of X_z and T_e.

If an allowance is made for wall sheath thickness, the field and electron temperature become dependent on the electron density. In this circumstance the longitudinal current for a specified electron density is readily calculable but the reverse problem is not susceptible to simple solution. The procedure may therefore be to calculate the current for a range of electron densities and to select from this information an electron density which corresponds to the specified current. Another method is that of successive approximations. Starting with an estimated value of electron density the field and electron temperature would be calculated. The specified current would be inserted in the mobility equation and a closer approximation to the electron density would be thus obtained. This sequence would be repeated until sufficiently accurate agreement was obtained between successive approximations.

15. Effect of Cumulative Ionisation Processes

In preceding sections the only mechanisms of ionisation which have been discussed are those which give a rate of production of ions which is either constant or proportional to the electron density. Broadly speaking, the latter case represents discharges in which the ions are formed by electron-atom collisions, whereas the former involves plasmas at sufficiently low electron temperature to avoid ionisation by collision, maintained by some external means of ionisation.

Other mechanisms of ionisation variously described as two-stage, stepwise or cumulative, proceed by way of one or more intermediate excited atomic states. It is necessary in evaluating the rate of ionisation thus occurring to make some estimate of the population of the various excited states. This will involve considerations of the rates of the various possible energy transitions between levels, by electron collision or spontaneous radiation, and of the rate of loss of excited atoms by diffusion to the walls. Here, data on the various collision cross-sections and spontaneous transitions may be culled from experimental results or quantum mechanical considerations. Phenomena of the resonance radiation imprisonment type may also affect the population of the various excited states. The population distribution over the tube cross-section for a given excited state will be governed by the specific relevant mechanisms of numerical accretion or depletion. These will, in general, differ from one state to another, so that an evident consequence of this more complicated

/process

process of ionisation is that the cross-sectional distributions of ion density and potential will depart from those investigated by Schottky and Tonks and Langmuir.

A further result of the intervention of intermediate excited states may be anticipated. The density of excited atoms is powerfully influenced by the rate of generation from the ground state by direct electron collision, and the frequency of transition from excited states to the ionised condition is proportional to the product of the population of the state and the electron density. It is therefore probable that the total rate of ionisation will contain a term which is proportional to the square of the electron density. Cross-sectional potential and ion distribution curves calculated by Tonks and Langmuir for ionisation which is proportional to electron density raised to powers zero and unity do not differ in major respects. Thus it has been argued that as an approximation it may be acceptable to assume that the major effect of the new process will be to modify the value of ν in the preceding analyses and render it current dependent. Although full rigour demands the setting up of new potential and ion density distributions, some workers have assumed that with ionisation proportional to n_{ea}^2, the cross-sectional distributions will be substantially similar to those for proportional ionisation.

It is of course possible to include suitable n_{ea}^2 terms in the complete plasma sheath differential equation but the solution presents difficulties and there appears to have been little progress in this direction.

16. Calculation of the Power Radiated in the Presence of Resonance Absorption

Let us consider a system involving only the ground state, a single excited state and the ionised state, and suppose that ionisation via the intermediate state does not occur. Resonance radiation absorption will result in the excitation of a second atom in place of the atom from whence the radiation was originally emitted. In due course, if the loss of excited atoms to the wall by diffusion is small, this second atom will radiate a similar photon and the total amount of radiant energy will be preserved. Within the prescribed limitations the radiated energy will not be affected by resonance absorption at normal currents, for which the ground state population is not depleted.

If now the effect of two-stage ionisation is taken into account, resonance radiation absorption will, by increasing the density of the excited state, cause an increase in the rate of ionisation (whose effects have already been discussed) and a reduction in the power radiated, below that corresponding to the rate of excitation by collision. This modification will affect the longitudinal field through the power balance equation.

A concept due to K.T. Compton[14] which has been used in the discussion of this problem is that of diffusion of the resonance radiation[36, 70]. According to this idea, the net transfer of radiant power at a given point is proportional to the gradient of the concentration of excited atoms. The power radiated from the tube will depend upon the value of this quantity at the edge of the tube. The

/calculation

calculation will require a solution of the density distribution of excited atoms. It may be noted that the concentration of excited atoms near the wall will probably be especially sensitive to the effects of loss of excited atoms by normal diffusion to the walls.

T. Holstein[24] has shown that the above treatment involves the assumption that there is a characteristic mean free path for photons, which implies that the radiation is subject to an exponential absorption law. He shows that owing to the finite line width, the normal mechanism of absorption will be selective in wavelength and therefore will not strictly obey this law. The escape of radiation cannot be regarded as a simple diffusion process. Holstein is mainly concerned with time decaying systems rather than with steady state equilibrium, and he has shown that the density of excited atoms at a given place and time is given by the sum of a series of eigen values, each comprising the product of a spatial distribution function and an exponential time decay factor. After a sufficiently long time the members of the series after the first will become negligible, and the system will degenerate to a characteristic spatial distribution obeying the exponential time law.

It has been argued that in a positive column showing strong resonance radiation imprisonment the corresponding population of excited atoms is built up to a level far exceeding that which would occur if all emitted photons were free to escape. Accordingly, the principal factor governing the spatial distribution of the excited atoms is thought to be the imprisonment process and not the original collision excitation process. Under these conditions Holstein's time

/asymptotic

asymptotic spatial distribution function and exponential decay factor may perhaps be applicable. This assumption is made by Kenty[53] but Waymouth and Bitter[108] incline to the view that the presence of high energy electrons and metastables may modify the characteristic spatial distribution.

These considerations illustrate the difficulties of calculation of the populations of the excited states, which directly affect the amount of radiated power and the rate of ionisation, and hence exert a more distant influence on other parameters of the discharge.

17. Kenty's Analysis of the Low Pressure Discharge in Mercury

Owing to the individuality of structure of energy levels in different types of atoms, further study of the effect on the discharge of cumulative excitation and ionisation processes tends to become restricted to specific cases. Kenty's analysis[53] of the low pressure mercury discharge serves as an example of the types of energy interchange which must be considered, and is concerned mainly with questions of population of the various levels and rates of transitions. Information on these matters is assembled from a variety of sources[1, 4, 9, 10, 28, 44, 52, 71, 72, 113]. The population of the 6^3P_1 state is estimated from the measured 2537 A° output and resonance radiation diffusion theory and is supported by measurements in decaying plasmas by Panevkin[76]. The populations of the 6^3P_0 and 6^3P_2 states are derived by comparison of absorption measurements for the triplet lines[29]. Electron temperature and concentration measurements are utilised in conjunction with experimental[2, 74] and quantum mechanical[41, 77, 110, 111] excitation cross-section curves to broaden the picture. In this way,

/estimates

estimates of rates of cumulative excitation and ionisation[57, 58] and quenching by electron collisions of the second kind are obtained. Although a brief discussion of the effect which these mechanisms have on the electrical characteristics of the discharge is given, the main value of this analysis is in indicating the relative magnitude of the various processes. This enables a simplification to be achieved by omission of minor factors, facilitating an integration of energy level transition concepts into the general electrical theory of the discharge.

From this point of view, the principal conclusions of Kenty's work are that in low pressure mercury substantially all the ionisation occurs via the intermediate levels 6^3P_1 and 6^3P_2, and that transistions involving the 6^3P_0, $6^3D_{3,2,1}$, 6^1P_1 and 7^3S_1 levels are relatively infrequent.

The importance of cumulative ionisation thus revealed probably accounts for the discrepancies noted in Killian's work, where the calculated rates of ionisation fell considerably short of those required by the plasma balance equation (Section 8). This is especially true of the set of results for the highest vapour pressure, if the solution for the high pressure form of the plasma balance equation be accepted. It is to be expected that resonance radiation absorption will operate to its greatest extent in this instance, and by raising the population of excited atoms, may facilitate ionisation by processes neglected in Killian's calculation. This explanation calls for comment on the low values of ν obtained from wall current measurements. Probes inserted in regions of low ion density may be subject to errors caused by ion drainage and field disturbance and this may possibly account for these discrepancies.

/Waymouth

18. Analysis of the Low Pressure Mercury Discharge by Waymouth and Bitter

Waymouth and Bitter[108] base their analysis of the low pressure mercury discharge in the presence of an inert gas on the simplifications suggested by Kenty's work[53]. They considered only the 6^1S_0, 6^3P_1, 6^3P_2 and ionised states of mercury and assumed that the inert gas was neither excited nor ionised. Their calculation consists essentially of three stages, namely computation of the densities of the excited states, evaluation of electron temperature through consideration of balance of production and loss of ions, and determination of power consumed, leading to the value of longitudinal field.

The populations of excited atoms are obtained by setting up, for each excited state, an equation expressing the equality of the rates of transition to and from the energy level. Processes involving electron collision are represented by terms involving electron density and a collison cross-section as factors. Such terms also contain a factor expressing the population of the relevant atomic state, which is conveniently expressed as a fraction f_R or f_M of the population which would exist under thermal equilibrium conditions. The suffices R and M refer to the radiating or metastable states respectively. The use of cross-sections for electron collisions of the second kind, involving transitions to ground level, is avoided by introducing Kenty's relationship between these transition rates and the hypothetical rates under thermal equilibrium conditions using the principle of detailed balancing. The required transition rates come out to be f_R or f_M times the transition rate from ground level to the excited state.

/The equations

The equations thus established consist wholly of terms of the described collision type with the exception of one term which is required to express the radiating transitions. The net number of excited atoms lost per second by radiating transition is taken to be proportional to the total number of excited atoms present and inversely proportional to τ, their mean effective lifetime. τ is expressed in terms of the tube diameter and mercury vapour pressure using a formula due to Mitchell and Zemansky[72], which takes into account the natural lifetime of the excited atoms and its effective prolongation by the repeated absorption and re-emission of photons within the gas. Due to the difficulties of radiation diffusion theory an adjustable coefficient is introduced at this stage. This constant is subject to final choice to fit the theory to the experimental data.

The two relationships thus obtained are treated as simultaneous equations for the determination of f_R and f_M. The solutions involve the factor $\overline{n_e}\tau$, the various collision cross-sections, the excitation energy values and the electron temperature. The latter two quantities enter by way of determining the thermal equilibrium populations of the excited states. The mean electron density over the tube cross-section $\overline{n_e}$ and effective lifetime of the excited atoms at the radiating level enter in conjunction as the consequence of the single term of disparate form in the original equations.

By using numerical values of excitation energy and collision cross-sections (the latter from Kenty's sources) the fractions f_R and f_M expressing the population densities are obtained as known functions of

/electron

electron temperature and the quantity $\bar{n}_e\tau$. It should be noted that the mean collision cross-sections themselves are functions of T_e. The form of the functions f_R and f_M indicates that f_R tends to be proportional to $\bar{n}_e\tau$ for low values of the latter, with f_M tending to a constant. On the other hand, under conditions of high electron density or marked self absorption of radiation, f_R and f_M, and therefore the populations of the excited states themselves, tend to become dependent solely upon T_e.

The second part of the analysis consists of the equating of the production and loss rates of ions. Ionisation is assumed to occur only from the 6^3P_1 and 6^3P_2 states. The production rate per electron is derived for a given T_e and initial state by integration of an assumed collision cross-section function over the electron energy range, taking into account the population of the initial excited state. Lack of basic data on cross-sections of this type necessitates the assumption that the section is proportional to the electron energy excess over the threshold value, and a further adjustable constant is introduced, which is chosen to obtain the best overall agreement of theory and experiment. These calculations yield a value of ν which is a function of T_e and (by virtue of the factors f_R and f_M) of $\bar{n}_e\tau$. This is the counterpart of the "ion generation equation" of Tonks and Langmuir.

The loss rate is calculated by Schottky theory making the usual boundary simplification. The influence of the presence of the inert gas is here manifest in the value ascribed to the mobility μ_{Hg}. and

/mean free

mean free path of the mercury ions, which justifies the use of the "high pressure" Schottky theory.

Equating the rates of production and loss of ions leads to the relationship

$$\frac{1}{2}\left(\frac{2.4}{r_\omega}\right)^2\left(\frac{\pi m_e}{2eV_I}\right)^{\frac{1}{2}}\left(\frac{\mu_{Hg}}{n_{Hg}\,\gamma}\right) = \eta_I^{-\frac{1}{2}}\left\{3f_R(\eta_{RI}+2)+5f_M(\eta_{MI}+2)\right\}\varepsilon^{-\eta_I} \qquad 64$$

where n_{Hg} = number of ground state mercury atoms/cc.

γ = quantity expressing the ionisation cross-section functions, including an adjustable factor.

$$\eta_I = eV_I / kT_e$$

$$\eta_{RI} = e(V_I - V_R)/kT_e$$

$$\eta_{MI} = e(V_I - V_M)/kT_e$$

The expression on the left hand side of this equation is given the symbol θ and may be written

$$\theta = C_7 \frac{\mu_{nHg}}{p_{Hg}\, p_g (r_\omega)^2} \qquad 65$$

where μ_{nHg} is the mobility of the ions at one millimetre pressure.

C_7 is the appropriate coefficient.

θ is dependent only on constants accessible to external control, being determined by the nature and pressure of the filling gas and the radius of the tube.

The right hand side of the equation is dependent on the electron temperature and the quantity $\overline{n_e\tau}$ through the expressions for f_e and f_M. By taking a series of values of T_e for various fixed values of $\overline{n_e\tau}$ the numerical value of the right hand side was calculated and the equation was expressed in graphical form (Fig. 2).

/A digression

A digression from Waymouth and Bitter's analysis is here made, to compare equation 64 with an analogous relationship. For this purpose, the left and right hand sides of equation 64 may be regarded as functions of $C_8 p' r_w$ and of $\left(\frac{V_{Te}}{V_I}, \frac{V_{Te}}{V_R}, \frac{V_{Te}}{V_M}\right)$ respectively and the similarity with Von Engel's universal relationship between $C_4 p r_w$ and $\frac{V_{Te}}{V_I}$ becomes evident. Here $C_8 = C_7^{-\frac{1}{2}}$ and $p' = (p_{Hg} p_g)^{\frac{1}{2}}$. The present equation is not amenable to presentation in the universal form owing to the large number of gas descriptive constants involved. A difference between the two equations is that the right hand side of the present form is also a function of $\overline{n_e}\tau$. Von Engel's relationship can be expressed as a single curve, whilst equation 64 requires a family of curves. A superficial resemblance in this respect is restored if the modification expressing the influence of the sheath thickness on Von Engel's curve is adopted. Again a series of curves is obtained, each being characterised by a parameter t dependent on the electron density (Section 7). Where the single curve is applicable the electron temperature is determined irrespective of the tube current, but in either instance in which a multiplicity of curves is obtained, T_e is dependent on current and additional relationships are required in order to obtain a solution.

These are the power balance and mobility equations. The latter is quite conventional but the power balance expression is worthy of comment in view of the special considerations relating to the multiplicity of excited levels. The wall loss is obtained using the Schottky expression for wall current, the energy of formation of the mercury ions being 10.5 volts and the sheath potential difference being assumed to be

/8 volts.

8 volts. This simplification evades the issues discussed in Section 15, but is justified by the authors on the grounds that small errors in this quantity are of little consequence, as the total wall loss is a minor factor under the conditions obtaining. The resultant expression indicates that this loss is proportional to $\overline{n_e} T_e$.

Loss by elastic collision is calculated from an expression due to Kenty, Easley and Barnes[54] involving an integration of a function including the collision cross-section, over the electron energy distribution. A simplification of the collision cross-section function was used to facilitate the integration. This loss was obtained as a known function of $\overline{n_e} T_e$ for various inert gases.

The power required for the excitation of the 6^3P levels depends on the net transition rates, which are shown by the earlier discussion on the population of these states to be a function of electron temperature and $\overline{n_e} \tau$. The net transition rates to the principal other radiating levels 7^3S, 6^3D and 6^1P_1 were estimated from collision cross-section data summarised by Kenty[53], and are also functions of T_e and $\overline{n_e} \tau$.

These detailed studies show that the total power consumption may be expressed by the sum of terms, all of which can be calculated for a range of values of T_e for selected values of $\overline{n_e} \tau$, and may thus be represented by a family of curves.

Waymouth and Bitter follow the successive approximation method of solution mentioned in Section 14. An estimated value of $\overline{n_e} \tau$ yields a value of T_e when substituted in equation 64. This pair of values can then be used to obtain a value of the longitudinal field from the power balance equations. Next, the field and current values may be

/substituted

substituted in the mobility equation, from which a revised value of n_{ea} is obtained. The whole procedure may then be repeated a sufficient number of times to give consistent results. In this way all the principal parameters of the discharge are determined simultaneously.

From the point of view of pure physics, the comparison between theory and experiment offered by Waymouth and Bitter is subject to criticism on the grounds that the choice of data considers engineering utility rather than suitability for the most direct testing of the analysis. The experimental measurements are made using alternating current and the various complicating factors thereby introduced detract from the value of the comparison. The radiation output of the experimental tubes is measured by conversion of the ultra-violet light to visible light by means of a phosphor. An additional formula is introduced to relate these quantities.

The experimental checks were made with tubes of three different diameters containing an argon/mercury mixture. Measurements were made for a range of currents at fixed bulb temperature, and for a range of temperatures at fixed current. The two adjustable constants in the theory were assigned values which ensured correspondence between the observed and calculated bulb temperature for maximum light output and between the observed and calculated electron temperature for selected operating conditions. In general, for electron temperature, positive column wattage and light output fairly good agreement was obtained from 20° - 60°C and from 100 to 600 m.A. for 1" and 1½" diameter tubes.

/Considerable

Considerable quantitative discrepancies occurred in the results for the $2\frac{1}{8}$" diameter tube. These were attributed to an unsatisfactory estimate of the effective mean life of the excited atoms, the value obtained from radiation diffusion theory being apparently three to six times too small. There were also appreciable quantitative discrepancies in the values of electric field and these were considered to arise from the alternating nature of the current. Maximum differences between the observed and calculated electron temperatures and densities were of the order of 5% and 10% respectively (D.C. measurements).

In view of the availability of two adjustable constants and the degree of correspondence attained between theory and experiment, it is not possible to regard the details of the analysis as fully vindicated. The general qualitative agreement probably justifies the broad interpretation of the various processes in the discharge.

Of the objections which can be raised in principle, the treatment of the escape of resonance radiation is probably the most serious, as has been acknowledged by the authors. Some of the difficulties of this problem have been discussed in Sections 15 and 16 and it merely remains to note here that great simplification has been introduced in the present analysis, and in spite of the use of the adjustable constants the predicted performance of the widest diameter tube is unsatisfactory. It is to be expected that inadequacies of the present treatment of radiation escapement will be most severe in large diameter tubes, in which imprisonment will operate to its greatest extent.

/The analysis

The analysis depends on the formation of averages over the tube cross-section of various quantities such as ion density, excited atom density and metastable density. These are obtained, without rigour, from plausible distribution laws. Again, the authors are conscious of the inadequacy of the assumptions but point out the extreme complication which is introduced by more rigorous treatment. The ion density would, for example, depend on the solution of two simultaneous non-linear second order differential equations.

Schottky's expression for rate of loss of ions by diffusion is only strictly true for ionisation proportional to electron density and its use here is another example of the rather drastic simplifications which have been made to evade the difficulties of rigorous treatment of the cross-sectional variation of parameters.

It is somewhat unprofitable to note without positive comments each of the many other stages at which some approximation, simplification or assumption is made. Mention may, however, be made of a certain element of internal inconsistency. In calculating the populations of the 6^3P_1, and 6^3P_2 levels no other excited levels have been considered, but in the estimation of power consumption, transitions to 7^3S and 6^3D states have been included. The results suggest that certain values of the governing parameters exist for which the energy expended in these transitions to higher levels may become appreciable in relation to the resonance energy radiated. In this case, the population of the 6^3P states might be expected to depart from the value calculated on the postulated basis, since the majority of the higher

/excitations

excitations probably originate from these levels.

The interest in Waymouth and Bitter's work lies in the relationship between the parameters which are displayed. In particular, the ion density and electron temperature assume ubiquitous importance, whereas the electric field plays a minor role. The latter fact is somewhat surprising in view of the familiar practice of presenting various discharge data as functions of X/p . This trend is associated with similarity concepts which are inapplicable in the present discharge involving cumulative processes. It might be debated whether the frequent occurrence of the electron temperature instead of the electric field is a device of analytical utility or a manifestation of inner physical significance. There can be no doubt that the concepts of electron temperature and mean electron energy are of high importance for intuitive insight into the positive column processes.

19. Some Experimental Checks on Waymouth's and Bitter's Theory

The writer has made measurements of electron temperature and density in a tube of 1½" diameter filled with a mixture of argon and mercury vapour. It is emphasised that this can only be regarded as a preliminary exploration, and the results are only of illustrative value.

The double probe technique of Johnson and Malter[48] was used, the probes being radially opposed. Observations were made at several values of arc current including the one used by Kenty, and Waymouth and Bitter, namely 0.42 amp. The mercury vapour pressure was also varied, using the technique of controlling the temperature of the coolest region of the tube, following the practice of Tonks and Langmuir (Section 8) and Druyvestyn and Warmoltz (Section 11). Gross separation of the gas

/components

components was avoided by reversal of the direct current at approximately ten second intervals but it was suspected that minor variations occurred. Thus the probe current tended to drift somewhat, between each reversal. Three procedures for the calculation of electron temperature are described by Johnson and Malter. Those involving obtaining a tangent to the probe characteristic at the origin and intercepts on the characteristic at two selected points were rejected, as placing too much weight on individual readings. The results were calculated by the full logarithmic plot method, which, although laborious, allowed equal weight to the individual data. This also allowed the detection of certain systematic departures from the expected linear form of the plot, which might be indicative of non-Maxwellian distribution. The latitude thus available in drawing the straight line of best fit considerably increases the random errors in the electron temperature values.

Two different sized probe pairs were used, in order to show up any disturbance which might be caused by the probes.

Fig. 3 shows the measured electron temperatures in comparison with the experimental results of Kenty, Easley and Barnes and the calculated data of Waymouth and Bitter. It should be noted that an adjustable constant has been selected to bring the latter into agreement at 40°C. This might with equal justification be chosen to bring the theoretical curve into correspondence with the present experimental data at this temperature. Thus the main valid points of comparison are the slope and the curvature of the characteristic. It may be anticipated that the

/correspondence

correspondence obtained will be of comparable nature to that shown in the published diagram. Waymouth and Bitter do not give theoretical curves for other currents but it is clear from their graph of θ against T_e that the latter should be insensitive to the electron density. The present results exemplify this, by demonstrating a change of electron temperature of the order of 500°K for a more than threefold increase in current.

Comparison of the electron temperatures measured with the large and small probes suggests that some disturbance of discharge conditions is occurring. Extrapolation of data to an infinitely small probe would yield rather higher values of T_e and would tend to bring the results into line with those of the other experimenters.

The electron densities were obtained by assuming their equality with the positive ion densities. The latter were estimated from the observed saturation positive ion currents, assuming a constant positive ion temperature of 330°K, and a sheath/plasma interface area equal to the probe surface area. The results for the two different probe sizes proved to be discordant and an estimate of the sheath area was obtained by assuming the sheath thickness to be the same for both probes. Taking the ratio of probe currents as equal to the ratio of sheath areas yields an equation from which the sheath thickness can be obtained. Unfortunately, at the time of the experimental work the convenience of this method was not appreciated and data are only available for one current (1 amp). Tentative results for other currents have been worked out using the same value of sheath area. These are shown in Fig. 4 in comparison with data from the other sources under discussion.

/A factor

A factor of 0.433 has been applied to convert the observed axial densities to mean cross-sectional values, assuming a distribution of the electrons according to the Bessel Law. The divergences in the results for electron density are greater than for electron temperature.

The values of positive ion mobility used by Waymouth and Bitter have been criticised by Chanin and Biondi[12]. Their direct measurements give a value for argon which is less than half the estimate used by the former investigators. The probable effect of this can be estimated from the θ, T_e curve of Waymouth and Bitter (Fig. 2). The reduction of μ_{Hg} will diminish the value of θ and will therefore necessitate a value of electron temperature or of $\bar{n}_e\tau$ lower than previously calculated. Both of these trends are consistent with the present experimental measurements. The latter effect could also bring the necessary value of τ more nearly into line with radiation diffusion theory estimates and diminish the major numerical discrepancy of the analysis.

A full experimental programme for the testing of the theory requires further careful selection and checking of the techniques for measuring the various parameters, together with comprehensive computations to evaluate the optimum values of the adjustable constants, which cannot be taken over from previous investigations.

The present limited investigation verifies the qualitative features of the theoretical analysis but requires refinement and extension in order to be capable of rendering valid quantitative comments.

20. Conclusion

It has been shown that the phenomena of the uniform positive column can be fairly adequately described qualitatively by available theories, and quantitative predictions at least to an order of magnitude are possible. The central concepts commonly include the ideas of quasi-neutrality, ambipolar diffusion and independent electron temperature. A Maxwellian distribution is usually assumed for the electrons, in some instances with rather weak justification dependent on insensitive probe measurements. The calculation of electric field is related to energy balance and here basic data tend to be rather sparse, measurements of collision cross-sections, absorption and diffusion of radiation being somewhat lacking. The numerous discharge parameters are linked by a system of equations whose rigorous solution presents mathematical difficulties. When sufficient data are available in the future, it seems likely that mechanical solutions will be obtained which will allow inclusion of a more comprehensive range of elementary phenomena with fewer simplifying approximations. The checking of theoretical predictions of this kind will probably require improved experimental techniques, amongst which will possibly be included the extended use of high frequency methods.

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/Appendix I

APPENDIX I

Symbolic Notation

Symbols Representing Quantities

A coefficient in complete plasma sheath equation.

$\mathcal{A}$ atomic weight.

C (with various numerical suffices) constants.

D diffusion coefficients (except when used as energy level designation).

E energies.

e electronic charge.

$\mathcal{E}$ excitation constant.

F electron reflection factor.

$\mathcal{F}$ (with various numerical suffices) functions.

f fraction of thermal equilibrium population.

G coefficient in the equation for rate of single stage ionisation

h dimensionless quantity analogous to radial current.

i currents.

J Bessel functions.

j current densities.

K Tonks' and Langmuir's geometry factor.

k Boltzmann's constant.

M number of ions generated/electron/volt.

cont.

m particle masses.

N total number of a certain kind of particles/unit length of tube.

n number of a certain kind of particles/unit volume.

P probability (except when used as energy level designation).

p pressures.

Q number of ion pairs produced/cc/sec. (source of constant ionisation).

q number of ion pairs produced/cc/sec.

r radial coordinates.

s dimensionless quantity analogous to radial coordinate.

T temperatures.

t wall sheath parameter

u number of ions lost to wall/cc/sec./electron/sec.

V voltages.

v velocities.

W power expended per unit length.

X electric fields.

x Schottky's ionisation efficiency factor.

y Tonks' and Langmuir's quantity (μ_p/e)

z longitudinal coordinates.

α radial coordinate conversion factor (s/r)

β slope of ionisation probability function.

γ quantity (including adjustable factor) expressing ionisation cross-section functions.

ε exponential.

ζ Townsend ionisation coefficient.

η quantities of the form (eV/kT_e)

cont.

θ Waymouth's and Bitter's parameter.

λ mean free paths.

μ mobilities.

ν number of ion pairs produced per second per electron.

ξ dimensionless quantity analogous to radial coordinate in sheath.

τ mean effective lifetime of 6^3P_1 state.

ϕ chosen fractional departure from quasi neutrality.

χ Morse's quantity similar to x

ψ Bessel variable $r(\nu/D_A)^{\frac{1}{2}}$

ω coefficient of r in Bessel variable, $(\nu/D_A)^{\frac{1}{2}}$

Qualifying Symbols (Suffices)

A ambipolar.

a axial.

c chaotic.

d drift.

e electron.

G ground state.

g inert gas.

Hg mercury.

h pertaining to the condition $n_{eh} = n_{ea}/2$

I ionised state.

M metastable state.

n normal pressure (1 mm. Hg.).

cont.

o pertaining to $\eta \to \infty$ or $d\eta/ds \to \infty$

P probe.

p positive ions.

R radiating state.

r radial.

s within the sheath.

v volume.

w wall.

z longitudinal.

α atoms.

ε excited state.

ρ radiation.

ϕ pertaining to the limit of applicability of the plasma solution.

/Appendix II

APPENDIX II

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