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Abstract

We implement some recent Monte Carlo estimators for option pricing and assess their performance in finite samples. We find that the accuracy of these estimators is remarkable, even when more exotic financial derivatives are considered. Finally, we implement the Glasserman and Yu (2004b) methodology to price Asian Bermudan options and basket options.

Key Words: American options, Monte Carlo method

JEL Classification: G10, G13

1. Introduction

Monte Carlo methods to price American style options seem to be now an active research area. The reason is mainly due to its suitability to price path dependent options and to solve high dimensional problems.

It is now standard to implement Monte Carlo methods using regression methods to price derivatives with American features. For example, Longstaff and Schwartz (2001) suggest using Least Squares approximation to approximate the option price on the continuation region and Monte Carlo methods to compute the option value (LS). Proof of asymptotic convergence of the option price estimator is derived under various assumptions and therefore more work is needed in this case.

Recently Clement et al (2002) undertake a theoretical analysis of the LS estimator, and show that the option price converges, in the limit, to the true option price. But the theoretical proof in Clement et al (2002) has some limitations in that it is based on a sequential rather than joint limit.

Glasserman et al (2004a) consider the limitations in Clement et al (2002) and prove convergence of the LS estimator as the number of paths and the number of polynomials functions increase together. Further assumption of martingales polynomials is required here.

Glasserman et al (2004b) (GY) implement a weighted Monte Carlo Estimator (WME) to price American derivatives and show that their estimator produces less disperse estimates of the option price. However, no finite-sample proof of convergence of the proposed estimator is provided in that study. Furthermore, proof of Theorem 1is based on a two period framework.

Applications of Monte Carlo estimators to price financial derivatives generally require using variance reduction techniques. One common feature of some of the studies cited above and other recent empirical ones is that they all have considered antithetic variates. As we shall see, particularly when pricing American style derivatives using one method rather than another makes the difference when determining the early exercise value.

In this paper we analyse the finite sample approximation of the LS (2001) and GY (2004b) by extending empirical studies such as Stentoft (2004). As shown in Glasserman and Yu (2004a) the choice of the basis function used in the regression is very important since (uniform) convergence of the option price to the true price can only be guaranteed if polynomials span the “true optimum”. To address this issue, we consider different basis functions and suggest possible “optimal polynomials”. We also discuss ways to implement variance reduction techniques in this context and study the contribution of these methods to variance reduction and bias. Finally, our study is the first empirical study on the WME as in Glasserman et al (2004b) and it also extends that methodology to price options on a maximum of n assets and Bermudan-Asian options. We show that, even when more difficult payoffs are considered, the WME estimator produces reasonably accurate prices.

2. The Least Squares Monte Carlo Methods

We consider a probability space $(\Omega, \mathrm{A}, \mathrm{P})$ and its discrete filtration $(F_i)_{i=0,\ldots,n}$, with n being an integer. Define with $X_0, X_1, \ldots X_n$ a R^d valued Markov chain representing the state variable recording all the relevant information on the price of a certain underlying asset. Assume that $V_i(x)$, $x \in R^d$, is the value of an option if exercised at time i under the state x. Using a dynamic programming framework the value of the option is given by:

$$V_i(x) = \sup_{\tau \in \Gamma} E[\Theta_\tau(X_\tau) \mid X_i = x] \tag{1}$$

with

$$V_n(x) = \Theta_n(x) \tag{2}$$

$$V_i(x) = \max\{\Theta_i(x), E[(V_{i+1}(X_{i+1}) \mid X_i = x]\} \tag{3}$$

To determine the option value V_0 one has to (i) approximate the conditional expectations in (3) in some ways, and (ii) obtain a numerical (Monte Carlo) evaluation of the latter.

If the option payoff is a square integrable function, then $V_i(.)$ will be a function spanning the Hilbert space and we can approximate the conditional expectations in (3) by the orthogonal projection on the space generated by a finite number of basis functions ϕ_{ik}, $i = 1,\ldots,n$ and $k = 0,1,\ldots,K$, such that

$$V_n(x) = \phi_n(x) \tag{5}$$

$$V_i(x) = \max\{\phi_i(x), E[(V_{i+1}(X_{i+1}) \mid X_i = x]\} \tag{6}$$

Using a simple regression approach:

$$V_{i+1}(X_{i+1}) \equiv \sum_{k=0}^{K} c_{i,k}\phi_{i,k}(X_i) + \varepsilon_{i+1} \qquad (7)$$

Therefore, we have transformed the dynamic programming scheme in (6) into a simple regression requiring the estimation of $K+1$ coefficients (7). At this point we need to evaluate the conditional expectation numerically. This can be done by simulating j paths of the Markov process X_i^j, with $j = 1,...,M$, and calculating, at each stopping time τ, recursively, the payoff $\mathrm{P}^{j}{*}_{i\tau} = \phi(\tau, X_i^j)$.

Remark. In Equation 7 we have included the error term ε_i. As pointed out in Grasserman and Yu (2004b), this is necessary for Equation 7 to hold at each i.

Assumption 1.

If $E(\varepsilon_{i+1}/X_i) = 0$ and $E[\phi_i(X_i)\phi_i(X_i)']$ is non-singular, then $\hat{V}_i \to V_i$ for all $i = 0,1,...,n$, where $\hat{V}_i$ is the estimated option price.

Proof of convergence in LS (2001) applies to the simplest possible case of only one exercise time and one state variable. Clement et al (2002) extend that proof to a multi period framework under the assumption that K is fixed. This would imply that the regression used is correct, therefore no sample bias is considered. GY (2004a) generalise the proof in Clement et al (2002) and show that the option price obtained by regression methods converges to the true price as $(K,M)_{\to\infty}$. However, martingales basis are considered in this case.

All the theoretical results mentioned above are very important, particularly from a theoretical point of view. However, for practical applications of these methodologies we are more concerned with their performance in finite sample.

In Equation (7) we have approximated the conditional expectation by using current basis functions (that is $\phi_i(X_i)$). However one would expect the option price at time $i+1$ to be more closely correlated with the basis function $\phi_{i+1}(X_{i+1})$ rather than $\phi_i(X_i)$. GY (2004b) develop a method based on Monte Carlo simulations where the conditional expectation is approximated by

$\phi_{i+1}(X_{i+1})$ rather than $\phi_i(X_i)$. They show that their Monte Carlo scheme has a regression representation given by:

$$\hat{V}_{i+1}(X^j_{i+1}) = \sum_{k=0}^{K} \varpi_{ik}\phi_{i+1,k}(X^j_{i+1}) + \hat{\varepsilon}_{i+1} \quad (8)$$

However an important assumption is necessary in this case:

Assumption 2. (Martingale property of basis function) $E(\phi_{i+1}(X_{i+1}) \mid X_i) = \phi_i(X_i)$, for all i.

GY (2004b) call this method regression later, since it involves using functions $\phi_{i+1}(X_{i+1})$. On the other hand, they call the LS (2001) method regression now since it uses functions $\phi_i(X_i)$. Although Theorem 1 in GY (2004b) provides a justification for using regression later as opposed to regression now, proof of that theorem is based on a single period framework. Furthermore, GY (2004b) neither provide an empirical application of their proposed estimator nor suggest ways of obtaining martingale basis.

3. A Simple Example

To motivate this study, in this section, we provide a simple example where we estimate early exercises values for American put options by crude Monte Carlo methods and using variance reduction techniques. Table 1 below shows the results.

Monte Carlo	EU-BS	Early Exercise Value	Binomial	Early Exercise	Difference
5.265	4.84	0.425	5.265	0.425	0.00%
6.234	5.96	0.274	6.244	0.284	1.00%
7.374	7.14	0.234	7.383	0.243	0.90%
Antithetic Variates					
5.261	4.84	0.421	5.265	0.425	0.40%
6.24	5.96	0.28	6.244	0.284	0.40%
7.384	7.14	0.244	7.383	0.243	-0.10%
Control Variates					
5.264	4.84	0.424	5.265	0.425	0.10%
6.246	5.96	0.286	6.244	0.284	-0.20%
7.387	7.14	0.247	7.383	0.243	-0.40%

Table 1. Monte Carlo refers to crude Monte Carlo method. EU-BS is the price of an equivalent

European option obtained by Black & Scholes formula. Binomial is the price of the option obtained by binomial methods. Early exercise refers to the estimates of the early exercise value.

We consider three in-the-money put options with strike \$45, initial price \$40, maturity seven months and risk free rate of interest 4.88% and volatilities 20%, 30% and 40% respectively. The last column shows the difference, in percentages, between estimates of the early exercise value by crude Monte Carlo, Monte Carlo implemented by variance reduction techniques, and Binomial methods. As we can see using variance reduction techniques reduces the bias by an order of 80% on average. This is likely to have a substantial impact on the estimate of the put option price.

4. Valuing American Put Options

Table 1 above shows that it is important to implement Monte Carlo methods with variance reduction techniques since, in this way, we can reduce the bias in the estimation of the early exercise value and achieve a more accurate price of the option. Therefore variance reduction techniques reduce the probability of generating sub-optimal exercise decisions. In this section we first apply the LS (2001) and GY (2004b) methods to price American style put options and thereafter implement the same methodologies by using different basis functions and different variance reduction techniques. As we pointed out above, we can only expect convergence of the estimated option price to the true price if polynomials used in the regression are "optimal polynomials".

We start with a simple application where we do not use variance reduction techniques. Prices reported are averages of 50 trials. We report standard errors and root mean square errors as a measure of the bias in the estimation of the conditional expectation in (6). As a benchmark, we consider the Binomial method with 20,000 time steps. Table 2 below shows the empirical results. To implement the GY (2004b) estimator we specify the following martingale basis under geometric Brownian motion and exponential polynomial:

$$\phi_{ik}(X_i) = (X_i)^k \exp - (kr + k(k-1)\sigma^{2/2})(t_i - t_0) \qquad (9)$$

On the other hand we could not find a valid martingale specification for polynomials when Laguerre basis were used. Finally, following GY (2004a), Hermite polynomials (H_k) define martingales as:

$$\phi_{ik}(X_i) = t^{k/2} H_k \qquad (10)$$

Longstaff-Schwartz (2001), Glasserman-Yu (2004b) Methods								
Exponential					Laguerre			BIN.
		2	3	4	2	3	4	
GY	0.2/0.0833	4.9968	4.997	4.997	0	0	0	5
	SE	[0.00122]	[0.00114]	[0.00095]	0	0	0	
	RMSE	[0.00331]	[0.00299]	[0.00328]	0	0	0	
LS	02/0.0833	4.9968	4.9967	4.997	4.995	4.996	4.996	5
	SE	[0.00021]	[0.00114]	[0.0011]	[0.000703]	[0.000407]	[0.000217]	
	RMSE	[0.00325]	[0.00326]	[0.00285]	[0.00535]	[0.00368]	[0.00376]	
GY	0.2/0.3333	5.0927	5.0857	5.082	0	0	0	5.087
	SE	[0.00749]	[0.0055]	[0.00678]	0	0	0	
	RMSE	[0.09267]	[0.1023]	[0.1009]	0	0	0	
LS	0.2/0.3333	5.0922	5.0856	5.0881	5.077	5.087	5.0852	5.087
	SE	[0.04687]	[0.00752]	[0.0082]	[0.00684]	[0.00647]	[0.00668]	
	RMSE	[0.0922]	[0.1005]	[0.10272]	[0.00098]	[0.00448]	[0.00184]	
GY	0.2/0.5833	5.2523	5.2614	5.2651	0	0	0	5.265
	SE	[0.00968]	[0.0066]	[0.0124]	0	0	0	
	RMSE	[0.2839]	[0.2935]	[0.29498]	0	0	0	
LS	0.2/0.5833	5.2489	5.2635	5.2641	5.2528	5.265	5.265	5.265
	SE	[0.00566]	[0.0114]	[0.008195]	[0.00677]	[0.009518]	[0.00838]	
	RMSE	[0.2865]	[0.2926]	[0.2871]	[0.01221]	[0.001313]	[0.000188]	
GY	0.3/0.0833	5.0597	5.0591	5.0611	0	0	0	5.06
	SE	[0.00591]	[0.005903]	[0.00711]	0	0	0	
	RMSE	[0.05975]	[0.06372]	[0.0685]	0	0	0	
LS	0.3/0.0833	5.0595	5.0591	5.0581	5.054	5.061	5.061	5.06
	SE	[0.00633]	[0.00541]	[0.0056]	[0.005489]	[0.006804]	[0.00679]	
	RMSE	[0.05951]	0.06372]	[0.06371]	[0.000651]	[0.004389]	[0.000094]	
GY	0.3/0.3333	5.6941	5.7042	5.7086	0	0	0	5.706
	SE	[0.01056]	[0.0098]	[0.0131]	0	0	0	
	RMSE	[0.7172]	[0.7300]	[0.73087]	0	0	0	

Table 2. Note that GY and LS are respectively the methodologies proposed by Longstaff and Swartz (2001) and Glasserman and Yu (2004b). SEs are standard errors and RMSEs are root mean square errors. Exponential and Laguerre are the polynomials used in this application. 2-4 refer to the number of basis used. BIN is the price of the option given by a Binomial method. The zeros refer to cases when we were not able to obtain martingales basis for a specific polynomial and therefore we could not implement the GY(2004b) method.

The first column of Table 2 shows the methodologies used (i.e. Glasserman and Yu, 2004b and Longstaff and Schwartz, 2001). In the second column we report the volatilities used to price the option and the time to expiry.[1] The strike of the option is assumed to be \$45 and the initial stock price \$40. Therefore we only consider in the money options. The risk free rate of interest is assumed to be 4.88% per year. Fifty time steps are considered in combination with 100,000 Monte Carlo replications. We consider two different polynomial basis, namely exponential and Laguerre. The numbers of basis used are 2, 3 and 4 bases. Following Brodie and Kaya (2004), the RMSE is defined as[2] $(bias^2 + \text{var}iance)^{1/2}$. Results in Table 2 favour Laguerre polynomials in quite few cases. Standard errors are in general small. The RMSE confirms what has been found in other studies, that is, the convergence implied by these estimators is not uniform. In fact, by increasing the number of basis one does not necessarily reduces the bias.

Table 2 continued

LS	0.3/0.3333	5.6941	5.6991	5.7034	5.689	5.699	5.706	5.706
	SE	[0.00918]	[0.0115]	[0.01309]	[0.01321]	[0.01385]	[0.00123]	
	RMSE	[0.7185]	[0.7279]	[0.73380]	[0.01735]	[0.006130]	[0.00467]	
GY	0.3/0.5833	6.2232	6.2379	6.2455	0	0	0	6.244
	SE	[0.0143]	[0.01952]	[0.01487]	0	0	0	
	RMSE	[1.268]	[1.2838]	[1.2875]	0	0	0	
LS	0.3/0.5833	6.2221	6.2314	6.2439	6.227	6.2427	6.234	6.244
	SE	[0.0068]	[0.01547]	[0.01326]	[0.01182]	[0.0133]	[0.01428]	
	RMSE	[1.2741]	[1.2866]	[1.2818]	[0.01678]	[0.00134]	[0.00592]	
GY	0.4/0.0833	5.2775	5.2864	5.2881	0	0	0	5.286
	SE	[0.01027]	[0.00855]	[0.00763]	0	0	0	
	RMSE	[0.28247]	[0.29144]	[0.2952]	0	0	0	
LS	0.4/0.0833	5.2758	5.2859	5.2832	5.2749	5.2889	5.2908	5.286
	SE	[0.00622]	[0.00874]	[0.01032]	[0.00089]	[0.00936]	[0.01071]	
	RMSE	[0.28319]	[0.29066]	[0.2958]	[0.0111]	[0.00287]	[0.00479]	
GY	0.4/0.3333	6.4911	6.5097	6.5131	0	0	0	6.51
	SE	[0.01522]	[0.01657]	[0.0123]	0	0	0	
	RMSE	[1.521]	[1.537]	[1.5386]	0	0	0	
LS	0.4/0.3333	6.4988	6.5006	6.5121	6.4954	6.511	6.514	6.51
	SE	[0.00732]	[0.0141]	[0.01898]	[0.01522]	[0.01479]	[0.01719]	
	RMSE	[1.522]	[1.532]	[1.5412]	[0.01459]	[0.00104]	[0.00370]	
GY	0.4/0.5833	7.3631	7.3781	7.382	0	0	0	7.383
	SE	[0.01906]	[0.02537]	[0.02382]	0	0	0	
	RMSE	[1.4086]	[1.4287]	[1.4321]	0	0	0	
LS	0.4/0.5833	7.3701	7.3760	7.3824	7.3621	7.376	7.374	7.383
	SE	[0.007797]	[0.02166]	[0.01782]	[0.02478]	[0.001341]	[0.01291]	
	RMSE	[1.4107]	[1.421]	[1.4301]	[0.02094]	[0.5611]	[0.00949]	

Following other studies such as LS (2001) and Stentoft (2004), in Table 3 below we have implemented these methodologies using standard antithetic variates. We price the same option (i.e. with the same parameters) as the one considered in Table 2. Although, as we mentioned above, antithetic variates have already been considered in other empirical studies using the LS (2001) method, they have never been used to implement the estimator proposed in GY (2004b). Therefore as far as we know the present study is the first empirical study to implement the GY (2004b) estimator to price financial derivatives.

[1] For example, 0.2/0.0833 should be read as 20% volatility and 1 month to expiry.
[2] Refer to Brodie and Kaya (2004) for further details.

Longstaff-Schwartz (2001), Glasserman and Yu (2004b) Methods								
Exponential					Laguerre			BIN.
		2	3	4	2	3	4	
GY	0.2/0.0833	4.996	4.997	4.997	0	0	0	5
SE		[0.000173]	[0.000259]	[0.0002537]	0	0	0	
RMSE		[0.003621]	[0.003447]	[0.003454]	0	0	0	
LS	02/0.0833	4.9966	4.996	4.997	4.994	4.996	4.996	5
SE		[0.000351]	[0.0002892]	[0.000279]	[0.000658]	[0.00039]	[0.000266]	
RMSE		[0.003424]	[0.003559]	[0.003459]	[0.00591]	[0.00367]	[0.003565]	
GY	0.2/0.3333	5.079	5.085	5.084	0	0	0	5.087
SE		[0.006513]	[0.00667]	[0.005471]	0	0	0	
RMSE		[0.008158]	[0.00207]	[0.00304]	0	0	0	
LS	0.2/0.3333	5.079	5.086	5.0865	5.082	5.084	5.084	5.087
SE		[0.006342]	[0.005806]	[0.00441]	[0.00650]	[0.00448]	[0.00529]	
RMSE		[0.008438]	[0.001535]	[0.000493]	[0.005289]	[0.00297]	[0.00281]	
GY	0.2/0.5833	5.251	5.261	5.262	0	0	0	5.265
SE		[0.006765]	[0.008092]	[0.00567]	0	0	0	
RMSE		[0.01367]	[0.003912]	[0.00305]	0	0	0	
LS	0.2/0.5833	5.254	5.259	5.2634	5.253	5.262	5.261	5.265
SE		[0.005042]	[0.006498]	[0.005579]	[0.00839]	[0.00571]	[0.00604]	
RMSE		[0.01113]	[0.00608]	[0.00158]	[0.01271]	[0.00296]	[0.004253]	
GY	0.3/0.0833	5.054	5.059	5.061	0	0	0	5.06
SE		[0.006252]	[0.004472]	[0.00404]	0	0	0	
RMSE		[0.00605]	[0.001031]	[0.000162]	0	0	0	

Table 3: Antithetic variates. GY and LS are respectively the methodologies proposed by Longstaff and Schwartz (2001) and Glasserman and Yu (2004b). SEs are standard errors and RMSEs are the root mean square errors. Exponential and Laguerre are the polynomials used in this application. 2-4 refer to the number of basis functions used. BIN is the price of the option given by the Binomial method. The zeros refer to cases when we were not able to obtain martingales basis for a specific polynomial and therefore we could not implement the GY (2004b) method.

Both the methodologies produce an accurate price of the option. Very small standard errors signal that estimates are accurate and not disperse. In general estimates of the option price given by the LS (2001) method seem to be less disperse than others. This result may not fully support Theorem 1 in Glasserman and Yu (2004b). Our result may imply that, once a multi period framework is considered, Theorem 1 in GY (2004b) no longer holds[3]. In fact, evidence of uniform convergence is much stronger when the LS (2001) method, in conjunction with Laguerre basis, is used than the GY (2004b) method. In fact, in this case, in general, the bias decreases as we increase the number of basis[4].

[3] This result might be due to the specific martingales bases used in this study. We shall look at this issue in more details in a separate study and suggest ways of designing martingales basis with bounded variance.

[4] Of course, we do not claim here that (uniform) convergence of the estimated option price to the true price is due to the variance reduction methodology employed. In fact, it may well be due to the polynomial chosen (i.e. Laguerre) in this empirical example.

Table 3 continued

LS	0.3/0.0833	5.055	5.059	5.068	5.054	5.059	5.061	5.06
SE		[0.00581]	[0.002595]	[0.00379]	[0.00579]	[0.00439]	[0.00432]	
RMSE		[0.00521]	[0.00122]	[0.000279]	[0.21134]	[0.000929]	[0.000554]	
GY	0.3/0.3333	5.691	5.702	5.707	0	0	0	5.706
SE		[0.01595]	[0.007062]	[0.00737]	0	0	0	
RMSE		[0.01595]	[0.00425]	[0.000681]	0	0	0	
LS	0.3/0.3333	5.693	5.704	5.704	5.690	5.702	5.705	5.706
SE		[0.007459]	[0.006206]	[0.00634]	[0.00823]	[0.00742]	[0.00562]	
RMSE		[0.01342]	[0.001563]	[0.001874]	[0.4254]	[0.004271]	[0.001261]	
GY	0.3/0.5833	6.228	6.24	6.239	0	0	0	6.244
SE		[0.00572]	[0.009582]	[0.00613]	0	0	0	
RMSE		[0.01583]	[0.00506]	[0.005132]	0	0	0	
LS	0.3/0.5833	6.229	6.235	6.238	6.224	6.239	6.240	6.244
SE		[0.00829]	[0.00758]	[0.00779]	[0.00999]	[0.00803]	[0.00647]	
RMSE		[0.01479]	[0.00873]	[0.006014]	[0.02019]	[0.00471]	[0.003669]	
GY	0.4/0.0833	5.275	5.285	5.289	0	0	0	5.286
SE		[0.008957]	[0.005144]	[0.00535]	0	0	0	
RMSE		[0.01085]	[0.0006409]	[0.002555]	0	0	0	
LS	0.4/0.0833	5.274	5.284	5.286	5.274	5.284	5.287	5.286
SE		[0.00565]	[0.005249]	[0.00546]	[0.00499]	[0.00643]	[0.00629]	
RMSE		[0.01239]	[0.001781]	[0.0001743]	[0.012403]	[0.00205]	[0.001198]	
GY	0.4/0.3333	6.494	6.509	6.509	0	0	0	6.51
SE		[0.008389]	[0.009061]	[0.00754]	0	0	0	
RMSE		[0.01627]	[0.000939]	[0.0002757]	0	0	0	
LS	0.4/0.3333	6.496	6.507	6.508	6.4923	6.506	6.51	6.51
SE		[0.007467]	[0.005948]	[0.00846]	[0.00734]	[0.000437]	[0.00709]	
RMSE		[0.01419]	[0.003064]	[0.001931]	[0.017695]	[0.004269]	[0.000199]	
GY	0.4/0.5833	7.366	7.371	7.378	0	0	0	7.383
SE		[0.00901]	[0.000234]	[0.00915]	0	0	0	
RMSE		[0.0167]	[0.0025]	[0.00469]	0	0	0	
LS	0.4/0.5833	7.361	6.379	7.384	7.366	7.376	7.384	7.383
SE		[0.009083]	[0.000342]	[0.00953]	[0.00836]	[0.000897]	[0.00720]	
RMSE		[0.02167]	[0.00291]	[0.000509]	[0.01692]	[0.006972]	[0.000523]	

To measure the impact of antithetic variates on the estimates of the option prices in Table 3, we have calculated the variance reduction (VR) factor, as the ratio of the estimate of naïve variance and the estimate of antithetic variate variance, for a reasonable sample of the options in Table 3. We have considered options with volatilities 20-40% and time to expiry 1 and 4 months. When exponential basis is used the VR factor ranges from 0.68 to 3.34 for GY (2004b) method and 0.96 to 6.62 for LS (2001) method. On the other hand when Laguerre basis is used the VR factor ranges from 0.10 to 7.47[5]. The biggest gain from using antithetic variates methods seems to come from implementing the LS (2001) method by using Laguerre basis.

4.1 Regression Methods and Moment Matching

One important issue when pricing derivatives by simulation is that we can confidently price a derivatives security if, in the first place, we can correctly simulate the dynamics of the underlying asset. The methodology we present below accomplishes this task.

We follow Boyle et al (1997) and consider the R^d valued Markov chain sequence $X_0, X_1, ... X_n$ and assume that we know the expectation $E(X) = \exp(-rt)X_0$. The sample mean process of the sequence above can be written as:

$$\bar{X} = \frac{1}{M}\sum_{j=1}^{M} X^j \qquad (11)$$

In finite sample we know that $E(X) \neq \bar{X}$. However we can adjust the simulated paths such that the following equality holds for all i:

$$\tilde{X}_i(t) = X_i(t) + E[X(t)] - \bar{X}(t) \qquad (12)$$

where $\tilde{X}_i(t)$ is the new simulated path after the transformation.

Consequently, we have that $E[\tilde{X}_i(t)] = E(X(t)]$ holds and the mean of the simulated sample path matches the population mean exactly. Apart from matching the first moment of the process, we can also match higher order moments such as variance for example. In this case one re-writes the process in (12) in the following form:

$$\tilde{X}_i(t) = [X_i(t) - \bar{X}(t)]\frac{\sigma_X}{s_X} + E[X(t)] \qquad (13)$$

[5] We have considered a sample of 30 options. Results are not reported to save space. We have also dropped in some cases some extreme values (i.e. very high or low VR factors) that might have generated outliers.

where σ_X and s_X are, respectively, the population and the sample variance.

One important drawback of the process in (12) is that sample paths are correlated and therefore it is unlikely that the initial and the simulated processes will have the same distribution. The correlation also makes estimates of standard errors meaningless. To overcome these drawbacks, in the empirical application, we have implemented the additive process in (12) to the standard Brownian motion process $B_i(t)$, in the following way:

$$\tilde{B}_i(t) = [B_i(t) - \bar{B}(t)] / \frac{s_B}{\sqrt{t}} \tag{13}$$

To preserve independence between sample paths, we have rescaled the increments of the process $B_i(t)$ as follows:

$$\tilde{Z}_i(t_k) = \sum_{j=1}^{j} \sqrt{t_j - t_{j-1}} \frac{Z_{ij} - \bar{Z}_j}{s_j}$$

where Z_{ij} are standard normal variables, and $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Z_{ij} - \bar{Z}_j)^2$.

4.2 Empirical Results

In this Section we show an application of moment matching when pricing one of the options considered in Table 2 and Table 3. We consider a put option with seven months to expiry, volatility 40%, initial stock price \$40. The rate of interest is 4.88%p.a. We set the number of steps equal to 50 in all the experiments we conduct. We compute standard errors and root mean squares errors for sample size of 16, 70, 300, 1000 based on 2000 simulations. Values are reported in log term.

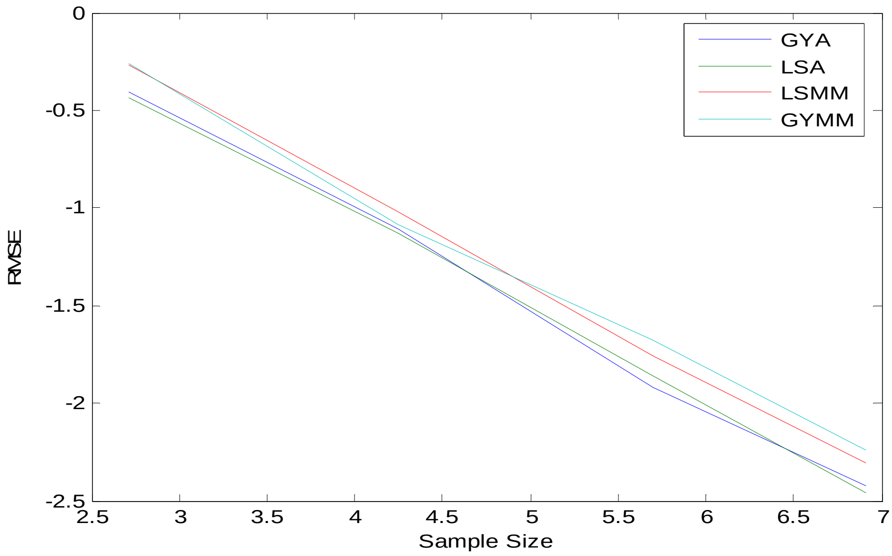

Figure 1 Standard errors versus sample size in pricing an American Put option with strike \$45 and initial stock price \$40.

In Figure 1 we have compared standard errors versus sample size for the GY (2004b) and LS (2001) methods implemented with antithetic variates (A) and moment matching (MM). Antithetic variates outperform moment matching in this case. Interestingly standard errors for GY (2004b) and LS (2004) methods are narrower when moment matching is used.

In Figure 2 we compare the root mean squares error for LS (2001) and GY (2004b) methods when implemented with the same variance reduction methods as in Figure 1.

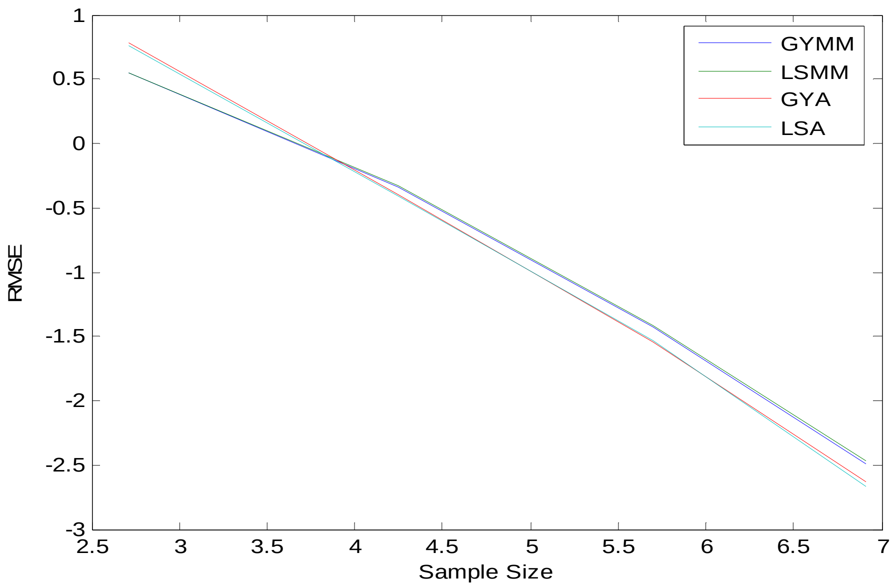

Figure 2 Root mean squares error versus sample size in pricing an American Put option with strike \$45 and initial stock price \$40.

It is interesting that the root mean squares errors for the LS (2001) and GY (2004b) methods are almost indistinguishable when implemented with the same variance reduction method. The lowest root mean square error is obtained when the Longstaff and Schwartz method is implemented with antithetic variates[6]. In the next section we shall present an alternative approach based on control variates to implement methodologies such as the LS (2001) and GY (2004b).

4.3 Regression Methods and Control Variates

The method of control variates is one of the most popular variance reduction techniques and has many analogies with moment matching. Applications of this method in finance for pricing, (Rubinstein, et al, 1985), or model calibration (Glasserman and Yu, 2003) are very common. In this section we implement the methodologies presented in Section 2 by using control variates. Suppose that, given a stopping time $\tau \in \Gamma(t,T)$, and the state variable X_i, we want to estimate the price of an option that, as in (1), can be found by solving the following conditional expectation:

$$V_i(x) = \sup_{\tau \in \Gamma} E[\Theta_\tau(X_\tau) \mid X_i = x] \qquad (14)$$

for the set of all possible stopping times τ. If we consider functions $\phi_k(x)$ and impose that:

Assumption 3.

For $i = 1,...,n-1$ $\phi_{ik}(x)$ is in $L^2(\phi(X_i))$; $\Pr(V_i(x) = \hat{V}_i) = 0$; Π_i denotes the orthogonal projection from $L^2(\Omega)$ onto the vector space generated by $\{\phi_1(x), \phi_2(x),..., \phi_k(x)\}$,

It follows that, as $(K,M)_{\to\infty}$, the sample estimator of the option:

$$\Pi_i \hat{V}_i = \frac{1}{M}\sum_{j=1}^{M} V_\tau^j = V_i \qquad (15)$$

---

[6] Note we do not claim that this is an universal result. In fact moment matching methods can also be implemented in other different ways. For example one could use moments for the stock price paths for the adjustment in (13) instead of using $\tilde{B}_i$.

The estimator in (15) therefore converges almost surely toward the price of the option given by (14)[7]. Define the path estimator of the option using control variates as follows:

$$\Pi_i \hat{Z}_i = \Pi_i \hat{V}_i + \lambda_i [\Pi_i \Upsilon^j - E_i (\Upsilon)] \qquad (16)$$

where λ_i is a previsible process in F with $E_F(\lambda_i) < \infty$ and Υ^j is a random variable for which we can compute the conditional expectation.

The sample estimator in (15) can be written as:

$$\Pi_i \tilde{V}_i = \frac{1}{m} \sum_{j=1}^{m} (Z_\tau^j) \qquad (17)$$

$$= \Pi_i \hat{V}_i + \lambda_i [\Pi_i Y^j - E_i (Y)] \qquad (18)$$

$$\lim_{j \to \infty} \lambda_i E_i [(\Pi_i Y^j - E_i (Y))] = 0 \quad (19)$$

Therefore the following result follows

$$E_i (\tilde{V}_i) = V_i \qquad (20)$$

From (16) it follows that $Var[\tilde{Z}_i(\lambda_i)]$, particularly we have $Var(\tilde{Z}_i) \le Var(\hat{V}_i)$ if

$$\lambda_i^* = -\frac{Cov[\hat{V}_i, Y_i]}{Var[Y_i]} \qquad (21)$$

Therefore efficiency can be gained by minimising λ_i in (16). To achieve this goal, we can use a simple Least Squares approach, that, we already use to compute estimates of the conditional

[7] See Clement et al (2002) amongst others.

expectation in (14). The estimation of λ_i, in this case, will introduce some bias, however this will vanish as the number of replications increases. As pointed out in Boyle, Broadie and Glasserman (1997), the estimator of λ_i need not be very precise to achieve a reduction of variance in the case of using only one control. It becomes instead important when multiple controls are introduced. In the empirical application in this paper we have fixed $\lambda_i = 1$ for all i.

4.4 Empirical Results

In this Section we implement the LS (2001) and GY (2004b) methodologies by using control variates. To implement the LS (2001) and GY (2004b) by control variates, we use the approach described above. Empirical results are reported in Table 4.

Longstaff-Schwartz (2001), Glasserman and Yu (2004a) Methods

Exponential					Laguerre			
		2	3	4	2	3	4	BIN.
GY	0.2/0.0833	4.999	4.996	4.997	0	0	0	5
SE		0.00689	0.00771	0.0077	0	0	0	
RMSE		0.00083	0.00395	0.003052	0	0	0	
LS	02/0.0833	4.996	4.995	4.995	4.996	4.9948	4.999	5
SE		0.00594	0.00871	0.00646	0.000519	0.0069	0.00682	
RMSE		0.00409	0.0048	0.005019	0.000373	0.000418	0.000101	
GY	0.2/0.3333	5.077	5.086	5.085	0	0	0	5.087
SE		0.01185	0.009746	0.0107	0	0	0	
RMSE		0.01015	0.000829	0.00194	0	0	0	
LS	0.2/0.3333	5.078	5.083	5.091	5.08	5.085	5.084	5.087
SE		0.00999	0.00777	0.0071	0.00522	0.00835	0.01024	
RMSE		0.00908	0.003678	0.00331	0.00659	0.002344	0.00274	
GY	0.2/0.5833	5.251	5.262	5.26	0	0	0	5.265
SE		0.0094	0.01269	0.0146	0	0	0	
RMSE		0.01514	0.002854	0.00531	0	0	0	
LS	0.2/0.5833	5.253	5.261	5.263	5.251	5.266	5.264	5.265
SE		0.0132	0.00957	0.0096	0.00968	0.010665	0.01412	
RMSE		0.01179	0.00373	0.00218	0.00149	0.0007153	0.000145	
GY	0.3/0.0833	5.056	5.056	5.059	0	0	0	5.06
SE		0.00819	0.00809	0.00811	0	0	0	
RMSE		0.003608	0.003745	0.0006428	0	0	0	
LS	0.3/0.0833	5.057	5.057	5.06	5.055	5.0595	5.063	5.06
SE		0.01011	0.00706	0.009211	0.0059	0.00786	0.00887	
RMSE		0.002734	0.00269	0.0004334	0.00504	0.000462	0.0003185	

Table 4. Control variates. SEs are standard errors and RMSEs are the root mean square errors. Exponential and Laguerre are the polynomials used in this application. 2-4 refers to the number of basis functions used. BIN is the price of the option given by a Binomial method. The zeros refer to cases when we were not able to obtain martingales baseis for a specific polynomial and therefore we were not able to obtain martingales basis and could not implement the GY (2004b) method.

We have implemented the method of control variates by sampling the price of a similar European option at each possible stopping time and setting the value of λ^* equal to 1.

Table 4 continued

GY	0.3/0.3333	5.691	5.705	5.702	0	0	0	5.706
SE		0.00701	0.00886	0.01252	0	0	0	
RMSE		0.01536	0.000733	0.004363	0	0	0	
LS	0.3/0.3333	5.688	5.707	5.706	5.692	5.702	5.704	5.706
SE		0.0117	0.00888	0.01082	0.00756	0.00952	0.00884	
RMSE		0.018356	0.000463	0.0004135	0.0144	0.00422	0.00171	
GY	0.3/0.5833	6.238	6.236	6.243	0	0	0	6.244
SE		0.00981	0.0146	0.00959	0	0	0	
RMSE		0.01636	0.00768	0.000871	0	0	0	
LS	0.3/0.5833	6.229	6.241	6.242	6.226	6.239	6.246	6.244
SE		0.01249	0.01081	0.01385	0.001369	0.0161	0.0156	
RMSE		0.01538	0.002699	0.001753	0.01792	0.000532	0.000152	
GY	0.4/0.0833	5.278	5.282	5.289	0	0	0	5.286
SE		0.00765	0.00704	0.008454	0	0	0	
RMSE		0.007675	0.00418	0.002976	0	0	0	
LS	0.4/0.0833	5.275	5.285	5.289	5.275	5.282	5.285	5.286
SE		0.01115	0.00978	0.00535	0.00875	0.00642	0.00705	
RMSE		0.010527	0.001442	0.002635	0.0115	0.003856	0.00102	
GY	04/0.3333	6.491	6.507	6.513	0	0	0	6.51
SE		0.01421	0.015	0.01444	0	0	0	
RMSE		0.0201	0.00264	0.003587	0	0	0	
LS	0.4/0.3333	6.491	6.504	6.51	6.496	6.513	6.51	6.51
SE		0.01076	0.0131	0.01243	0.0128	0.01025	0.0105	
RMSE		0.01554	0.005928	0.002473	0.0144	0.0003199	0.000489	
GY	0.4/0.5833	7.364	7.374	7.38	0	0	0	7.383
SE		0.01583	0.0111	0.0123	0	0	0	
RMSE		0.01915	0.008729	0.003454	0	0	0	
LS	0.4/0.5833	7.363	7.381	7.382	7.365	7.377	7.383	7.383
SE		0.01011	0.01657	0.0145	0.01286	0.013356	0.0162	
RMSE		0.02024	0.002854	0.001329	0.01776	0.005812	0.0004751	

Results in Table 4 show that control variates estimator produces a very accurate price regardless the function used in the regression. Three basis are sufficient to achieve a low RMSE. Even in this application, it is not always the case that the GY (2004b) method produces the smallest standard errors. This may further support what we pointed out in Section 3[8]. The LS (2001) method implemented with Laguerre bases seems to produce the lowest RMSE and the strongest evidence of uniform convergence. Finally, the VR factor, in this case, ranges between 0.45 and 5.

[8] The assumption of finite variance on the basis functions (see Assumption C1 in Glasserman and Yu, 2004b) may also be another reason why variation of the estimates of the option in this case is not as stable as Theorem 1 would suggest..

5 Valuing American Bermuda Asian Options

We consider the previous methodologies when pricing more complex options such as American Asian options and options written on a maximum of n assets. It is with this type of options that the LS (2001) and GY (2004b) methodologies become interesting.

As in Longstaff and Schwartz (2001), we consider pricing an American Asian option having also an initial lockout period. In order to use the options prices reported in Longstaff and Schwartz (2001) as benchmark, we consider an American call option that after an initial lock out period of three months can be exercised at any time up to maturity T. We assume $T = 2$ years. The average is the (continuous) arithmetic average of the underlying stock price calculated over the lock out period. We implement the LS (2001) and GY (2004b) methodologies by using control variates method. The choice of the control in this case falls, obviously, on the price of an equivalent geometric option. Therefore, we use the methodology described above and choose the price of a geometric average option as a control. As in Longstaff and Schwartz (2001) the strike price is \$100, the risk free rate of interest 6% and volatility 20%. We use different scenarios for the stock prices (S) and assume 200 steps for both stock price and average. The results are reported in Table 5:

American Bermudan Asian Options (LS 2001 Method)

S	Expon. m = 30,000	Lagu.	Expon. M = 50,000	Lagu.	Expon. m = 75,000	Lagu.
80	0.9211	0.9218	0.937	0.945	0.9422	0.950
90	3.080	3.106	3.210	3.312	3.320	3.312
100	7.492	7.522	7.679	7.845	7.843	7.873
110	13.23	13.89	14.188	14.234	14.355	14.501
120	20.09	21.2	22.081	22.111	22.189	22.311

Table 5. S is the stock price, m the number of simulations, while Expon. and Lagu. are respectively exponential and Laguerre basis functions.

As in Longstaff and Schwartz (2001), we use the first eight Laguerre basis[9] and 50,000 replications. In our application, we have also used exponential basis. Using finite difference methods to price these options LS (2001) report option prices equal to \$0.949 (\$80), \$3.267 (\$90), \$7.889 (\$100), \$14.538 (\$110) and \$22.423 (\$120)[10]. In general, our results support those reported in Tables 3 of Longstaff and Schwartz (2001). That is the LS (2001) method produces a very accurate price of the option. If we calculate the early exercise value in this case and compare it with what reported in LS (2001) for the same options but using antithetic variates, we have that, for Laguerre bases and,

[9] That is, first two Laguerre bases on the stock price and average plus their cross products including an intercept.
[10] Number in the brackets are initial stock prices and the initial average value for the stock price is assumed to be 90.

$m = 75{,}000$, differences in the early exercise values in LS (2001) ranges between 0.007 and 0.050, while in the present study the range is between 0.001 and 0.042. This is in line with what we pointed out at the beginning. The choice and the correct implementation of variance reduction techniques is important when pricing option with American features since it reduces the probability of generating sub-optimal strategies.

In Table 6, we extend the Glasserman and Yu (2004b) method to price American Asian options. We use Hermite basis (ϕ_{KH}) to satisfy Assumption 2 as follows, $f_k \phi_{KH}$, with $f_k = t^{k/2}$. The method seems to underestimate the option price.

American Bermudan Asian Options (GY, 2004b Method)

	Hermite		
S	**m = 30,000**	**M = 50,000**	**M = 75,000**
80	0.925	0.936	0.940
90	3.188	3.310	3.166
100	7.521	7.544	7.563
110	13.83	14.223	14.321
120	20.11	21.645	22.022

Table 6. S is the stock price, m the number of simulations.

However, in general, more work is necessary to implement this method since the choice of a martingale basis might be fundamental. On the other hand it seems that this fundamental problem has become more, to use Chris Rogers`s words, "an art than a science". As pointed out above we shall address this important issue in a separate study.

6 Valuing American Basket Options

Finally, we consider an additional high dimensional problem. We consider an American call option written on a maximum of five risky assets paying a proportional dividend. We assume that each asset return is independent from the other. Once again, we use the same parameter specifications as in Longstaff and Schwartz (2001) and Broadie and Glaserman (1997) such that we can use prices reported in these papers as benchmark. We implement the methodologies by using antithetic variates.

Broadie and Glasserman (1997) use stochastic mesh to solve this type of problems and report confidence interval for the option prices. We consider three different options with initial stock prices of 90,100, and 110 respectively. The assets pay a 10% proportional dividend, the strike price of the option is 100, the risk free rate of interest is 10% and volatility is 20%. Confidence intervals reported in Brodie and Glasserman (1997) are [16.602, 16.710] when the initial asset value is 90; [26.101, 26.211] with initial asset value of 100, and finally [36.719, 36.842] when the initial value is 110. The option prices in Longstaff and Schwartz (2001) are respectively, 16.657, 26.182, and 36.812 and they all fall within the Broadie and Glasserman `s confidence interval above.

American Basket Option (LS 2001 Method)

	Expon.	Hermite	Expon.	Hermite	Expon.	Hermite
S		**m = 30,000**		**M = 50,000**		**m = 75,000**
90	16.6895	16.677	16.6555	16.6171	16.6632	16.642
100	26.1758	26.1744	26.1708	26.1033	26.0804	26.12
110	36.7697	36.7642	36.7826	36.7482	36.8214	36.748

Table 7. S is the stock price, m the number of simulations, while Expon. and Hermite are respectively exponential and Hermite basis functions.

American Basket Options (GY, 2004b Method)

		Hermite	
S	**m =30.**	**m =50.**	**m = 75.**
90	16.5935	16.623	16.4759
100	26.0789	26.181	25.6802
110	36.286	36.71	36.1032

Table 8. S is the stock price, m the number of simulations.

We note that regardless of the number of replications or basis functions used, we achieve, in all cases, a price that lies within the above interval. We have also extended the GY (2004b) method to price basket options (see Table 8). As in the previous application, we have used Hermite polynomials to satisfy Assumption 1 in GY (2004b). We note that option prices estimates fall within the Broadie and Glasserman `s confidence interval when 50,000 paths are considered. Therefore the martingale basis used in this case seems to be appropriate.

8. Conclusions

From an academic and even a practitioner`s point of view, pricing American options still remains an interesting research area, particularly when Monte Carlo methods is employed. This is mainly due to the flexibility of this method to accommodate high dimensional features.

Recently, Longstaff and Schwartz (2001) and Glasserman and Yu (2004b) propose two option pricing estimators based on Monte Carlo simulations. The general objective of this paper is to undertake an empirical analysis to investigate the finite sample approximations of these estimators. Apart from the above specified objective, we also (i) estimate the bias induced by these estimators, (ii) suggest an "optimal" polynomial function, (iii) extend these methodologies by implementing various variance reduction techniques. Finally, this is the first empirical study on the estimator proposed in Glasserman and Yu (2004b) and it extends that method to solve high dimensional problems.

One general result in the literature on pricing American options by Monte Carlo methods (regression methods) is that Monte Carlo methods generate sub-optimal policies when used to estimate early exercises values and consequently they generate estimated prices that are below the true price (see for example LS, 2001, and Clement et al, 2002, for a discussion). Rogers (2002) formulate the problem in Equation (3) as the dual and show that one can use a martingale approach to reduce the probability of choosing sub-optimal policies when determining the early exercise value. However, this approach requires designing an optimal martingale and there is no clear cut rule yet on how to do that. In this study we point out that variance reduction techniques, if correctly implemented, can help us to reduce the probability of generating sub-optimal policies.

Overall, we find that option prices estimates by LS (2001) and GY (2004b) methodologies are accurate regardless the type of option considered. A large part of the sample bias can be eliminated with an acceptable number of replications (i.e. 100,000). However, in general, the LS (2001) estimator performs the best. With this estimator we found Laguerre polynomials and control variates to out-perform the others[11]. Therefore, in practical applications, we recommend using Laguerre polynomials. In general, a number of basis equal to three, 100,000 replication and control variate seem to be the right combination to achieve a substantial level of accuracy.

[11] We have also considered Hermite polynomials but the results were not satisfactory and therefore were not reported in this study. However, results are available upon request.

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