Cohomology of groups with normal Engel subgroups

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November 27, 2019

Abstract

The paper contains several results on the structure of 1-cocycles and the first cohomology group for a pair of representations of a connected group with a normal Engel subgroup.

1 Introduction

Through the paper $G$ is a connected locally compact group.

Let $\pi$ be a representation of $G$ on a Hilbert space $X$ and $L$ be a $\pi$-invariant subspace of $X$. Set $\mathfrak{N} = L^\perp$. Denote by $B(\mathfrak{N}, L)$ the space of all bounded operators from $\mathfrak{N}$ to $L$, $B(\mathfrak{N}) = B(\mathfrak{N}, \mathfrak{N})$ and $B(L) = B(L, L)$. Then

$$X = L \oplus \mathfrak{N} \text{ and } \pi(g) = \begin{pmatrix} \lambda(g) & \xi(g) \\ 0 & U(g) \end{pmatrix} \text{ for } g \in G,$$

(1.1)

where $\lambda$ and $U$ are representations of $G$ on $L$ and $\mathfrak{N}$, respectively, and the map $\xi: G \to B(\mathfrak{N}, L)$ is a $(\lambda, U)$-cocycle, i.e.,

$$\xi(gh) = \lambda(g)\xi(h) + \xi(g)U(h) \text{ for } g, h \in G.$$

It is a $(\lambda, U)$-coboundary if $\xi(g) = \lambda(g)T - TU(g)$ for all $g \in G$ and some $T \in B(\mathfrak{N}, L)$.

The representation $\pi$ is called an extension of $\lambda$ by $U$. If $\xi$ in (1.1) is $(\lambda, U)$-coboundary then there is a $\pi$-invariant subspace $H$ such that $X = L + H$, so that $L$ has a $\pi$-invariant complement.

Denote by $Z(G, \lambda, U)$ the set of all $(\lambda, U)$-cocycles and by $B(G, \lambda, U)$ the set of all $(\lambda, U)$-coboundaries. Then $H^1(G, \lambda, U) = Z(G, \lambda, U)/B(G, \lambda, U)$ is the first cohomology group of $G$ related to the representations $\lambda, U$. If each $(\lambda, U)$-cocycle is a $(\lambda, U)$-coboundary then $H^1(G, \lambda, U) = 0$. If $\dim L = 1$ and the representation $\lambda = \iota$ is the trivial representation on $L$ ($\iota(g) \equiv 1_L$), then $H^1(G, \iota, U)$ is the standard cohomology group of $U$ (see [G]).

Maps $\lambda: G \to B(L)$ and $U: G \to B(\mathfrak{N})$ are spectrally disjoint at $h \in G$ (see [KS1]), if

$$\text{Sp}(\lambda(h)) \cap \text{Sp}(U(h)) = \emptyset.$$

In [KS1] we investigated the cohomology group $H^1(G, \lambda, U)$ in the case when $G$ is an Engel group (in particular, a nilpotent group). We showed in Corollaries 2.9 and 2.16 there that if $\lambda$ and $U$ are spectrally, or sectionally spectrally disjoint then $H^1(G, \lambda, U) = 0$. We applied this to describe extensions of the elementary representations $\lambda$ of $N$ by unitary representations $U$ (see (1.1)).

In this paper we consider the much more general case that $G$ contains a connected normal subgroup $N$ and impose conditions on the restrictions $\pi^N, \xi^N, \lambda^N$ and $U^N$ of $\pi, \xi, \lambda$ and $U$ to $N$. 

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We show in Corollary 2.3 that if 1) $U$ has an upper triangular form and $\lambda$ is spectrally disjoint with each diagonal $U_i$ at some $h^i \in N$, and 2) $\mathcal{H}^1(N, \lambda^N, U^N) = 0$, then $\mathcal{H}^1(G, \lambda, U) = 0$. If $N$ is an Engel group and $U$ has a finite upper triangular form, then $\mathcal{H}^1(G, \lambda, U) = 0$ if only condition 1) holds (see Corollary 2.4).

In Theorem 2.5 we describe the structure of $(\lambda, U)$-cocycles in the case when 1) $\mathcal{F} = \bigoplus_{i=1}^{\infty} \mathcal{F}_i$, where all $\mathcal{F}_i$ are $U$-invariant, 2) $\lambda$ is spectrally disjoint with each $U_i = U|_{\mathcal{F}_i}$ at some $h^i \in N$ and 3) $\mathcal{H}^1(N, \lambda^N, U^N) = 0$ for $i \in \mathbb{N}$.

Let $\lambda = \iota$ be the trivial representation of $G$. Then the description of all $(\iota, U)$-cocycles is given in Corollary 2.7, where condition 2) above is replaced by the following condition: 2') $1 \notin \text{Sp}(U_i(h^i))$ for all $i$ and some $h^i \in N$. If, in addition, $N$ is an Engel group, then condition 3) above holds automatically. Finally, if $G = N$ is a nilpotent group, then all conditions 1), 2'), 3) in Corollary 2.7 can be replaced by only one condition: $\mathcal{F}_e := \{x \in \mathcal{F}: U(g)x = x \text{ for all } g \in G\} = \{0\}$ (see Theorem 2.10).

2 Spectrally disjoint representations. Trivial cohomology groups.

For $g \in G$ and $h \in N$, we have $gh = h_g g$, where $h_g = ghg^{-1} \in N$. The following lemma is an important tool in the proofs of further results.

**Lemma 2.1** Let a map $\mu: G \to B(L)$ and a representation $U: G \to B(\mathcal{F})$ be spectrally disjoint at some $h \in N$. Then $\text{Sp}(\lambda(h)) \cap \text{Sp}(U(h_g)) = \emptyset$ for each $g \in G$. So

1. If $\mu(h)Y = YU(h_g)$ for some $Y \in B(\mathcal{F}, L)$ and some $g \in G$, then $Y = 0$.
2. If $Y \mu(h) = U(h_g)Y$ for some $Y \in B(L, \mathcal{F})$ and some $g \in G$, then $Y = 0$.

Let $U$ be a representation of $G$ on $\mathcal{F}$ and let $\{H_i\}_{i=0}^{k}, H_0 = \{0\}$ and $k \leq \infty$, be a nest of $U$-invariant subspaces of $\mathcal{F}$: $H_i \subset H_{i+1}$ and $\mathcal{F} = \bigcup_i H_i$. Set $\mathcal{F}_i = H_{i+1} \ominus H_i$ and let $U_i$ be the representation on $\mathcal{F}_i$ generated by $U$. Then

$$\mathcal{F} = \bigoplus_{i=1}^{k} \mathcal{F}_i \quad \text{and} \quad U = \begin{pmatrix} U_1 & U_{12} & U_{13} & \cdots \\ 0 & U_2 & U_{23} & \cdots \\ 0 & 0 & U_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$ 

We write $U = [U_i]_{i=1}^{k}$. (2.1) \(h2.5\)

Denote by $\lambda$ a representation of $G$ on a Hilbert space $L$, and by $\lambda^N$ and $U^N$ the restrictions of $\lambda$ and $U$ to the subgroup $N$.

**Proposition 2.2** Let $\lambda$ be spectrally disjoint with each $U_i$ at some $h^i \in N$. Suppose that the restriction $\xi^N$ of a $(\lambda, U)$-cocycle $\xi$ to $N$ is a $(\lambda^N, U^N)$-coboundary, i.e.,

$$\xi(h) = \lambda(h)T - TU(h) \text{ for all } h \in N \text{ and some } T \in B(\mathcal{F}, L).$$

Then $\xi$ is a $(\lambda, U)$-coboundary: $\xi(g) = \lambda(g)T - TU(g)$ for $g \in G$, and $T$ is a unique such operator.

Proposition 2.2 yields
Corollary 2.3 Let $U = [U_i]_{i=1}^k$ and $\lambda$ be spectrally disjoint with each $U_i$ at some $h^i \in N$.

(i) If $\mathcal{H}^1(N,\lambda^N, U_i) = 0$ then $\mathcal{H}^1(G,\lambda, U) = 0$.

(ii) If $k < \infty$ and all $\mathcal{H}^1(N,\lambda^N, U_i^N) = 0$, $i = 1, \ldots, k$, then $\mathcal{H}^1(G,\lambda, U) = 0$.

In Corollary 2.3(ii) two conditions are required:

1) spectral disjointness of $\lambda$ and all $U_i$;
2) $\mathcal{H}^1(N,\lambda^N, U_i^N) = 0$, $1 \leq i \leq k < \infty$.

We will consider now the case when only condition 1) is needed.

Each $h \in G$ defines a map $\text{ad}_h$ on $G$: $\text{ad}_h(g) = ghg^{-1}h^{-1}$ for $g \in G$. We say that $h$ is an Engel element if, for each $g \in G$, there is $n_g \in \mathbb{N}$ such that $\text{ad}_h^{n_g}(g) = e$; $G$ is an Engel group, if it consists of Engel elements. Nilpotent groups are Engel groups, while solvable ones are not always Engel.

Corollary 2.4 Let $U = [U_i]_{i=1}^k$, $k < \infty$. If $\lambda$ is spectrally disjoint with each $U_i$ at Engel element $h_i$ of $N$, then $\mathcal{H}^1(G,\lambda, U) = 0$.

For $k = \infty$, the condition that all $\mathcal{H}^1(N,\lambda^N, U_i^N) = 0$ is not sufficient for $\mathcal{H}^1(G,\lambda, U) = 0$.

We consider now a particular case of decomposition (2.1). Using Proposition 2.2, we have

Proposition 2.5 Let $\lambda, U$ be representations of $G$ on $L$ and $\mathcal{H}$, respectively. Let $\mathcal{H} = \bigoplus_{i=1}^\infty \mathcal{H}_i$, where all $\mathcal{H}_i$ are $U$-invariant. Let $U_i = U|_{\mathcal{H}_i}$. Suppose that 1) $\lambda$ be spectrally disjoint with each $U_i$ at some $h^i \in N$; and 2) $\mathcal{H}^1(N,\lambda^N, U_i^N) = 0$ for $i \in \mathbb{N}$. Then

(i) $\mathcal{H}^1(G,\lambda, U_i) = 0$ for all $i \in \mathbb{N}$.

(ii) A map $\xi: G \to B(\mathcal{H}, L)$ is a $(\lambda, U)$-cocycle if and only if there are unique operators $T_i \in B(\mathcal{H}_i, L)$ such that

$$\xi(g)Q_i = \lambda(g)T_i - T_iU_i(g)$$

for all $g \in G$ and $i \in \mathbb{N}$, where $Q_i$ are the projections on $\mathcal{H}_i$. It is a $(\lambda, U)$-coboundary if and only if $T = \{T_i\}_{i=1}^\infty \in B(\mathcal{H}, L)$.

Let $\mathcal{H} = \bigoplus_{i=1}^\infty \mathcal{H}_i$ and all $\mathcal{H}_i$ be $U^N$-invariant. Below we give some conditions for all $\mathcal{H}_i$ to be also $U$-invariant.

Proposition 2.6 Let $U$ be a weakly continuous representation of $G$ on $\mathcal{H}$ and let $\mathcal{H} = \bigoplus_{i=1}^\infty \mathcal{H}_i$, where all $\mathcal{H}_i$ are invariant for $U^N$. If $U^N|_{\mathcal{H}_i} = \omega_i1_{\mathcal{H}_i}$ for some distinct unitary characters $\{\omega_i\}_{i=1}^\infty$ on $N$, then all $\mathcal{H}_i$ are $U$-invariant.

Let now dim $L = 1$, i.e., $L = \mathbb{C} e$. Then each operator in $B(\mathcal{H}, L)$ has form $z \otimes e$, where $(z \otimes e)x = (x, z)_e e$ for all $x \in \mathcal{H}$ and some $z \in \mathcal{H}$. Every map $\xi: G \to B(\mathcal{H}, L)$ has form $\xi(g) = r(g) \otimes e$ for some map $r: G \to \mathcal{H}$.

If $\lambda = 1$ is the trivial representation of $G$ on $L$, then Proposition 2.5 has a simpler form.

Corollary 2.7 Let $U$ be a representation of $G$ on $\mathcal{H}$ and $\mathcal{H} = \bigoplus_{i=1}^\infty \mathcal{H}_i$, where all $\mathcal{H}_i$ are $U$-invariant. Set $U_i = U|_{\mathcal{H}_i}$. Let 1) $1 \notin \text{Sp}(U_i(h^i))$ for all $i$ and some $h^i \in N$, and let 2) $\mathcal{H}^1(G,\iota^N, U_i^N) = 0$ for all $i \in \mathbb{N}$. Then

(i) $\mathcal{H}^1(G,\iota, U_i) = 0$ for all $i \in \mathbb{N}$.
(ii) A map $\xi: G \to r(g) \otimes e$ is a $(\iota, U)$-cocycle if and only if there are unique $y_i \in \mathfrak{H}_i$ such that

$$r(g) = \bigoplus_{i=1}^{\infty} (y_i - U^*_i(g) y_i) \in \mathfrak{H}_i, \text{ i.e., } \sum_{i=1}^{\infty} \|y_i - U^*_i(g)y_i\|^2 < \infty \text{ for } g \in G. \quad (2.2)$$

It is a $(\iota, U)$-coboundary if and only if the operator $(y_i \otimes e)^{\infty}_{i=1} \in B(\mathfrak{H}_i, L),$ i.e., $\sum_{i=1}^{\infty} \|y_i\|^2 < \infty.$

If $N$ has property $(T)$ ($\iota$ is isolated in the set $\tilde{N}$ of all irreducible representations of $N$), then $\mathcal{H}^2(N, \iota, V) = \{0\}$ for any unitary representation $V$ [G]. So Corollary 2.7 yields

\[ \text{C2.5} \]

**Corollary 2.8** Let $G, U$ and $\mathfrak{H}$ be as in Corollary 2.7 and $U^N$ be unitary. Let 1) $1 \not\in \text{Sp}(U_i(h^i))$ for all $i$ and some $h^i \in N$, and 2) $N$ have property $(T)$. Then the results of Corollary 2.7 hold.

Using Corollary 2.4, we obtain the following generalization of Corollary 2.7 without the assumption that $N$ has property $(T)$.

\[ \text{C2.6} \]

**Corollary 2.9** Let $G, U$ and $\mathfrak{H}$ be as in Corollary 2.7. If $1 \not\in \text{Sp}(U_i(h^i))$ for all $i$ and some Engel elements $h^i \in N$, then the results of Corollary 2.7 hold.

Let $G = N$ be a nilpotent group and $U$ be a unitary representation of $N$ on $\mathfrak{H}$. Assume that $\mathfrak{H}_e := \{x \in \mathfrak{H}: U(g)x = x \text{ for all } g \in G\} = \{0\}$. Then it follows from Proposition 3.3 [KS1] that there is a decomposition $\mathfrak{H}_e = \bigoplus_{i=1}^{k} \mathfrak{H}_i$, $k \leq \infty$, such that all $\mathfrak{H}_i$ are $U$-invariant and the representation $\iota$ is spectrally disjoint with each $U_i = U|_{\mathfrak{H}_i}$ at some $h^i \in N$, i.e., $1 \not\in \text{Sp}(U_i(h^i))$. Therefore, since $N$ is an Engel group, Corollary 2.9 yields

\[ \text{T2.5} \]

**Theorem 2.10** Let $N$ be a nilpotent connected group and $U$ be a unitary representation of $N$ on $\mathfrak{H}$. If $\mathfrak{H}_e = \{0\}$ then there is a decomposition $\mathfrak{H}_e = \bigoplus_{i=1}^{k} \mathfrak{H}_i$, $k \leq \infty$, such that all $\mathfrak{H}_i$ are $U$-invariant, $\mathcal{H}^1(N, \iota, U_i) = 0$ for all $i$ and (2.2) describes $(\iota, U)$-cocycles.

Let $U$ be a unitary representation of $G$ on $\mathfrak{H}$. Let a normal subgroup $N$ of $G$ be nilpotent, connected and locally compact. By Theorem 2.10, $\mathfrak{H}_e = \bigoplus_{i=1}^{k} \mathfrak{H}_i$ for $k \leq \infty$, where $\mathfrak{H}_i$ are $U^N$-invariant subspaces, and $\mathcal{H}^1(N, \iota, U^N_i) = 0$ for all $i$. If $U^N|_{\mathfrak{H}_i} = \omega_i 1_{\mathfrak{H}_i}$ for some distinct unitary characters $\{\omega_i\}_{i=1}^{k}$, then by Proposition 2.6, all $\mathfrak{H}_i$ are $U$-invariant and $\mathcal{H}^1(G, \iota, U_i) = 0$ for all $i$.

Some applications of the obtained results to spectral continuity and representations in the spaces with indefinite metric will be considered in subsequent publications.

**References**


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