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Abstract

Mixtures of distributions have been applied to contingent claim pricing as a way of extending the Black and Scholes (1973) assumption of lognormally distributed assets. The pricing framework presented here delivers preference free contingent claim pricing formulae and extends the literature in two ways: First, we widen the set of distributions used in the mixture by assuming that the terminal price of the underlying security has a mixture of transformed-normal distributions. Second, we show that the components of the mixture do not need to have the same density as long as they belong to the family of transformed-normal distributions. Our framework is developed in a discrete time equilibrium economy. It is strongly related to Camara (2003) and is consistent with the sufficient conditions of Heston (1993) and Schroder (2004). We show that by restricting the value of some distributional parameters, it is possible to obtain a risk neutral valuation relationship for the pricing of contingent claims when the terminal price of the underlying asset has a mixture of transformed normal distributions. An interesting aspect of the mixtures of distributions, and in particular of the framework developed here, is that the actual and the risk neutral distributions might not have the same shape. This fact could help to explain the non-monotonic pricing kernel obtained by Jackwerth and Rubinstein (1996), Brown and Jackwerth (2004), Ait-Sahalia and Lo (1998) among others.

Keywords: Mixture of distributions, transformed-normal distribution, risk neutral valuation relationship

JEL classification: G13.

A Note on the Pricing of Contingent Claims with a Mixture of Distributions in a Discrete-Time General Equilibrium Framework

1 Introduction

Mixtures of distributions have been widely applied to contingent claim pricing as a way of extending the Black and Scholes (1973) assumption of lognormally distributed assets. This is because mixtures usually cover a larger area in the skewness and kurtosis plane than the lognormal distribution, which is limited to a single line on this plane.[1] Contingent claim pricing models based on mixtures usually assume that the underlying security price can be represented by a mixture of two or more lognormal densities, which means that the components of the mixture have the same distribution.[2]

The pricing framework presented here delivers preference free contingent claim pricing formulae and extends the literature in two ways: First, we widen the set of distributions used in the mixture by assuming that the terminal price of the underlying security has a mixture of transformed-normal distributions.[3] Second, we show that the components of the mixture do not need to have the same density as long as they belong to the family of transformed-normal distributions.

This framework, developed in a discrete time equilibrium economy, is strongly related to Camara (2003) and is consistent with the sufficient conditions of Heston (1993) and Schroder (2004). It is shown that by restricting the value of some distributional parameters, it is possible to obtain a risk neutral valuation relationship for the pricing of contingent claims when the terminal price of the underlying asset has a mixture of transformed normal distributions. Although the idea of restricting distributional parameters is not new, it provides a simple and systematic way of achieving risk neutrality for the broad family of transformed distributions.

An interesting aspect of the mixtures of distributions, and in particular of the framework developed here, is that the actual and the risk neutral distributions might not have the same shape. This fact could help to explain the results of Jackwerth and Rubinstein (1996), Brown and Jackwerth (2004), Ait-Sahalia and Lo (1998) among others, which show a non-monotonic pricing kernel.

This paper is divided as follows: First we introduce the basic economy and the general form for the pricing kernel. Then, in section two, we provide the framework for the pricing of European contingent claims and illustrate its application through several examples, which return new option pricing formulae. In section three we briefly discuss

[1] In addition to pricing options, mixture of distributions have also been applied to "back out" risk neutral densities implied by the price of traded options. The literature in this particular field is large and a comprehensive survey on the methods for extracting risk neutral densities may be found in Söderlind and Svensson (1997) and Bahra (1997).

[2] See Ritchey (1990) and Melick and Thomas (1997) for instance.

[3] The transformed-normal distribution is introduced in equation (4).

the implications of mixtures on the pricing kernel and conclude.

2 The forward price equilibrium relationship

Consider a risk-averse representative investor in a complete market setting[4] that maximises her expected utility of future wealth, $\max E\left[U\left(\tilde{W}\right)\right]$. In equilibrium, the forward price of an underlying asset is given by[5]

$$F\left(\tilde{S}\right) = E^P\left[m\left(\tilde{S}\right)\tilde{S}\right], \tag{1}$$

where

$$m\left(\tilde{S}\right) = \frac{E^P\left[U'\left(\tilde{W}\right) \mid \tilde{S}\right]}{E^P\left[U'\left(\tilde{W}\right)\right]}, \tag{2}$$

is the asset specific pricing kernel, the superscript P of $E(\cdot)$ means the expectation is taken with respect to the actual probability, $\tilde{S}$ is the payoff of the underlying asset, and $U'(\cdot)$ is the representative investor's marginal utility function.

Assume that $\tilde{S}$ has a mixture of distributions

$$f\left(\tilde{S}\right) = \sum_i \alpha_i f_{\tilde{S},i}\left(\tilde{S}\right),\ i \geq 1, \tag{3}$$

where α_i is the weight on the i^{th} component with $\sum_i \alpha_i = 1,\ \alpha_i \geq 0\ \forall i$ and $f_{\tilde{S},i}\left(\tilde{S}\right)$ is the i^{th} density function of $\tilde{S}$.

The random payoff $\tilde{S}$ has a transformed normal distribution if[6]

$$h_{\tilde{S},i}\left(\tilde{S}\right) = \mu_{\tilde{S},i} + \sigma_{\tilde{S},i} z, \tag{4}$$

where $h_{\tilde{S},i}\left(\tilde{S}\right)$ is a strictly monotonic function, z is a standard normal random variable, $\mu_{\tilde{S},i} \in \Re$ and $\sigma_{\tilde{S},i} \in \Re^+$ are the location and the scale parameter of the i^{th} component respectively. Then, considering the equation above the densities of $\tilde{S}$ are

$$f_{\tilde{S},i}\left(\tilde{S}\right) = \frac{1}{\sigma_{\tilde{S},i}\sqrt{2\pi}}\left|h'_{\tilde{S},i}\left(\tilde{S}\right)\right| \exp\left[-\frac{1}{2\sigma^2_{\tilde{S},i}}\left(h_{\tilde{S},i}\left(\tilde{S}\right) - \mu_{\tilde{S},i}\right)^2\right]. \tag{5}$$

Now, assume that $\tilde{W}$ has a transformed-normal distribution

$$h_{\tilde{W}}\left(\tilde{W}\right) = \mu_{\tilde{W}} + \sigma_{\tilde{W}} z, \tag{6}$$

[4] Please note that this framework is consistent with a dynamically incomplete market.

[5] These results are obtained from the first order condition for a maximum and from the application of conditional expectation proprieties. For a detailed derivation see Huang and Litzenberger (1988).

[6] Note that if $h_{\tilde{S}}\left(\tilde{S}\right)$ has a normal distribution, then $\tilde{S}$ has a transformed normal distribution. For instance, if $h_{\tilde{S}}\left(\tilde{S}\right) = \ln\left(\tilde{S}\right)$ then $\tilde{S}$ has a lognormal distribution.

Note that the functions $h_{\tilde{S}}\left(\tilde{S}\right)$ and $h_{\tilde{W}}\left(\tilde{W}\right)$ do not have to be the same i.e. the distribution of $\tilde{S}$ and $\tilde{W}$ can be different as long as they are in accordance with equation (4).

Finally, assume that the representative investor's marginal utility function is given by

$$U'\left(\tilde{W}\right) = \exp\left[\gamma h_{\tilde{W}}\left(\tilde{W}\right)\right], \qquad (7)$$

where the constant γ is the risk-aversion parameter.

Given these assumptions, it is shown in the appendix that the asset specific pricing kernel has the following form

$$m\left(\tilde{S}\right) = \sum_i \frac{\alpha_i}{\sigma_{\tilde{S},i}} \exp\left[-\frac{1}{2\sigma_{\tilde{S},i}}\left(h_{\tilde{S}}\left(\tilde{S}\right) - \mu_{\tilde{S},i} - \gamma\rho_i\sigma_{\tilde{W}}\sigma_{\tilde{S},i}\right)^2\right]\phi^{-1}, \qquad (8)$$

where ρ_i is the correlation coefficient of the i^{th} density of $\tilde{S}$ with wealth, $\sum_i \alpha_i = 1$, $\alpha_i \geq 0\ \forall i$, and

$$\phi = \sum_i \frac{\alpha_i}{\sigma_{\tilde{S},i}} \exp\left[-\frac{1}{2\sigma_{\tilde{S},i}}\left(h_{\tilde{S}}\left(\tilde{S}\right) - \mu_{\tilde{S},i}\right)^2\right].$$

From equation (1) the forward price of the underlying security is then given by

$$\begin{aligned} F &= \int \tilde{S} m\left(\tilde{S}\right) f\left(\tilde{S}\right) d\tilde{S} \\ &= \int \tilde{S} m\left(\tilde{S}\right) \sum_i \alpha_i f_{\tilde{S},i}\left(\tilde{S}\right) d\tilde{S} \\ &= \int \tilde{S} \sum_i \alpha_i \hat{f}_{\tilde{S},i}\left(\tilde{S}\right) d\tilde{S} \\ &= \int \tilde{S} \hat{f}\left(\tilde{S}\right) d\tilde{S} \qquad (9) \end{aligned}$$

where $\hat{f}_{\tilde{S},i}\left(\tilde{S}\right)$ differs from $f_{\tilde{S},i}\left(\tilde{S}\right)$ only by the location parameter. That is, $\hat{f}_{\tilde{S},i}\left(\tilde{S}\right)$ has the same density and the same scale parameter as $f_{\tilde{S},i}\left(\tilde{S}\right)$, but with location parameter $\mu_{\tilde{S},i} + \gamma\rho_i\sigma_{\tilde{W}}\sigma_{\tilde{S},i}$.[7]

The density $\hat{f}\left(\tilde{S}\right)$ is defined as the risk adjusted density, which is given by the product of the actual distribution and the asset specific pricing kernel. The term "risk adjusted" comes from the fact that this density contains parameters that are related to the investor's preference and wealth. In the transformed normal case this adjustment shifts the location parameter of the actual density according to investor's risk preferences.

In the next section we introduce the contingent claim pricing framework and provide several applications that show the use and flexibility of our framework.

[7] This is obtained by the direct multiplication of $m\left(\tilde{S}\right)$ by $f\left(\tilde{S}\right)$.

2.1 The contingent claim pricing framework

Let $\nu\left(\tilde{S}\right)$ be the contingent claim payoff function. Then, using the same equilibrium arguments as in equation (9), the price of a contingent claim written on the forward price of $\tilde{S}$ is given by

$$\begin{aligned} V\left[F\left(\tilde{S}\right)\right] &= E^P\left[m\left(\tilde{S}\right)\nu\left(\tilde{S}\right)\right] \\ &= \int \nu\left(\tilde{S}\right)\hat{f}\left(\tilde{S}\right)d\tilde{S}. \end{aligned} \tag{10}$$

Since the density $\hat{f}\left(\tilde{S}\right)$ contains parameters related to investor preferences, if we wanted to price contingent claims using the risk adjusted density we would have to take into account these parameters, which would involve the estimation of several unobservable parameters.[8]

One way of avoiding this problem is to work in a risk neutral setting, which means that all assets in the economy would have to have the same rate of return regardless of their risk since investors would be insensitive to risk. Here, this would mean having to work with a risk-aversion parameter of zero, $\gamma = 0$.

Although it would be very convenient to simply make this assumption, it would be very difficult to justify on economic grounds since investors are clearly risk averse. Thus, to keep the risk neutral setting without making unjustifiable assumptions, one can try to replace these preference parameters by "observable" or "marketable" parameters, such as asset and bond prices. The usual way of doing this is by inverting[9] equation (9) and expressing the location parameter, $\mu_{\tilde{S},i} + \gamma\rho_i\sigma_{\tilde{W}}\sigma_{\tilde{S},i}$, as a function of F. (See Brennan (1979) and Camara (2003) for instance)

Nevertheless, when the terminal value of $\tilde{S}$ is given by a mixture distribution, there are i densities and as a consequence i "location parameters" (i.e. there is a set of i parameters related to preference and wealth) and thus, it is not possible to invert equation (9) anymore. If F is the only price available, i.e. if there are not any other securities or derivative securities prices available that are related to F, then further assumptions are needed.

Here we fix the location parameters of all i densities so as to be the same. That is, we assume that

$$\mu_{\tilde{S},i} + \gamma\rho_i\sigma_{\tilde{W}}\sigma_{\tilde{S},i} = \hat{\mu},\ \forall i, \tag{11}$$

which allows us to write $\hat{\mu}$ as a function of F.

Although this assumption restricts the range of skewness and kurtosis of the mixture, it does not reduce the model's ability of capturing the terminal distribution of $\tilde{S}$, mainly considering the family of transformed normal distributions, which contains several high moment distributions.

[8] As Merton (1973, p.161) points out, "... the expected return is not directly observable and estimates from past data are poor because of nonstationarity. It also implies that attempts to use the option price to estimate expected returns on the stock or risk-preferences of investors are doomed to failure".

[9] Note however that it is not always possible to invert equation (9).

Thus, if it is possible to solve equation (9) for $\hat{\mu}$ and equation (11) holds, then equation (10) can be rewritten as

$$V\left[F\left(\tilde{S}\right)\right] = E^{Q}\left[\nu\left(\tilde{S}\right)\right],$$

where the superscript Q of $E(\cdot)$ means that the expectation is taken with respect to the risk neutral probability and the price of the contingent claim is a martingale under the measure Q.

In the following examples we show the application of the above framework to the pricing of European call options. It is assumed that the terminal value of $\tilde{S}$ has a mixture of transformed normal distributions as in equation (4) and that equation (11) holds.

Example 1 *(The S_U and lognormal mixture): Assume that $\tilde{S}$ has a mixture of a S_U distribution and a lognormal distribution.*[10] *Then, the forward price in equation (9) is given by*

$$\begin{aligned} F &= \int \tilde{S}\left[\alpha_1 m_1\left(\tilde{S}\right) f_1\left(\tilde{S}\right) + \alpha_2 m_2\left(\tilde{S}\right) f_2\left(\tilde{S}\right)\right] d\tilde{S} \\ &= \int \tilde{S}\left[\alpha_1 \hat{f}_1\left(\tilde{S}\right) + \alpha_2 \hat{f}_2\left(\tilde{S}\right)\right] d\tilde{S}, \end{aligned} \tag{12}$$

where

$$\begin{aligned} m_1\left(\tilde{S}\right) &= \frac{\alpha_1}{\phi\sigma_{\tilde{S},1}} e^{-\frac{1}{2\sigma_{\tilde{S},1}}\left(\ln\left(\tilde{S}\right)-\mu_{\tilde{S},1}-\gamma\rho_1\sigma_{\tilde{W}}\sigma_{\tilde{S},1}\right)^2} \\ m_2\left(\tilde{S}\right) &= \frac{\alpha_2}{\phi\sigma_{\tilde{S},2}} e^{-\frac{1}{2\sigma_{\tilde{S},2}}\left(\sinh^{-1}\left(\tilde{S}\right)-\mu_{\tilde{S},2}-\gamma\rho_2\sigma_{\tilde{W}}\sigma_{\tilde{S},2}\right)^2} \end{aligned}$$

$$\phi = \frac{\alpha_1}{\sigma_{\tilde{S},1}} e^{-\frac{1}{2\sigma_{\tilde{S},1}}\left(\ln\left(\tilde{S}\right)-\mu_{\tilde{S},1}\right)^2} + \frac{\alpha_2}{\sigma_{\tilde{S},2}} e^{-\frac{1}{2\sigma_{\tilde{S},2}}\left(\sinh^{-1}\left(\tilde{S}\right)-\mu_{\tilde{S},2}\right)^2}$$

$$\hat{f}_1\left(\tilde{S}\right) = \frac{1}{\sigma_{\tilde{S},1}\sqrt{2\pi}\tilde{S}} e^{-\frac{1}{2\sigma^2_{\tilde{S},1}}\left[\ln\left(\tilde{S}\right)-\left(\mu_{\tilde{S},1}+\gamma\rho_1\sigma_{\tilde{W}}\sigma_{\tilde{S},1}\right)\right]^2} \tag{13}$$

$$\hat{f}_2\left(\tilde{S}\right) = \frac{1}{\sigma_{\tilde{S},2}\sqrt{2\pi}\sqrt{\tilde{S}+1}} e^{-\frac{1}{2\sigma^2_{\tilde{S},2}}\left[\sinh^{-1}\left(\tilde{S}\right)-\left(\mu_{\tilde{S},2}+\gamma\rho_2\sigma_{\tilde{W}}\sigma_{\tilde{S},2}\right)\right]^2}. \tag{14}$$

The value of F is thus given by[11]

$$F = \alpha_1 e^{\mu_{\tilde{S},1}+\gamma\rho_1\sigma_{\tilde{W}}\sigma_{\tilde{S},1}+\frac{1}{2}\sigma^2_{\tilde{S},1}} + \alpha_2 e^{\frac{1}{2}\sigma^2_{\tilde{S},2}} \sinh\left(\mu_{\tilde{S},2}+\gamma\rho_2\sigma_{\tilde{W}}\sigma_{\tilde{S},2}\right).$$

Since by assumption $\mu_{\tilde{S},1}+\gamma\rho_1\sigma_{\tilde{W}}\sigma_{\tilde{S},1} = \mu_{\tilde{S},2}+\gamma\rho_2\sigma_{\tilde{W}}\sigma_{\tilde{S},2} = \hat{\mu}$, the above equation becomes

$$F = \alpha_1 e^{\hat{\mu}+\frac{1}{2}\sigma^2_{\tilde{S},1}} + \frac{1}{2}\alpha_2 e^{\frac{1}{2}\sigma^2_{\tilde{S},2}}\left(e^{\hat{\mu}} - e^{-\hat{\mu}}\right),$$

[10] For the S_U distribution $h\left(\tilde{S}\right) = \sinh^{-1}\left(\tilde{S}\right)$. For a detailed discussion see Johsnon (1949).

[11] Recall that $\sinh(y) = 0.5e^{y} - 0.5e^{-y}$.

which allows us to solve for the variable $\hat{\mu}$

$$\hat{\mu} = -\ln\left[-F + \left(F^2 + 2\alpha_1\alpha_2 \exp\left(\frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2\right) + \alpha_2^2 \exp\left(\sigma_2^2\right)\right)^{\frac{1}{2}}\right] + \ln\left(a_2\right) + \frac{1}{2}\sigma_2^2.$$

Once we obtain an expression for $\hat{\mu}$ we can solve for the price of an option. Let $\nu\left(\tilde{S}\right) = \max\left(\tilde{S} - K, 0\right)$, where K is the exercise price. From equation (10), the price of a call option is then

$$\begin{aligned} V\left[F\left(\tilde{S}\right)\right] &= E^P\left[m\left(\tilde{S}\right)\nu\left(\tilde{S}\right)\right] \\ &= \int \max\left(\tilde{S} - K, 0\right)\left[\alpha_1 \hat{f}_1\left(\tilde{S}\right) + \alpha_2 \hat{f}_2\left(\tilde{S}\right)\right] d\tilde{S}, \end{aligned}$$

which, after substituting for equations (13) and (14) and simplifying, yields the option pricing formula

$$\begin{aligned} V\left[F\left(\tilde{S}\right)\right] &= \frac{\alpha_1\alpha_2}{A} e^{\frac{1}{2}\sigma^2_{\tilde{S},1} + \frac{1}{2}\sigma^2_{\tilde{S},2}} N\left(d1\right) - \alpha_1 K N\left(d2\right) \\ &\quad + \frac{\alpha_2^2}{2A} e^{\sigma^2_{\tilde{S},2}} N\left(d3\right) - \frac{1}{2}\alpha_2 K N\left(d4\right) - \frac{1}{2} A N\left(d5\right), \end{aligned} \tag{15}$$

where

$$\begin{aligned} d1 &= \frac{\ln\left(\frac{a_2}{AK}\right) + 0.5\sigma_2^2}{\sigma_{\tilde{S},1}} + \sigma_{\tilde{S},1} \\ d2 &= d1 - \sigma_{\tilde{S},1} \\ d3 &= \frac{\ln\left(\frac{a_2}{A}\right) - \sinh^{-1}\left(K\right) + \frac{1}{2}\sigma_2^2}{\sigma_{\tilde{S},2}} + \sigma_{\tilde{S},2} \\ d4 &= d3 - \sigma_{\tilde{S},2} \\ d5 &= d4 - \sigma_{\tilde{S},2} \\ A &= -F + \left[F^2 + 2\alpha_1\alpha_2 \exp\left(\frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2\right) + \alpha_2^2 \exp\left(\sigma_2^2\right)\right]^{\frac{1}{2}}. \end{aligned}$$

Example 2 *(The displaced lognormal and negative-skewed lognormal mixture): Assume that $\tilde{S}$ has a mixture of a displaced lognormal distribution and a negative-skewed lognormal distribution.*[12] *Then, as in example 1*

$$F = \int \tilde{S}\left[\alpha_1 \hat{f}_1\left(\tilde{S}\right) + \alpha_2 \hat{f}_2\left(\tilde{S}\right)\right] d\tilde{S}, \tag{16}$$

where

$$\begin{aligned} m_1\left(\tilde{S}\right) &= \frac{\alpha_1}{\phi\sigma_{\tilde{S},1}} \exp\left[-\frac{1}{2\sigma_{\tilde{S},1}}\left(\ln\left(\tilde{S} - \delta_1\right) - \mu_{\tilde{S},1} - \gamma\rho_1\sigma_{\tilde{W}}\sigma_{\tilde{S},1}\right)^2\right] \\ m_2\left(\tilde{S}\right) &= \frac{\alpha_2}{\phi\sigma_{\tilde{S},2}} \exp\left[-\frac{1}{2\sigma_{\tilde{S},2}}\left(\ln\left(\delta_2 - \tilde{S}\right) - \mu_{\tilde{S},2} - \gamma\rho_2\sigma_{\tilde{W}}\sigma_{\tilde{S},2}\right)^2\right] \end{aligned}$$

[12] See for instance Rubinstein (1983) and Stapleton and Subrahmanyam (1984) for the displaced lognormal and for the negative skewed lognormal respectively.

$$\phi = \frac{\alpha_1}{\sigma_{\tilde{S},1}} e^{-\frac{1}{2\sigma_{\tilde{S},1}}\left(\ln\left(\tilde{S}-\delta_1\right)-\mu_{\tilde{S},1}\right)^2} + \frac{\alpha_2}{\sigma_{\tilde{S},2}} e^{-\frac{1}{2\sigma_{\tilde{S},2}}\left(\ln\left(\delta_2-\tilde{S}\right)-\mu_{\tilde{S},2}\right)^2}$$

$$\hat{f}_1\left(\tilde{S}\right) = \frac{1}{\sigma_{\tilde{S},1}\sqrt{2\pi}\left(\tilde{S}-\delta_1\right)} e^{-\frac{1}{2\sigma^2_{\tilde{S},1}}\left[\ln\left(\tilde{S}-\delta_1\right)-\left(\mu_{\tilde{S},1}+\gamma\rho_1\sigma_{\tilde{W}}\sigma_{\tilde{S},1}\right)\right]^2} \quad (17)$$

$$\hat{f}_2\left(\tilde{S}\right) = \frac{1}{\sigma_{\tilde{S},2}\sqrt{2\pi}\left(\delta_2-\tilde{S}\right)} e^{-\frac{1}{2\sigma^2_{\tilde{S},2}}\left[\ln\left(\delta_2-\tilde{S}\right)-\left(\mu_{\tilde{S},2}+\gamma\rho_2\sigma_{\tilde{W}}\sigma_{\tilde{S},2}\right)\right]^2}. \quad (18)$$

with $\delta_1 < \tilde{S} < \delta_2$.

The value of F *is given by*

$$F = \alpha_1\left(e^{\hat{\mu}+\frac{1}{2}\sigma^2_{\tilde{S},1}} + \delta_1\right) + \alpha_2\left(\delta_2 - e^{\hat{\mu}+\frac{1}{2}\sigma^2_{\tilde{S},2}}\right),$$

and solving for $\hat{\mu}$

$$\hat{\mu} = \ln\left[\frac{F-\alpha_1\delta_1-\alpha_2\delta_2}{\alpha_1 e^{\frac{1}{2}\sigma^2_{\tilde{S},1}} - \alpha_2 e^{\frac{1}{2}\sigma^2_{\tilde{S},2}}}\right].$$

Letting $\nu\left(\tilde{S}\right) = \max\left(\tilde{S}-K,0\right)$, *equation (10) becomes*

$$\begin{aligned} V\left[F\left(\tilde{S}\right)\right] &= E^P\left[m\left(\tilde{S}\right)\nu\left(\tilde{S}\right)\right] \\ &= \int \max\left(\tilde{S}-K,0\right)\left[\alpha_1\hat{f}_1\left(\tilde{S}\right)+\alpha_2\hat{f}_2\left(\tilde{S}\right)\right]d\tilde{S}, \end{aligned}$$

which, after substituting for equations (17) and (18) and simplifying the resulting equation, yields the option pricing formula

$$\begin{aligned} V\left[F\left(\tilde{S}\right)\right] &= \alpha_1 A e^{\frac{1}{2}\sigma^2_{\tilde{S},1}} N(d1) - \alpha_1\left(K-\delta_1\right)N(d2) \\ &\quad +\alpha_2\left(\delta_2-K\right)N(d3) - \alpha_2 A e^{\frac{1}{2}\sigma^2_{\tilde{S},2}} N(d4), \end{aligned} \quad (19)$$

where

$$\begin{aligned} A &= \frac{F-\alpha_1\delta_1-\alpha_2\delta_2}{\alpha_1 e^{\frac{1}{2}\sigma^2_{\tilde{S},1}} - \alpha_2 e^{\frac{1}{2}\sigma^2_{\tilde{S},2}}} \\ d1 &= \frac{\ln\left(A/\left(K-\delta_1\right)\right)}{\sigma_{\tilde{S},1}} + \sigma_{\tilde{S},1} \\ d2 &= d1 - \sigma_{\tilde{S},1} \\ d3 &= \frac{\ln\left(\left(\delta_2-K\right)/A\right)}{\sigma_{\tilde{S},2}} \\ d4 &= d3 - \sigma_{\tilde{S},2} \end{aligned}$$

Example 3 *(The mixture of tree lognormal distributions): Assume that* $\tilde{S}$ *has a mixture of three lognormal distributions. Then*

$$F = \int \tilde{S}\left[\alpha_1\hat{f}_1\left(\tilde{S}\right)+\alpha_2\hat{f}_2\left(\tilde{S}\right)+\alpha_3\hat{f}_3\left(\tilde{S}\right)\right]d\tilde{S}, \quad (20)$$

where

$$\begin{aligned}
m_1\left(\tilde{S}\right) &= \frac{\alpha_1}{\phi\sigma_{\tilde{S},1}}\exp\left[-\frac{1}{2\sigma_{\tilde{S},1}}\left(\ln\left(\tilde{S}\right)-\mu_{\tilde{S},1}-\gamma\rho_1\sigma_{\tilde{W}}\sigma_{\tilde{S},1}\right)^2\right] \\
m_2\left(\tilde{S}\right) &= \frac{\alpha_2}{\phi\sigma_{\tilde{S},2}}\exp\left[-\frac{1}{2\sigma_{\tilde{S},2}}\left(\ln\left(\tilde{S}\right)-\mu_{\tilde{S},2}-\gamma\rho_2\sigma_{\tilde{W}}\sigma_{\tilde{S},2}\right)^2\right] \\
m_3\left(\tilde{S}\right) &= \frac{\alpha_3}{\phi\sigma_{\tilde{S},3}}\exp\left[-\frac{1}{2\sigma_{\tilde{S},3}}\left(\ln\left(\tilde{S}\right)-\mu_{\tilde{S},3}-\gamma\rho_3\sigma_{\tilde{W}}\sigma_{\tilde{S},3}\right)^2\right]
\end{aligned}$$

$$\phi = \frac{\alpha_1}{\sigma_{\tilde{S},1}}e^{-\frac{1}{2\sigma_{\tilde{S},1}}\left(\ln\left(\tilde{S}\right)-\mu_{\tilde{S},1}\right)^2} + \frac{\alpha_2}{\sigma_{\tilde{S},2}}e^{-\frac{1}{2\sigma_{\tilde{S},2}}\left(\ln\left(\tilde{S}\right)-\mu_{\tilde{S},2}\right)^2} + \frac{\alpha_3}{\sigma_{\tilde{S},3}}e^{-\frac{1}{2\sigma_{\tilde{S},3}}\left(\ln\left(\tilde{S}\right)-\mu_{\tilde{S},3}\right)^2}$$

and for $i = 1, 2, 3$

$$\hat{f}_i\left(\tilde{S}\right) = \frac{1}{\sigma_{\tilde{S},i}\sqrt{2\pi}\tilde{S}}e^{-\frac{1}{2\sigma^2_{\tilde{S},i}}\left[\ln\left(\tilde{S}\right)-\left(\mu_{\tilde{S},i}+\gamma\rho_i\sigma_{\tilde{W}}\sigma_{\tilde{S},i}\right)\right]^2}.$$

In this case, the value of F *is given by*

$$F = \alpha_1 e^{\hat{\mu}+\frac{1}{2}\sigma^2_{\tilde{S},1}} + \alpha_2 e^{\hat{\mu}+\frac{1}{2}\sigma^2_{\tilde{S},2}} + \alpha_2 e^{\hat{\mu}+\frac{1}{2}\sigma^2_{\tilde{S},3}},$$

which after solving for $\hat{\mu}$

$$\hat{\mu} = \ln\left[\frac{F}{\alpha_1 e^{\frac{1}{2}\sigma^2_{\tilde{S},1}} + \alpha_2 e^{\frac{1}{2}\sigma^2_{\tilde{S},2}} + \alpha_2 e^{\frac{1}{2}\sigma^2_{\tilde{S},3}}}\right].$$

Letting $\nu\left(\tilde{S}\right) = \max\left(\tilde{S}-K, 0\right)$, *equation (10) becomes*

$$\begin{aligned}
V\left[F\left(\tilde{S}\right)\right] &= E^P\left[m\left(\tilde{S}\right)\nu\left(\tilde{S}\right)\right] \\
&= \int \max\left(\tilde{S}-K, 0\right)\left[\alpha_1\hat{f}_1\left(\tilde{S}\right)+\alpha_2\hat{f}_2\left(\tilde{S}\right)+\alpha_3\hat{f}_3\left(\tilde{S}\right)\right]d\tilde{S},
\end{aligned}$$

which, after substituting for the appropriate densities and for $\hat{\mu}$ *and simplifying, yields the option pricing formula*

$$V\left[F\left(\tilde{S}\right)\right] = \sum_i \alpha_i A e^{\frac{1}{2}\sigma^2_{\tilde{S},i}}N(d1) - \alpha_i K N(d2), \tag{21}$$

where

$$\begin{aligned}
d1 &= \frac{\ln(A/K)}{\sigma_{\tilde{S},i}} + \sigma_{\tilde{S},i} \\
d2 &= d1 - \sigma_{\tilde{S},i} \\
A &= \frac{F}{\alpha_1 e^{\frac{1}{2}\sigma^2_{\tilde{S},1}} + \alpha_2 e^{\frac{1}{2}\sigma^2_{\tilde{S},2}} + \alpha_2 e^{\frac{1}{2}\sigma^2_{\tilde{S},3}}}.
\end{aligned}$$

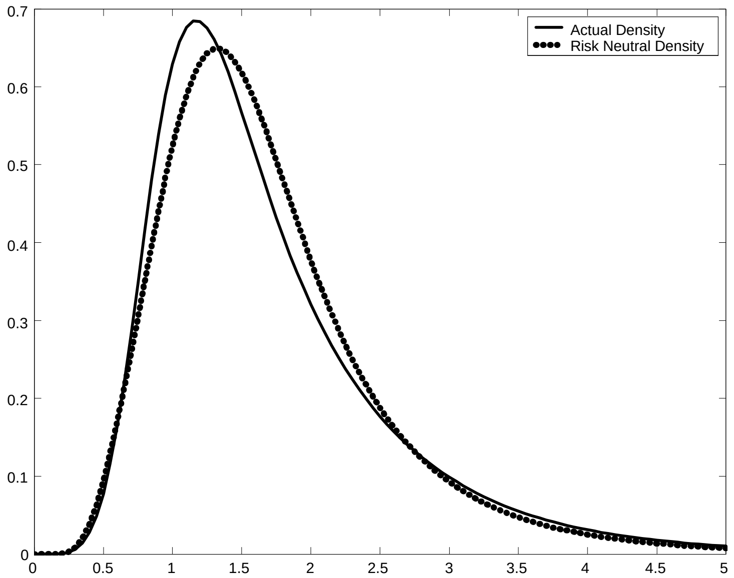

Figure 1: Actual density as a mixture of two lognormals with parameters $\alpha_1 = 0.75$, $\mu_1 = 0.5$, $\sigma_1 = 0.5$, $\alpha_2 = 0.25$, $\mu_2 = 0.2$, $\sigma_2 = 0.3$ and risk neutral density with parameter $\hat{\mu} = 0.44$ calculated according to equation (11). The forward price is set to the expected value of the actual distribution, $F = 1.72$.

3 Discussion and Conclusion

This paper extends the literature on the pricing of contingent claims with a mixture of distributions by allowing its components to have transformed-normal distributions. These components do not have to have the same density, and this provides the model with additional flexibility. The only requirement is that they must belong to the transformed-normal family. By introducing a restriction on the value of some distributional parameters, we show that it is possible to achieve a risk neutral valuation relationship.

It is interesting to note that when the underlying asset price distribution is given by a mixture of distributions, actual and risk neutral distributions may not have the same shape. A direct consequence of this feature is that the pricing kernel may not be monotonic. As a simple example, assume that the underlying asset distribution is given by a mixture of two lognormal densities. Assuming that equation (11) holds, figure (1) depicts the actual and the risk neutral densities. It is easy to see that the two densities cross each other three times, resulting in a non-monotonic pricing kernel. As in Brown and Jackwerth (2004), Figure (2) shows the pricing kernel as the ratio of the risk neutral and actual densities, which cross each other three times implying a non-monotonic pricing kernel.

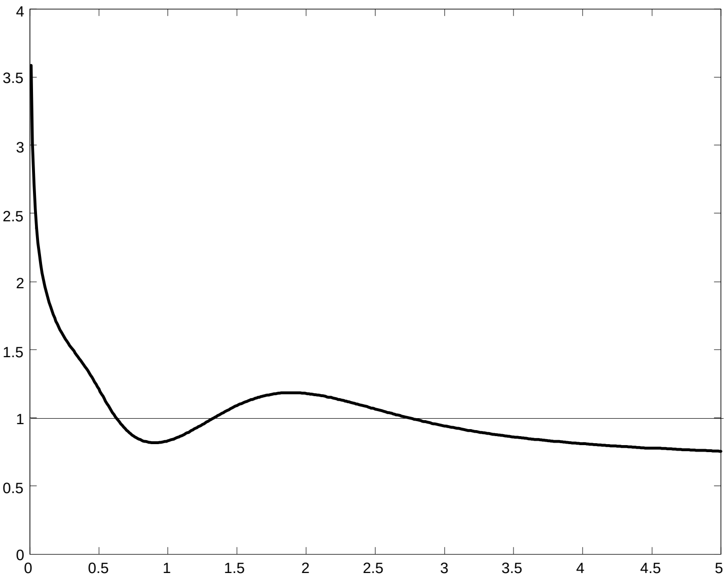

Figure 2: The pricing kernel as the ratio of the risk neutral and the actual densities.

Appendix

Proof. First we solve for the denominator of equation (2). Since (i) $h_{\tilde{W}}\left(\widetilde{W}\right)$ is normally distributed and (ii) $U'\left(\widetilde{W}\right)$ is lognormally distributed (see equation (7)), we can use the definition of the expected value of a lognormal random variable. Thus, $E^P\left[U'\left(\tilde{W}\right)\right] = \exp\left(\gamma\mu_{\tilde{W}} + \gamma^2 0.5\sigma^2_{\tilde{W}}\right)$.

Second we solve for the numerator of equation (2). This requires additional steps. Let $f\left(\tilde{W} \mid \tilde{S}\right)$ be the conditional density of $\tilde{W}$ given $\tilde{S}$ and $i = 1, 2, \ldots, I$. Then

$$\begin{aligned} E^P\left[U'\left(\tilde{W} \mid \tilde{S}\right)\right] &= \int_{\tilde{W}} U'\left(\tilde{W}\right) f\left(\tilde{W} \mid \tilde{S}\right) d\tilde{W} \\ &= \int_{\tilde{W}} U'\left(\tilde{W}\right) \sum_i \alpha_i f_i\left(\tilde{W} \mid \tilde{S}\right) d\tilde{W} \\ &= \sum_i \alpha_i E^P_{f_i}\left[U'\left(\tilde{W}\right) \mid \tilde{S}\right]. \end{aligned}$$

The last equality comes from

$$\begin{aligned} \int_{\tilde{W}} U'\left(\tilde{W}\right) \sum_i \alpha_i f_i\left(\tilde{W} \mid \tilde{S}\right) d\tilde{W} &= \alpha_1 E^P_{f_1}\left[U'\left(\tilde{W}\right) \mid \tilde{S}\right] + \cdots + \alpha_I E^P_{f_I}\left[U'\left(\tilde{W}\right) \mid \tilde{S}\right] \\ &= \sum_i \alpha_i E^P_{f_i}\left[U'\left(\tilde{W}\right) \mid \tilde{S}\right], \end{aligned}$$

where

$$f_i\left(\tilde{W} \mid \tilde{S}\right) = \frac{f_i\left(\tilde{W}, \tilde{S}\right)}{\sum_i \alpha_i f_i\left(\tilde{S}\right)},$$

$\sum_i \alpha_i = 1$, $\alpha_i \geq 0$ $\forall i$, $f_i\left(\tilde{W}, \tilde{S}\right)$ is i^{th} joint-density of $\tilde{W}$ and $\tilde{S}$, and $f_i\left(\tilde{S}\right)$ is the i^{th} density of $\tilde{S}$.

Thus, given $U'\left(\widetilde{W}\right) = \exp\left[\gamma h_{\tilde{W}}\left(\widetilde{W}\right)\right]$, we obtain

$$E^P\left[U'\left(\tilde{W} \mid \tilde{S}\right)\right] = \sum_i \frac{\frac{\alpha_i}{\sigma_i}\exp\left[A_i\right]}{\sum_i \frac{\alpha_i}{\sigma_i}\exp\left[-\frac{1}{2}\left(\frac{h_{\tilde{S}}\left(\tilde{S}\right)-\mu_{\tilde{S},i}}{\sigma_{\tilde{S},i}}\right)^2\right]},$$

where

$$A_i = \gamma\mu_{\tilde{W}} + \gamma\rho_i\sigma_{\tilde{W}}\left(\frac{h_{\tilde{S}}\left(\tilde{S}\right) - \mu_{\tilde{S},i}}{\sigma_{\tilde{S},i}}\right) - \frac{1}{2}\left(\frac{h_{\tilde{S}}\left(\tilde{S}\right) - \mu_{\tilde{S},i}}{\sigma_{\tilde{S},i}}\right)^2 + \left(1 - \rho_i^2\right)\frac{1}{2}\gamma^2\sigma_{\tilde{W}}^2$$

Finally, substituting these results into equation (2) yields equation (8). ■

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