Cohomology of groups with normal Engel subgroups

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Abstract

The paper contains several results on the structure of 1-cocycles and the first cohomology group for a pair of representations of a connected group with a normal Engel subgroup.

1 Introduction

Through the paper G is a *connected locally compact group*.

Let π be a representation of G on a Hilbert space X and L be a π-invariant subspace of X. Set $\mathfrak{H} = L^{\perp}$. Denote by $B(\mathfrak{H},L)$ the space of all bounded operators from $\mathfrak{H}$ to L, $B(\mathfrak{H}) = B(\mathfrak{H},\mathfrak{H})$ and $B(L) = B(L,L)$. Then

$$X = L \oplus \mathfrak{H} \text{ and } \pi(g) = \begin{pmatrix} \lambda(g) & \xi(g) \\ 0 & U(g) \end{pmatrix} \text{ for } g \in G, \tag{1.1}$$ 1

where λ and U are representations of G on L and $\mathfrak{H}$, respectively, and the map ξ: $G \to B(\mathfrak{H},L)$ is a (λ,U) *cocycle*, i.e.,

$$\xi(gh) = \lambda(g)\xi(h) + \xi(g)U(h) \text{ for } g,h \in G.$$

It is a (λ,U)-*coboundary* if $\xi(g) = \lambda(g)T - TU(g)$ for all $g \in G$ and some $T \in B(\mathfrak{H},L)$.

The representation π is called an *extension of* λ *by* U. If ξ in (1.1) is (λ,U)-coboundary then there is a π-invariant subspace H such that $X = L \dotplus H$, so that L has a π-invariant complement.

Denote by $\mathcal{Z}(G,\lambda,U)$ the set of all (λ,U)-cocycles and by $\mathcal{B}(G,\lambda,U)$ the set of all (λ,U)-coboundaries. Then $\mathcal{H}^1(G,\lambda,U) = \mathcal{Z}(G,\lambda,U)/\mathcal{B}(G,\lambda,U)$ is the *first cohomology group* of G related to the representations λ, U. If each (λ,U)-cocycle is a (λ,U)-coboundary then $\mathcal{H}^1(G,\lambda,U) = 0$. If $\dim L = 1$ and the representation $\lambda = \iota$ is the trivial representation on L ($\iota(g) \equiv \mathbf{1}_L$), then $\mathcal{H}^1(G,\iota,U)$ is the standard cohomology group of U (see [G]).

Maps λ*:* $G \to B(L)$ and U*:* $G \to B(\mathfrak{H})$ are *spectrally disjoint* at $h \in G$ (see [KS1]), if

$$\mathrm{Sp}(\lambda(h)) \cap \mathrm{Sp}(U(h)) = \varnothing.$$

In [KS1] we investigated the cohomology group $\mathcal{H}^1(G,\lambda,U)$ in the case when G is an Engel group (in particular, a nilpotent group). We showed in Corollaries 2.9 and 2.16 there that if λ and U are spectrally, or sectionally spectrally disjoint then $\mathcal{H}^1(G,\lambda,U) = 0$. We applied this to describe extensions of the elementary representations λ of N by unitary representations U (see (1.1)).

In this paper we consider the much more general case that G contains a *connected normal subgroup* N and impose conditions on the restrictions π^N, ξ^N, λ^N and U^N of π, ξ, λ and U to N.

We show in Corollary 2.3 that if 1) U has an upper triangular form and λ is spectrally disjoint with each diagonal U_i at some $h^i \in N$, and 2) $\mathcal{H}^1(N, \lambda^N, U^N) = 0$, then $\mathcal{H}^1(G, \lambda, U) = 0$. If N is an Engel group and U has a *finite* upper triangular form, then $\mathcal{H}^1(G, \lambda, U) = 0$ if only condition 1) holds (see Corollary 2.4).

In Theorem 2.5 we describe the structure of (λ, U)-cocycles in the case when 1) $\mathfrak{H} = \oplus_{i=1}^{\infty} \mathfrak{H}_i$, where all $\mathfrak{H}_i$ are U-invariant, 2) λ is spectrally disjoint with each $U_i = U|_{\mathfrak{H}_i}$ at some $h^i \in N$ and 3) $\mathcal{H}^1(N, \lambda^N, U_i^N) = 0$ for $i \in \mathbb{N}$.

Let $\lambda = \iota$ be the trivial representation of G. Then the description of all (ι, U)-cocycles is given in Corollary 2.7, where condition 2) above is replaced by the following condition: $2')$ $1 \notin \mathrm{Sp}(U_i(h^i))$ for all i and some $h^i \in N$. If, in addition, N is an Engel group, then condition 3) above holds automatically. Finally, if $G = N$ is a nilpotent group, then all conditions 1), 2'), 3) in Corollary 2.7 can be replaced by only one condition: $\mathfrak{H}_e := \{x \in \mathfrak{H}\colon\ U(g)x = x \text{ for all } g \in G\} = \{0\}$ (see Theorem 2.10).

2 Spectrally disjoint representations. Trivial cohomology groups.

For $g \in G$ and $h \in N$, we have $gh = h_g g$, where $h_g = ghg^{-1} \in N$. The following lemma is an important tool in the proofs of further results.

L1.1 **Lemma 2.1** *Let a map $\mu\colon G \to B(L)$ and a representation $U\colon G \to B(\mathfrak{H})$ be spectrally disjoint at some $h \in N$. Then $\mathrm{Sp}(\lambda(h)) \cap \mathrm{Sp}(U(h_g)) = \varnothing$ for each $g \in G$. So*

1) *If $\mu(h)Y = YU(h_g)$ for some $Y \in B(\mathfrak{H}, L)$ and some $g \in G$, then $Y = 0$.*
2) *If $Y\mu(h) = U(h_g)Y$ for some $Y \in B(L, \mathfrak{H})$ and some $g \in G$, then $Y = 0$.*

Let U be a representation of G on $\mathfrak{H}$ and let $\{H_i\}_{i=0}^k$, $H_0 = \{0\}$ and $k \le \infty$, be a nest of U-invariant subspaces of $\mathfrak{H}$: $H_i \subset H_{i+1}$ and $\mathfrak{H} = \overline{\cup_i H_i}$. Set $\mathfrak{H}_i = H_{i+1} \ominus H_i$ and let U_i be the representation on $\mathfrak{H}_i$ generated by U. Then

$$\mathfrak{H} = \sum_{i=1}^{k} \oplus \mathfrak{H}_i \text{ and } U = \begin{pmatrix} U_1 & U_{12} & U_{13} & \cdots \\ 0 & U_2 & U_{23} & \cdots \\ 0 & 0 & U_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \text{ We write } U = [U_i]_{i=1}^k. \tag{2.1}$$ h2.5

Denote by λ a representation of G on a Hilbert space L, and by λ^N and U^N the restrictions of λ and U to the subgroup N.

P2.1 **Proposition 2.2** *Let λ be spectrally disjoint with each U_i at some $h^i \in N$. Suppose that the restriction ξ^N of a (λ, U)-cocycle ξ to N is a (λ^N, U^N)-coboundary, i.e.,*

$$\xi(h) = \lambda(h)T - TU(h) \text{ for all } h \in N \text{ and some } T \in B(\mathfrak{H}, L).$$

Then ξ is a (λ, U)-coboundary: $\xi(g) = \lambda(g)T - TU(g)$ for $g \in G$, and T is a unique such operator.

Proposition 2.2 yields

C2.1 **Corollary 2.3** *Let $U = [U_i]_{i=1}^k$ and λ be spectrally disjoint with each U_i at some $h^i \in N$.*

(i) *If $\mathcal{H}^1(N, \lambda^N, U^N) = 0$ then $\mathcal{H}^1(G, \lambda, U) = 0$.*

(ii) *If $k < \infty$ and all $\mathcal{H}^1(N, \lambda^N, U_i^N) = 0$, $i = 1, ..., k$, then $\mathcal{H}^1(G, \lambda, U) = 0$.*

In Corollary 2.3(ii) two conditions are required:

1) spectral disjointness of λ and all U_i;

2) $\mathcal{H}^1(N, \lambda^N, U_i^N) = 0$, $1 \le i \le k < \infty$.

We will consider now the case when only condition 1) is needed.

Each $h \in G$ defines a map ad_h on G: $\mathrm{ad}_h(g) = ghg^{-1}h^{-1}$ for $g \in G$. We say that h is an *Engel element* if, for each $g \in G$, there is $n_g \in \mathbb{N}$ such that $\mathrm{ad}_h^{n_g}(g) = e$; G is an *Engel group*, if it consists of Engel elements. Nilpotent groups are Engel groups, while solvable ones are not always Engel.

C2 **Corollary 2.4** *Let $U = [U_i]_{i=1}^k$, $k < \infty$. If λ is spectrally disjoint with each U_i at Engel element h_i of N, then $\mathcal{H}^1(G, \lambda, U) = 0$.*

For $k = \infty$, the condition that all $\mathcal{H}^1(N, \lambda^N, U_i^N) = 0$ is not sufficient for $\mathcal{H}^1(G, \lambda, U) = 0$.

We consider now a particular case of decomposition (2.1). Using Proposition 2.2, we have

T1 **Proposition 2.5** *Let λ, U be representations of G on L and $\mathfrak{H}$, respectively. Let $\mathfrak{H} = \oplus_{i=1}^\infty \mathfrak{H}_i$, where all $\mathfrak{H}_i$ are U-invariant. Set $U_i = U|_{\mathfrak{H}_i}$. Suppose that 1) λ be spectrally disjoint with each U_i at some $h^i \in N$; and 2) $\mathcal{H}^1(N, \lambda^N, U_i^N) = 0$ for $i \in \mathbb{N}$. Then*

(i) *$\mathcal{H}^1(G, \lambda, U_i) = 0$ for all $i \in \mathbb{N}$.*

(ii) *A map ξ: $G \to B(\mathfrak{H}, L)$ is a (λ, U)-cocycle if and only if there are unique operators $T_i \in B(\mathfrak{H}_i, L)$ such that*

$$\xi(g)Q_i = \lambda(g)T_i - T_iU_i(g) \text{ for all } g \in G \text{ and } i \in \mathbb{N},$$

where Q_i are the projections on $\mathfrak{H}_i$. It is a (λ, U)-coboundary if and only if $T = \{T_i\}_{i=1}^\infty \in B(\mathfrak{H}, L)$.

Let $\mathfrak{H} = \oplus_{i=1}^\infty \mathfrak{H}_i$ and all $\mathfrak{H}_i$ be U^N-invariant. Below we give some conditions for all $\mathfrak{H}_i$ to be also U-invariant.

P4.1 **Proposition 2.6** *Let U be a weakly continuous representation of G on $\mathfrak{H}$ and let $\mathfrak{H} = \oplus_{i=1}^\infty \mathfrak{H}_i$, where all $\mathfrak{H}_i$ are invariant for U^N. If $U^N|_{\mathfrak{H}_i} = \omega_i \mathbf{1}_{\mathfrak{H}_i}$ for some distinct unitary characters $\{\omega_i\}_{i=1}^\infty$ on N, then all $\mathfrak{H}_i$ are U-invariant.*

Let now $\dim L = 1$, i.e., $L = \mathbb{C}e$. Then each operator in $B(\mathfrak{H}, L)$ has form $z \otimes e$, where $(z \otimes e)x = (x, z)_{\mathfrak{H}} e$ for all $x \in \mathfrak{H}$ and some $z \in \mathfrak{H}$. Every map ξ: $G \to B(\mathfrak{H}, L)$ has form $\xi(g) = r(g) \otimes e$ for some map r: $G \to \mathfrak{H}$. Each $T \in B(\mathfrak{H}_i, L)$ has form $T = y \otimes e$ for some $y \in \mathfrak{H}_i$.

If $\lambda = \iota$ is the trivial representation of G on L, then Proposition 2.5 has a simpler form.

C3.1n **Corollary 2.7** *Let U be a representation of G on $\mathfrak{H}$ and $\mathfrak{H} = \oplus_{i=1}^\infty \mathfrak{H}_i$, where all $\mathfrak{H}_i$ are U-invariant. Set $U_i = U|_{\mathfrak{H}_i}$. Let 1) $1 \notin \mathrm{Sp}(U_i(h^i))$ for all i and some $h^i \in N$, and let 2) $\mathcal{H}^1(G, \iota^N, U_i^N) = 0$ for all $i \in \mathbb{N}$. Then*

(i) *$\mathcal{H}^1(G, \iota, U_i) = 0$ for all $i \in \mathbb{N}$.*

(ii) *A map* $\xi\colon G\to r(g)\otimes e$ *is a* (ι,U)*-cocycle if and only if there are unique* $y_i\in\mathfrak{H}_i$ *such that*

$$r(g)=\oplus_{i=1}^{\infty}(y_i-U_i^*(g)y_i)\in\mathfrak{H},\ \ i.e.,\ \ \sum_{i=1}^{\infty}\|y_i-U_i^*(g)y_i\|^2<\infty\ for\ g\in G. \qquad (2.2)$$ 1.5

It is a (ι,U)*-coboundary if and only if the operator* $(y_i\otimes e)_{i=1}^{\infty}\in B(\mathfrak{H},L)$, *i.e.,* $\sum_{i=1}^{\infty}\|y_i\|^2<\infty$.

If N has property (T) (ι is isolated in the set $\widehat{N}$ of all irreducible representations of N), then $\mathcal{H}^1(N,\iota,V)=\{0\}$ for any unitary representation V [G]. So Corollary 2.7 yields

C2.5 **Corollary 2.8** *Let* G,U *and* $\mathfrak{H}$ *be as in Corollary* 2.7 *and* U^N *be unitary. Let* 1) $1\notin\mathrm{Sp}(U_i(h^i))$ *for all* i *and some* $h^i\in N$, *and* 2) N *have property* (T). *Then the results of Corollary* 2.7 *hold.*

Using Corollary 2.4, we obtain the following generalization of Corollary 2.7 without the assumption that N has property (T).

C2.6 **Corollary 2.9** *Let* G,U *and* $\mathfrak{H}$ *be as in Corollary* 2.7. *If* $1\notin\mathrm{Sp}(U_i(h^i))$ *for all* i *and some Engel elements* $h^i\in N$, *then the results of Corollary* 2.7 *hold.*

Let $G=N$ be a nilpotent group and U be a unitary representation of N on $\mathfrak{H}$. Assume that $\mathfrak{H}_e:=\{x\in\mathfrak{H}\colon U(g)x=x \text{ for all } g\in G\}=\{0\}$. Then it follows from Proposition 3.3 [KS1] that there is a decomposition $\mathfrak{H}=\oplus_{i=1}^k\mathfrak{H}_i$, $k\le\infty$, such that all $\mathfrak{H}_i$ are U-invariant and the representation ι is spectrally disjoint with each $U_i=U|_{\mathfrak{H}_i}$ at some $h^i\in N$, i.e., $1\notin\mathrm{Sp}(U_i(h^i))$. Therefore, since N is an Engel group, Corollary 2.9 yields

T2.5 **Theorem 2.10** *Let* N *be a nilpotent connected group and* U *be a unitary representation of* N *on* $\mathfrak{H}$. *If* $\mathfrak{H}_e=\{0\}$ *then there is a decomposition* $\mathfrak{H}=\oplus_{i=1}^k\mathfrak{H}_i$, $k\le\infty$, *such that all* $\mathfrak{H}_i$ *are* U*-invariant,* $\mathcal{H}^1(N,\iota,U_i)=0$ *for all* i *and* (2.2) *describes* (ι,U)*-cocycles.*

Let U be a unitary representation of G on $\mathfrak{H}$. Let a normal subgroup N of G be nilpotent, connected and locally compact. By Theorem 2.10, $\mathfrak{H}=\oplus_{i=1}^k\mathfrak{H}_i$ for $k\le\infty$, where $\mathfrak{H}_i$ are U^N-invariant subspaces, and $\mathcal{H}^1(N,\iota,U_i^N)=0$ for all i. If $U^N|_{\mathfrak{H}_i}=\omega_i\mathbf{1}_{\mathfrak{H}_i}$ for some distinct unitary characters $\{\omega_i\}_{i=1}^k$, then by Proposition 2.6, all $\mathfrak{H}_i$ are U-invariant and $\mathcal{H}^1(G,\iota,U_i)=0$ for all i.

Some applications of the obtained results to spectral continuity and representations in the spaces with indefinite metric will be considered in subsequent publications.

References

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