FROM LOMONOSOV LEMMA TO RADICAL APPROACH IN JOINT SPECTRAL RADIUS THEORY

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To the memory of Victor Lomonosov, a man who moved mountains in Mathematics

Abstract. In this paper we discuss the infinite-dimensional generalizations of the famous theorem of Berger-Wang (generalized Berger-Wang formulas) and give an operator-theoretic proof of I. Morris's theorem about coincidence of three essential joint spectral radius, related to these formulas. Further we develop Banach-algebraic approach based on the theory of topological radicals, and obtain some new results about these radicals.

1. Introduction

1.1. **Banach-algebraic consequences of Lomonosov Lemma.** The famous Lomonosov Lemma [Lom73] states:

If an algebra A of operators on a Banach space X contains a non-zero compact operator T then either A has a non-trivial closed invariant subspace (IS, for brevity) *or it contains a compact operator with a non-zero point in spectrum.*

An immediate consequence of this result is that *any algebra of compact quasinilpotent operators has an IS*; the standard technique gives then that such an algebra is triangularizable.

M. G. Krein proposed to call compact quasinilpotent operators *Volterra operators*; respectively, a set of operators is called *Volterra* if all its elements are Volterra operators. Thus *any Volterra algebra has an IS*. This result was extended by the second author [Sh84] as follows:

Any Volterra algebra A has an IS which is also invariant for all operators commuting with A (such subspaces are called *hyperinvariant*).

Besides of Lomonosov's technique the proof used estimations of the norms of products for elements of a Volterra algebra A; in fact, it was proved in [Sh84] that *the joint spectral radius $\rho(M)$ of any finite set $M \subseteq A$ equals* 0, i.e. A is *finitely quasinilpotent.*

This result can be considered as an application of the invariant subspace theory to the theory of joint spectral radius. Conversely, the second part of the proof in [Sh84] is an application of the joint spectral radius technique to the invariant subspace theory (again via Lomonosov's theorem about Volterra algebras): if $M = \{T_1, ..., T_n\}$ and $\rho(M) = 0$ then

$$\rho\left(\sum_{i=1}^{n} T_i S_i\right) = 0$$

for all operators S_i commuting with every operator from M. So the algebra generated by a Volterra algebra A and its commutant has a non-zero Volterra ideal. The interaction of these theories remained to be fruitful in subsequent studies.

The notion of the joint spectral radius of a bounded subset M in a normed algebra A was introduced by Rota and Strang [RS60]. To give precise definition, let us set $\|M\| = \sup\{\|a\| : a \in M\}$, the *norm* of M, and $M^n = \{a_1 \cdots a_n : a_1, \dots, a_n \in M\}$, the *n-power* of M. The number

$$\rho(M) := \lim \|M^n\|^{1/n} = \inf \|M^n\|^{1/n}$$

is called a (*joint*) *spectral radius* of M. If $\rho(M) = 0$ then we say that M is *quasinilpotent.*

In [Tur99] the third author, using the joint spectral radius approach, obtained the solution of Volterra semigroup problem posed by Heydar Radjavi: it was proved in [Tur99] that *any Volterra semigroup generates a Vollterra algebra* and, therefore, has an IS by Lomonosov's theorem. Further, in [ST00] it

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was proved that *any Volterra Lie algebra has an IS*; this result can be regarded as an infinite-dimensional extension of Engel Theorem, playing the fundamental role in the theory of finite-dimensional Lie algebras.

One of the main technical tools obtained and applied in [ST00] was an infinite-dimensional extension of the Berger-Wang Theorem [BW92], a fundamental result of the finite dimensional linear dynamics [Jun09]. This theorem establishes the equality

$$\rho(M) = r(M), \tag{1.1}$$

for any bounded set M of matrices, where

$$r(M) := \limsup \{\rho(a) : a \in M^n\}^{1/n};$$

the number $r(M)$ called a *BW-radius* of M. In [ST00] the equality (1.1) was proved for any precompact set M of compact operators on an *infinite-dimensional* Banach space.

To see the importance of validity of (1.1) for precompact sets of compact operators, note that it easily implies that if G is a Volterra semigroup then $\rho(M) = 0$, for each precompact $M \subset G$ (because clearly $r(M) = 0$). This result proved in [Tur99] played a crucial role in the solution of Volterra semigroup problem. But it should be said that the proof of (1.1) in [ST00] used the results of [Tur99].

Other results on invariant subspaces of operator semigroups, Lie algebras and Jordan algebras were obtained on this way in [ST00, ST'05, KeST9].

1.2. **The generalized BW-formula.** To move further we have to introduce some "essential radii" $\rho_e(M)$, $\rho_f(N)$ and $\rho_\chi(M)$ of a set M of operators on a Banach space X. They are defined in the same way as $\rho(M)$ but by using seminorms $\|\cdot\|_e$, $\|\cdot\|_f$ and $\|\cdot\|_\chi$, instead of the operator norm $\|\cdot\|$.

Let $B(X)$ be the algebra of all bounded linear operators on X, and $K(X)$ the ideal of all compact operators. The essential norm $\|T\|_e$ of an operator $T \in B(X)$ is just the norm of the image $T + K(X)$ of T in the quotient $B(X)/K(X)$; in other words

$$\|T\|_e = \inf\{\|T - S\| : S \text{ is a compact operator}\}.$$

Similarly

$$\|T\|_f = \inf\{\|T - S\| : S \text{ is a finite rank operator}\}.$$

The Hausdorff norm $\|T\|_\chi$ is defined as $\chi(TX_\odot)$, the Hausdorff measure of noncompactness of the image of the unit ball $X_\odot$ of X under T. Recall that, for any bounded subset E of X, the value $\chi(E)$ is equal to the infimum of such $t > 0$ that E has a finite t-net.

It is easy to check that $\|T\|_\chi \le \|T\|_e \le \|T\|_f$ and therefore

$$\rho_\chi(M) \le \rho_e(M) \le \rho_f(M), \tag{1.2}$$

for each bounded set $M \subset B(X)$. The number $\rho_\chi(M)$ is called the *Hausdorff radius*, $\rho_e(M)$ the *essential radius*, and $\rho_f(M)$ the *f-essential radius* of M.

In what follows, for a set M in a normed algebra A and a closed ideal J of A, we write M/J for the image of M in A/J under the canonic quotient map $q_J : A \longrightarrow A/J$:

$$M/J := q_J(M).$$

So we write $\rho_e(M) = \rho(M/K(X))$. This reflects the fact that essential radius $\rho_e(M)$ is the usual joint spectral radius of the image of M in the Calkin algebra $B(X)/K(X)$.

In [ST02] the following extension of (1.1) to precompact sets of general (not necessarily compact) operators was obtained:

$$\rho(M) = \max\{\rho_\chi(M), r(M)\}. \tag{1.3}$$

It was proved under assumption that X is reflexive (or, more generally, that M consists of weakly compact operators). We call this equality the *generalized BW-formula*.

Furthermore, in the short communication [ST01] a Banach algebraic version of the generalized BW-formula was announced (see (4.1) below) which, being applied to the algebra $B(X)$, shows that

$$\rho(M) = \max\{\rho_e(M), r(M)\}. \tag{1.4}$$

for all Banach spaces. The proof of (1.3) in full generality was firstly presented in the arXive publication [ST08*]; the journal version appeared in [ST12].

Several months after presentation of [ST08*], I. Morris in arXive publication [M09*] gave another proof of (1.3) based on the multiplicative ergodic theorem of Tieullen [Th87] and deep technique of the theory of cohomology of dynamical systems. The main result of [M09*] establishes an equality similar to (1.3) for operator valued cocycles of dynamical systems. It was also proved in [M09*] that

$$\rho_\chi(M) = \rho_e(M) = \rho_f(M) \tag{1.5}$$

for any precompact set $M \subset B(X)$. The journal publication of these results appeared in [Mor12].

Here we give another, operator-theoretic proof of (1.5) and then discuss related Banach-algebraic results and constructions connected with the different joint spectral radius formulas. It will be shown that topological radicals present a convenient tool in the search of an optimal joint spectral radius formula.

2. Coincidence of Hausdorff and essential radii

In this section we are going to prove the equality $\rho_\chi(M) = \rho_e(M)$, for any precompact set in $B(X)$; the proof of the equality $\rho_e(M) = \rho_f(M)$ will be presented in the next section.

2.1. **An estimation of the Hausdorff radius for multiplication operators.** At the beginning we transfer some results of [ST00] from operators to elements of the Calkin algebra $B(X)/K(X)$. We use the following link of Hausdorff norm with the Hausdorff measure of non-compactness:

Lemma 2.1. *Let M be a precompact subset of $B(X)$. Then $\chi(MW) \le \|M\|_\chi \|W\|$ for any bounded subset W of X, and $\|M\|_\chi = \chi(MX_\odot)$.*

The inequality in Lemma 2.1 was obtained in [ST00, Lemma 5.2]. The equality $\|M\|_\chi = \chi(MX_\odot)$ is obvious for a finite $M \subseteq B(X)$ by definition, due to $\chi(G \cup K) = \max\{\chi(G), \chi(K)\}$ for bounded subsets of X If M is precompact then $\|M\|_\chi = \sup\{\|N\|\,\chi\colon N \subseteq M \text{ is finite}\}$, and the result follows.

For $T \in B(X)$, let L_T and R_T denote the left and right multiplication operators on $B(X)$: $\mathrm{L}_T P = TP$ and $\mathrm{R}_T P = PT$ for each $P \in B(X)$.

For $M \subseteq B(X)$, put $\mathrm{L}_M := \{\mathrm{L}_T\colon T \in M\}$ and $\mathrm{R}_M := \{\mathrm{R}_T\colon T \in M\}$. If M is a set in a Banach algebra A we define L_M and R_M similarly. By [ST12, Lemma 2.1],

$$r(\mathrm{L}_M \mathrm{R}_M) = r(M)^2 \text{ and } \rho(\mathrm{L}_M \mathrm{R}_M) = \rho(M)^2 \tag{2.1}$$

for every bounded set M in A.

Lemma 2.2. *Let M be a bounded subset of $B(X)$. Then*

$$\left\|\mathrm{L}_{M/K(X)} \mathrm{R}_{M/K(X)}\right\|_\chi \le 16 \|M\|_\chi \|M/K(X)\|.$$

Proof. Let $T, S \in B(X)$. It is clear that

$$\begin{aligned} \left\|\mathrm{L}_{T/K(X)} \mathrm{R}_{S/K(X)}\right\|_\chi &= \chi\left((T/K(X))\,(B(X)/K(X))_\odot\,(S/K(X))\right) \\ &\le \chi\left(T\,(B(X))_\odot\,S\right) = \|\mathrm{L}_T \mathrm{R}_S\|_\chi. \end{aligned}$$

By [ST00, Lemma 6.4], $\|\mathrm{L}_T \mathrm{R}_S\|_\chi \le 4\left(\|T^*\|_\chi \|S\| + \|S\|_\chi \|T\|\right)$ for any $T, S \in B(X)$. As $\|T^*\|_\chi \le 2\|T\|_\chi$ by [GM65], and $\|T - P\|_\chi = \|T\|_\chi$, $\|S - F\|_\chi = \|S\|_\chi$, for any $P, F \in K(X)$, we obtain that

$$\begin{aligned} \left\|\mathrm{L}_{T/K(X)} \mathrm{R}_{S/K(X)}\right\|_\chi &\le \inf_{P,F \in K(X)} \|\mathrm{L}_{T-P} \mathrm{R}_{S-F}\|_\chi \\ &\le 8 \inf_{P,F \in K(X)} \left(\|T\|_\chi \|S - F\| + \|S\|_\chi \|T - P\|\right) \\ &= 8\left(\|T\|_\chi \|S/K(X)\| + \|S\|_\chi \|T/K(X)\|\right). \end{aligned}$$

Therefore

$$\begin{aligned} \left\|\mathrm{L}_{M/K(X)} \mathrm{R}_{M/K(X)}\right\|_\chi &\le 8 \sup_{T,S \in M} \left(\|T\|_\chi \|S/K(X)\| + \|S\|_\chi \|T/K(X)\|\right) \\ &\le 16 \sup_{T \in M} \|T\|_\chi \sup_{S \in M} \|S/K(X)\| = 16 \|M\|_\chi \|M/K(X)\|. \end{aligned}$$

□

2.2. **Semigroups in the Calkin algebra.** Let $M \subseteq B(X)$, and let $\mathrm{SG}(M)$ be the semigroup generated by M. The same notation is used if M is a subset of an arbitrary Banach algebra.

Proposition 2.3. *Let M be a precompact subset of $B(X)$. If $\mathrm{SG}(M/K(X))$ is bounded and $\rho_\chi(M) < 1$ then $\mathrm{SG}(M/K(X))$ is precompact.*

Proof. Let $G_n = \cup \left\{M^k/K(X) : k > n\right\}$ for each $n \geq 0$. As $L_{M^k/K(X)}R_{M^k/K(X)}$ is a precompact set in $B(B(X)/K(X))$, then, by Lemmas 2.1 and 2.2,

$$\begin{aligned}\chi(G_{2k}) &= \chi\left(\left(M^k/K(X)\right)G_0\left(M^k/K(X)\right)\right) = \chi\left(\mathrm{L}_{M^k/K(X)}\mathrm{R}_{M^k/K(X)}G_0\right)\\ &\leq \left\|\mathrm{L}_{M^k/K(X)}\mathrm{R}_{M^k/K(X)}\right\|_\chi \|G_0\| \leq 16\left\|M^k\right\|_\chi \left\|M^k/K(X)\right\| \|G_0\|\\ &\leq \left(16\|G_0\|^2\right)\left\|M^k\right\|_\chi .\end{aligned}$$

As $\rho_\chi(M) < 1$, there is $m > 0$ such that $\|M^m\|_\chi < 1/2$. Then for $n > 2km$, we have that

$$\chi(G_n) \leq \chi(G_{2km}) \leq \left(16\|G_0\|^2\right)(1/2)^k \to 0 \text{ under } k \to \infty.$$

This shows that $\chi(G_n) \to 0$ under $n \to 0$. As $\mathrm{SG}(M/K(X))\backslash G_n$ is precompact,

$$\chi(\mathrm{SG}(M/K(X))) = \chi(G_n)$$

for every n. Therefore $\chi(\mathrm{SG}(M/K(X))) = 0$, i.e. $\mathrm{SG}(M/K(X))$ is precompact. □

Let A be a Banach algebra and $M \subseteq A$. Let $\mathrm{LIM}(M)$ be the set of limit points of all sequences (a_k) with $a_k \in M^{n_k}$, $n_k \to \infty$ when $k \to \infty$. It follows from [ST00, Corollary 6.12] that *if $\rho(M) = 1$ and $\mathrm{SG}(M)$ is precompact then $\mathrm{LIM}(M) = \mathrm{LIM}(M)^2$ and it has a non-zero idempotent.* We will use this fact in the proof of Theorem 2.5 (the part Case 1).

An element $a \in A$ is called *n-leading* for M if $a \in M^n$ and $\|a\| \geq \left\|\cup_{k<n}M^k\right\|$; a sequence $(a_k) \subseteq A$ is called *leading* for M, if a_k is n_k-leading for M, where $n_k \to \infty$, and $\|a_k\| \to \infty$ under $k \to \infty$.

Let $\mathrm{ld}^n(M)$ be the set of all n-leading elements for M, $\mathrm{ld}(M) = \cup_{n\geq 2}\mathrm{ld}^n(M)$ and $\mathrm{ld}_{[1]}(M) = \{a/\|a\| : a \in \mathrm{ld}(M)\}$.

Lemma 2.4. *Let M be a precompact set of $B(X)$. If $\|M^m\|_\chi \|M^m/K(X)\| < 1$, for some $m > 0$ then $\mathrm{ld}_{[1]}(M/K(X))$ is precompact.*

Proof. Let $G_n = \left\{a/\|a\| : a \in \mathrm{ld}^i(M/K(X)), i \geq n\right\}$ for any $n > 0$. Let $n = 2km+j$, where $0 \leq j < 2m$. Then, for $N = M^m$ and $B_{(1)} = (B(X)/K(X))_\odot$, we obtain that

$$G_n \subseteq \left(N^k/K(X)\right)B_{(1)}\left(N^k/K(X)\right). \tag{2.2}$$

Indeed, if $T/K(X) \in \mathrm{ld}^i(M/K(X))$ where $i \geq n$, then

$$T = T_1T_2T_3/K(X)$$

for some $T_1/K(X), T_3/K(X) \in N^k/K(X)$ and $T_2/K(X) \in M^{i-2km}/K(X)$. As $T/K(X)$ is an i-leading element for $M/K(X)$, then

$$\|T_2/K(X)\| \leq \|T/K(X)\|.$$

This proves (2.2).

Let $t = \|N\|_\chi \|N/K(X)\|$. As $N^k/K(X)$ is a precompact set, we get from Lemmas 2.1 and 2.2 that

$$\begin{aligned}\chi(G_n) &\leq \chi\left(\mathrm{L}_{N^k/K(X)}\mathrm{R}_{N^k/K(X)}B_{(1)}\right) \leq \left\|\mathrm{L}_{N^k/K(X)}\mathrm{R}_{N^k/K(X)}\right\|_\chi\\ &\leq 16\left\|N^k\right\|_\chi \left\|N^k/K(X)\right\| \leq 16\left(\|N\|_\chi \|N/K(X)\|\right)^k = 16t^k\end{aligned}$$

whence $\chi(G_n) \to 0$ under $n \to \infty$. As $\mathrm{ld}_{[1]}(M/K(X))\backslash G_n$ is precompact,

$$\chi\left(\mathrm{ld}_{[1]}(M/K(X))\right) = \chi(G_n)$$

for every n. Therefore $\chi\left(\mathrm{ld}_{[1]}(M/K(X))\right) = 0$, i.e., $\mathrm{ld}_{[1]}(M/K(X))$ is a precompact set. □

2.3. **Hausdorff radius equals essential radius.**

Theorem 2.5. *Let M be a precompact subset of $B(X)$. Then*

$$\rho_e(M) = \rho_\chi(M). \tag{2.3}$$

Proof. Let $\rho_e(M) = 1$. Assume, aiming at the contrary, that $\rho_\chi(M) < 1$. We consider two cases.

Case 1. $\mathrm{SG}(M/K(X))$ *is bounded.*

By Proposition 2.3, $\mathrm{SG}(M/K(X))$ is precompact. As $\rho(M/K(X)) = 1$, then $\mathrm{LIM}(M/K(X))$ has a non-zero idempotent by [ST00, Corollary 6.12]. On the other hand, let $T \in \mathrm{SG}(M)$ be an arbitrary operator such that $T/K(X) \in \mathrm{LIM}(M/K(X))$. Then there is a sequence (T_k) with $T_k/K(X) \in (M/K(X))^{n_k}$ for $n_k \to \infty$ and $T_k/K(X) \to T/K(X)$ under $k \to \infty$. Hence $\|T_k - T\|_\chi \to 0$ under $k \to \infty$. As $q := \rho_\chi(M) < 1$ then $\|T_k\|_\chi \le q^{n_k} \to 0$ under $n_k \to \infty$. So T is a compact operator. Hence $\mathrm{LIM}(M/K(X)) = (0)$, a contradiction. This shows that $\rho_\chi(M) = \rho_e(M)$ holds in Case 1.

Case 2. $\mathrm{SG}(M/K(X))$ *is not bounded.*

It follows easily from definition that in this case there exists a leading sequence for $M/K(X)$. Let $(T_k/K(X))_{k=1}^\infty$ be such a sequence. For brevity, set $a_k = T_k/K(X)$ for each k. Then

$$G := \{a_k/\|a_k\| \colon k \in \mathbb{N}\} \subseteq \mathrm{ld}_{[1]}(M/K(X)).$$

Let $\rho_\chi(M) = t_1 < 1$ and $t_1 < t_2 < 1$. It follows from the condition $\rho_e(M) = 1$ that, for any $\varepsilon > 0$ with $t_2(1+\varepsilon) < 1$, there is $n_1 > 0$ such that $\|M^n/K(X)\|^{1/n} < 1 + \varepsilon$ for all $n > n_1$, and also there is $n_2 > 0$ such that $\|M^n\|_\chi^{1/n} < t_2$. Then

$$\|M^n\|_\chi \|M^n/K(X)\| < (t_2(1+\varepsilon))^n < 1$$

for any $n > \max\{n_1, n_2\}$. By Lemma 2.4, G is precompact. Let $b := S/K(X)$ be a limit point of G. One may assume that

$$\|b - a_k/\|a_k\|\| \to 0 \text{ under } k \to \infty.$$

It is clear that $\|b\| = 1$. We have

$$\begin{aligned} \|S\|_\chi &\le \|S - T_k/\|a_k\|\|_\chi + \|T_k/\|a_k\|\|_\chi \\ &\le \|b - a_k/\|a_k\|\| + \|T_k\|_\chi/\|a_k\|. \end{aligned} \tag{2.4}$$

As $\rho_\chi(M) < 1$, $\left\{\|T_k\|_\chi \colon k \in \mathbb{N}\right\}$ is a bounded set. As $\|a_k\| \longrightarrow_k \infty$, we get $\|T_k\|_\chi/\|a_k\| \to_k 0$. We obtain from (2.4) that $\|S\|_\chi = 0$, i.e., S is a compact operator. Hence immediately $b = 0$, a contradiction. Thus $\rho_e(M) = \rho_\chi(M)$ in any case. ☐

3. Banach-algebraic approach to the joint spectral radius formulas

3.1. BW**-ideals.** Now we present the Banach-algebraic approach to the formulas for the joint spectral radius. Let us consider a Banach algebra A instead of $B(X)$. Let $BW(A)$ denote the set of all closed ideals J of A such that

$$\rho(M) = \max\{\rho(M/J), r(M)\} \text{ for all precompact } M \subseteq A. \tag{3.1}$$

The ideals J for which (3.1) holds are called *BW-ideals*. Clearly, if $I \subset J$, $J \in BW(A)$ then $I \in BW(A)$. It is known that $BW(A)$ has maximal elements; moreover it was proved in [ST12, Lemma 5.2] that if $J = \overline{\cup J_\lambda}$ where (J_λ) is a linearly ordered set of BW-ideals of A then $J \in BW(A)$.

Let us call an increasing transfinite sequence $(J_\alpha)_{\alpha \le \gamma}$ of closed ideals in a Banach algebra A an *increasing transfinite chain of closed ideals* if $J_\beta = \overline{\cup_{\alpha<\beta} J_\alpha}$ for any limit ordinal $\beta \le \gamma$, and a decreasing transfinite sequence $(I_\alpha)_{\alpha \le \gamma}$ – a *decreasing transfinite chain of closed ideals* if $I_\beta = \cap_{\alpha<\beta} I_\alpha$ for any limit ordinal $\beta \le \gamma$.

By [ST12], if $I \subset J$ are closed ideals of A, $I \in BW(A)$ and $J/I \in BW(A/I)$ then $J \in BW(A)$. This implies the transfinite stability for BW-ideals.

Proposition 3.1. *If in increasing transfinite chain $(J_\alpha)_{\alpha \le \gamma}$ of closed ideals in a Banach algebra A the ideal J_0 belongs to $BW(A)$ and $J_{\alpha+1}/J_\alpha \in BW(A/J_\alpha)$, for all α, then $J_\gamma \in BW(A)$.*

Every BW-ideal J of a Banach algebra turns out to be a *Berger-Wang algebra*in the sense that the equality

$$\rho(M) = r(M) \tag{3.2}$$

holds for any precompact set M of J. It follows from (3.2) and [Tur85, Proposition 3.5] that *every semigroup consisting of quasinilpotent elements of a Berger-Wang algebra generates a finitely quasinilpotent subalgebra.*

Since the Jacobson radical $\mathrm{Rad}\,(A)$ of every Banach algebra A consists of quasinilpotents, then for a Berger-Wang Banach algebra A, $\mathrm{Rad}\,(A)$ *is compactly quasinilpotent, i.e.,* $\rho\,(M) = 0$ for any precompact set M of $\mathrm{Rad}\,(A)$.

3.2. **First Banach-algebraic formulas for the joint spectral radius.** A natural analogue of compact operators in the Banach algebra context was proposed by K. Vala [Val64] who proved that the map $T \mapsto S_1 T S_2$ on the algebra $B(X)$ is compact if and only the operators S_1 and S_2 are compact. So an element a of a normed algebra A is called *compact* (*finite rank*) if the operator $\mathrm{L}_a\mathrm{R}_a$: $x \mapsto axa$ on A is compact (finite rank). A normed algebra A is called *bicompact* if all operators $\mathrm{L}_a\mathrm{R}_b$: $x \mapsto axb$ $(a, b \in A)$ are compact. An ideal of A is called *bicompact* if it is bicompact as a normed algebra.

It follows from [ST12, Corollary 4.8] that for every bicompact ideal J of A the equality (3.1) holds. Since, by [Val64], $K\,(X)$ is a bicompact ideal of $B\,(X)$, this result widely extends the generalized BW-formula (1.4) (which is the same as (1.3) by virtue of Theorem 2.5).

A normed algebra A is called *hypocompact* (*hypofinite*) if every non-zero quotient A/J has a non-zero compact (finite rank) element. An ideal is *hypocompact* (*hypofinite*) if it is hypocompact (hypofinite) as a normed algebra.

Each bicompact algebra is hypocompact, and any hypocompact ideal can be composed from bicompact blocks:

Proposition 3.2. [ST12, Proposition 3.8] *For any hypocompact closed ideal I of a Banach algebra A, there is a transfinite increasing chain $(J_\alpha)_{\alpha \le \gamma}$ of closed ideals of A such that $J_1 = (0)$ and $J_\gamma = I$, and every quotient space $J_{\alpha+1}/J_\alpha$ is a bicompact ideal of A/J_α.*

Theorem 3.3. [ST12, Theorem 4.11] *The formula (3.1) holds for every hypocompact closed ideal J of A.*

Indeed, as every closed bicompact ideal of a Banach algebra A is a BW-ideal, the result follows from Propositions 3.1 and 3.2.

A Banach algebra is called *scattered* if its elements have countable spectra. It follows from [ST14, Theorem 8.15] that every hypocompact algebra is scattered.

3.3. **Compact quasinilpotence, and coincidence of essential and f-essential joint spectral radii.** Recall that a Banach algebra A is *compactly quasinilpotent* if $\rho(M) = 0$ for any precompact subset M of A.

The following result shows that any compactly quasinilpotent ideal can be considered as inessential when one calculates the joint spectral radius.

Theorem 3.4. [ST05, Theorem 4.18] $\rho\,(M) = \rho\,(M/J)$ *for each compactly quasinilpotent ideal and precompact set $M \subset A$.*

In particular all compactly quasinilpotent ideals are BW-ideals.

Theorem 3.5. [ST12, Theorem 3.14] *If a Banach algebra A is hypocompact and consists of quasinilpotents then it is compactly quasinilpotent.*

The following result shows that the reverse inclusion fails.

Proposition 3.6. *There are compactly quasinilpotent Banach algebras without non-zero hypocompact ideals.*

Proof. Let V be the algebra $\ell^1(w)$, where the weight $w = (w_k)_{k=1}^\infty$ satisfies the condition

$$\lim\,(w_{k+1}/w_k) = 0 \tag{3.3}$$

(for instance, one can take $w_k = 1/k^k$). It follows easily from (3.3) that such a weight is radical, that is $\lim_{k\to\infty} w_k^{1/k} = 0$. Therefore all elements of V are quasinilpotent.

Let A be the projective tensor product $V \widehat{\otimes} B$ of V and any commutative Banach algebra B without non-zero compact elements (for instance, one may take for B the algebra $C\,[0.1]$ of continuous functions on $[0.1] \subseteq \mathbb{R}$).

Let us write elements $v \in V$ as

$$v = \sum_{k=1}^{\infty} \lambda_k e_k, \tag{3.4}$$

where e_k is the sequence $(\alpha_1, \alpha_2, ...)$ with $\alpha_i = 1$ if $i = k$, and 0 otherwise. It follows that $e_k \neq 0$ for all k, so that V has no non-zero nilpotents. Indeed, if $v^m = 0$ and λ_n is the first non-zero coefficient in the expansion (3.4) then clearly $e_{mn} = 0$, a contradiction.

To see that the algebra V is compact, note that the set of all compact elements in any Banach algebra is closed and with each element contains the algebra generated by it. So it suffices to show that the element e_1 is compact, because V is topologically generated by e_1.

Let $V_\odot$ be the unit ball of V:

$$V_\odot = \left\{ \sum_{k=1}^{\infty} \lambda_k e_k : \sum_{k=1}^{\infty} |\lambda_k| w_k \leq 1 \right\}.$$

We are going to show that L_{e_1} is a compact operator. For this, it suffices to show that, for each $\varepsilon > 0$, the set $e_1 V_\odot$ contains a finite ε-net.

For each n, let P_n be the natural projection on the linear span of $\{e_1, ..., e_n\}$, and let $K_n = P_n V_\odot$ and $K_n^\perp = (1 - P_n)V_\odot$. Then

$$e_1 V_\odot \subset e_1 K_n + e_1 K_n^\perp.$$

The set $e_1 K_n$ is compact for each n, so in any case it contains a finite $(\varepsilon/2)$-net. Now it suffices to show that $\|e_1 a\| \leq \varepsilon/2$ for each $a \in K_n^\perp$ if n is sufficiently large.

By (3.3), there is n such that $w_{k+1} < \varepsilon w_k/2$ for all $k \geq n$. Then for each $a = \sum_{k=n+1}^{\infty} \lambda_k e_k \in K_n^\perp$, we have

$$\begin{aligned} \|e_1 a\| &= \left\| \sum_{k=n+1}^{\infty} \lambda_k e_{k+1} \right\| = \sum_{k=n+1}^{\infty} |\lambda_k| w_{k+1} < \varepsilon \left(\sum_{k=n+1}^{\infty} |\lambda_k| w_k \right) /2 = \varepsilon \|a\|/2 \\ &\leq \varepsilon/2. \end{aligned}$$

Thus $e_1 V_\odot$ contains a finite ε-net for every $\varepsilon > 0$, i.e., $e_1 V_\odot$ is a compact set, whence V is a compact algebra consisting of quasinilpotents.

Applying Theorem 3.5 we see that the algebra V is compactly quasinilpotent. By [ST05, Theorem 4.29], the same is true for the tensor product of V and any Banach algebra. Thus the algebra A is compactly quasinilpotent. It remains to show that A is not hypocompact. Each element of A has the form

$$a = \sum_{k=1}^{\infty} e_k \otimes b_k, \text{ where } \sum_{k=1}^{\infty} \|b_k\| w_k < \infty.$$

Suppose that an element $c \in A$ is compact. Since A is commutative, the operator $\mathrm{L}_{c^2} = \mathrm{L}_c \mathrm{R}_c$ is compact. Thus setting $a = c^2$ we have that the set $aA_\odot$ is precompact. Let $B_\odot$ be the unit ball of B. If $aA_\odot$ is a precompact subset of A then, in particular, the set

$$\{a(e_1 \otimes b) \colon b \in B_\odot\} = \left\{ \sum_{k=1}^{\infty} e_{k+1} \otimes b_k b \colon b \in B_\odot \right\}$$

is precompact. In particular, all sets

$$E_k := \{e_{k+1} \otimes b_k b \colon b \in B_\odot\}$$

are precompact because the natural projection of $V \widehat{\otimes} B$ onto the subspace $e_j \otimes B$ is bounded. Each set E_k is homeomorphic to $b_k B_\odot$ whence b_k is a compact element of B. Since B has no non-zero compact elements, $b_k = 0$ for any $k > 0$, whence $a = 0$, i.e. $c^2 = 0$.

Let us show that $c = 0$. Indeed, if $c \neq 0$ let

$$c = \sum_{k=1}^{\infty} e_k \otimes d_k,$$

and let d_m be the first non-zero element among all d_k. Then

$$0 = c^2 = \sum_{k=2m}^{\infty} e_k \otimes \sum_{i+j=k} d_i d_j \text{ whence } d_m^2 = 0.$$

Therefore d_m is a compact element of B (since B is commutative, $d_m b d_m = d_m^2 b = 0$, for all $b \in B$). Since B has no non-zero compact elements, $d_m = 0$, a contradiction.

We proved that A has no non-zero compact elements. It follows that A has no bicompact and hypocompact ideals. $\square$

Theorem 3.7. *Let A be a Banach algebra. Then there are the largest hypocompact ideal $\mathcal{R}_{\mathrm{hc}}(A)$, the largest hypofinite ideal $\mathcal{R}_{\mathrm{hf}}(A)$, the largest compactly quasinilpotent ideal $\mathcal{R}_{\mathrm{cq}}(A)$ and the largest scattered ideal $\mathcal{R}_{\mathrm{sc}}(A)$.*

For the proofs see [ST12, Corollary 3.10], [ST05, Theorem 4.18], and [ST14, Theorem 8.10].

Now we return to our initial problem.

Theorem 3.8. *Let M be a precompact subset of $B(X)$. Then*

$$\rho_f(M) = \rho_e(M). \tag{3.5}$$

Proof. As $K(X)$ is a bicompact algebra by [Val64], the algebra $K(X)/\overline{F(X)}$ is also bicompact. As spectral projections of compact operators are in $F(X)$, it is easy to see that $K(X)/\overline{F(X)}$ consists of quasinilpotents. Then it is compactly quasinilpotent by Theorem 3.5. Therefore $K(X)/\overline{F(X)}$ is a compactly quasinilpotent ideal of $B(X)/\overline{F(X)}$. Using Theorem 3.4 applied to $J = K(X)/\overline{F(X)}$, we have that

$$\begin{aligned}\rho_f(M) &= \rho\left(M/\overline{F(X)}\right) = \rho\left(\left(M/\overline{F(X)}\right)/\left(K(X)/\overline{F(X)}\right)\right)\\ &= \rho(M/K(X)) = \rho_e(M)\end{aligned}$$

for a precompact subset M of $B(X)$. $\square$

3.4. **The largest BW-ideal problem and topological radicals.** Let A be a Banach algebra. As it was already noted, the set of all BW-ideals has maximal elements. However, it is not known whether $\overline{I+J} \in BW(A)$ if $I, J \in BW(A)$. So the problem of existence of the largest BW-ideal is open.

On the other hand, the largest BW-ideal problem disappears if one only consideres ideals defined by some natural properties — as, for example, the ideals $\mathcal{R}_{\mathrm{hc}}(A)$, $\mathcal{R}_{\mathrm{hf}}(A)$, $\mathcal{R}_{\mathrm{cq}}(A)$ and $\mathcal{R}_{\mathrm{sc}}(A)$ defined in Theorem 3.7. To formulate this precisely we turn to the theory of topological radicals. We recall some definitions and results of this theory; a reader can refer to the works [Dix97, ST05, KST09, ST10, KST12, ST12, ST14, CT16] for additional information.

In what follows the term *ideal* will mean *a two-sided ideal.* In general, radicals can be defined on classes of rings and algebras; the topological radicals are defined on classes of normed algebras. A radical is an *ideal map*, i.e., a map that assigns to each algebra its ideal, while a topological radical is a *closed ideal map*, it assigns to a normed algebra its closed ideal. In correspondence with our subject here we restrict our attention to the class of all Banach algebras.

We begin with the most important and convenient class of topological radicals. A *hereditary topological radical* on the class of all Banach algebras is a closed ideal map $\mathcal{P}$ which assigns to each Banach algebra A a closed two-sided ideal $\mathcal{P}(A)$ of A and satisfies the following conditions:

(H1) $f(\mathcal{P}(A)) \subset \mathcal{P}(B)$ for a continuous surjective homomorphism $f : A \longrightarrow B$;
(H2) $\mathcal{P}(A/\mathcal{P}(A)) = (0)$;
(H3) $\mathcal{P}(J) = J \cap \mathcal{P}(A)$ for any ideal J of A.

It can be seen from (H2) that every radical $\mathcal{P}$ accumulates some special property in the ideal $\mathcal{P}(A)$ of an algebra A which is called *the $\mathcal{P}$-radical* of A.

For the proof of the following theorem see [ST05, Theorem 4.25], [ST10, Theorems 3.58 and 3.59], [ST14, Section 8].

Theorem 3.9. *The maps $\mathcal{R}_{\mathrm{cq}}$: $A \longmapsto \mathcal{R}_{\mathrm{cq}}(A)$, $\mathcal{R}_{\mathrm{hc}}$: $A \longmapsto \mathcal{R}_{\mathrm{hc}}(A)$, $\mathcal{R}_{\mathrm{hf}}$: $A \longmapsto \mathcal{R}_{\mathrm{hf}}(A)$ and $\mathcal{R}_{\mathrm{sc}}$: $A \longmapsto \mathcal{R}_{\mathrm{sc}}(A)$ are hereditary topological radicals.*

The maps $\mathcal{R}_{\mathrm{hc}}$, $\mathcal{R}_{\mathrm{hf}}$, $\mathcal{R}_{\mathrm{cq}}$ and $\mathcal{R}_{\mathrm{sc}}$ are called the *hypocompact, hypofinite, compactly quasinilpotent* and *scattered radical* respectively.

It follows immediately from Axiom (H3) that hereditary radicals satisfy the conditions:

(I1) $\mathcal{P}(\mathcal{P}(A)) = \mathcal{P}(A)$;
(I2) $\mathcal{P}(J)$ of an ideal J of A is an ideal of A which is contained in the radical $\mathcal{P}(A)$.

If a closed ideal map $\mathcal{P}$ on the class of all Banach algebras satisfies (H1), (H2) and, instead of (H3), also (I1) and (I2) then $\mathcal{P}$ is called a *topological radical* (see [Dix97]).

If an ideal map [a closed ideal map] $\mathcal{P}$ satisfies (H1), it is called a *preradical* [a *topological preradical*].

A closed ideal map $\mathcal{P}$ is called an *under topological radical* (UTR) if it satisfies all axioms of topological radicals, besides possibly of (H2), and an *over topological radical* (OTR) if it satisfies all axioms, apart from possibly of (I1) (see [Dix97, Definition 6.2])).

Given a preradical $\mathcal{P}$, an algebra A is called *$\mathcal{P}$-radical* if $A = \mathcal{P}(A)$, and *$\mathcal{P}$-semisimple* if $\mathcal{P}(A) = 0$. It follows from (H1) for a topological preradical $\mathcal{P}$ that $\mathcal{P}$-radical and $\mathcal{P}$-semisimple algebras are invariant with respect to topological isomorphisms.

Let $\mathcal{P}$ be a topological radical. It follows easily from the definition that quotients of $\mathcal{P}$-radical algebras are $\mathcal{P}$-radical, and ideals of $\mathcal{P}$-semisimple algebras are $\mathcal{P}$-semisimple. Moreover, the class of all $\mathcal{P}$-radical ($\mathcal{P}$-semisimple) algebras is stable with respect to extensions: *If J is a $\mathcal{P}$-radical* ($\mathcal{P}$-semisimple) *ideal* of *A and the quotient A/J is $\mathcal{P}$-radical* ($\mathcal{P}$-semisimple) *then A itself is also $\mathcal{P}$-radical* ($\mathcal{P}$-semisimple).

The proof of the following properties *of transfinite stability* can be found in [ST14, Theorem 4.18].

Proposition 3.10. *Let $\mathcal{P}$ be a topological radical, A a Banach algebra, and let $(I_\alpha)_{\alpha\le\gamma}$ and $(J_\alpha)_{\alpha\le\gamma}$ be decreasing and increasing transfinite chains of closed ideals of A. Then*

(1) *If A/I_1 and all quotients $I_\alpha/I_{\alpha+1}$ are $\mathcal{P}$-semisimple then A/I_γ is $\mathcal{P}$-semisimple;*
(2) *If J_1 and all quotients $J_{\alpha+1}/J_\alpha$ are $\mathcal{P}$-radical then J_γ is $\mathcal{P}$-radical.*

4. Around joint spectral radius formulas and radicals

4.1. **Comparison of joint spectral radius formulas.** It follows from Theorem 3.3 that for any Banach algebra A and precompact set $M \subseteq A$, the equality

$$\rho(M) = \max\{\rho(M/\mathcal{R}_{\mathrm{hc}}(A)), r(M)\} \tag{4.1}$$

holds. Since $\mathcal{R}_{\mathrm{hf}}(A) \subseteq \mathcal{R}_{\mathrm{hc}}(A)$, we certainly have

$$\rho(M) = \max\{\rho(M/\mathcal{R}_{\mathrm{hf}}(A)), r(M)\}, \tag{4.2}$$

for any precompact set in A. Obviously $\overline{F(X)} \subseteq \mathcal{R}_{\mathrm{hf}}(B(X)) \subseteq \mathcal{R}_{\mathrm{hc}}(B(X))$, so the inequalities

$$\rho(M/\mathcal{R}_{\mathrm{hc}}(B(X))) \le \rho(M/\mathcal{R}_{\mathrm{hf}}(B(X))) \le \rho_f(M) = \rho_e(M) \tag{4.3}$$

are always true for all precompact $M \subset B(X)$; recall that $\rho_f(M) = \rho_e(M)$ by Theorem 3.8.

The inequality $\rho(M/\mathcal{R}_{\mathrm{hc}}(B(X))) \le \rho_f(M)$ in (4.3) can be strict. For example, if X is an Argyros-Haydon space then $\rho(M/\mathcal{R}_{\mathrm{hc}}(B(X))) = 0$ for each precompact set $M \subseteq B(X)$ while $\rho_f(M)$ can be non-zero by virtue of semisimplicity $K(X)$). This shows that *even in the operator case the joint spectral radius formula* (4.1) *is stronger than the generalized BW-formula* (1.3).

In general, the inequality $\rho(M/\mathcal{R}_{\mathrm{hc}}(A)) \le \rho(M/\mathcal{R}_{\mathrm{hf}}(A))$ can be also strict. To see this, let V be the radical compact Banach algebra $\ell_1(w)$ considered in Proposition 3.6. As we saw, V has no non-zero nilpotent elements. Therefore the only finite-rank element v in V is 0. Indeed, the multiplication operator $\mathrm{L}_v\mathrm{R}_v = \mathrm{L}_{v^2}$ is quasinilpotent. So if it has finite rank then it is nilpotent:

$$\mathrm{L}_{v^2}^m = 0.$$

Applying the operator $\mathrm{L}_{v^2}^m$ to v we have that $v^{2m+1} = 0$, i.e., v is nilpotent, whence $v = 0$.

Let now A be the unitization of V. Since all finite-rank elements of A must lie in V, it follows from the above that

$$\mathcal{R}_{\mathrm{hf}}(V) = \mathcal{R}_{\mathrm{hf}}(A) = (0)\,.$$

On the other hand, A is hypocompact, whence $A = \mathcal{R}_{\mathrm{hc}}(A)$. For $M = \{1\}$ we have that

$$\rho(M/\mathcal{R}_{\mathrm{hc}}(A)) = 0 \ne 1 = \rho(M/\mathcal{R}_{\mathrm{hf}}(A)).$$

As usual, in the class of all C*-algebras the situation is simpler.

Theorem 4.1. *If A is a C*-algebra then $\mathcal{R}_{\mathrm{hf}}(A) = \mathcal{R}_{\mathrm{hc}}(A)$.*

Proof. Indeed, if an element a of A is compact then a^*a is compact and therefore its spectral projections are finite-rank elements and therefore belong to $\mathcal{R}_{\mathrm{hf}}(A)$. Since a^*a is a limit of linear combinations of its spectral projections we have that

$$a^*a \in \mathcal{R}_{\mathrm{hf}}(A).$$

But it is known (see for example [Ped18, Proposition 1.4.5]) that the closed ideal generated by a^*a contains a. Thus $\mathcal{R}_{\mathrm{hf}}(A)$ contains all compact elements of A. Now let $(J_\alpha)_{\alpha\le\gamma}$ be an increasing transfinite chain of closed ideals with bicompact quotients, and $J_\gamma = \mathcal{R}_{\rm hc}(A)$. Assume by induction that J_α are contained in $\mathcal{R}_{\rm hf}(A)$ for all $\alpha < \lambda$. If the ordinal λ is limit then clearly J_λ is also contained in $\mathcal{R}_{\rm hf}(A)$. Otherwise, we have that $\lambda = \beta + 1$ for some β. If $a \in J_\lambda$ then a/J_β is a compact element of A/J_β whence $a/J_\beta \in \mathcal{R}_{\rm hf}(A/J_\beta)$ and $a \in \mathcal{R}_{\rm hf}(A)$. Therefore, by induction, $\mathcal{R}_{\rm hc}(A) = J_\gamma = \mathcal{R}_{\rm hf}(A)$. □

The class of hypocompact C*-algebras is contained in the class of all GCR algebras (algebras of type I) and this inclusion is strict: it suffices to note that even the algebra $C([0,1])$ is not hypocompact. Moreover, there is an analogue of (1.3) that holds for all C*-algebras A satisfying some natural restrictions on the space $\mathrm{Prim}(A)$ of all primitive ideals of A:

$$\rho(M) = \max\{\rho(M/\mathcal{R}_{\rm gcr}(A)), r(M)\}, \tag{4.4}$$

where $\mathcal{R}_{\rm gcr}(A)$ is the largest GCR ideal of A. The map $A \longmapsto \mathcal{R}_{\rm gcr}(A)$ is a hereditary topological radical on the class of all C*-algebras. It follows from (4.4) that any GCR-algebra is a Berger-Wang algebra. The proof and more information can be found in [ST14, Section 10].

Apart from (4.1), another version of the joint spectral radius formula was established in [ST08*]:

$$\rho(M) = \max\{\rho^\chi(M), r(M)\} \tag{4.5}$$

holds for every precompact set M in A, where $\rho^\chi(M)$ is defined as $\rho_\chi(\mathrm{L}_M\mathrm{R}_M)^{1/2}$. Unlike $\rho(M/\mathcal{R}_{\rm hc}(A))$ and $\rho(M/\mathcal{R}_{\rm hf}(A))$, the value $\rho_\chi(M)$ is not of the form $\rho(M/J)$, but it deserves some interest because it is natural to regard $\rho^\chi(M)$ as a Banach algebraic analogue of $\rho_\chi(M)$. By Theorem 2.5 and [ST12, Lemma 4.7],

$$\rho^\chi(M) = \rho_\chi(\mathrm{L}_M\mathrm{R}_M)^{1/2} = \rho_e(\mathrm{L}_M\mathrm{R}_M)^{1/2} \le \rho\,(M/J)^{1/2}\,\rho\,(M)^{1/2} \tag{4.6}$$

for every precompact set M in A and every bicompact ideal J of A.

In general, $\rho^\chi(M) \ne \rho(M/\mathcal{R}_{\rm hc}(A))$. Indeed, if X is an Argyros-Haydon space [AH11] then $B(X)$ is a one-dimensional extension of $K(X)$. So the algebra $B(X)$ is hypocompact and $\rho(M/\mathcal{R}_{\rm hc}(B(X))) = 0$ for each precompact set $M \subseteq B(X)$. On the other hand, for $M = \{1\}$, we see that $\mathrm{L}_M\mathrm{R}_M$ is the identity operator on the infinite-dimensional space $B(X)$, whence $\rho_\chi(\mathrm{L}_M\mathrm{R}_M) = 1$ and $\rho^\chi(M) = 1$.

4.2. BW**-radicals.** In line with the above discussion we are looking for such radicals $\mathcal{P}$ that $\mathcal{P}(A)$ is a BW-ideal for each A; it is natural to call them *BW-radicals*. Clearly, we are interested in "large" BW-radicals, so that we have to compare them.

The order for ideal maps, in particular for topological radicals, is introduced in the usual way: $\mathcal{P} \le \mathcal{R}$ means that $\mathcal{P}(A) \subseteq \mathcal{R}(A)$ for every algebra A. For instance, it is obvious that

$$\mathcal{R}_{\rm hf} \le \mathcal{R}_{\rm hc} \text{ and } \mathcal{R}_{\rm cq} \le \text{ Rad},$$

where Rad is the *Jacobson radical* $A \longmapsto \mathrm{Rad}(A)$ (recall that for a Banach algebra A, $\mathrm{Rad}(A)$ can be defined as the largest ideal of A consisting of quasinilpotents). It is well known that Rad is hereditary.

As usual, we write $\mathcal{P} < \mathcal{R}$ if $\mathcal{P} \le \mathcal{R}$ and there is an algebra A such that $\mathcal{P}(A) \ne \mathcal{R}(A)$. For example,

$$\mathcal{R}_{\rm hf} < \mathcal{R}_{\rm hc} < \mathcal{R}_{\rm sc} \text{ and Rad } < \mathcal{R}_{\rm sc}.$$

It is known that, for any family $\mathcal{F}$ of topological radicals, there exists the smallest upper bound $\vee\mathcal{F}$ and the largest lower bound $\wedge\mathcal{F}$ of $\mathcal{F}$ in the class of all topological radicals; clearly, $\vee\mathcal{F}$ and $\wedge\mathcal{F}$ need not belong to $\mathcal{F}$ itself. If $\mathcal{F} = \{\mathcal{P}, \mathcal{R}\}$, we write $\mathcal{P} \vee \mathcal{R}$ for $\vee\mathcal{F}$ and $\mathcal{P} \wedge \mathcal{R}$ for $\wedge\mathcal{F}$. We will describe later a constructive way for obtaining the radicals $\vee\mathcal{F}$ and $\wedge\mathcal{F}$.

The following theorem establishes that there is the largest BW-radical.

Theorem 4.2. [ST12, Theorem 5.9] *Let $\mathcal{F}$ be the family of all BW-radicals and $\mathcal{R}_{\rm bw} = \vee\mathcal{F}$. Then $\mathcal{R}_{\rm bw}$ is a BW-radical*; *any topological radical $\mathcal{P} \le \mathcal{R}_{\rm bw}$ is a BW-radical.*

The proof uses the structure of radical ideals in $\vee\mathcal{F}$, and transfinite stability of the class of BW-ideals (see Proposition 3.1).

To show the utility of $\mathcal{R}_{\rm bw}$, consider the following example. It follows from Theorems 3.3 and 3.4 that $\mathcal{R}_{\rm hc}$ and $\mathcal{R}_{\rm cq}$ are BW-radicals. So, for any Banach algebra A, $\mathcal{R}_{\rm hc}(A)$ and $\mathcal{R}_{\rm cq}(A)$ are BW-ideals. They can differ; moreover, it can be deduced from Proposition 3.6 that there is a Banach algebra A such that $\mathcal{R}_{\rm hc}(A)$ and $\mathcal{R}_{\rm cq}(A)$ are both non-zero, but have zero intersection. The existence of $\mathcal{R}_{\rm bw}$ implies that $\overline{\mathcal{R}_{\rm hc}(A) + \mathcal{R}_{\rm cq}(A)}$ is a BW-ideal, because both summands are contained in $\mathcal{R}_{\rm bw}(A)$. Now one can further extend this BW-ideal by building an increasing transfinite chain (J_α) of closed ideals such that

- $J_0 = (0)$ and $J_{\alpha+1}/J_\alpha = \overline{\mathcal{R}_{\rm hc}(A/J_\alpha) + \mathcal{R}_{\rm cq}(A/_\alpha)}$ for all α.

In the correspondence with Proposition 3.1 we conclude that all J_α are BW-ideals. It is obvious that there is an ordinal γ such that $J_{\gamma+1}=J_\gamma$. It turns out that $J_\gamma=(\mathcal{R}_{\rm hc}\vee\mathcal{R}_{\rm cq})(A)$. To see it and much more, we consider the details of a construction of radicals $\vee\mathcal{F}$ and $\wedge\mathcal{F}$ in the following subsection. Of course, we have that $\mathcal{R}_{\rm hc}\vee\mathcal{R}_{\rm cq}\le\mathcal{R}_{\rm bw}$, so that the formula

$$\rho(M)=\max\{\rho(M/(\mathcal{R}_{\rm hc}\vee\mathcal{R}_{\rm cq})(A)),r(M)\} \tag{4.7}$$

is valid, for any precompact set M in A.

It seems that in the Banach algebra context the best candidate for the joint spectral radius formula is

$$\rho(M)=\max\{\rho(M/\mathcal{R}_{\rm bw}(A)),r(M)\}. \tag{4.8}$$

But a priori there can exist a Banach algebra A with non-trivial BW-ideals and with $\mathcal{R}_{\rm bw}(A))=0$ — the disadvantage of formula (4.8) is that the largest BW-radical is defined not directly, since the family of BW-radicals is not completely described. However, in radical context the formula (4.8) is certainly optimal. In particular, it is stronger than formula (4.1) because the largest BW-radical contains the hypocompact radical for any Banach algebra, and the inclusion can be strict as the above example shows.

In what follows we gather some facts for the better understanding of the nature of the radical $\mathcal{R}_{\rm bw}$.

4.3. **Procedures and operations.** Here we describe some ways to construct radicals from preradicals that only partially satisfy the axioms.

Procedures are mappings from one class of ideal maps to another class of ideal maps. The important examples are the following. If $\mathcal{P}$ and $\mathcal{R}$ are topological preradicals satisfying (I1) and (I2), for any algebra A, let $(I_\alpha)_{\alpha\le\gamma}$ and $(J_\alpha)_{\alpha\le\delta}$ be transfinite chains such that

$$J_\alpha=A,\ J_{\alpha+1}=\mathcal{P}(J_\alpha);\quad I_0=(0),\ I_{\alpha+1}=q_{I_\alpha}^{-1}(\mathcal{R}(A/I_\alpha)), \tag{4.9}$$

where $q_{I_\alpha}:A\longrightarrow A/I_\alpha$ is the standard quotient map. Then the maps $\mathcal{P}_{(\alpha)^\circ}$: $A\longmapsto J_\alpha$ and $\mathcal{R}_{(\alpha)^*}$: $A\longmapsto I_\alpha$ are topological preradicals satisfying (I1) and (I2). So $\mathcal{P}\longmapsto\mathcal{P}_{(\alpha)^\circ}$ and $\mathcal{R}\longmapsto\mathcal{R}_{(\alpha)^*}$ are procedures (α-*superposition* and α-*convoluton procedures*). The transfinite chains of ideals in (4.9) stabilize at some steps $\gamma=\gamma(A)$ and $\delta=\delta(A)$, that is,

$$I_\gamma=I_{\gamma+1}\ \text{ and }\ J_{\delta+1}=J_\delta.$$

Set $\mathcal{P}^\circ$: $A\longmapsto J_\delta$ and $\mathcal{R}^*$: $A\longmapsto I_\gamma$. Then $\mathcal{P}\longmapsto\mathcal{P}^\circ$ and $\mathcal{R}\longmapsto\mathcal{R}^*$ are called *superposition* and *convolution procedures*, respectively; $\mathcal{P}^\circ$ *satisfies* (I1) and $\mathcal{R}^*$ *satisfies* (H2) (see [Dix97, Theorems 6.6 and 6.10]).

The following two ways of getting new ideal maps are very useful in the theory. If $\mathcal{F}$ is a family of UTRs then

$$\mathtt{H}_{\mathcal{F}}\colon A\longmapsto\mathtt{H}_{\mathcal{F}}(A):=\overline{\sum_{\mathcal{R}\in\mathcal{F}}\mathcal{R}(A)}$$

is a UTR; if $\mathcal{F}$ consists of OTRs then

$$\mathtt{B}_{\mathcal{F}}\colon A\longmapsto\mathtt{B}_{\mathcal{F}}(A):=\bigcap_{\mathcal{R}\in\mathcal{F}}\mathcal{R}(A)$$

is an OTR (see [ST14, Theorem 4.1]).

Now we extend the action of operations $\vee$ and $\wedge$ introduced in the preceding subsection. Let $\mathcal{F}$ be a family of topological preradicals satisfying (I1) and (I2). Set

$$\vee\mathcal{F}=(\mathtt{H}_{\mathcal{F}})^*\ \text{ and }\ \wedge\mathcal{F}=(\mathtt{B}_{\mathcal{F}})^\circ. \tag{4.10}$$

Then $\vee\mathcal{F}$ is the smallest OTR larger than or equal to each $\mathcal{P}\in\mathcal{F}$; and $\wedge\mathcal{F}$ is the largest UTR smaller than or equal to each $\mathcal{P}\in\mathcal{F}$. In particular, if $\mathcal{F}$ consists of UTRs then $\vee\mathcal{F}$ is the smallest topological radical that is no less than each $\mathcal{P}\in\mathcal{F}$; if $\mathcal{F}$ consists of OTRs then $\wedge\mathcal{F}$ is the largest topological radical that does not exceed each $\mathcal{P}\in\mathcal{F}$ (see [ST14, Remark 4.2 and Corollary 4.3]).

Theorem 4.3. [ST14, Theorem 8.15] $\mathrm{Rad}\vee\mathcal{R}_{\rm hc}=\mathcal{R}_{\rm sc}$.

If a family $\mathcal{F}$ consists of hereditary topological radicals then

$$\wedge\mathcal{F}=\mathtt{B}_{\mathcal{F}}$$

is the largest hereditary topological radical that does not exceed each $P\in\mathcal{F}$ (see [ST12, Lemma 3.2]).

As $\mathcal{R}_{\rm hc}$ and Rad are hereditary topological radicals then it follows from Theorem 3.5 that $\mathtt{B}_{\{\mathcal{R}_{\rm hc},\mathrm{Rad}\}}$ is a hereditary topological radical $\mathcal{R}_{\rm hc}\wedge\mathrm{Rad}$ and $\mathcal{R}_{\rm hc}\wedge\mathrm{Rad}\le\mathcal{R}_{\rm cq}$. It follows from Proposition 3.6 that

$$\mathtt{B}_{\{\mathcal{R}_{\rm hc},\mathrm{Rad}\}}=\mathcal{R}_{\rm hc}\wedge\mathrm{Rad}<\mathcal{R}_{\rm cq}. \tag{4.11}$$

4.4. **Convolution and superposition operations.** In this subsection we prove two useful lemmas.

For an ideal map $\mathcal{P}$ and a closed ideal I of a Banach algebra A, it is convenient to define an ideal $\mathcal{P} * I$ of A by setting

$$\mathcal{P} * I = q_I^{-1}(\mathcal{P}(A/I))$$

where q_I: $A \longrightarrow A/I$ is the standard quotient map. Clearly, $I \subseteq \mathcal{P} * I$. If $\mathcal{P}$ and $\mathcal{R}$ are topological preradicals satisfying (I1) and (I2), define the *convolution* $\mathcal{P} * \mathcal{R}$ and *superposition* $\mathcal{P} \circ \mathcal{R}$ by

$$\mathcal{P} * \mathcal{R}(A) = q_{\mathcal{R}}^{-1}(\mathcal{P}(A/\mathcal{R}(A))) \text{ and } \mathcal{P} \circ \mathcal{R}(A) = \mathcal{P}(\mathcal{R}(A)) \tag{4.12}$$

for every algebra A, where $q_{\mathcal{R}}$: $A \longrightarrow A/\mathcal{R}(A)$ is the standard quotient map. Then $\mathcal{P} * \mathcal{R}$ and $\mathcal{P} \circ \mathcal{R}$ are topological preradicals satisfying (I1) and (I2) (see [ST14, Subsection 4.2]); the convolution operation for preradicals is associative (see [ST14, Lemma 4.10]). If $\mathcal{P}$ and $\mathcal{R}$ are UTRs then so is $\mathcal{P} * \mathcal{R}$; if $\mathcal{P}$ and $\mathcal{R}$ are OTRs then so is $\mathcal{P} \circ \mathcal{R}$ (see [ST14, Corollary 4.11]).

We underline that one may define the convolution $\mathcal{P} * \mathcal{R}$ as above if $\mathcal{P}$ is an ideal map and $\mathcal{R}$ is a closed ideal map.

Lemma 4.4. *If $\mathcal{P}$ is a preradical, $\mathcal{R}$ and $\mathcal{S}$ are closed ideal maps and $\mathcal{R} \le \mathcal{S}$, then $\mathcal{P} * \mathcal{R} \le \mathcal{P} * \mathcal{S}$ and $\mathtt{H}_{\{\mathcal{P},\mathcal{R}\}} \le \mathcal{P} * \mathcal{S}$.*

Proof. Let A be a Banach algebra, $J = \mathcal{R}(A)$ and $I = \mathcal{S}(A)$. Let $q_J : A \longrightarrow A/J$, $q_I : A \longrightarrow A/I$ and $q : A/J \longrightarrow A/I$ be the standard quotient maps. Then $q \circ q_J = q_I$ and $q(\mathcal{P}(A/J)) \subseteq (\mathcal{P}(A/I))$. Therefore

$$\mathcal{P} * \mathcal{R}(A) = q_J^{-1}(\mathcal{P}(A/J)) \subseteq q_J^{-1} q^{-1} q(\mathcal{P}(A/J)) \subseteq q_I^{-1}(\mathcal{P}(A/I)) = \mathcal{P} * \mathcal{S}(A).$$

Hence $\mathcal{P} * \mathcal{R} \le \mathcal{P} * \mathcal{S}$.

Further, $\mathcal{R}(A) = J \subseteq I$ and $q_I(\mathcal{P}(A)) \subseteq \mathcal{P}(A/I)$ whence $\mathcal{P}(A) \subseteq q_I^{-1}(\mathcal{P}(A/I))$. Hence

$$\mathtt{H}_{\{\mathcal{P},\mathcal{R}\}}(A) = \overline{\mathcal{P}(A) + \mathcal{R}(A)} \subseteq q_I^{-1}(\mathcal{P}(A/I)) = \mathcal{P} * \mathcal{S}(A),$$

i.e., $\mathtt{H}_{\{\mathcal{P},\mathcal{R}\}} \le \mathcal{P} * \mathcal{S}$. □

The implication $\mathcal{P} \le \mathcal{S} \Longrightarrow \mathcal{P} * \mathcal{R} \le \mathcal{S} * \mathcal{R}$ is obvious.

Lemma 4.5. *If $\mathcal{P}$ and $\mathcal{R}$ are UTRs then the radical $\mathcal{P} \vee \mathcal{R}$ is equal to $(\mathcal{P} * \mathcal{R})^*$; if $\mathcal{P}$ and $\mathcal{R}$ are OTRs then the radical $\mathcal{P} \wedge \mathcal{R}$ is equal to $(\mathcal{P} \circ \mathcal{R})^\circ$.*

Proof. Let $\mathcal{P}$ and $\mathcal{R}$ be UTRs. By Lemma 4.4, $\mathrm{H}_{\{\mathcal{P},\mathcal{R}\}} \le \mathcal{P} * \mathcal{R} \le (\mathcal{P} * \mathcal{R})^*$ whence

$$\mathcal{P} \vee \mathcal{R} = \left(\mathtt{H}_{\{\mathcal{P},\mathcal{R}\}}\right)^* \le (\mathcal{P} * \mathcal{R})^{**} = (\mathcal{P} * \mathcal{R})^*.$$

On the other hand, $\mathcal{P} * \mathcal{R} \le \mathtt{H}_{\{\mathcal{P},\mathcal{R}\}} * \mathcal{R} \le \mathtt{H}_{\{\mathcal{P},\mathcal{R}\}} * \mathtt{H}_{\{\mathcal{P},\mathcal{R}\}}$ by Lemma 4.4. Therefore

$$(\mathcal{P} * \mathcal{R})^* \le \left(\mathtt{H}_{\{\mathcal{P},\mathcal{R}\}} * \mathtt{H}_{\{\mathcal{P},\mathcal{R}\}}\right)^* = \left(\left(\mathtt{H}_{\{\mathcal{P},\mathcal{R}\}}\right)_{(2)^*}\right)^* = \left(\mathtt{H}_{\{\mathcal{P},\mathcal{R}\}}\right)^* = \mathcal{P} \vee \mathcal{R}.$$

Let $\mathcal{P}$ and $\mathcal{R}$ be OTRs. Then $(\mathcal{P} \circ \mathcal{R})^\circ \le \mathcal{P} \circ \mathcal{R} \le \mathtt{B}_{\{\mathcal{P},\mathcal{R}\}}$ whence

$$(\mathcal{P} \circ \mathcal{R})^\circ \le \left(\mathtt{B}_{\{\mathcal{P},\mathcal{R}\}}\right)^\circ = \mathcal{P} \wedge \mathcal{R}.$$

On the other hand, $\mathtt{B}_{\{\mathcal{P},\mathcal{R}\}} \circ \mathtt{B}_{\{\mathcal{P},\mathcal{R}\}} \le \mathcal{P} \circ \mathtt{B}_{\{\mathcal{P},\mathcal{R}\}} \le \mathcal{P} \circ \mathcal{R}$. Therefore

$$\mathcal{P} \wedge \mathcal{R} = \left(\mathtt{B}_{\{\mathcal{P},\mathcal{R}\}}\right)^\circ = \left(\mathtt{B}_{\{\mathcal{P},\mathcal{R}\}} \circ \mathtt{B}_{\{\mathcal{P},\mathcal{R}\}}\right)^\circ \le (\mathcal{P} \circ \mathcal{R})^\circ.$$

□

4.5. **Scattered *BW*-radical.** Here we will show that the restriction of $\mathcal{R}_{\mathrm{bw}}$ to the class of scattered algebras is closely related to radicals of somewhat less mysterious nature. Namely it coincides with the topological radical $\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}}$ constructed earlier.

Theorem 4.6. *Let A be a scattered Banach algebra. Then $\mathcal{R}_{\mathrm{bw}}(A) = (\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}})(A)$.*

Proof. Clearly, $\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}} \le \mathcal{R}_{\mathrm{bw}}$. Let $I = (\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}})(A)$, $J = \mathcal{R}_{\mathrm{bw}}(A)$, $B = A/I$ and $K = J/I$. Then B is a scattered, $\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}}$-semisimple algebra and $K \subseteq \mathcal{R}_{\mathrm{bw}}(B)$ is a closed ideal of B. Assume to the contrary that $K \neq (0)$.

As K is a BW-ideal, it is a Berger-Wang algebra. So

$$\mathrm{Rad}(K) = \mathcal{R}_{\mathrm{cq}}(K) = K \cap \mathcal{R}_{\mathrm{cq}}(B) \text{ (we used heredity of } \mathcal{R}_{\mathrm{cq}}).$$

But $\mathcal{R}_{\mathrm{cq}}(B) \subseteq (\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}})(B) = (0)$. Therefore $\mathrm{Rad}(K) = (0)$, whence K is a semisimple algebra.

Since B is scattered, K is also scattered. By Barnes' Theorem [Bar68] (see also [Aup91, Theorem 5.7.8] with another proof) K has a non-zero socle. Since the socle is generated by finite-rank projections, it is a hypocompact (even hypofinite) ideal and, therefore, is contained in $\mathcal{R}_{\mathrm{hc}}(K)$. Since $\mathcal{R}_{\mathrm{hc}}$ is a hereditary radical then

$$(0) \neq \mathcal{R}_{\mathrm{hc}}(K) = K \cap \mathcal{R}_{\mathrm{hc}}(B).$$

But $\mathcal{R}_{\mathrm{hc}}(B) = (0)$, a contradiction. Hence $K = (0)$, i.e., $J = I$. □

Theorem 4.7. *The radical $\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}}$ is hereditary.*

Proof. Set $\mathcal{P} = \mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}}$. Let A be a Banach algebra and I its closed ideal. If $\mathcal{P}(I) \neq I \cap \mathcal{P}(A)$, let $B = A/\mathcal{P}(I)$, $J = I/\mathcal{P}(I)$ and $K = \mathcal{P}(A)/\mathcal{P}(I)$. Then J is a $\mathcal{P}$-semisimple ideal of B. Therefore

$$\mathcal{R}_{\mathrm{hc}}(J) = \mathcal{R}_{\mathrm{cq}}(J) = (0) \tag{4.13}$$

and K is an ideal of B contained in $\mathcal{P}(B)$. Hence $K \in BW(B)$. As $\mathcal{P}(B) \subseteq \mathcal{R}_{\mathrm{sc}}(B)$ and $\mathcal{R}_{\mathrm{sc}}$ is hereditary, K is a scattered algebra.

Let $L = J \cap K$. Then L is a non-zero ideal of B. As L is an ideal of K, $L \in BW(B)$ and L is scattered. As L is an ideal of J, it follows from (4.13) that

$$\mathcal{R}_{\mathrm{hc}}(L) = \mathcal{R}_{\mathrm{cq}}(L) = (0). \tag{4.14}$$

As L is a Berger-Wang algebra, it follows from (3.2) that $\mathrm{Rad}(L) \subseteq \mathcal{R}_{\mathrm{cq}}(L)$. Therefore L is a semisimple non-zero Banach algebra by (4.14). By Barnes' Theorem [Bar68], L has non-zero socle $\mathrm{soc}(L)$, i.e., $\mathcal{R}_{\mathrm{hc}}(L) \neq (0)$, a contradiction. □

Theorem 4.8. $\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}} = \mathcal{R}_{\mathrm{bw}} \circ \mathcal{R}_{\mathrm{sc}} = \mathcal{R}_{\mathrm{bw}} \wedge \mathcal{R}_{\mathrm{sc}}$.

Proof. Let A be a Banach algebra and $I = \mathcal{R}_{\mathrm{sc}}(A)$. Then I is a scattered algebra and $\mathcal{R}_{\mathrm{bw}}(I) = (\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}})(I)$ by Theorem 4.6. As $\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}} \leq \mathcal{R}_{\mathrm{sc}}$ and $\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}} \leq \mathcal{R}_{\mathrm{bw}}$, we obtain that

$$\begin{aligned}(\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}})(A) &= (\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}})((\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}})(A)) \\ &\subseteq (\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}})(\mathcal{R}_{\mathrm{sc}}(A)) \subseteq \mathcal{R}_{\mathrm{bw}}(\mathcal{R}_{\mathrm{sc}}(A)) \\ &= \mathcal{R}_{\mathrm{bw}}(I) = (\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}})(I) \\ &\subseteq (\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}})(A),\end{aligned}$$

i.e., $\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}} = \mathcal{R}_{\mathrm{bw}} \circ \mathcal{R}_{\mathrm{sc}}$.

It is clear that $\mathcal{R}_{\mathrm{bw}} \circ \mathcal{R}_{\mathrm{sc}} \leq \mathcal{R}_{\mathrm{bw}}$ and $\mathcal{R}_{\mathrm{bw}} \circ \mathcal{R}_{\mathrm{sc}} \leq \mathcal{R}_{\mathrm{sc}}$. As $\mathcal{R}_{\mathrm{bw}} \circ \mathcal{R}_{\mathrm{sc}}$ is a topological radical, then $\mathcal{R}_{\mathrm{bw}} \circ \mathcal{R}_{\mathrm{sc}} \leq \mathcal{R}_{\mathrm{bw}} \wedge \mathcal{R}_{\mathrm{sc}}$. By Lemma 4.5,

$$\mathcal{R}_{\mathrm{bw}} \wedge \mathcal{R}_{\mathrm{sc}} = (\mathcal{R}_{\mathrm{bw}} \circ \mathcal{R}_{\mathrm{sc}})^{\circ} \leq \mathcal{R}_{\mathrm{bw}} \circ \mathcal{R}_{\mathrm{sc}} \leq \mathcal{R}_{\mathrm{bw}} \wedge \mathcal{R}_{\mathrm{sc}}.$$

Therefore $\mathcal{R}_{\mathrm{bw}} \circ \mathcal{R}_{\mathrm{sc}} = \mathcal{R}_{\mathrm{bw}} \wedge \mathcal{R}_{\mathrm{sc}}$. □

Let us call $\mathcal{R}_{\mathrm{sbw}} := \mathcal{R}_{\mathrm{sc}} \wedge \mathcal{R}_{\mathrm{bw}}$ the *scattered BW-radical*.

4.6. **The centralization procedure.** Our next aim is to remove the frame of the class of scattered algebras by adding commutative algebras and forming transfinite extensions. For this in the theory of topological radicals there exists a special procedure.

Let $\sum_a(A)$ be the sum of all commutative ideals of A, and let $\sum_\beta(A)$ be the sum of all nilpotent ideals. The maps $\sum_a$ and $\sum_\beta$ are preradicals on the class of Banach algebras.

Note that the ideals $\sum_a(A)$ and $\sum_\beta(A)$ can be non-closed.

If A is semiprime then $\sum_a(A)$ is the largest central ideal of A (see [ST14, Lemma 5.1]). Let $\mathcal{P}$ be a closed ideal map on the class of Banach algebras. Define an ideal map $\mathcal{P}^a$ by setting

$$\mathcal{P}^a = \sum\nolimits_a * \mathcal{P}.$$

Let $\sum_\beta \leq \mathcal{P}$. Then $\mathcal{P}^a(A)$ is the largest ideal of A commutative modulo $\mathcal{P}(A)$, and if $\mathcal{P}$ is a topological radical then, by [ST14, Theorem 5.3], $\mathcal{P}^a$ is a UTR.

Proposition 4.9. *Let $\mathcal{F}$ be a family of topological radicals, let $\sum_\beta \leq \mathcal{P} \in \mathcal{F}$ and $\mathcal{G} = \mathcal{F} \backslash \{\mathcal{P}\}$. Then $(\mathtt{H}_{\mathcal{F}})^a \leq \mathcal{P}^a * \mathtt{H}_{\mathcal{G}}$ and $(\mathtt{H}_{\mathcal{F}})^{a*} = (\mathcal{P}^a * \mathtt{H}_{\mathcal{G}})^*$.*

Proof. Let $\mathcal{T} = \mathtt{H}_{\mathcal{G}}$. Then $\mathcal{T}$ is a UTR. As the convolution operation is associative then

$$\mathcal{P}^a * \mathcal{T} = \left(\sum\nolimits_a * \mathcal{P}\right) * \mathcal{T} = \sum\nolimits_a * (\mathcal{P} * \mathcal{T}) = (\mathcal{P} * \mathcal{T})^a . \tag{4.15}$$

By Lemma 4.4, $\mathtt{H}_{\mathcal{F}} \leq \mathcal{P} * \mathcal{T}$. Then

$$(\mathtt{H}_{\mathcal{F}})^a \leq (\mathcal{P} * \mathcal{T})^a . \tag{4.16}$$

Let $\mathcal{R} = (\mathtt{H}_{\mathcal{F}})^{a*}$ and $\mathcal{S} = (\mathcal{P} * \mathcal{T})^{a*}$. It follows from (4.16) that $(\mathtt{H}_{\mathcal{F}})^a \leq (\mathcal{P} * \mathcal{T})^a \leq \mathcal{S}$ and, therefore,

$$\mathcal{R} = (\mathtt{H}_{\mathcal{F}})^{a*} \leq \mathcal{S}^* = \mathcal{S}.$$

On the other hand, $\mathtt{H}_{\mathcal{F}} \leq \mathcal{R}$ whence $\mathcal{P} * \mathcal{T} \leq \mathcal{R} * \mathcal{R} = \mathcal{R}$, $(\mathcal{P} * \mathcal{T})^a \leq \mathcal{R}^a = \mathcal{R}$ and

$$\mathcal{S} = (\mathcal{P} * \mathcal{T})^{a*} \leq \mathcal{R}^* = \mathcal{R}.$$

□

Sometimes $\mathcal{P}^a$ is a topological radical if $\mathcal{P}$ is a topological radical. We have the following

Theorem 4.10. [ST12, Theorem 5.13] *$\mathcal{R}_{\mathrm{cq}}^a$ is a hereditary BW-radical.*

Corollary 4.11. [ST12, Corollary 5.15] *$\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}}^a$ is a BW-radical.*

4.7. **Centralization of BW-radicals and continuity of the joint spectral radius.**

Lemma 4.12. *$\mathcal{R}_{\mathrm{bw}}^a(A)$ and $\mathcal{R}_{\mathrm{sbw}}^a(A)$ are BW-ideals for every Banach algebra A.*

Proof. Indeed, $\rho(M) = \max\{\rho(M/\mathcal{R}_{\mathrm{bw}}(A)), r(M)\}$ for every precompact set M in A by definition of $\mathcal{R}_{\mathrm{bw}}$. Let $B = A/\mathcal{R}_{\mathrm{bw}}(A)$ and $N = M/\mathcal{R}_{\mathrm{bw}}(A)$. As $\sum_\beta \leq \mathcal{R}_{\mathrm{cq}} \leq \mathcal{R}_{\mathrm{bw}}$, B is semiprime and $\sum_a(B)$ is the largest central ideal of B. It is clear that $\sum_a(B)$ is closed. By [ST12, Lemma 5.5],

$$\rho(N) = \max\{\rho(N/\textstyle\sum_a(B)), r_1(N)\}$$

where $r_1(N) = \sup\{\rho(a) \colon a \in N\} \leq r(N)$. Hence $\sum_a(B) = \mathcal{R}_{\mathrm{bw}}^a(A)/\mathcal{R}_{\mathrm{bw}}(A)$ and $\mathcal{R}_{\mathrm{bw}}(A)$ are BW-ideals. By Proposition 3.1, BW-ideals are stable with respect to extensions. So $\mathcal{R}_{\mathrm{bw}}^a(A)$ is a BW-ideal.
As $\mathcal{R}_{\mathrm{sbw}}^a(A) \subseteq \mathcal{R}_{\mathrm{bw}}^a(A)$, then $\mathcal{R}_{\mathrm{sbw}}^a(A)$ is also a BW-ideal. □

Theorem 4.13. *$\mathcal{R}_{\mathrm{sbw}}^{a*}$ is a BW-radical and $\mathcal{R}_{\mathrm{bw}}^a = \mathcal{R}_{\mathrm{bw}}$.*

Proof. Let $\mathcal{P}$ be $\mathcal{R}_{\mathrm{sbw}}$ or $\mathcal{R}_{\mathrm{bw}}$. Clearly, $\sum_\beta \leq \mathcal{P}$. Let A be a Banach algebra, and let $(J_\alpha)_{\alpha \leq \gamma+1}$ be an increasing transfinite chain of closed ideals of A such that $J_1 = \mathcal{P}^a(A)$ and $J_{\gamma+1} = J_\gamma$, and $J_{\alpha+1}/J_\alpha = \mathcal{P}^a(A/J_\alpha)$ for all $\alpha \leq \gamma$. By Lemma 4.12, ideals J_1 and $J_{\alpha+1}/J_\alpha$ are BW-ideals. By Proposition 3.1, $\mathcal{P}^{a*}(A)$ is a BW-ideal for every Banach algebra A, that is, $\mathcal{P}^{a*}$ is a BW-radical.
As $\mathcal{R}_{\mathrm{bw}}$ is the largest BW-radical, then $\mathcal{R}_{\mathrm{bw}}^{a*} \leq \mathcal{R}_{\mathrm{bw}}$. We obtain that

$$\mathcal{R}_{\mathrm{bw}} \leq \mathcal{R}_{\mathrm{bw}}^a \leq \mathcal{R}_{\mathrm{bw}}^{a*} \leq \mathcal{R}_{\mathrm{bw}}$$

whence $\mathcal{R}_{\mathrm{bw}}^a = \mathcal{R}_{\mathrm{bw}}$. □

This formally gives the following

Corollary 4.14. *Any Banach algebra A commutative modulo the radical $\mathcal{R}_{\mathrm{bw}}(A)$ is $\mathcal{R}_{\mathrm{bw}}$-radical.*

Proof. Let $B = A/\mathcal{R}_{\mathrm{bw}}(A)$. Clearly $B = \mathcal{R}^a(B)$ for every topological radical $\mathcal{R}$. Therefore

$$B = \mathcal{R}_{\mathrm{bw}}^a(B) = \mathcal{R}_{\mathrm{bw}}(B) = \mathcal{R}_{\mathrm{bw}}(A/\mathcal{R}_{\mathrm{bw}}(A)) = (0)$$

whence $A = \mathcal{R}_{\mathrm{bw}}(A)$. □

Theorem 4.15. $\mathcal{R}_{\mathrm{sbw}}^{a*} = \mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}}^a$.

Proof. Let $\mathcal{S} = ((\mathcal{R}_{\mathrm{cq}} * \mathcal{R}_{\mathrm{hc}})^a)^*$. By Lemma 4.5 and formula (4.15) applied to $\mathcal{P} = \mathcal{R}_{\mathrm{cq}}$ and $\mathcal{T} = \mathcal{R}_{\mathrm{hc}}$, we have that

$$\mathcal{R}_{\mathrm{sbw}} = \mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}} \leq \mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}}^a = \left(\mathcal{R}_{\mathrm{cq}}^a * \mathcal{R}_{\mathrm{hc}}\right)^* = ((\mathcal{R}_{\mathrm{cq}} * \mathcal{R}_{\mathrm{hc}})^a)^* = \mathcal{S}.$$

Let A be a Banach algebra, and let $I = \mathcal{S}(A)$. By Lemma 4.5,

$$\mathcal{R}_{\mathrm{sbw}}^a(A) \subseteq \mathcal{S}^a(A) = q_I^{-1}\left(\textstyle\sum_a(A/I)\right).$$

As $(\mathcal{R}_{\mathrm{cq}} * \mathcal{R}_{\mathrm{hc}})^a * \mathcal{S} = \mathcal{S}$ then $\mathcal{R}_{\mathrm{hc}}(A/I) = (0)$. Indeed, if $\mathcal{R}_{\mathrm{hc}}(A/I) \neq (0)$ then

$$\mathcal{R}_{\mathrm{hc}} * \mathcal{S}(A) = q_I^{-1}(\mathcal{R}_{\mathrm{hc}}(A/I))$$

differs from I and

$$\left(\mathcal{R}_{\mathrm{cq}} * \mathcal{R}_{\mathrm{hc}}\right)^{a} * \mathcal{S}(A)=\left(\mathcal{R}_{\mathrm{cq}}^{a} * \mathcal{R}_{\mathrm{hc}}\right) * \mathcal{S}(A)=\mathcal{R}_{\mathrm{cq}}^{a} *\left(\mathcal{R}_{\mathrm{hc}} * \mathcal{S}\right)(A) \neq I=\mathcal{S}(A),$$

a contradiction. Therefore $\mathcal{R}_{\mathrm{hc}} * \mathcal{S}=\mathcal{S}$.

Similarly, we obtain that $\mathcal{R}_{\mathrm{cq}}(A/I)=(0)$ and $\mathcal{R}_{\mathrm{cq}} * \mathcal{S}=\mathcal{S}$. Then

$$\begin{aligned}\mathcal{R}_{\mathrm{sbw}}^{a}(A) \subseteq \mathcal{S}^{a}(A) & =\sum\nolimits_{a} * \mathcal{S}(A)=\sum\nolimits_{a} *\left(\mathcal{R}_{\mathrm{cq}} *\left(\mathcal{R}_{\mathrm{hc}} * \mathcal{S}\right)\right)(A) \\ & =\left(\sum\nolimits_{a} *\left(\mathcal{R}_{\mathrm{cq}} * \mathcal{R}_{\mathrm{hc}}\right)\right) * \mathcal{S}(A)=\left(\mathcal{R}_{\mathrm{cq}} * \mathcal{R}_{\mathrm{hc}}\right)^{a} * \mathcal{S}(A) \\ & =\mathcal{S}(A),\end{aligned}$$

i.e., $\mathcal{R}_{\mathrm{sbw}}^{a} \leq \mathcal{S}$. As $\mathcal{S}$ is a topological radical,

$$\mathcal{R}_{\mathrm{sbw}}^{a*}=\left(\mathcal{R}_{\mathrm{sbw}}^{a}\right)^{*} \leq \mathcal{S}^{*}=\mathcal{S}=\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}}^{a}.$$

On the other hand, as $\mathcal{R}_{\mathrm{cq}} * \mathcal{R}_{\mathrm{hc}} \leq\left(\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}}\right) *\left(\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}}\right)=\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}}=\mathcal{R}_{\mathrm{sbw}}$,

$$\left(\mathcal{R}_{\mathrm{cq}} * \mathcal{R}_{\mathrm{hc}}\right)^{a} \leq \mathcal{R}_{\mathrm{sbw}}^{a}$$

and then

$$\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}}^{a}=\mathcal{S}=\left(\left(\mathcal{R}_{\mathrm{cq}} * \mathcal{R}_{\mathrm{hc}}\right)^{a}\right)^{*} \leq\left(\mathcal{R}_{\mathrm{sbw}}^{a}\right)^{*}=\mathcal{R}_{\mathrm{sbw}}^{a*}.$$

□

We will mention now an application of this result to the problem of continuity of joint spectral radius. Let us recall the required definitions. Consider the function $M \longmapsto \rho(M)$ for bounded sets M of a Banach algebra A. This function is upper continuous (see [ST00, Theorem 3.1]), that is,

$$\limsup \rho\left(M_{n}\right) \leq \rho(M) \tag{4.17}$$

when M_{n} converges to M in the Hausdorff metric. The set M is a *point of continuity of the joint spectral radius* if $\rho\left(M_{n}\right) \rightarrow \rho(M)$ for every sequence $\left(M_{n}\right)$ convergent to M.

Corollary 4.16. *Let M be a precompact set in a Banach algebra A. If $\rho\left(M/\mathcal{R}_{\mathrm{sbw}}^{a*}(A)\right)<\rho(M)$ then M is a point of continuity of the joint spectral radius.*

Proof. By virtue of Theorem 4.15 it is sufficient to remark that M is a point of continuity of the joint spectral radius if $\rho\left(M/\left(\mathcal{R}_{\mathrm{hc}} \vee \mathcal{R}_{\mathrm{cq}}^{a}\right)(A)\right)<\rho(M)$ by [ST12, Theorem 6.3]. □

The following corollary is a consequence of [ST12, Corollary 6.4] and Theorem 4.15.

Corollary 4.17. *Let A be a Banach algebra, and let G be a semigroup in $\mathcal{R}_{\mathrm{sbw}}^{a*}(A)$. If G consists of quasinilpotent elements of A then the closed subalgebra $\overline{A(G)}$ generated by G is compactly quasinilpotent.*

Proof. As G consists of quasinilpotent elements, $r(M)=0$ for every precompact set M in G. As $\mathcal{R}_{\mathrm{sbw}}^{a*} \leq \mathcal{R}_{\mathrm{bw}}$, then $\rho(M)=r(M)$ for every precompact set M in $\mathcal{R}_{\mathrm{sbw}}^{a*}(A)$. Hence $\rho(M)=0$ for every precompact set M in G. As it was described above (see, for instance, [Tur85, Proposition 3.5]), $A(G)$ is finitely quasinilpotent. It follows from Corollary 4.16 that ρ is continuous at any precompact set in $\mathcal{R}_{\mathrm{sbw}}^{a*}(A)$. As the closure $\overline{A(G)}$ is contained in $\mathcal{R}_{\mathrm{sbw}}^{a*}(A)$, and each compact subset of $\overline{A(G)}$ is a limit of a net of finite subsets of $A(G)$, the algebra $\overline{A(G)}$ is compactly quasinilpotent. □

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