GEOMETRIC METHOD FOR GLOBAL STABILITY AND REPULSION IN KOLMOGOROV SYSTEMS

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Abstract. A class of autonomous Kolmogorov systems that are dissipative and competitive with the origin as a repellor are considered when each nullcline surface is either concave or convex. Geometric method is developed by using the relative positions of the upper and lower planes of the nullcline surfaces for global asymptotic stability of an interior or a boundary equilibrium point. Criteria are also established for global repulsion of an interior or a boundary equilibrium point on the carrying simplex. This method and the theorems can be viewed as a natural extension of those results for Lotka-Volterra systems in the literature.

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1. Introduction

In this paper, we consider differential systems of the form

$$\dot{x}_i = x_i F_i(x), \quad i \in I_N = \{1, 2, \dots, N\}, \tag{1}$$

known as Kolmogorov systems. Since such systems typically model populations of species, genes, molecules, and so on, where each x_i denotes the population size and F_i the intrinsic growth rate of the ith species, the phase space for the study of (1) is restricted to the first orthant $\mathbb{R}^N_+$ or an invariant subset of $\mathbb{R}^N_+$. We assume that $F : \mathbb{R}^N_+ \to \mathbb{R}^N$ is at least C^1. Some particular class of examples include Lotka-Volterra systems where each F_i is an affine function,

$$\dot{x}_i = r_i x_i(1 - a_{i1}x_1 - \cdots - a_{iN}x_N), \quad i \in I_N, \tag{2}$$

Gompertz models where each F_i has the form $F_i(y) = r_i \ln \frac{1}{y}$,

$$\dot{x}_i = r_i x_i \ln \frac{1}{a_{i1}x_1 + \cdots + a_{iN}x_N}, \quad i \in I_N, \tag{3}$$

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Leslie/Gower (or Atkinson/Allen) models where each F_i has the form $F_i(y) = c_i\left(\frac{1+r_i}{r_i+y} - 1\right)$,

$$\dot{x}_i = c_i x_i \left(\frac{1+r_i}{r_i + a_{i1}x_1 + \cdots + a_{iN}x_N} - 1\right), \quad i \in I_N, \tag{4}$$

Ricker models where each F_i has the form $F_i(y) = c_i(e^{r_i(1-y)} - 1)$,

$$\dot{x}_i = c_i x_i(\exp[r_i(1 - a_{i1}x_1 - \cdots - a_{iN}x_N)] - 1), \quad i \in I_N. \tag{5}$$

There is an extensive literature in population ecology and dynamical systems on (1) and its various particular cases. To name a few, Hirsch [2, 3, 4] investigated the dynamics of competitive and cooperative systems and Zeeman [14] studied bifurcations in competitive Lotka-Volterra systems focusing on three-dimensional cases. Hirsch [4] showed that competitive dissipative systems with the origin as a repellor has a global attractor Σ on $\mathbb{R}^N_+ \setminus \{0\}$, where Σ is homeomorphic to the standard $(N-1)$-simplex Δ^{N-1} by radial projection. Zeeman [14] called Σ carrying simplex and used geometric analysis of nullclines of Lotka-Volterra systems to classify three-dimensional systems with stable nullclines into 33 classes, the dynamics of each class has a clear description on Σ. Similar to [14], Jiang, Niu and Zhu [11] did a complete classification of nullcline stable competitive three-dimensional Gompertz models. Jiang and Niu [10] further extended such classification to three-dimensional competitive systems with linearly determined nullclines including (2)–(5) and more. For a wider survey, see the above articles and the references cited therein.

Here we are concerned with the asymptotic behaviour of (1) when there is an equilibrium point $p \in \mathbb{R}^N_+$ that is globally attracting or repelling. For Lotka-Volterra systems, a criteron by Lyapunov function method is known for global asymptotic stability of a boundary or interior equilibrium point (see Theorem 3.2.1 in [12]). For competitive Lotka-Volterra systems, Zeeman and Zeeman [15] developed the split Lyapunov function method and provided sufficient conditions for an interior equilibrium point to be a global attractor or a global repellor. Hou and Baigent [5, 1] further developed the split Lyapunov function method and extended the above results for global attraction or repulsion to a boundary as well as interior equilibrium point of Lotka-Volterra systems that may not be competitive. In [6], the authors applied the Lyapunov function method and the split Lyapunov function method to dissipative systems (1) with both 0 and ∞ as repellors in $\mathbb{R}^N_+$, and obtained criteria for global asymptotic stability or global repulsion of an equilibrium point. These results can be viewed as further extension of [15, 5, 1] from Lotka-Volterra systems to Kolmogorov systems (1). Yu, Wang and Lu [13] obtained sufficient conditions for global stability of three-dimensional competitive Gompertz models. For Lotka-Volterra systems, there are also results for global repulsion or attraction by methods that are not using Lyapunov functions. For example, Hou used geometric method for global attraction [7, 8] and global repulsion [9].

In this paper, we are going to deal with a class of competitive dissipative systems (1) that has a carrying simplex Σ and each nullcline is a concave or convex surface. By using geometric analysis of such nullclines, we provide sufficient conditions for an equilibrium point $p \in \mathbb{R}^N_+ \setminus \{0\}$ to be globally asymptotically stable or for p to be globally repelling.

The rest of the paper is organised as follows: 2. System description and notation. 3. Geometric method for global stability. 4. Proof of Theorem 3.1. 5. Geometric method for global repulsion. 6. Proof of Theorem 5.1. 7. Conclusion.

2. System description and notation

For convenience, we rewrite system (1) as

$$\dot{x} = f(x) \equiv D[x]F(x), \quad x \in \mathbb{R}^N_+, \tag{6}$$

where $D[x] = \mathrm{diag}[x_1, \dots, x_N]$ and $F \in C^1(\mathbb{R}^N_+, \mathbb{R}^N)$. Let $\mathrm{int}\mathbb{R}^N_+$ denote the interior of $\mathbb{R}^N_+$. For any $x, y \in \mathbb{R}^N$, we write $x \ll y$ or $y \gg x$ if $y - x \in \mathrm{int}\mathbb{R}^N_+$, $x \leq y$ or $y \geq x$ if $y - x \in \mathbb{R}^N_+$, and $x < y$ or $y > x$ if $x \leq y$ but $x \neq y$. We view each $x \in \mathbb{R}^N$ as a column vector and use x^T as the transpose of x. With a slight abuse of notation, we shall use 0 for scalar and vector zero as well as the origin in $\mathbb{R}^N$.

Throughout the paper we assume that (6) meets the following assumptions:

- (A1) $F(0) \gg 0$ so that the origin 0 is a repellor.
- (A2) The system is dissipative: there is a compact invariant set that attracts uniformly each compact set of initial points.
- (A3) The system is competitive: $\frac{\partial F_i}{\partial x_j} \leq 0$ for all $i, j \in I_N$ with $i \neq j$.
- (A4) $\frac{\partial F_i}{\partial x_j}(p) < 0$ for every fixed point $p \in \mathbb{R}^N_+ \setminus \{0\}$ and all $i, j \in I_N$.

Then the basin of repulsion of 0 in $\mathbb{R}^N_+$, $Br(0) = \{x \in \mathbb{R}^N_+ : \alpha(x) = \{0\}\}$, is a bounded open set of $\mathbb{R}^N_+$ and $\Sigma = \partial Br(0) \setminus Br(0)$ is known as the carrying simplex. The theorem below (see Theorem 1.7 in [4] or Theorem 2.1 in [14]) describes the dynamics of (6) in terms of Σ.

Theorem 2.1. *Under the assumptions (A1)–(A4), every trajectory in $\mathbb{R}^N_+ \setminus \{0\}$ is asymptotic to one in Σ, and Σ is a Lipschitz submanifold homeomorphic to the unit simplex in $\mathbb{R}^N_+$ by radial projection.*

Now we explain some concepts that will be used later. Let $G \in C^1(\mathbb{R}^N_+, \mathbb{R})$ such that, for some α in the range of G,

$$\Gamma = \{x \in \mathbb{R}^N_+ : G(x) = \alpha\} \tag{7}$$

is a connected $(N-1)$-dimensional surface restricted to $\mathbb{R}^N_+$. Suppose that $\mathbb{R}^N_+$ is divided into three mutually exclusive connected subsets Γ^-, Γ and Γ^+ with $0 \in \Gamma^-$. Then a point

$x \in \mathbb{R}^N_+$ is said to be *below* (*on* or *above*) Γ if $x \in \Gamma^-$ ($x \in \Gamma$ or $x \in \Gamma^+$); a nonempty set $S \subset \mathbb{R}^N_+$ is said to be *below* (*on* or *above*) Γ if $S \subset \Gamma^- \cup \Gamma$ ($S \subset \Gamma$ or $S \subset \Gamma^+ \cup \Gamma$); $S \subset \mathbb{R}^N_+$ is said to be *strictly below* (*above*) Γ if $S \subset \Gamma^-$ ($S \subset \Gamma^+$).

The function G is said to be *convex* if $G(sx + (1-s)y) \geq sG(x) + (1-s)G(y)$ holds for any two points x, y in its domain and all $s \in [0, 1]$. For a surface Γ with the division of $\mathbb{R}^N_+$ into Γ^-, Γ and Γ^+, Γ is said to be *convex* (*concave*) if for any distinct points $x, y \in \Gamma$, the line segment $\overline{xy} = \{tx + (1-t)y : 0 \leq t \leq 1\}$ is contained in $\Gamma^- \cup \Gamma$ ($\Gamma^+ \cup \Gamma$). Recall that a nonempty set $S \subset \mathbb{R}^N_+$ is called convex if $\overline{xy} \subset S$ for all $x, y \in S$. From these concepts we obtain some observations summarised in the following proposition.

Proposition 2.2. *Assume that Γ defined by (7) divides $\mathbb{R}^N_+$ into Γ^-, Γ and Γ^+ as described above. Then the following statements are true.*

(i) *If Γ is a plane in $\mathbb{R}^N_+ \setminus \{0\}$ then it is both convex and concave.*

(ii) *The surface Γ is convex if and only if the set $\Gamma^- \cup \Gamma$ is convex; Γ if concave if and only if $\Gamma \cup \Gamma^+$ is convex.*

(iii) *If the function G is convex with $G(0) = \max_{x \in \mathbb{R}^N_+} G(x)$, then Γ is also convex for any $\alpha < G(0)$ in the range of G.*

(iv) *If the function $-G$ is convex with $G(0) = \min_{x \in \mathbb{R}^N_+} G(x)$, then Γ is also convex for any $\alpha > G(0)$ in the range of G.*

(v) *If the function G is convex with $G(0) = \min_{x \in \mathbb{R}^N_+} G(x)$, then Γ is concave for any $\alpha > G(0)$ in the range of G.*

(vi) *If the function $-G$ is convex with $G(0) = \max_{x \in \mathbb{R}^N_+} G(x)$, then Γ is concave for any $\alpha < G(0)$ in the range of G.*

The proof of Proposition 2.2 can be found in the Appendix at the end of this paper.

For any point $u \in \Gamma$, the tangent plane of Γ at u is

$$T_u(\Gamma) = \{x \in \mathbb{R}^N_+ : \nabla G(u)(x-u) = 0\} \tag{8}$$

if $\nabla G(u) \neq 0$, where $\nabla G(u) = (\frac{\partial G}{\partial u_1}, \ldots, \frac{\partial G}{\partial u_N})$ is viewed as a row vector. Denote the positive half x_i-axis by J_i for all $i \in I_N$. Next, we assume that Γ intersects at least one of the half axes J_i. If $\Gamma \cap J_i \neq \emptyset$, we assume that R_i is the unique intersection point, i.e. $\Gamma \cap J_i = \{R_i\}$. If $\Gamma \cap J_i = \emptyset$, we say that the point R_i does not exist. Now let $L(\Gamma)$ be the $(N-1)$-dimensional plane in $\mathbb{R}^N_+$ determined by these intersection points: If R_i exists then $R_i \in L(\Gamma)$, if R_i does not exist then $L(\Gamma)$ is parallel to the half axis J_i. Then the relative positions of $T_u(\Gamma), \Gamma$ and $L(\Gamma)$ are clear from the proposition below when Γ is convex or concave.

Proposition 2.3. *(a) Suppose Γ given by (7) is convex. Then Γ is above $L(\Gamma)$ but below $T_u(\Gamma)$ for any $u \in \Gamma$. (b) Suppose Γ is concave and, if $\Gamma \cap J_j = \emptyset$ for some $j \in I_N$, for* *any point* $w \in \Gamma$*, the half line* $L_{(w)j}$ *passing through* w *and parallel to* J_j *is contained in* $\Gamma \cup \Gamma^+$*. Then* Γ *is below* $L(\Gamma)$ *and, for any* $u \in \Gamma$ *with* $\nabla G(u)u \neq 0$*,* Γ *is above* $T_u(\Gamma)$*.*

The proof of Proposition 2.3 is also left to the Appendix.

For each $i \in I_N$, the ith nullcline surface of (6) is defined by

$$\Gamma_i = \{x \in \mathbb{R}^N_+ : F_i(x) = 0\}. \tag{9}$$

If $\mathbb{R}^N_+$ is divided into three mutually exclusive connected subsets Γ_i^-, Γ_i, Γ_i^+ then the assumptions (A1)–(A3) imply that $\dot{x}_i > 0$ for $x \in \Gamma_i^-$ and $\dot{x}_i < 0$ for $x \in \Gamma_i^+$. The ith coordinate plane is denoted by

$$\pi_i = \{x \in \mathbb{R}^N_+ : x_i = 0\}. \tag{10}$$

For any $u, v \in \mathbb{R}^N_+$ with $u \leq v$, $i \in I_N$, and $I \subset I_N$, define

$$[u, v] = \{x \in \mathbb{R}^N_+ : u \leq x \leq v\}, \tag{11}$$

$$\mathbb{R}^N_+(u) = \{x \in \mathbb{R}^N_+ : x \geq u\}, \tag{12}$$

$$\pi_i(u) = \{x \in \mathbb{R}^N_+(u) : x_i = u_i\}, \tag{13}$$

$$S(u, v_i) = \{x \in \mathbb{R}^N_+(u) : x_i \geq v_i\}, \tag{14}$$

$$S^0(u, v_i) = \{x \in \mathbb{R}^N_+(u) : x_i > v_i\}, \tag{15}$$

$$C_I^0 = \{x \in \mathbb{R}^N_+ : \forall i \in I, x_i = 0; \forall j \in I_N \setminus I, x_j > 0\}, \tag{16}$$

$$\mathbb{R}_I = \{x \in \mathbb{R}^N_+ : \forall j \in I_N \setminus I, x_j > 0\}. \tag{17}$$

Then $\mathbb{R}^N_+(0) = \mathbb{R}^N_+$, $\pi_i(0) = \pi_i$, $C^0_{I_N} = \{0\}$, $C^0_\emptyset = \mathrm{int}\mathbb{R}^N_+ = \mathbb{R}_\emptyset$ and $\mathbb{R}_{I_N} = \mathbb{R}^N_+$. For any nonempty set $A \subset \mathbb{R}^N$ and $\varepsilon > 0$, the ε-neighbourhood of A is denoted by

$$\mathcal{B}(A, \varepsilon) = \{x \in \mathbb{R}^N : \|x - a\| < \varepsilon \text{ for some } a \in A\}. \tag{18}$$

Suppose $p \in C_I^0$ with $I \neq I_N$ is an equilibrium point. Then $p \in \Sigma$. We say that p is *globally attracting* if $\lim_{t\to+\infty} x(x_0, t) = p$ for all $x_0 \in \mathbb{R}_I$; p is *globally repelling* if for all $x_0 \in (\Sigma \setminus \{p\}) \cap \mathbb{R}_I$, we have $\omega(x_0) \subset (\cup_{j \in I_N \setminus I} \pi_j) \cap \Sigma$ and $\alpha(x_0) = \{p\}$; p is called *globally asymptotically stable* if p is globally attracting and p is locally asymptotically stable in $\mathbb{R}^N_+$. Note that since Σ is a global attractor of (6) in $\mathbb{R}^N_+ \setminus \{0\}$, if p is globally repelling, p is essentially repelling on $\Sigma \cap \mathbb{R}_I$. So we also say that p is globally repelling on Σ.

3. Geometric method for global stability

In this section, we assume that $p \in \mathbb{R}^N_+ \setminus \{0\}$ is a nontrivial equilibrium point of (6) with support $J = \{j \in I_N : p_j > 0\}$, i.e. $p \in C_I^0$ for $I = I_N \setminus J \neq I_N$. Then p is an interior equilibrium if $J = I_N$ or on the boundary $\partial\mathbb{R}^N_+$ if J is a proper subset of I_N. We call p *saturated* if $F_i(p) \leq 0$ for all $i \in I_N$. Then, from the fact that $F_i(p)$ is an eigenvalue of the

Jacobian matrix $Df(p)$ if $i \in I_N \setminus J$, it follows that a necessary condition for p to be stable is that p must be saturated.

Now assume that p is a saturated equilibrium point. For each $i \in I_N$, if $F_i(p) = 0$ then p is on the ith nullcline surface

$$\Gamma_i = \{x \in \mathbb{R}^N_+ : F_i(x) = 0\} \tag{19}$$

and Γ_i at p has a tangent plane

$$L_i(p) = \{x \in \mathbb{R}^N_+ : \nabla F_i(p)(x - p) = 0\} \tag{20}$$

as $\nabla F_i(p) \neq 0$ by (A4). We assume that on each positive half x_i-axis, J_i, (6) has a unique equilibrium point R_i, i.e. $\Gamma_i \cap J_i = \{R_i\}$. Assume also that each Γ_i has at most one intersection point R_{ij} with J_j for each $j \in I_N$ ($R_{ii} = R_i$). Let $\tilde{L}_i$ be the plane in $\mathbb{R}^N_+$ determined by the intersection points R_{ij} of Γ_i with the coordinate axes: If Γ_i intersects J_j at a point R_{ij} then $R_{ij} \in \tilde{L}_i$; if R_{ij} does not exist then $\tilde{L}_i$ is parallel to J_j. Note that (A4) implies that $\nabla F_i(p) \ll 0$ and $p > 0$ so $\nabla F_i(p)p < 0$. If $\Gamma_i \cap J_j = \emptyset$, then, for any point $w \in \Gamma_i$, since $F_i(w) = 0$ and $\frac{\partial F_i}{\partial x_j} \leq 0$ by (A3), $F_i(x)$ is nonincreasing on $L_{(w)j}$ so $F_i(x) \leq F_i(w) = 0$ for all $x \in L_{(w)j}$. Thus, $L_{(w)j} \subset \Gamma_i \cup \Gamma_i^+$. Then, if Γ_i is convex or concave, from Proposition 2.3 we see that Γ_i is between $L_i(p)$ and $\tilde{L}_i$: if Γ_i is convex then it is below $L_i(p)$ but above $\tilde{L}_i$; if Γ_i is concave then it is above $L_i(p)$ but below $\tilde{L}_i$.

Suppose $F_i(p) < 0$ for some $i \in I_N \setminus J$. Then p is above Γ_i so the plane $L_i(p)$ is not tangent to Γ_i. If Γ_i is concave then it is below $\tilde{L}_i$. However, if Γ_i is convex, we may further assume that F_i is a convex function with $F_i(0) = \max_{x \in \mathbb{R}^N_+} F_i(x)$ so that, by Proposition 2.2 (iii), both Γ_i and the surface $\{x \in \mathbb{R}^N_+ : F_i(x) = F_i(p)\}$ are convex surfaces and the former is below the latter. Note that $L_i(p)$ is tangent to $\{x \in \mathbb{R}^N_+ : F_i(x) = F_i(p)\}$ at p so $\{x \in \mathbb{R}^N_+ : F_i(x) = F_i(p)\}$ is below $L_i(p)$. Then Γ_i is also below $L_i(p)$. Hence, we can always find a plane above Γ_i if p is above Γ_i.

Now for each $k \in I_N$, define an upper plane L^u_k by $L^u_k = \tilde{L}_k$ if Γ_k is concave or $L^u_k = L_k(p)$ if Γ_k is convex and, for each $j \in J$, define a lower plane L^l_j by $L^l_j = L_j(p)$ if Γ_j is concave or $L^l_j = \tilde{L}_j$ if Γ_j if convex. Then each convex or concave Γ_i is below L^u_i for all $i \in I_N$ but is above L^l_i for all $i \in J$.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $N \times N$ matrices with real entries such that

$$L^u_i = \{x \in \mathbb{R}^N_+ : (Ax)_i = 1\}, \quad i \in I_N, \tag{21}$$

$$L^l_i = \{x \in \mathbb{R}^N_+ : (Bx)_i = 1\}, \quad i \in J. \tag{22}$$

Then the entries of A and B can be determined as follows. First, suppose Γ_i is concave, so we have $L^u_i = \tilde{L}_i$ and $L^l_i = L_i(p)$. If Γ_i intersects the half axis J_j at the point R_{ij} with $r_{ij} > 0$ as its jth component, then $a_{ij} r_{ij} = 1$ so $a_{ij} = \frac{1}{r_{ij}}$; if Γ_i does not intersect J_j then

$a_{ij} = 0$. So a_{ij} is defined by

$$a_{ij} = \begin{cases} 0 & \text{if } \Gamma_i \text{ does not intersect } J_j, \\ \frac{1}{r_{ij}} & \text{if the } j\text{th component of } R_{ij} \text{ is } r_{ij}. \end{cases} \tag{23}$$

Since $L_i(p)$ has the equation $\nabla F_i(p)(x-p) = 0$, we have $\nabla F_i(p)x = \nabla F_i(p)p$ so $(Bx)_i = 1$ with

$$(Bx)_i = (\nabla F_i(p)p)^{-1}\nabla F_i(p)x, \tag{24}$$

i.e. $(\nabla F_i(p)p)^{-1}\nabla F_i(p)$ is taken to be the ith row of B. If Γ_i is convex then $L_i^u = L_i(p)$ and $L_i^l = \tilde{L}_i$. In this case, we have

$$(Ax)_i = (\nabla F_i(p)p)^{-1}\nabla F_i(p)x, \tag{25}$$

i.e. $(\nabla F_i(p)p)^{-1}\nabla F_i(p)$ is taken to be the ith row of A, and b_{ij} is given by

$$b_{ij} = \begin{cases} 0 & \text{if } \Gamma_i \text{ does not intersect } J_j, \\ \frac{1}{r_{ij}} & \text{if the } j\text{th component of } R_{ij} \text{ is } r_{ij}. \end{cases} \tag{26}$$

Note from (A4) that $\frac{\partial F_i}{\partial x_j}(p) < 0$ for all $i, j \in I_N$ so that $\nabla F_i(p)p = \sum_{j=1}^N \frac{\partial F_i}{\partial x_j}(p)p_j < 0$. Thus, $(\nabla F_i(p)p)^{-1}\nabla F_i(p) \gg 0$. Then, from (A1)–(A4), (23)–(26) and the assumptions we see that

$$\forall i, j \in I_N, a_{ii} > 0 \text{ and } a_{ij} \geq 0. \tag{27}$$

Let

$$Y = \left(\frac{1}{a_{11}}, \dots, \frac{1}{a_{NN}}\right)^T. \tag{28}$$

For any subset $K \subset I_N$ and $u \in \mathbb{R}^N$, the point $u^K \in \mathbb{R}^N$ is defined by

$$u_i^K = \begin{cases} u_i & \text{if } i \in K, \\ 0 & \text{if } i \in I_N \setminus K. \end{cases} \tag{29}$$

We are now in a position to state the main result of this section in geometric terms.

Theorem 3.1. *Assume that the following conditions hold.*

(a) *System (6) has a saturated equilibrium point $p \in \mathbb{R}_+^N \setminus \{0\}$ with support $J \subset I_N$.*

(b) *For each $i \in I_N$, the nullcline surface Γ_i is either concave or convex, and if $F_i(p) < 0$ with Γ_i convex, the function F_i is also convex with $F_i(0) = \max_{x \in \mathbb{R}_+^N} F_i(x)$.*

(c) *For each $i \in J$, either the point $Y^{I_N \setminus \{i\}}$ is below L_i^l or the set $L_i^l \cap [0, Y^{I_N \setminus \{i\}}]$ is strictly above L_j^u for all $j \in I_N \setminus \{i\}$.*

Then p is globally attracting. If, in addition, all eigenvalues of the Jacobian matrix $Df(p)$ have negative real parts, then p is globally asymptotically stable.

Remark 1. If p is a boundary equilibrium point with $F_i(p) < 0$ for some $i \in I_N \setminus J$ and Γ_i is convex, p is above Γ_i. As $F_i(0) > 0$ and F_i is continuous, there is a number $s_i \in (0,1)$ such that $F_i(s_ip) = 0$. Since $s_ip \in \Gamma_i$, $L_i(s_ip)$ is a tangent plane of Γ_i at s_ip. By the convexity of Γ_i, Γ_i is below $L_i(s_ip)$. Thus, as an alternative to the part of the condition (b) in Theorem 3.1, instead of requiring F_i to be a convex function, we may define $L_i^u = L_i(s_ip)$ with

$$(Ax)_i = (\nabla F_i(s_ip)s_ip)^{-1}\nabla F_i(s_ip)x$$

and require the inequalities in (27) hold.

Remark 2. Since L_i^l is described by the equation $(Bx)_i = 1$ and L_j^u by the equation $(Ax)_j = 1$, condition (c) in Theorem 3.1 is ensured by the following inequalities: For each $i \in J$, either

$$(BY^{I_N\setminus\{i\}})_i < 1 \tag{30}$$

or

$$\forall j \in I_N \setminus \{i\}, \max\left\{0, \frac{b_{ij}}{a_{jj}}(1 - (AY^{I_N\setminus\{i,j\}})_j)\right\} < 1 - (BY^{I_N\setminus\{i,j\}})_i. \tag{31}$$

Indeed, it is obvious that (30) holds if and only if $Y^{I_N\setminus\{i\}}$ is below L_i^l. Since L_i^u and L_i^l have equations

$$\begin{aligned}(Ax)_i \equiv a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{iN}x_N &= 1,\\ (Bx)_i \equiv b_{i1}x_1 + b_{i2}x_2 + \cdots + b_{iN}x_N &= 1\end{aligned}$$

respectively and L_i^l is below L_i^u, we must have

$$\forall i, j \in I_N, a_{ij} \leq b_{ij}.$$

If $Y^{I_N\setminus\{i\}}$ is not below L_i^l, then $Y^{I_N\setminus\{i\}}$ is on or above L_i^l so $(BY^{I_N\setminus\{i\}})_i \geq 1$. If (31) holds, then

$$(AY^{I_N\setminus\{i,j\}})_i \leq (BY^{I_N\setminus\{i,j\}})_i < 1$$

so $Y^{I_N\setminus\{i,j\}}$ is below L_i^l for all $j \in I_N \setminus \{i\}$. Thus, the line segment $[Y^{I_N\setminus\{i,j\}}, Y^{I_N\setminus\{i\}}]$ and the plane L_i^l have a unique intersection point Q_j with $\frac{1}{b_{ij}}(1 - (BY^{I_N\setminus\{i,j\}})_i)$ as its jth component. From (31) we obtain

$$\frac{1}{a_{jj}}(1 - (AY^{I_N\setminus\{i,j\}})_j) < \frac{1}{b_{ij}}(1 - (BY^{I_N\setminus\{i,j\}})_i).$$

If the expression on the left-hand side of the above inequality is negative, then $[Y^{I_N\setminus\{i,j\}}, Y^{I_N\setminus\{i\}}]$ is strictly above L_j^u so Q_j is above L_j^u. Otherwise, since the expression on the left-hand side of the above inequality is the jth component of the intersection point of the plane L_j^u with the line segment $[Y^{I_N\setminus\{i,j\}}, Y^{I_N\setminus\{i\}}]$, the above inequality shows that Q_j is above L_j^u for all $j \in I_N \setminus \{i\}$. In particular, $Y^{I_N\setminus\{i\}}$ is above L_j^u for every $j \in I_N \setminus \{i\}$. For each $k \in I_N \setminus \{i, j\}$,

$$(AY^{I_N\setminus\{i,j\}})_k \geq a_{kk}Y_k = 1$$

so $Y^{I_N\setminus\{i,j\}}$ is on or above L_k^u. Thus, $tY^{I_N\setminus\{i,j\}}+(1-t)Y^{I_N\setminus\{i\}}$ is above L_k^u for all $t\in[0,1)$ and $k\in I_N\setminus\{i,j\}$. Therefore, for all $j,k\in I_N\setminus\{i\}$, Q_j is above L_k^u. Since $[0,Y^{I_N\setminus\{i\}}]\cap L_i^l$ is the convex hull determined by Q_j for all $j\in I_N\setminus\{i\}$, $[0,Y^{I_N\setminus\{i\}}]\cap L_i^l$ is strictly above L_k^u for all $k\in I_N\setminus\{i\}$.

For a particular class of systems (6) when each Γ_i is a plane, condition (b) in Theorem 3.1 is met as Γ_i is both concave and convex. Since $\Gamma_i, \tilde{L}_i$ (and $L_i(p)$ if $p\in\Gamma_i$) will coincide, in condition (c) we shall use Γ_i instead of L_i^u and L_i^l.

Corollary 3.2. *Assume that the following conditions hold.*

(a) *System (6) has a saturated equilibrium point $p\in\mathbb{R}_+^N\setminus\{0\}$ with support $J\subset I_N$.*

(b) *For each $i\in I_N$, the nullcline surface Γ_i is a plane.*

(c) *For each $i\in J$, either the point $Y^{I_N\setminus\{i\}}$ is below Γ_i or the set $\Gamma_i\cap[0,Y^{I_N\setminus\{i\}}]$ is strictly above Γ_j for all $j\in I_N\setminus\{i\}$.*

Then p is globally attracting. If, in addition, all eigenvalues of the Jacobian matrix $Df(p)$ have negative real parts, then p is globally asymptotically stable.

Remark 3. When each Γ_i is a plane in $\mathbb{R}_+^N$ with equation $(Ax)_i=1$, from Remark 2 we see that condition (c) in Corollary 3.2 is guaranteed by the following inequalities: For each $i\in J$, either

$$(AY^{I_N\setminus\{i\}})_i<1 \tag{32}$$

or

$$\forall j\in I_N\setminus\{i\},\ \max\left\{0,\frac{a_{ij}}{a_{jj}}(1-(AY^{I_N\setminus\{i,j\}})_j)\right\}<1-(AY^{I_N\setminus\{i,j\}})_i. \tag{33}$$

Example 3.3. Consider the Ricker model (5) with $N=3$, $r_i>0, c_i>0$ and

$$A=\begin{pmatrix} 1 & \frac14 & \frac12\\ \frac12 & 1 & \frac14\\ \frac14 & \frac12 & 1\end{pmatrix}.$$

It has an interior equilibrium $p=(\frac47,\frac47,\frac47)^T$ and $Y=(1,1,1)^T$. Since $(AY^{\{2,3\}})_1=\frac14+\frac12=\frac34<1$, $(AY^{\{1,3\}})_2=\frac14+\frac12=\frac34<1$ and $(AY^{\{1,2\}})_3=\frac14+\frac12=\frac34<1$, (32) holds. Then, from Corollary 3.2 and Remark 3, p is globally attracting. In addition, if each eigenvalue of $Df(p)$ has a negative real part, then p is globally asymptotically stable.

Note that the conditions (32) and (33) can be applied to any one of the systems (2)–(5). In particular, for Lotka-Volterra system (2), these are consistent with the conditions given in [7].

Example 3.4. Consider the system

$$\dot{x}_i=x_i[1-a_1\ln(1+x_i)-a_2\ln(1+x_{i+1})-\cdots-a_N\ln(1+x_{i+N-1})]=x_iF_i(x),$$

for $i \in I_N$ and $x \in \mathbb{R}^N_+$, where the a_j are positive constants and $x_{j+N} = x_j$. The system has an interior equilibrium point $p = p_0(1, \ldots, 1)^T$ with

$$p_0 = e^{1/(a_1+\cdots+a_N)} - 1.$$

Since $\ln(1 + su + (1 - s)v) \geq s \ln(1 + u) + (1 - s) \ln(1 + v)$ for $u \geq 0$, $v \geq 0$ and $0 \leq s \leq 1$, each F_i satisfies

$$\forall x, y \in \mathbb{R}^N_+, \quad F_i(sx + (1 - s)y) \leq sF_i(x) + (1 - s)F_i(y).$$

This shows that $\Gamma_i = \{x \in \mathbb{R}^N_+ : F_i(x) = 0\}$ is concave. Then

$$\frac{\partial F}{\partial x}(p) = -\frac{1}{1 + p_0} \begin{pmatrix} a_1 & a_2 & \cdots & a_N \\ a_N & a_1 & \cdots & a_{N-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix}.$$

By (24),

$$B = \left(D\left[\frac{\partial F}{\partial x}(p)p\right]\right)^{-1} \frac{\partial F}{\partial x}(p) = p_0^{-1}\left(\sum_{i=1}^N a_i\right)^{-1} \begin{pmatrix} a_1 & a_2 & \cdots & a_N \\ a_N & a_1 & \cdots & a_{N-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix}.$$

The intersection points of Γ_1 with the coordinate axes are

$$(e^{1/a_1} - 1, 0, \ldots, 0)^T, (0, e^{1/a_2} - 1, \ldots, 0)^T, \ldots, (0, \ldots, 0, e^{1/a_N} - 1)^T.$$

Thus, from (23),

$$A = \begin{pmatrix} a_1' & a_2' & \cdots & a_N' \\ a_N' & a_1' & \cdots & a_{N-1}' \\ \cdots & \cdots & \cdots & \cdots \\ a_2' & a_3' & \cdots & a_1' \end{pmatrix}, \quad a_i' = \frac{1}{e^{1/a_i} - 1} > 0.$$

Clearly, $a_{ij} > 0$ for all $i, j \in I_N$ so (27) holds. By (28), $Y = (e^{1/a_1} - 1)(1, \ldots, 1)^T$. Note that A and B are both circulant matrices. Then (30) becomes

$$\left(\sum_{i=2}^N a_i\right)(e^{1/a_1} - 1) < \left(\sum_{i=1}^N a_i\right)(e^{1/(a_1+\cdots+a_N)} - 1) \tag{34}$$

and (31) becomes

$$\max\left\{0, p_0^{-1}\left(\sum_{i=1}^N a_i\right)^{-1} a_j(e^{1/a_1} - 1)\left[1 - (e^{1/a_1} - 1) \sum_{k \in I_N \setminus \{1,j\}} a_k'\right]\right\}$$

$$< \quad 1 - (e^{1/a_1} - 1)p_0^{-1}\left(\sum_{i=1}^N a_i\right)^{-1} \sum_{k \in I_N \setminus \{1,j\}} a_k, \quad \forall j \in I_N \setminus \{1\}. \tag{35}$$

By Remark 2 and Theorem 3.1, if either (34) or (35) holds, then p is globally attracting. We observe that for fixed $a_1 > 0$, (34) holds when $a_2, \ldots, a_N$ are small enough.

4. Proof of Theorem 3.1

The proof of Theorem 3.1 is divided into three steps.

Proof of Theorem 3.1. **Step 1.** We first show that $\omega(x_0) \subset [0, Y]$ for all $x_0 \in \mathbb{R}^N_+$. For each $i \in I_N$ and every $x \in \mathbb{R}^N_+$ with $x_i > Y_i$, x is above L^u_i. Since L^u_i is above Γ_i, we have $x \in \Gamma^+_i$ so $\dot{x}_i = x_i F_i(x) < 0$ due to $F_i(0) > 0$ by (A1). Thus, for any $\delta > 0$, the flow of the system will be transversal to the plane $x_i = Y_i + \delta$ downwardly, so $\omega(x_0)$ is strictly below the plane $x_i = Y_i + \delta$ for all $x_0 \in \mathbb{R}^N_+$. Therefore, $\omega(x_0) \subset [0, Y]$ for all $x_0 \in \mathbb{R}^N_+$.

Step 2. Assume that $\omega(x_0) \subset [u, v] \subset [0, Y]$ for all $x_0 \in \mathbb{R}_I$. If for v' with $v'_i = u_i$ for some $i \in J$ and $v'_j = v_j$ for all $j \in I_N \setminus \{i\}$, either v' is below L^l_i or $[u, v'] \cap L^l_i$ is strictly above L^u_j for all $j \in I_N \setminus \{i\}$, we show the existence of $\delta > 0$ such that $\omega(x_0) \subset [\tilde{u}, v]$ for all $x_0 \in \mathbb{R}_I$, where $\tilde{u}_i = u_i + \delta \le v_i$ and $\tilde{u}_j = u_j$ for all $j \in I_N \setminus \{i\}$.

If v' is below L^l_i, then $[u, v']$ is strictly below L^l_i. By the compactness of $[u, v']$, there is a $\delta > 0$ such that the set $\mathcal{B}([u, v'], 2\delta) \cap \mathbb{R}^N_+$ is strictly below L^l_i. As L^l_i is below Γ_i, $\mathcal{B}([u, v'], 2\delta) \cap \mathbb{R}^N_+$ is strictly below Γ_i so any solution in $\mathcal{B}([u, v'], 2\delta) \cap \mathbb{R}^N_+ \setminus \pi_i$ satisfies $x'_i(t) = x_i(t) F_i(x(t)) > 0$, i.e. $x_i(t) \uparrow$. We show that, for $\tilde{u}$ with $\tilde{u}_i = u_i + \delta$ and $\tilde{u}_j = u_j$ for all $j \in I_N \setminus \{i\}$,

$$\omega(x_0) \subset [\tilde{u}, v], \quad \forall x_0 \in \mathbb{R}_I. \tag{36}$$

Suppose (36) is not true so $\omega(x_0) \cap [u, v''] \neq \emptyset$ for some $x_0 \in \mathbb{R}_I$, where $v''_i = v'_i + \delta = u_i + \delta$ and $v''_j = v_j$ for all $j \in I_N \setminus \{i\}$. As $\omega(x_0) \subset [u, v]$ and $\omega(x_0)$ is compact, there is a $y^0 \in \omega(x_0)$ such that $y_i \ge y^0_i$ for all $y \in \omega(x_0)$. If $y^0_i > 0$ then $y^0 \in \mathcal{B}([u, v'], 2\delta) \cap \mathbb{R}^N_+ \setminus \pi_i$ so $x_i(y^0, t)$ is increasing for $|t|$ small enough. Thus, $x_i(y^0, t) < y^0_i$ for t close to 0 from left. As the whole orbit $\gamma(y^0)$ is contained in $\omega(x_0)$, this contradicts $y_i \ge y^0_i$ for all $y \in \omega(x_0)$. Hence, we must have $y^0_i = 0$, so $u_i = 0$ and $y^0 \in \omega(x_0) \cap [u, v'] \subset \pi_i$. But $x_i(t) \uparrow$ in $\mathcal{B}([u, v'], 2\delta) \cap \mathbb{R}^N_+ \setminus \pi_i$ means that $[u, v']$ repels the solutions away so $[u, v'] \cap \omega(x_0) = \emptyset$, a contradiction to $y^0 \in \omega(x_0) \cap [u, v']$. Therefore, we must have (36).

Now suppose $[u, v'] \cap L^l_i$ is strictly above L^u_j for all $j \in I_N \setminus \{i\}$ (see Figure 1 (a) for illustration). Consider the plane

$$L'_i = \{x \in \mathbb{R}^N_+ : \sum_{j \in I_N \setminus \{i\}} b_{ij} x_j + b_{ii}(u_i + \delta) = 1\}.$$

Then L'_i is parallel to the x_i-axis and passes through $L^l_i \cap \{x \in \mathbb{R}^N_+ : x_i = u_i + \delta\}$. Note that for each $x \in L^l_i \cap [u, v'']$,

$$\sum_{j \in I_N \setminus \{i\}} b_{ij} x_j + b_{ii}(u_i + \delta) \ge (Bx)_i = 1,$$

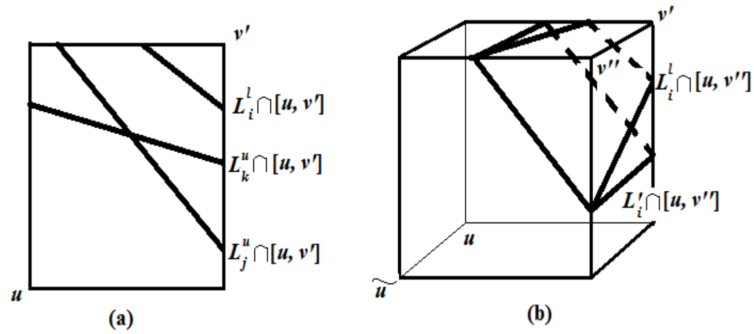

FIGURE 1. (a) Illustration of condition (c) in theorem 3.1 for $N = 3$ and distinct i, j, k in $\{1, 2, 3\}$. (b) Illustration of L_i^l and L_i' for $N = 3$.

so x is on or above L_i'. Thus, $L_i^l \cap [u, v'']$ is above L_i' (see Figure 1 (b) for illustration). Also, we have

$$\lim_{\delta \to 0} \sup_{x \in L_i' \cap [u, v'']} \{ \inf_{y \in L_i^l \cap [u, v']} \|y - x\| \} = 0.$$

Hence, $\forall \varepsilon > 0$, $\exists \delta_0 > 0$ such that $\forall \delta \in (0, \delta_0]$,

$$\sup_{x \in L_i' \cap [u, v'']} \{ \inf_{y \in L_i^l \cap [u, v']} \|y - x\| \} < \varepsilon.$$

So $\forall x \in L_i' \cap [u, v'']$, $\exists y \in L_i^l \cap [u, v']$ such that $\|y - x\| < \varepsilon$. Therefore, for any $\varepsilon > 0$, there is $\delta_0 > 0$ such that for $\delta \in (0, \delta_0]$,

$$L_i' \cap [u, v''] \subset \mathcal{B}(L_i^l \cap [u, v'], \varepsilon).$$

Since $L_i^l \cap [u, v']$ is strictly above L_j^u for all $j \in I_N \setminus \{i\}$, for $\varepsilon > 0$ small enough the set $\mathcal{B}(L_i^l \cap [u, v'], \varepsilon)$ is also strictly above L_j^u for all $j \in I_N \setminus \{i\}$. Thus, for $\delta \in (0, \delta_0]$, $L_i' \cap [u, v'']$ is strictly above L_j^u for all $j \in I_N \setminus \{i\}$. As L_i' is parallel to the x_i-axis and $a_{ji} \geq 0$ in the equation $(Ax)_j = 1$ for L_j^u, if $(Ax)_j > 1$ then $(Ax')_j > 1$ for x' with $x_i' \geq x_i$ and $x_k' = x_k$ for $k \in I_N \setminus \{i\}$. Hence, $L_i' \cap [u, v]$ is strictly above L_j^u for all $j \in I_N \setminus \{i\}$. This shows that each solution $x(t)$ in $[u, v]$ satisfies $x_j(t) \downarrow$ for all $j \in I_N \setminus \{i\}$ as long as $x(t)$ is on or above L_i' and $x_j(t) \not\equiv 0$. Therefore, for any solution $x(t)$ staying in a very small vicinity of $[u, v]$, for $t \geq T$, once it goes below L_i' it will stay below L_i' forever. Since $\omega(x_0) \subset [u, v]$ for all $x_0 \in \mathbb{R}_I$, $\omega(x_0)$ must be strictly below L_i'. The subset of $[u, v'']$ strictly below L_i' is also strictly below L_i^l.

We show that (36) holds. Suppose (36) is not true. Then $\omega(x_0) \cap [u, v''] \neq \emptyset$ for some $x_0 \in \mathbb{R}_I$. Since $\omega(x_0) \subset [u, v]$, there is $y^0 \in \omega(x_0) \cap [u, v'']$ such that $y_i \geq y_i^0$ for all $y \in \omega(x_0)$. If $u_i = 0$, as the subset of $[u, v'] \subset \pi_i$ below L_i' is strictly below Γ_i, this set

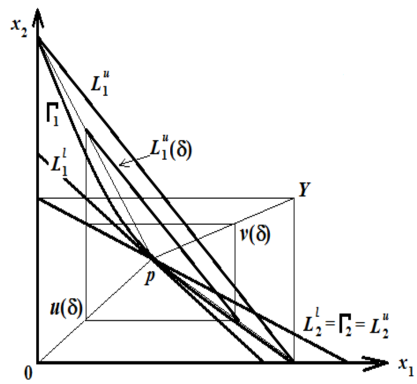

FIGURE 2. Illustration of $L_i^u, L_i^l, L_i^u(\delta)$ and $L_i^l(\delta)$ for $N = 2$.

repels the solutions in $\mathbb{R}_I$ away from π_i. Since $\omega(x_0)$ is strictly below L_i', we must have $\omega(x_0) \cap \pi_i = \emptyset$, so $y_i^0 > 0$. Then, since y^0 is below Γ_i, $x_i(y^0, t) < y_i^0$ for $t < 0$ close enough to 0. As $\gamma(y^0) \subset \omega(x_0)$, this contradicts $y_i \geq y_i^0$ for all $y \in \omega(x_0)$. Therefore, (36) holds.

Step 3. Let $u(s) = sp$ and $v(s) = sp + (1 - s)Y$ for $s \in [0, 1]$. We show that $\omega(x_0) \subset [u(s), v(s)]$ for all $x_0 \in \mathbb{R}_I$ and all $s \in [0, 1]$. Thus, $\omega(x_0) = [u(1), v(1)] = \{p\}$ for all $x_0 \in \mathbb{R}_I$ and the conclusion of Theorem 3.1 holds.

From step 1 we know that $\omega(x_0) \subset [0, Y] = [u(0), v(0)]$ for all $x_0 \in \mathbb{R}_+^N$. By step 2 and condition (c), there is a $\delta \in (0, 1)$ such that $\omega(x_0) \subset [u(\delta), Y]$ for all $x_0 \in \mathbb{R}_I$. Now define an affine map $m_\delta : \mathbb{R}_+^N \to \mathbb{R}_+^N(u(\delta))$ by $m_\delta(x) = \delta p + (1 - \delta)x$. Then $m_\delta(p) = p$ and $m_\delta(x) - p = (1 - \delta)(x - p)$. Thus, m_δ maps the line segment $\overline{xp}$ to $(1 - \delta)\overline{xp}$, $[0, Y]$ to $[u(\delta), v(\delta)]$, and each $L_i(p)$ to $L_i(p) \cap \mathbb{R}_+^N(u(\delta))$. Now consider the set

$$C_j = \{sp + (1 - s)y : \forall y \in (\partial \mathbb{R}_+^N) \cap \tilde{L}_j, \forall s \in [0, 1]\}$$

Then C_j is a cone surface with p as the vertex and $\tilde{L}_j$ as its base. The map m_δ maps C_j to $C_j \cap \mathbb{R}_+^N(u(\delta))$ and $\tilde{L}_j$ in $\mathbb{R}_+^N$ to a plane $\hat{L}_j$ in $\mathbb{R}_+^N(u(\delta))$. Note that $C_j \cap \mathbb{R}_+^N(u(\delta))$ is a cone with p as its vertex and $\hat{L}_j$ as its base (see Figure 2 for illustration). Thus, if Γ_j is concave (convex) then both $\tilde{L}_j$ and $\hat{L}_j$ are above (below) Γ_j. Hence, m_δ maps L_i^l and L_j^u in $\mathbb{R}_+^N$ to planes $L_i^l(\delta)$ and $L_j^u(\delta)$ in $\mathbb{R}_+^N(u(\delta))$ for each $i \in J$ and all $j \in I_N \setminus \{i\}$. Due to the nature of the map m_δ projecting points along straight lines towards p, the relative positions of L_i^l and L_j^u in $\mathbb{R}_+^N$ are preserved for the planes $L_i^l(\delta)$ and $L_j^u(\delta)$ in $\mathbb{R}_+^N(u(\delta))$ for each $i \in J$ and all $j \in I_N \setminus \{i\}$.

For each $i \in I_N$, Γ_i in $\mathbb{R}_+^N(u(\delta))$ is below $L_i^u(\delta)$. The intersection point R_{ii} of L_i^u with the x_i-axis has $\frac{1}{a_{ii}}$ as its ith component and 0 as other components. The point R_{ii} is mapped to $m_\delta(R_{ii}) = \delta p + (1-\delta)R_{ii}$, which has $\delta p_i + \frac{1-\delta}{a_{ii}}$ as its ith component and δp_j as the jth component for $j \neq i$. Since $L_i^u(\delta)$ is below the plane $x_i = \delta p_i + \frac{1-\delta}{a_{ii}}$, by the reasoning similar to that in step 1 we see that $\omega(x_0)$ is below this plane for all $x_0 \in \mathbb{R}_I$. As $v(\delta) = m_\delta(Y) = \delta p + (1-\delta)Y$ with $v_i(\delta) = \delta p_i + \frac{1-\delta}{a_{ii}}$, we have $\omega(x_0) \subset [u(\delta), v(\delta)]$ for all $x_0 \in \mathbb{R}_I$.

For each $i \in J$, let $v_i(\delta)' = u_i(\delta)$ and $v_j(\delta)' = v_j(\delta)$ for $j \in I_N \setminus \{i\}$. Then condition (c) and the nature of m_δ imply that either $v(\delta)'$ is below $L_i^l(\delta)$ or $L_i^l(\delta) \cap [u(\delta), v(\delta)']$ is strictly above $L_j^u(\delta)$ for all $j \in I_N \setminus \{i\}$. From step 2 again, we can always replace $\delta \in (0,1)$ by a larger one. Repetition of the above process shows that $\omega(x_0) \subset [u(\delta), v(\delta)]$ holds for all $x_0 \in \mathbb{R}_I$ and all $\delta \in (0,1)$. Taking the limit $\delta \to 1^-$, we obtain $\omega(x_0) = \{p\}$. $\square$

5. Geometric method for global repulsion

In this section, we assume that p is an interior equilibrium point, so $p \in \text{int}\mathbb{R}_+^N$. When each nullcline surface Γ_i is concave or convex, we define the planes L_i^u and L_i^l in the same way as in section 3 for $i \in I_N$. Then each Γ_i is below L_i^u but above L_i^l,

$$L_i^l = \{x \in \mathbb{R}_+^N : (Bx)_i = 1\}, \quad L_i^u = \{x \in \mathbb{R}_+^N : (Ax)_i = 1\}.$$

Thus,

$$\forall i, j \in I_N, a_{ij} \leq b_{ij}. \tag{37}$$

We also assume that for each $i \in I_N$, the intersection point of L_i^l with the positive half x_i-axis is above L_j^u for all $j \in I_N \setminus \{i\}$. Then $a_{ji}/b_{ii} > 1$ so

$$\forall i \in I_N, \forall j \in I_N \setminus \{i\}, a_{ji} > b_{ii} > 0. \tag{38}$$

From (37) and (38) we have

$$a_{jj} \leq b_{jj} < a_{ij} \leq b_{ij} \ (i \neq j).$$

For any $k \in I_N \setminus \{i\}$, as $b_{ii} < a_{ji}$, if $b_{ik} > a_{jk}$ then the system of simultaneous equations

$$b_{ii}x_i + b_{ik}x_k = 1, \ a_{ji}x_i + a_{jk}x_k = 1 \tag{39}$$

has a solution

$$x_i = \frac{a_{jk} - b_{ik}}{b_{ii}a_{jk} - b_{ik}a_{ji}} > 0, \ x_j = \frac{b_{ii} - a_{ji}}{b_{ii}a_{jk} - b_{ik}a_{ji}} > 0.$$

This shows that L_j^u and L_i^l restricted to $\cap_{m \in I_N \setminus \{i,k\}} \pi_m$ has a unique intersection point. This is obviously true for $k = j$ as $b_{ij} > a_{jj}$. If $b_{ik} \leq a_{jk}$ then (39) has no solution with $x_i > 0$. Thus, the largest possible ith component of the points in $L_i^l \cap L_j^u$ is

$$\max\left\{ \frac{b_{ik} - a_{jk}}{a_{ji}b_{ik} - a_{jk}b_{ii}} : k \in I_N \setminus \{i\} \text{ if } b_{ik} > a_{jk} \right\}.$$

Now define $U \gg 0$ by

$$U_i = \max\left\{ \frac{b_{ik} - a_{jk}}{a_{ji}b_{ik} - a_{jk}b_{ii}} : j,k \in I_N \setminus \{i\} \text{ if } b_{ik} > a_{jk} \right\}, i \in I_N. \tag{40}$$

For a surface Γ in $\mathbb{R}^N_+$, we call it *strongly balanced* if for all distinct points $u, v \in \Gamma$, neither $u - v$ nor $v - u$ is in $\mathbb{R}^N_+$.

Theorem 5.1. *Assume that the following conditions hold.*

(a) *System (6) has an interior equilibrium point $p \in \mathrm{int}\mathbb{R}^N_+$.*

(b) *For each $i \in I_N$, the nullcline surface Γ_i is strongly balanced and either convex or concave.*

(c) *For each $i \in I_N$, the intersection point of L^l_i with the positive half x_i-axis is above L^u_j for all $j \in I_N \setminus \{i\}$.*

(d) *For each $i \in I_N$, either $\pi_i \cap [0, U]$ or $L^u_i \cap \pi_i \cap [0, U]$ is strictly below L^l_j for all $j \in I_N \setminus \{i\}$, where U is defined by (40).*

Then p is globally repelling.

Remark 4. Let $D \subset \mathbb{R}^N_+$ be a bounded region such that $\Gamma_i \subset D$ for all $i \in I_N$. Then, by Proposition 2.2 (iii) and (vi), the requirement of each Γ_i to be concave or convex in part of condition (b) in Theorem 5.1 is guaranteed if each function F_i or $-F_i$ is a convex function in D with $F_i(0) = \max_{x \in \mathbb{R}^N_+} F_i(x)$. The requirement that each Γ_i is strongly balanced is ensured by the following:

$$\forall i, j \in I_N, \forall u \in D, \frac{\partial F_i}{\partial x_j}(u) < 0. \tag{41}$$

Indeed, (41) implies that $F_i(x)$ is strictly increasing in each x_j for $x \in D$. Thus, for any $u, v \in D$ with $u < v$, we have $F_i(u) > F_i(v)$, so it is impossible to have both $u \in \Gamma_i$ and $v \in \Gamma_i$.

Remark 5. The algebraic condition equivalent to condition (c) in Theorem 5.1 is (38). Then from (40) we see that conditions (a)–(c) guarantee the existence of $U \gg 0$: U_i is the maximum of the ith components of all the possible intersection points of L^l_i with $\cup_{j \in I_N \setminus \{i\}} L^u_j$. Note that each set $L^u_i \cap \pi_i \cap [0, U]$ in condition (d), if not empty, is a convex hull which is determined by linear combinations of a finite number of vertices $v_{i1}, \ldots, v_{im}$. Thus, $L^u_i \cap \pi_i \cap [0, U]$ is strictly below L^l_j for all $j \in I_N \setminus \{i\}$ if and only if each vertex v_{ik} is below L^l_j for all $j \in I_N \setminus \{i\}$. This will be clear from Figure 3 in Example 5.4 later.

Remark 6. Under the conditions of Theorem 5.1, from Theorem 2.1 we see that p is a saddle point with a one-dimensional stable manifold $W^s(p)$ and $(N-1)$-dimensional unstable manifold $W^u(p) = \mathrm{int}\Sigma \setminus \{p\}$. Thus, for each $x_0 \gg 0$, we have $\omega(x_0) = \{p\}$ if $x_0 \in W^s(p)$ and $\omega(x_0) \subset \partial\Sigma$ if $x_0 \notin W^s(p)$. For each $x_0 \in \mathrm{int}\Sigma \setminus \{p\}$, we have $\omega(x_0) \subset \partial\Sigma$ and $\alpha(x_0) = \{p\}$.

For a particular class of systems (6) when each Γ_i is a plane, so $\Gamma_i = L_i^u = L_i^l$, it is both concave and convex. Then condition (c) of Theorem 5.1 guarantees that each Γ_i is strongly balanced. Thus, condition (b) is redundant and Theorem 5.1 is simplified as follows.

Corollary 5.2. *Assume that the following conditions hold.*

(a) *System (6) has an equilibrium point* $p \in \text{int}\mathbb{R}_+^N$.

(b) *For each* $i \in I_N$, *the nullcline surface* Γ_i *is a plane.*

(c) *Each axial equilibrium point* R_i *is above* Γ_j *for all* $j \in I_N \setminus \{i\}$.

(d) *For each* $i \in I_N$, *either* $\pi_i \cap [0, U]$ *or* $\Gamma_i \cap \pi_i \cap [0, U]$ *is strictly below* Γ_j *for all* $j \in I_N \setminus \{i\}$, *where* U *is given by (40) with* $L_i^l = L_i^u = \Gamma_i$.

Then p *is globally repelling.*

Example 5.3. Consider the system (4) with $r_i > 0$, $c_i > 0$ and $a_{ij} > 0$ for all $i, j \in I_N$. Suppose $p \in \text{int}\mathbb{R}_+^N$ is an interior equilibrium point. Then each Γ_i is a plane,

$$\forall i \in I_N, \Gamma_i = L_i^l = L_i^u = \{x \in \mathbb{R}_+^N : a_{i1}x_1 + \cdots + a_{iN}x_N = 1\}.$$

Assume that each axial equilibrium R_i is above Γ_j for all $j \in I_N \setminus \{i\}$, i.e.

$$\forall i, j \in I_N \, (i \neq j), a_{ji} > a_{ii} > 0.$$

Define $U \gg 0$ by

$$\forall i \in I_N, U_i = \max\left\{\frac{a_{ik} - a_{jk}}{a_{ji}a_{ik} - a_{jk}a_{ii}} : j, k \in I_N \setminus \{i\} \text{ if } a_{ik} > a_{jk}\right\}.$$

Then, by Corollary 5.2, p is globally repelling if either $\pi_i \cap [0, U]$ or $\Gamma_i \cap \pi_i \cap [0, U]$ is strictly below Γ_j for all $i \in I_N$ and $j \in I_N \setminus \{i\}$.

Note that the above result for (4) is also true for other systems in (2)–(5). In particular, for Lotka-Volterra system (2), this result is consistent with [9].

Example 5.4. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1(1 - 2ax_1 - ax_1^2 - x_2 - x_3) = x_1F_1(x), \\ \dot{x}_2 &= x_2(1 - x_1 - 2ax_2 - ax_2^2 - x_3) = x_2F_2(x), \\ \dot{x}_3 &= x_3(1 - x_1 - x_2 - 2ax_3 - ax_3^2) = x_3F_3(x), \end{aligned} \tag{42}$$

where $a > 0$ is a constant. The system has an interior equilibrium point $p = p_0(1, 1, 1)^T$ with p_0 satisfying $ap_0^2 + 2(a + 1)p_0 = 1$, so

$$p_0 = \frac{1}{a}[\sqrt{a + (a + 1)^2} - (a + 1)] = \frac{1}{\sqrt{a^2 + 3a + 1} + a + 1}.$$

Then

$$\begin{aligned}
\Gamma_1 &= \{x \in \mathbb{R}^3_+ : 2ax_1 + ax_1^2 + x_2 + x_3 = 1\}, \\
\Gamma_2 &= \{x \in \mathbb{R}^3_+ : x_1 + 2ax_2 + ax_2^2 + x_3 = 1\}, \\
\Gamma_1 &= \{x \in \mathbb{R}^3_+ : x_1 + x_2 + 2ax_3 + ax_3^2 = 1\}, \\
L_1(p) &= 2a(1+p_0)x_1 + x_2 + x_3 = 1 + ap_0^2, \\
L_2(p) &= x_1 + 2a(1+p_0)x_2 + x_3 = 1 + ap_0^2, \\
L_3(p) &= x_1 + x_2 + 2a(1+p_0)x_3 = 1 + ap_0^2.
\end{aligned}$$

As Γ_1 intersects the axes at $(\sqrt{1+\frac{1}{a}} - 1, 0, 0)^T$, $(0,1,0)^T$, $(0,0,1)^T$ respectively and $(\sqrt{1+\frac{1}{a}} - 1)^{-1} = a(\sqrt{1+\frac{1}{a}} + 1)$, we have

$$\tilde{L}_1 = \{x \in \mathbb{R}^3_+ : a(\sqrt{1+\frac{1}{a}} + 1)x_1 + x_2 + x_3 = 1\}.$$

Similarly,

$$\begin{aligned}
\tilde{L}_2 &= \{x \in \mathbb{R}^3_+ : x_1 + a(\sqrt{1+\frac{1}{a}} + 1)x_2 + x_3 = 1\}, \\
\tilde{L}_3 &= \{x \in \mathbb{R}^3_+ : x_1 + x_2 + a(\sqrt{1+\frac{1}{a}} + 1)x_3 = 1\}.
\end{aligned}$$

Note that

$$\forall i \in I_3, \forall x, y \in \mathbb{R}^3_+, F_i(sx + (1-s)y) \geq sF_i(x) + (1-s)F_i(y)$$

so F_1, F_2 and F_3 are convex functions with $F_i(0) = \max_{x \in \mathbb{R}^3_+} F_i(x)$ for $i \in I_3$. By Proposation 2.2 (iii), Γ_1, Γ_2 and Γ_3 are convex. Then $L_i^l = \tilde{L}_i$ and $L_i^u = L_i(p)$ for $i \in I_3$. Since $\frac{\partial F_i}{\partial x_i} = -2a - 2ax_i < 0$ and $\frac{\partial F_i}{\partial x_j} = -1 < 0$ for $i, j \in I_3$ $(i \neq j)$, by Remark 4 each Γ_i is strongly balanced. Thus, conditions (a) and (b) of Theorem 5.1 are fulfilled.

If $a \in (0, 0.3]$, then we have $p_0 < \frac{1}{2}$ so $p_0^2 < \frac{1}{4}$ and

$$\sqrt{1+\frac{1}{a}} - 1 \geq \sqrt{\frac{1.3}{0.3}} - 1 > \frac{4.3}{4} \geq 1 + \frac{1}{4}a > 1 + ap_0^2.$$

Thus, the equilibrium point $(\sqrt{1+\frac{1}{a}} - 1, 0, 0)^T$, which is the intersection point of L_1^l with the positive half x_1-axis, is above L_2^u and L_3^u. By symmetry, condition (c) of Theorem 5.1 is met.

To check condition (d), we need to find $U \gg 0$ given by (40). The point in $L_1^l \cap L_2^u \cap \pi_3$ is given by the solution of

$$a(\sqrt{1+\frac{1}{a}} + 1)x_1 + x_2 = 1,\ x_1 + 2a(1+p_0)x_2 = 1 + ap_0^2,$$

which has the components

$$x_1 = \frac{1+ap_0^2-2a(1+p_0)}{1-2a^2(\sqrt{1+\frac{1}{a}}+1)(1+p_0)},\ x_2 = \frac{1-a(\sqrt{1+\frac{1}{a}}+1)(1+ap_0^2)}{1-2a^2(\sqrt{1+\frac{1}{a}}+1)(1+p_0)},\ x_3 = 0.$$

The point in $L_1^l \cap L_3^u \cap \pi_3$ is given by the solution of

$$a(\sqrt{1+\frac{1}{a}}+1)x_1 + x_2 = 1,\ x_1 + x_2 = 1 + ap_0^2,$$

which has the components

$$x_1 = \frac{ap_0^2}{1-a(\sqrt{1+\frac{1}{a}}+1)},\ x_2 = \frac{1-a(\sqrt{1+\frac{1}{a}}+1)(1+ap_0^2)}{1-a(\sqrt{1+\frac{1}{a}}+1)},\ x_3 = 0.$$

The point in $L_1^l \cap L_2^u \cap \pi_2$ is the same as that in $L_1^l \cap L_3^u \cap \pi_3$ with the swap of x_2 and x_3, and the point in $L_1^l \cap L_3^u \cap \pi_2$ is the same as that in $L_1^l \cap L_2^u \cap \pi_3$ with the swap of x_2 and x_3. We can easily check that the function

$$f(s) = \frac{ap_0^2 + s}{1-a(\sqrt{1+\frac{1}{a}}+1) + a(\sqrt{1+\frac{1}{a}}+1)s}$$

is increasing in s, so

$$\frac{1+ap_0^2-2a(1+p_0)}{1-2a^2(\sqrt{1+\frac{1}{a}}+1)(1+p_0)} = f(1-2a(1+p_0)) > f(0) = \frac{ap_0^2}{1-a(\sqrt{1+\frac{1}{a}}+1)}.$$

Then, by (40), U_1 is the maximum of the first component of the points in $L_1^l \cap L_2^u \cap \pi_3$, $L_1^l \cap L_3^u \cap \pi_3$, $L_1^l \cap L_2^u \cap \pi_2$ and $L_1^l \cap L_3^u \cap \pi_2$. Thus,

$$U_1 = \frac{1+ap_0^2-2a(1+p_0)}{1-2a^2(\sqrt{1+\frac{1}{a}}+1)(1+p_0)}$$

and by symmetry, $U_2 = U_3 = U_1$.

Next, we derive a condition on a so that condition (d) of Theorem 5.1 is satisfied. The set $L_1^u \cap \pi_1 \cap [0, U]$ is the line segment $\overline{AB}$ (see Figure 3) on the plane $x_1 = 0$ with $A(u_0, U_3)$ and $B(U_2, u_0)$, where

$$u_0 = 1 + ap_0^2 - U_3 = \frac{2a(1+p_0)[1-a(\sqrt{1+\frac{1}{a}}+1)(1+ap_0^2)]}{1-2a^2(\sqrt{1+\frac{1}{a}}+1)(1+p_0)}.$$

The point in $L_2^l \cap L_1^u \cap \pi_1$ is given by the solution of

$$a(\sqrt{1+\frac{1}{a}}+1)x_2 + x_3 = 1,\ x_2 + x_3 = 1 + ap_0^2,$$

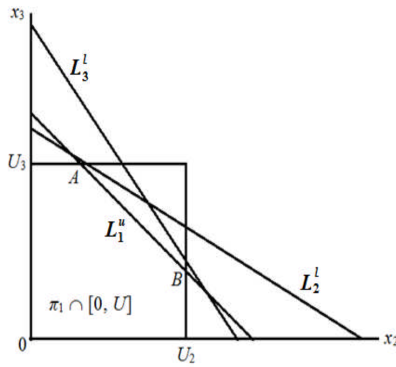

FIGURE 3. Illustration of $L_1^u \cap \pi_1 \cap [0, U], L_2^l \cap \pi_1$ and $L_3^l \cap \pi_1$.

which has the components

$$x_1 = 0, x_2 = \frac{ap_0^2}{1 - a(\sqrt{1 + \frac{1}{a}} + 1)}, \; x_3 = \frac{1 - a(\sqrt{1 + \frac{1}{a}} + 1)(1 + ap_0^2)}{1 - a(\sqrt{1 + \frac{1}{a}} + 1)}.$$

As the axial fixed point $(0, \sqrt{1 + \frac{1}{a}} - 1, 0)^T$, which is the intersection point of L_2^l with the positive half x_2-axis, is above L_1^u, any point in $L_2^l \cap \pi_1$ with $\frac{ap_0^2}{1-a(\sqrt{1+\frac{1}{a}}+1)} < x_2 \leq \sqrt{1 + \frac{1}{a}} - 1$ is above L_1^u. Thus, $L_1^u \cap \pi_1 \cap [0, U]$ is strictly below L_2^l if $u_0 > \frac{ap_0^2}{1-a(\sqrt{1+\frac{1}{a}}+1)}$ (see Figure 3). Note that $p_0 < \frac{1}{2}$ so $\frac{1+p_0}{p_0^2} > 6$. Then, if $a \in (0, 0.3]$ is small enough to satisfy

$$12[1 - a(\sqrt{1 + \frac{1}{a}} + 1)(1 + ap_0^2)][1 - a(\sqrt{1 + \frac{1}{a}} + 1)] + 2a^2(\sqrt{1 + \frac{1}{a}} + 1)(1 + p_0) \geq 1, \tag{43}$$

$L_1^u \cap \pi_1 \cap [0, U]$ is strictly below L_2^l. Similarly, (43) also ensures that $L_1^u \cap \pi_1 \cap [0, U]$ is strictly below L_3^l. By symmetry, (43) guarantees condition (d) of Theorem 5.1. Therefore, p is a global repellor if $a \leq 0.3$ and satisfies (43).

6. PROOF OF THEOREM 5.1

To prepare for the proof of Theorem 5.1, we present five lemmas below, of which the first three reveal some general properties of (6) under certain conditions and the last two are closely related to the conditions of Theorem 5.1.

Lemma 6.1. *If the ith axial fixed point R_i is above (below) Γ_j for all $j \in I_N \setminus \{i\}$, then R_i is an attractor in $\mathbb{R}^N_+$ (a repellor in Σ).*

Proof. Note that $\nabla(x_iF_i(x))|_{x=R_i} \ll 0$ by (A4),

$$\frac{\partial x_jF_j(x)}{\partial x_k}|_{x=R_i} = 0, \ \frac{\partial x_jF_j(x)}{\partial x_j}|_{x=R_i} = F_j(R_i), \ k \neq j \neq i.$$

Thus, the eigenvalue of $\frac{\partial D[x]F(x)}{\partial x}|_{x=R_i}$ with an eigenvector on x_i-axis is negative and the $F_j(R_i)$ are eigenvalues of $\frac{\partial D[x]F(x)}{\partial x}|_{x=R_i}$ with an eigenvector transversal to the x_i-axis. If R_i is above (below) Γ_j for all $j \in I_N \setminus \{i\}$, then $\frac{\partial D[x]F(x)}{\partial x}|_{x=R_i}$ has N negative eigenvalues ($N-1$ positive eigenvalues with eigenvectors transversal to the x_i-axis) so R_i is an attractor in $\mathbb{R}^N_+$ (a repellor in Σ). □

Lemma 6.2. *Assume that each Γ_i is strongly balanced. For any $u \in \mathbb{R}^N_+ \setminus \{0\}$ with support $I \subset I_N$, if there is a nonempty $I_0 \subset I$ such that u is below Γ_j for all $j \in I_0$ but is on Γ_k for all $k \in I \setminus I_0$, then $u \in Br(0)$.*

Proof. If $I_0 = I$, then $x(u,t)$ is below Γ_j for all $j \in I$ and sufficiently small $|t|$. Since each Γ_j is strongly balanced, by the monotone property of competitive systems, we have $x(u,t_2) < x(u,t_1) < u$ for all $t_2 < t_1 < 0$. Then there is a $q \in \mathbb{R}^N_+$ with $q < u$ such that $\alpha(u) = \{q\}$ so q is an equilibrium point. We show that $q = 0$ so that $u \in Br(0)$.

For each $i \in I$, since $u_i > 0$, $F_i(u) > 0$, $F_i(0) > 0$ by (A1), and $F_i(v) < 0$ for sufficiently large $|v|$ by (A3), if $F_i(q) \leq 0$, then the continuity of F_i ensures the existence of $q', u' \in \Gamma_i$ satisfying $q \leq q' < u < u'$. This contradicts the assumption that Γ_i is strongly balanced. Therefore, we must have $F_i(q) > 0$. Since q is an equilibrium point, we have $D[q]F(q) = 0$ so $q = 0$ and $u \in Br(0)$.

If $I_0 \neq I$, let $u(\varepsilon) > 0$ be defined by

$$u_j(\varepsilon) = u_j \text{ for } j \notin I \setminus I_0, u_k(\varepsilon) = u_k - \varepsilon \text{ for } k \in I \setminus I_0$$

for sufficiently small $\varepsilon > 0$. Then $u(\varepsilon) < u$. As each Γ_i is strongly balanced, we have $F_i(u(\varepsilon)) > 0$ so $u(\varepsilon)$ is below Γ_i for all $i \in I$. Thus, from the case of $I_0 = I$, we have

$$u(\varepsilon) \in Br(0) \text{ and } F_i(x(u(\varepsilon),t)) > 0, \ \ \forall t < 0, \ \forall i \in I$$

for each sufficiently small $\varepsilon > 0$. Then, by continuous dependence on initial values, we have $F_i(x(u,t)) \geq 0$ for all $t < 0$ and $i \in I$. But $\dot{x}_j(u,0) = u_jF_j(u) > 0$ for $j \in I_0$, so $x_j(u,t) < u_j$ for all $t < 0$ and $j \in I_0$. Then $x(u,t) < u$ for all $t < 0$. Since each Γ_i is strongly balanced, we have $F_i(x(u,t)) > 0$ for all $t < 0$ and $i \in I$. Hence, $x(u,t) \in Br(0)$ for all $t < 0$ so $u \in Br(0)$. □

Lemma 6.3. *Assume that each Γ_i is strongly balanced. For any $u \in \mathbb{R}^N_+ \setminus \{0\}$ with support $I \subset I_N$, if there is a nonempty $I_0 \subset I$ such that u is above Γ_j for all $j \in I_0$ but is on Γ_k for all $k \in I \setminus I_0$, then $u \in Br(\infty)$.*

The proof of Lemma 6.3 is similar to that of Lemma 6.2 so we omit it here.

Lemma 6.4. *Assume that each Γ_i is strongly balanced. Assume the existence of $u \in \mathbb{R}^N_+$, $i \in I_N$ and $v_i > u_i$ such that either $S^0(u, v_i)$ or $\Gamma_i \cap S^0(u, v_i)$ is strictly above Γ_j for all $j \in I_N \setminus \{i\}$. Then, for each $x_0 \in \mathbb{R}^N_+ \setminus \{0\}$, if $\omega(x_0) \subset \mathbb{R}^N_+(u)$ then either $\omega(x_0) \subset \mathbb{R}^N_+(u) \setminus S^0(u, v_i)$ or $\omega(x_0) = \{R_i\}$ (so $R_i \in \mathbb{R}^N_+(u)$). Moreover, if the whole trajectory $\gamma(x_0)$ is in $\Sigma \cap \mathbb{R}^N_+(u)$ for some $x_0 \in \mathrm{int}\Sigma$, then either $\omega(x_0) = \{R_i\}$ and $\alpha(x_0) \subset \Sigma \cap (\mathbb{R}^N_+(u) \setminus S^0(u, v_i))$ or $\gamma(x_0) \subset \Sigma \cap (\mathbb{R}^N_+(u) \setminus S^0(u, v_i))$.*

Proof. If $S^0(u, v_i)$ is strictly above Γ_i then it is strictly above Γ_j for all $j \in I_N$. By Lemma 6.3, $S^0(u, v_i) \subset Br(\infty)$ so $\Sigma \cap S^0(u, v_i) = \emptyset$. If $\omega(x_0) \subset \mathbb{R}^N_+(u)$ for some $x_0 \in \mathbb{R}^N_+ \setminus \{0\}$, as $\omega(x_0) \subset \Sigma$ by Theorem 2.1, we have $\omega(x_0) \cap S^0(u, v_i) = \emptyset$ so $\omega(x_0) \subset \mathbb{R}^N_+(u) \setminus S^0(u, v_i)$.

Now suppose $\Gamma_i \cap S^0(u, v_i) \neq \emptyset$. Since this set is strictly above Γ_j for all $j \in I_N \setminus \{i\}$, either $S^0(u, v_i)$ contains no equilibrium point or, if $R_i \in S^0(u, v_i)$, the axial equilibrium point R_i is the unique equilibrium point in $S^0(u, v_i)$ and, by Lemma 6.1, R_i is an attractor. By Lemma 6.3, any point on or above Γ_i in $S^0(u, v_i) \setminus \{R_i\}$ belongs to $Br(\infty)$, so it can be neither an ω-limit point nor an α-limit point.

For any $x_0 \in \mathbb{R}^N_+ \setminus \{0\}$, if $\omega(x_0) \subset \mathbb{R}^N_+(u)$ with $\omega(x_0) \cap S^0(u, v_i) \neq \emptyset$, we show that $\omega(x_0) = \{R_i\}$. Indeed, if $R_i \in \omega(x_0) \cap S^0(u, v_i)$, then R_i is the unique ω-limit point in $Ba(R_i)$. As $\omega(x_0)$ is connected, we must have $\omega(x_0) = \{R_i\}$. Now suppose $R_i \notin \omega(x_0) \cap S^0(u, v_i)$. Then $\omega(x_0) \cap S^0(u, v_i)$ is strictly below Γ_i so it contains no equilibrium point. For any point $q \in \omega(x_0) \cap S^0(u, v_i)$, $x(q, t) \in \omega(x_0)$ for all $t \in \mathbb{R}$ and $x_i(q, t)$ is increasing as long as $x(q, t) \in \omega(x_0) \cap S^0(u, v_i)$. Thus,

$$x(q, t) \in \omega(x_0) \cap S(u, q_i) \subset \omega(x_0) \cap S^0(u, v_i)$$

and $x_i(q, t)$ is increasing for all $t \geq 0$. Since $\omega(x_0) \cap S(u, q_i)$ is compact and strictly below Γ_i, we have

$$\delta_0 = \min\{F_i(x) : x \in \omega(x_0) \cap S(u, q_i)\} > 0 \tag{44}$$

so

$$x_i(q, t) = q_i \exp\left(\int_0^t F_i(x(q, s))ds\right) \geq q_i e^{\delta_0 t}, \forall t \geq 0. \tag{45}$$

This leads to the unboundedness of $x(q, t)$ for $t \geq 0$, a contradiction to $x(q, t) \in \omega(x_0) \cap S(u, q_i)$. Hence, we have shown that the case $R_i \notin \omega(x_0) \cap S^0(u, v_i) \neq \emptyset$ does not exist. Therefore, for any $x_0 \in \mathbb{R}^N_+ \setminus \{0\}$ with $\omega(x_0) \subset \mathbb{R}^N_+(u)$, we have either $\omega(x_0) \subset \mathbb{R}^N_+(u) \setminus S^0(u, v_i)$ or $\omega(x_0) = \{R_i\}$ so $R_i \in \mathbb{R}^N_+(u)$.

Next, we suppose $\gamma(x_0) \subset \Sigma \cap \mathbb{R}^N_+(u)$ for some $x_0 \in \mathrm{int}\Sigma$. If $\gamma(x_0) \cap \Sigma \cap S^0(u, v_i) \neq \emptyset$, as any point on or above Γ_i (except R_i) belongs to $Br(\infty)$ and $R_i \notin \gamma(x_0)$, $\gamma(x_0) \cap \Sigma \cap S^0(u, v_i)$ is strictly below Γ_i. Thus, $x_i(x_0, t)$ is increasing as long as $x(x_0, t) \in S^0(u, v_i)$. By $\gamma(x_0) \subset \Sigma \cap \mathbb{R}^N_+(u)$, there is a $t_1 \in \mathbb{R}$ such that $x(x_0, t) \in S^0(u, v_i)$ for $t > t_1$ but $x(x_0, t) \in \mathbb{R}^N_+(u) \setminus S^0(u, v_i)$ for $t \leq t_1$. Thus, $\alpha(x_0) \subset \Sigma \cap (\mathbb{R}^N_+(u) \setminus S^0(u, v_i))$ and, as $x_i(x_0, t)$

is increasing for $t > t_1$, $\omega(x_0) \subset S^0(u, v_i)$. It then follows from the previous paragraph that $\omega(x_0) = \{R_i\}$. If $\gamma(x_0) \cap \Sigma \cap S^0(u, v_i) = \emptyset$ then $\gamma(x_0) \subset \Sigma \cap (\mathbb{R}^N_+(u) \setminus S^0(u, v_i))$. □

Lemma 6.5. *Under the conditions of Theorem 5.1, assume the existence of $u \in \mathbb{R}^N_+$, $i \in I_N$ and $v > u$ with $v_i > u_i$ such that u is below Γ_i and either $\pi_i(u) \cap [u, v]$ or $\Gamma_i \cap \pi_i(u) \cap [u, v]$ is below Γ_j for all $j \in I_N \setminus \{i\}$ and $R_i \notin [u, v]$. Then there is a $\delta \in (0, v_i - u_i)$ such that if $\omega(x_0) \subset [u, v]$ for some $x_0 \gg 0$, then either $\omega(x_0) \subset \pi_i$ (and $u_i = 0$) or $\omega(x_0) \subset [u', v]$, where $u'_i = u_i + \delta$ and $u'_j = u_j$ for all $j \in I_N \setminus \{i\}$. Moreover, if $\gamma(x_0) \subset \Sigma \cap [u, v]$ for some $x_0 \in \mathrm{int}\Sigma$, then either $\omega(x_0) \subset \pi_i$ and $\alpha(x_0) \subset \Sigma \cap [u', v]$ or $\gamma(x_0) \subset \Sigma \cap [u', v]$.*

Proof. Let $D = \{x \in \pi_i(u) \cap [u, v] : F_i(x) \geq 0\}$. Since u is below Γ_i, we have $F_i(u) > 0$ so $u \in D$ and $D \neq \emptyset$. By the assumption, D is strictly below Γ_j for all $j \in I_N \setminus \{i\}$, so R_i is the only possible nontrivial equilibrium in D. But $D \subset [u, v]$ and $R_i \notin [u, v]$. Hence, $R_i \notin D$ so D conatins no nontrivial equilibrium point. Since each point in D is on or below Γ_i, by Lemma 6.2 $D \subset Br(0)$. Since $Br(0)$ is open in $\mathbb{R}^N_+$, D is compact, and F_i is continuous, there is a small $\delta \in (0, v_i - u_i)$ such that the set

$$S = \{x \in [u, v] : x_i \leq u_i + \delta, F_i(x) \geq -\delta\}$$

is a subset of $Br(0)$. Thus, any nontrivial point in S is neither an ω-limit point nor an α-limit point. So $\omega(x_0) \cap S = \emptyset$ if $x_0 \neq 0$. Now suppose $\omega(x_0) \subset [u, v]$ and $\omega(x_0) \cap ([u, v] \setminus [u', v]) \neq \emptyset$ for some $x_0 \in \mathbb{R}^N_+ \setminus \{0\}$. We show that $\omega(x_0) \subset \pi_i$ so $u_i = 0$. Since $y \in \omega(x_0) \cap ([u, v] \setminus [u', v])$ implies $y \notin S$ so $F_i(y) < -\delta$, by the compactness of $\omega(x_0)$ and the continuity of F_i, there is an $\varepsilon > 0$ such that

$$\forall z \in \mathcal{B}(\omega(x_0), \varepsilon) \text{ with } z_i \leq u_i + \delta + \varepsilon, F_i(z) \leq -\delta/2. \tag{46}$$

By definition of $\omega(x_0)$ and the assumption $\omega(x_0) \cap ([u, v] \setminus [u', v]) \neq \emptyset$, there is a $T > 0$ such that $x_i(x_0, T) < u_i + \delta + \varepsilon$ and $x(x_0, t) \in \mathcal{B}(\omega(x_0), \varepsilon)$ for all $t \geq T$. Then, by (46),

$$x_i(x_0, t) = x_i(x_0, T) \exp\left(\int_T^t F_i(x(x_0, s))ds\right) \leq x_i(x_0, T)e^{-\delta(t-T)/2} \tag{47}$$

for $t > T$ as long as $x_i(x_0, t) < u_i + \delta + \varepsilon$. This shows that $\lim_{t\to+\infty} x_i(x_0, t) = 0$, so $\omega(x_0) \subset \pi_i$ and $u_i = 0$.

Finally, suppose $\gamma(x_0) \subset \Sigma \cap [u, v]$ for some $x_0 \in \mathrm{int}\Sigma$. If $\gamma(x_0) \cap \Sigma \cap ([u, v] \setminus [u', v]) \neq \emptyset$, then, as $\Sigma \cap Br(0) = \emptyset$ so $\gamma(x_0) \cap S = \emptyset$, from the definition of S we see that $F_i(x(x_0, t)) < -\delta$ so $x_i(x_0, t)$ is decreasing as long as $x_i(x_0, t) \leq u'_i$. This shows the existence of $T \in \mathbb{R}$ such that $x_i(x_0, t) \leq u'_i$ and $F_i(x(x_0, t)) < -\delta$ for all $t \geq T$ but $x_i(x_0, t) > u'_i$ for $t < T$. Therefore, $\alpha(x_0) \subset \Sigma \cap [u', v]$ and, from (47), $\omega(x_0) \subset \pi_i$ and $u_i = 0$. If $\gamma(x_0) \cap \Sigma \cap ([u, v] \setminus [u', v]) = \emptyset$ then $\gamma(x_0) \subset \Sigma \cap [u', v]$. □

With the help of Lemmas 6.1–6.5, we are now in a position to prove Theorem 5.1.

Proof of Thorem 5.1. For each $i \in I_N$, by condition (c) and (40) we see that $L^l_i \cap S^0(0, U_i)$ is strictly above L^u_j for all $j \in I_N \setminus \{i\}$. From condition (b) we know that Γ_i is above L^l_i

and Γ_j is below L_j^u. So $\Gamma_i \cap S^0(0, U_i)$ is strictly above Γ_j for all $j \in I_N \setminus \{i\}$. By Lemma 6.4 with $u = 0$ and $v_i = U_i$, for each $x_0 \in \mathbb{R}_+^N \setminus \{0\}$ we have either $\omega(x_0) \subset [0, U]$ or $\omega(x_0) = \{R_i\}$ for some $i \in I_N$. Moreover, for each $x_0 \in \text{int}\Sigma$, we have either $\omega(x_0) = \{R_i\}$ and $\alpha(x_0) \subset \Sigma \cap [0, U]$ for some $i \in I_N$ or $\gamma(x_0) \subset \Sigma \cap [0, U]$.

By condition (d), for each $i \in I_N$, either $\pi_i \cap [0, U]$ or $L_i^u \cap \pi_i \cap [0, U]$ is strictly below L_j^l for all $j \in I_N \setminus \{i\}$ so either $\pi_i \cap [0, U]$ or $\Gamma_i \cap \pi_i \cap [0, U]$ is strictly below Γ_j for all $j \in I_N \setminus \{i\}$. Note that 0 is below Γ_i by (A1). From (40) and condition (c) we know that the intersection point R_{ii} of L_i^l with the positive half x_i-axis satisfies $R_{ii} \notin [0, U]$. As the ith axial equilibrium R_i is on or above L_i^l whereas $R_{ii} \in L_i^l$, we must have $R_i \notin [0, U]$. Then, by Lemma 6.5 with $[u, v] = [0, U]$, there is a $\delta \in (0, 1)$ such that for all $x_0 \in \mathbb{R}_+^N \setminus \{0\}$ we have either $\omega(x_0) \subset \pi_i$ for some $i \in I_N$ or $\omega(x_0) \subset [\delta p, U]$. Further, for all $x_0 \in \text{int}\Sigma$, we have either $\omega(x_0) \subset \pi_i \cap \Sigma$ and $\alpha(x_0) \subset [\delta p, U] \cap \Sigma$ for some $i \in I_N$ or $\gamma(x_0) \subset [\delta p, U] \cap \Sigma$.

Now define an affine map $m_\delta : \mathbb{R}_+^N \to \mathbb{R}_+^N(\delta p)$ by

$$m_\delta(x) = \delta p + (1 - \delta)x.$$

Then $m_\delta(0) = \delta p$, $m_\delta(p) = p$, $m_\delta(U) = \delta p + (1 - \delta)U$. Let $[u(\delta), v(\delta)] = [m_\delta(0), m_\delta(U)]$. Then m_δ maps $[0, U]$ to $[u(\delta), v(\delta)]$, each $L_i(p)$ to $L_i(p) \cap \mathbb{R}_+^N(u(\delta))$, and each $\tilde{L}_j$ to a plane $\hat{L}_j$ in $\mathbb{R}_+^N(u(\delta))$. Note that $\tilde{L}_j$ is the convex hull of the vertex set $\{V_{j1}, \ldots, V_{jN}\}$, i.e.

$$\tilde{L}_j = \{s_1 V_{J1} + \cdots s_N V_{jN} : \forall i \in I_N, s_i \geq 0, s_1 + \cdots + s_N = 1\},$$

and $\hat{L}_j$ is the convex hull of the vertex set $\{m_\delta(V_{j1}), \ldots, m_\delta(V_{jN})\}$. Since each Γ_i is concave or convex, Γ_i is between $L_i(p)$ and $\tilde{L}_i$, so $\Gamma_i \cap \mathbb{R}_+^N(u(\delta))$ is between $L_i(p) \cap \mathbb{R}_+^N(u(\delta))$ and $\hat{L}_i$, the one above $\Gamma_i \cap \mathbb{R}_+^N(u(\delta))$ is denoted by $L_i^u(\delta)$ and the one below $\Gamma_i \cap \mathbb{R}_+^N(u(\delta))$ is denoted by $L_i^l(\delta)$. Then it follows from the radial projection feature of m_δ (centred at p) that the relationship between the positions of the $L_j^l(\delta)$, $L_j^u(\delta)$, p and $[u(\delta), v(\delta)]$ in $\mathbb{R}_+^N(u(\delta))$ is exactly the same as that of the L_j^l, L_j^u, p and $[0, U]$ in $\mathbb{R}_+^N$. Thus, for each $i \in I_N$, $L_i^l(\delta) \cap S^0(u(\delta), v_i(\delta))$ is strictly above $L_j^u(\delta)$ for all $j \in I_N \setminus \{i\}$ so $\Gamma_i \cap S^0(u(\delta), v_i(\delta))$ is strictly above Γ_j for all $j \in I_N \setminus \{i\}$. Following the conclusion from the previous paragraph and by Lemma 6.4, for each $x_0 \in \mathbb{R}_+^N \setminus \{0\}$ we have either $\omega(x_0) \subset [u(\delta), v(\delta)]$ or $\omega(x_0) \subset \pi_k$ for some $k \in I_N$. Furthermore, for each $x_0 \in \text{int}\Sigma$, we have either $\omega(x_0) \subset \Sigma \cap \pi_k$ and $\alpha(x_0) \subset \Sigma \cap [u(\delta), v(\delta)]$ for some $k \in I_N$ or $\gamma(x_0) \subset \Sigma \cap [u(\delta), v(\delta)]$.

From condition (d) and the feature of m_δ we see that for each $i \in I_N$, $L_i^u(\delta) \cap \pi_i(u(\delta)) \cap [u(\delta), v(\delta)]$ is strictly below $L_j^l(\delta)$ for all $j \in I_N \setminus \{i\}$. By Lemma 6.5 again and repeating the above process, we obtain $\delta_1 \in (\delta, 1)$ so that $[u(\delta), v(\delta)]$ can be replaced by $[u(\delta_1), v(\delta_1)]$ in the above conclusion. Since this process can be repeated as long as $\delta_1 < 1$, by taking the supremum of such δ_1, we obtain the conclusion with $[u(1), v(1)] = \{p\}$. Therefore, for each $x_0 \in \mathbb{R}_+^N \setminus \{0\}$, we have either $\omega(x_0) \subset \pi_i$ for some $i \in I_N$ or $\omega(x_0) = \{p\}$; for each $x_0 \in \text{int}\Sigma$, we have either $\omega(x_0) \subset \Sigma \cap \pi_i$ for some $i \in I_N$ and $\alpha(x_0) = \{p\}$ or $\gamma(x_0) = \{p\}$ so $x_0 = p$. $\square$

7. Conclusion

So far by using geometric analysis, we have obtained a sufficient condition (Theorem 3.1) for a boundary or an interior equilibrium point p to be globally asymptotically stable. We have also derived a sufficient condition (Theorem 5.1) for an interior equilibrium point to be globally repelling on Σ. These results can be applied to a class of systems (6) when each nullcline surface Γ_i is concave or convex, so that an upper plane L_i^u above Γ_i and a lower plane L_i^l below Γ_i can be defined. Then, geometric conditions of the theorems are formed by using the relative positions of the L_i^u and the L_j^l on the boundary $\partial\mathbb{R}_+^N$ within a set $[0, V]$.

Note that Theorem 5.1 for global repulsion cannot be applied to a boundary equilibrium point $p \in \mathbb{R}_+^N \setminus \{0\}$ with support J a proper subset of I_N. However, it can be applied to the $|J|$-dimensional subsystem

$$\dot{x}_i = x_i F_i(x),\ i \in J,\ x \in \cap_{k \in I_N \setminus J} \pi_k \tag{48}$$

as p is an interior equilibrium of (48). If p is globally repelling for the $|J|$-dimensional subsystem (48) and there is a saturated boundary equilibrium point p_0 that is globally attracting for system (6), then it might be possible for p to be globally repelling on Σ.

Theorem 7.1. *Assume that the following conditions hold:*

(a) *The kth axial equilibrium point R_k of (6) is saturated for some $k \in I_N$.*

(b) *For each $i \in I_N$, the nullcline surface Γ_i is either concave or convex. If Γ_i is convex with $F_i(R_k) < 0$ then the function F_i is also convex with $F_i(0) = \max_{x \in \mathbb{R}_+^N} F_i(x)$.*

(c) *For all $i, j \in I_N \setminus \{k\}$, the intersection point R_{ki} of L_k^l with the positive half x_i-axis is above L_j^u.*

(d) *System (6) has an equilibrium $p \in \mathbb{R}_+^N$ with support $J = I_N \setminus \{k\}$ and p as an interior equilibrium point of the subsystem*

$$\dot{x}_i = x_i F_i(x),\ i \in J,\ x \in \pi_k, \tag{49}$$

is globally repelling on $\Sigma \cap \pi_k$.

(e) *Any α limit set $\alpha(x_0)$ consists of a single equilibrium point if $\alpha(x_0) \subset \Sigma \cap \pi_k \cap (\cup_{j \in J} \pi_j)$.*

(f) *The unstable manifold $W^u(q)$ for each equilibrium q in $\Sigma \cap \pi_k \cap (\cup_{j \in J} \pi_j)$ is a subset of $\cup_{j \in J} \pi_j$.*

Then p is globally repelling on Σ and R_k is globally attracting. Moreover, if R_k is above Γ_i for all $i \in J$, then R_k is globally asymptotically stable.

Proof. From condition (c) we know that either Y^J is below L_k^l or $L_k^l \cap [0, Y] \cap \pi_k$ is strictly above L_j^u for all $j \in I_N \setminus \{k\}$, where Y is defined by (28). By conditions (a)–(c) and

Theorem 3.1, R_k is globally attracting. Thus, we have $\omega(x_0) = \{R_k\}$ for any $x_0 \in \Sigma \cap \mathbb{R}_J$. In particular, R_k attracts the compact set $\Sigma_\delta = \{x \in \Sigma : x_k = \delta\}$ for sufficiently small $\delta > 0$. Condition (c) and Lemma 6.3 imply that $\Sigma \cap \pi_k$ is strictly below Γ_k. Thus, Σ_δ is strictly below Γ_k for sufficiently small $\delta > 0$. Since $x_k(t)$ is increasing as long as $x(t)$ is below Γ_k, we have shown that $\alpha(x_0) \subset \Sigma \cap \pi_k$ for $x_0 \in \Sigma_\delta$ and, hence, for all $x_0 \in \Sigma \cap \mathbb{R}_J \setminus \{R_k\}$. As p repels on $\Sigma \cap \pi_k$ by condition (d) and p is below Γ_k, p is a repellor on Σ. Thus, for any $\alpha(x_0) \subset \Sigma \cap \pi_k$, we have either $\alpha(x_0) = \{p\}$ or $\alpha(x_0) \subset \Sigma \cap \pi_k \cap (\cup_{j \in J} \pi_j)$. By condition (e) we know that, as $t \to -\infty$, $x(x_0, t)$ converges to p or an equilibrium point in $\Sigma \cap \pi_k \cap (\cup_{j\in J}\pi_j)$ for $x_0 \in \Sigma \cap \mathbb{R}_J \setminus \{R_k\}$. Now we claim that $\alpha(x_0) = \{p\}$ for all $x_0 \in \mathrm{int}\Sigma$. Indeed, $x_0 \gg 0$ so $x_0 \notin \pi_i$ for all $i \in I_N$. By condition (f), $x_0 \notin W^u(q)$ for any equilibrium point $q \in \Sigma \cap \pi_k \cap (\cup_{j\in J}\pi_j)$. Thus, $\alpha(x_0) \not\subset \Sigma \cap \pi_k \cap (\cup_{j\in J}\pi_j)$ so $\alpha(x_0) = \{p\}$. Therefore, p is globally repelling on Σ. Finally, if R_k is above Γ_i for all $i \in J$, then the Jacobian matrix $Df(R_k)$ has N negative eigenvalues, so R_k is globally asymptotically stable. □

Example 7.2. Consider the system

$$
\begin{aligned}
\dot{x}_1 &= x_1(1 - 2ax_1 - ax_1^2 - x_2 - x_3 - x_4) = x_1F_1(x),\\
\dot{x}_2 &= x_2(1 - x_1 - 2ax_2 - ax_2^2 - x_3 - x_4) = x_2F_2(x),\\
\dot{x}_3 &= x_3(1 - x_1 - x_2 - 2ax_3 - ax_3^2 - x_4) = x_3F_3(x),\\
\dot{x}_4 &= x_4(2 - 3ax_1 - 3ax_2 - 3ax_3 - x_4) = x_4F_4(x),
\end{aligned} \tag{50}
$$

where $a \in (0, 0.3]$ is a constant satisfying (43). The 3-dimensional subsystem on π_4 is the system (42) in Example 5.4. So $p = (p_0, p_0, p_0, 0)^T$ is globally repelling on $\Sigma \cap \pi_4$. This shows that system (50) satisfies condition (d) of Theorem 7.1. The axial equilibrium point $R_4 = (0, 0, 0, 2)^T$ is above Γ_1, Γ_2 and Γ_3 so it is saturated. Clearly, Γ_4 is a plane and F_1, F_2, F_3 are convex (see Example 5.4 in section 5). Thus, (50) meets conditions (a) and (b) of Theorem 7.1. The intersection points of L_1^u, L_2^u, L_3^u and L_4^l with the positive half x_1-axis are $(\frac{1+ap_0^2}{2a(1+p_0)}, 0, 0, 0)^T, (1, 0, 0, 0)^T, (1, 0, 0, 0)^T$ and $R_{41} = (\frac{2}{3a}, 0, 0, 0)^T$ respectively. As $p_0 < \frac{1}{2}$ and $a \le 0.3$, we have

$$
1 < \frac{1 + ap_0^2}{0.9} \le \frac{1 + ap_0^2}{2a(1 + p_0)} < \frac{1}{2a} + \frac{1}{2[(\frac{1}{p_0})^2 + \frac{1}{p_0}]} < \frac{1}{2a} + \frac{1}{12} = \frac{6 + a}{12a} < \frac{2}{3a}.
$$

Thus, R_{41} is above L_1^u, L_2^u and L_3^u. By symmetry, R_{42} and R_{43} are also above L_1^u, L_2^u and L_3^u. This shows that (50) satisfies condition (c) of Theorem 7.1. To check conditions (e) and (f), we note that the phase portrait on $\Sigma \cap \pi_4$ is given by Figure 4. From the flow on $\Sigma \cap \pi_4$ we see that any $\alpha(x_0) \subset \Sigma \cap \pi_4$ must consist of a single equilibrium point. Thus, condition (e) of Theorem 7.1 holds for (50). Since $\Sigma \cap \pi_4$ is strictly below Γ_4, for any equilibrium point $q \in \Sigma \cap \pi_4 \cap (\pi_1 \cup \pi_2 \cup \pi_3)$, $Df(q)$ has an eigenvector in $\pi_1 \cup \pi_2 \cup \pi_3$ transverse to π_4 corresponding to the positive eigenvalue $F_4(q)$. By the invariance of each π_i, we have $W^u(q) \subset (\pi_1 \cup \pi_2 \cup \pi_3)$. Thus, (50) satisfies condition (f) of Theorem 7.1. Then, by Theorem 7.1, R_4 is globally asymptotically stable and p is globally repelling on Σ.

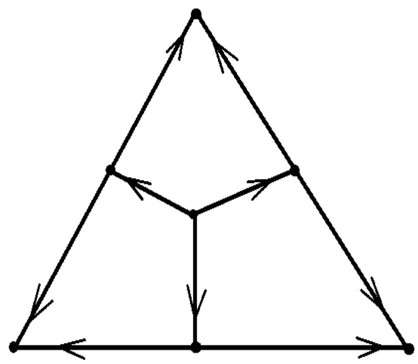

FIGURE 4. Phase portrait for system (50) on $\Sigma \cap \pi_4$.

Discussion. For any equilibrium $p \in \Sigma$ with support $J \subset I_N$, we call p *saturated in reversed time* if $F_i(p) \geq 0$ for all $i \in I_N$. As $F_i(p)$ is an eigenvalue of the Jacobian matrix $Df(p)$ if $i \in I_N \setminus J$, it follows that a necessary condition for p to be a repellor on Σ is that p must be saturated in reversed time. Combining Theorems 5.1 and 7.1, we have obtained sufficient conditions for an equilibrium p saturated in reversed time to be globally repelling on Σ if p has at most one zero component (i.e. $|J| \geq N-1$). However, if p has more than one zero components, our theorems for global repulsion are not applicable. Does the geometric method used here still have the power to deal with the problem of global repulsion when $|J| < N-1$? This is left as an open problem.

APPENDIX

The proofs of Propositions 2.2 and 2.3 are given below.

Proof of Proposition 2.2. (i) and (ii) are straightforward from the definitions of convexity and concavity of a surface and convexity of a set.

(iii) Taking any $x, y \in \Gamma$ with $\alpha < G(0)$ in the range of G, by the convexity of G we have

$$\forall s \in [0,1], G(sx + (1-s)y) \geq sG(x) + (1-s)G(y) = s\alpha + (1-s)\alpha = \alpha. \tag{51}$$

Since $0 \in \Gamma^-$ and $G(0) > \alpha$, we must have $G(w) > \alpha$ for all $w \in \Gamma^-$. So $sx+(1-s)y \in \Gamma^- \cup \Gamma$ for all $s \in [0,1]$. This shows that Γ is convex.

(iv) Since $-G$ is convex, for any $\alpha > G(0)$ in the range of G and $x, y \in \Gamma$, we have

$$\forall s \in [0,1], -G(sx + (1-s)y) \geq -sG(x) - (1-s)G(y) = -s\alpha - (1-s)\alpha = -\alpha. \tag{52}$$

So $G(sx + (1-s)y) \leq \alpha$ for all $s \in [0,1]$. Then $G(0) < \alpha$ and $0 \in \Gamma^-$ imply $G(w) < \alpha$ for all $w \in \Gamma^-$. Thus, $sx + (1-s)y \in \Gamma^- \cup \Gamma$ for all $s \in [0,1]$, so Γ is convex.

(v) Since G is convex, (51) holds for all $x, y \in \Gamma$ with $\alpha > G(0)$ in the range of G. Since $0 \in \Gamma^-$ and $G(0) < \alpha$, we must have $G(w) < \alpha$ for all $w \in \Gamma^-$ and $G(w) \geq \alpha$ for all $w \in \Gamma \cup \Gamma^+$. It then follows from (51) that $\overline{xy} \subset \Gamma \cup \Gamma^+$, so Γ is concave.

(vi) Since $-G$ is convex, (52) holds for all $x, y \in \Gamma$ with $\alpha < G(0)$ in the range of G. Since $0 \in \Gamma^-$ and $G(0) > \alpha$, we must have $G(w) > \alpha$ for all $w \in \Gamma^-$ and $G(w) \leq \alpha$ for all $w \in \Gamma \cup \Gamma^+$. It then follows from (52) that $\overline{xy} \subset \Gamma \cup \Gamma^+$. This shows the concavity of Γ. □

Proof of Proposition 2.3. Note that the sign of $G(x) - \alpha$ on Γ^- is opposite to that on Γ^+. We first assume $G(x) - \alpha < 0$ for $x \in \Gamma^-$ and $G(x) - \alpha > 0$ for $x \in \Gamma^+$.

(a) From the convexity of Γ and Proposition 2.2 (ii), $\Gamma^- \cup \Gamma$ is a convex set. So, for each $x \in \Gamma^- \cup \Gamma \setminus \{u\}$, $\overline{xu} \subset \Gamma^- \cup \Gamma$. Thus,

$$\forall s \in [0,1], G(sx + (1-s)u) - G(u) = G(u + s(x-u)) - \alpha \leq 0. \tag{53}$$

From this it follows that

$$D_{\overrightarrow{ux}} G(u) = \lim_{s \to 0^+} \frac{1}{s}[G(u + s(x-u)) - G(u)] \leq 0.$$

Since the directional derivative of G satisfies

$$D_{\overrightarrow{ux}} G(u) = \nabla G(u) \frac{(x-u)}{\|x-u\|} = \frac{1}{\|x-u\|} \nabla G(u)(x-u),$$

we obtain $\nabla G(u)(x-u) \leq 0$ for all $x \in \Gamma^- \cup \Gamma$. This shows that $\Gamma^- \cup \Gamma$ is on one side of $T_u(\Gamma)$. As $0 \in \Gamma^-$ and 0 is below $T_u(\Gamma)$, the set $\Gamma^- \cup \Gamma$ is below $T_u(\Gamma)$ and so is Γ.

To show that Γ is above $L(\Gamma)$, we need only show that $L(\Gamma)$ is below Γ, i.e. $L(\Gamma) \subset \Gamma^- \cup \Gamma$. If R_i, R_j exist for some distinct $i, j \in I_N$, as $R_i, R_j \in \Gamma$, by the convexity of Γ and Proposition 2.2 (i), $\overline{R_i R_j} \subset (\Gamma^- \cup \Gamma) \cap L(\Gamma)$. If R_i exists but R_j does not exist, then $J_j \subset \Gamma^-$. Let $Q_j \in J_j$ with v_j as its jth component. Then $\overline{R_i Q_j} \subset \Gamma^- \cup \Gamma$. As the half line $L_{(R_i)j}$ passing through R_i and parallel to J_j lies in $L(\Gamma)$, by the definition of $L(\Gamma)$, and is the limit of $\overline{R_i Q_j}$ as $v_j \to +\infty$, we also have $L_{(R_i)j} \subset (\Gamma^- \cup \Gamma) \cap L(\Gamma)$. This shows that each one-dimensional edge of $L(\Gamma)$ is contained in $\Gamma^- \cup \Gamma$. Since $\Gamma^- \cup \Gamma$ is convex and $L(\Gamma)$ is both convex and concave, for any $x, y \in (\Gamma^- \cup \Gamma) \cap L(\Gamma)$, we must have $\overline{xy} \subset (\Gamma^- \cup \Gamma) \cap L(\Gamma)$. As each two-dimensional face of $L(\Gamma)$ consists of $\overline{xy}$ with x, y taking all the points in two one-dimensional edges, all two-dimensional faces of $L(\Gamma)$ are contained in $\Gamma^- \cup \Gamma$. Repeating this process a finite number of times, we obtain $L(\Gamma) \subset \Gamma^- \cup \Gamma$. Hence, $L(\Gamma)$ is below Γ.

(b) By the concavity of Γ and Proposition 2.2 (ii), $\Gamma \cup \Gamma^+$ is convex. So, for any $x \in \Gamma \cup \Gamma^+ \setminus \{u\}$, we have $\overline{xu} \subset \Gamma \cup \Gamma^+$. Thus,

$$\forall s \in [0,1], G(sx + (1-s)u) - G(u) = G(u + s(x-u)) - \alpha \geq 0, \tag{54}$$

from which follows $D_{\overrightarrow{ux}}G(u) \geq 0$. As $D_{\overrightarrow{ux}}G(u) = \frac{1}{\|x-u\|}\nabla G(u)(x-u)$, we obtain $\nabla G(u)(x-u) \geq 0$ for all $x \in \Gamma \cup \Gamma^+$. Thus, $\Gamma \cup \Gamma^+$ is on one side of $T_u(\Gamma)$. We shall see that

$$\forall w \in \Gamma, \mathbb{R}^N_+(w) = \{x \in \mathbb{R}^N_+ : x \geq w\} \subset \Gamma \cup \Gamma^+. \tag{55}$$

So, from this follows $\mathbb{R}^N_+(u) \subset \Gamma \cup \Gamma^+$ since $u \in \Gamma$. As $2u \in \mathbb{R}^N_+(u)$ so $2u \in \Gamma \cup \Gamma^+$, we have $\nabla G(u)(2u-u) = \nabla G(u)u \geq 0$. This together with $\nabla G(u)u \neq 0$ implies that $\nabla G(u)u > 0$ and $\nabla G(u)(0-u) < 0$. Thus, $\Gamma \cup \Gamma^+$ is on one side of $T_u(\Gamma)$ but 0 is on the other side of $T_u(\Gamma)$. Since 0 is below $T_u(\Gamma)$ by definition, $\Gamma \cup \Gamma^+$ is above $T_u(\Gamma)$ and so is Γ.

To show that Γ is below $L(\Gamma)$, we need only show that $L(\Gamma)$ is above Γ, i.e. $L(\Gamma) \subset \Gamma \cup \Gamma^+$. For this purpose, we first show (55). We claim that $L_{(w)i} \subset \Gamma \cup \Gamma^+$ for all $i \in I_N$. Indeed, if R_i does not exist, then the half line $L_{(w)i}$ lies in $\Gamma \cup \Gamma^+$ by assumption. If R_i exists, then, for any $Q_i \in J_i$ with v_i as its ith component and $Q_i > R_i$, by the convexity of $\Gamma \cup \Gamma^+$ and $Q_i, w \in \Gamma \cup \Gamma^+$, we have $\overline{wQ_i} \subset \Gamma \cup \Gamma^+$. Since $L_{(w)i}$ is the limit of $\overline{wQ_i}$ as $v_i \to +\infty$, we also have $L_{(w)i} \subset \Gamma \cup \Gamma^+$. Then it follows from the convexity of $\Gamma \cup \Gamma^+$ that

$$L_{(w)i} \times L_{(w)j} = \{sx + (1-s)y : x \in L_{(w)i}, y \in L_{(w)j}, s \in [0,1]\} \subset \Gamma \cup \Gamma^+$$

for all $i, j \in I_N$. Since $\mathbb{R}^N_+(w) = L_{(w)1} \times L_{(w)2} \times \cdots \times L_{(w)N}$, repeating the above process a finite number of times, we have shown (55).

Now if R_i, Rj exist for some distinct $i, j \in I_N$, $\overline{R_iR_j}$ is a one-dimensional edge of $L(\Gamma)$ and $\overline{R_iR_j} \subset \Gamma \cup \Gamma^+$ by the convexity of $\Gamma \cup \Gamma^+$. If R_i exists but R_j does not exist, then $L_{(R_i)j}$ is a one-dimensional edge of $L(\Gamma)$ and $L_{(R_i)j} \subset \Gamma \cup \Gamma^+$ by assumption. Thus, every one-dimensional edge of $L(\Gamma)$ is contained in $\Gamma \cup \Gamma^+$. Then, following the same reasoning as we did in part (a), we obtain $L(\Gamma) \subset \Gamma \cup \Gamma^+$, so $L(\Gamma)$ is above Γ.

The proof is complete under the assumption $G(x) - \alpha < 0$ for $x \in \Gamma^-$ and $G(x) - \alpha > 0$ for $x \in \Gamma^+$. If $G(x) - \alpha > 0$ for $x \in \Gamma^-$ and $G(x) - \alpha < 0$ for $x \in \Gamma^+$, the above proof is still valid after swapping "$\leq$" and "$\geq$" in (53), (54) and some related inequalities. □

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References

[1] S. Baigent and Z. Hou, Global stability of interior and boundary fixed points for Lotka-Volterra systems, *Differential Equations Dynam. Systems* **20** (2012), 53–66.

[2] M. W. Hirsch, Systems of differential equations that are competitive or coorperative. I: Limit sets, *SIAM Journal on Mathematical Analysis* **13** (1982), 167–179.

[3] M. W. Hirsch, Systems of differential equations that are competitive or coorperative. II: Convergence almost everywhere, *SIAM Journal on Mathematical Analysis* **16** (1985), 423–439.

[4] M. W. Hirsch, Systems of differential equations that are competitive or coorperative. III: Competing species, *Nonlinearity* **1** (1988), 51-71.

[5] Z. Hou and S. Baigent, Fixed point global attractors and repellors in competitive Lotka-Volterra systems, *Dynamical Systems* **26** (2011), 367–390.

[6] Z. Hou and S. Baigent, Global stability and repulsion in autonomous Kolmogorov systems, *Communications in Pure and Applied Analysis* **14** No. **3** (2015), 1205–1238.

[7] Z. Hou, Global attractor in autonomous competitive Lotka-Volterra systems, *Proc. Amer. Math. Soc.* **127** No. **12** (1999), 3633-3642.

[8] Z. Hou, Global attractor in competitive Lotka-Volterra systems, *Math. Nachr.* **282** No. **7** (2009), 995–1008.

[9] Z. Hou, Geometric method for a global repellor in competitive Lotka-Volterra systems, *Nonlinear Analysis* **71** (2009), 3587–3595.

[10] J. Jiang and L. Niu, On the validity of Zeeman's classification for three-dimensional competitive differential equations with linearly determined nullclines, *J. Differential Equations* **263** (2017), 7753–7781.

[11] J. Jiang, L. Niu and D. Zhu, On the complete classification of nullcline stable competitive three-dimensional Gompertz models, *Nonlinear Analysis: RWA* **20** (2014), 21–35.

[12] Y. Takeuchi, "Global dynamical properties of Lotka-Volterra systems", World Scientific, Singapore, 1996.

[13] Y. Yu, W. Wang and Z. Lu, Global stability of Gompertz model of three competitive populations, *J. Math. Anal. Appl.* **334** (2007), 333–348.

[14] M. L. Zeeman, Hopf bifurcations in competitive three-dimensional Lotka-Volterra systems, *Dynamics and Stability of Systems* **8** (1993), 189–217.

[15] E. C. Zeeman and M. L. Zeeman, From local to global behavior in competitive Lotka-Volterra systems, *Trans. Amer. Math. Soc.* **355** (2003), 713–734.

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