GEOMETRIC METHOD FOR GLOBAL STABILITY AND REPULSION IN KOLMOGOROV SYSTEMS

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Abstract. A class of autonomous Kolmogorov systems that are dissipative and competitive with the origin as a repellor are considered when each nullcline surface is either concave or convex. Geometric method is developed by using the relative positions of the upper and lower planes of the nullcline surfaces for global asymptotic stability of an interior or a boundary equilibrium point. Criteria are also established for global repulsion of an interior or a boundary equilibrium point on the carrying simplex. This method and the theorems can be viewed as a natural extension of those results for Lotka-Volterra systems in the literature.

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1. Introduction

In this paper, we consider differential systems of the form

\[ \dot{x}_i = x_i F_i(x), \quad i \in I_N = \{1, 2, \ldots, N\}, \tag{1} \]

known as Kolmogorov systems. Since such systems typically model populations of species, genes, molecules, and so on, where each \( x_i \) denotes the population size and \( F_i \) the intrinsic growth rate of the \( i \)th species, the phase space for the study of (1) is restricted to the first orthant \( \mathbb{R}_+^N \) or an invariant subset of \( \mathbb{R}_+^N \). We assume that \( F : \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N \) is at least \( C^1 \). Some particular class of examples include Lotka-Volterra systems where each \( F_i \) is an affine function,

\[ \dot{x}_i = r_i x_i (1 - a_{i1} x_1 - \cdots - a_{iN} x_N), \quad i \in I_N, \tag{2} \]

Gompertz models where each \( F_i \) has the form \( F_i(y) = r_i \ln \frac{1}{y} \),

\[ \dot{x}_i = r_i x_i \ln \frac{1}{a_{i1} x_1 + \cdots + a_{iN} x_N}, \quad i \in I_N, \tag{3} \]
Leslie/Gower (or Atkinson/Allen) models where each $F_i$ has the form $F_i(y) = c_i \left( \frac{1+r_i}{r_i+y} - 1 \right)$,

\begin{equation}
\dot{x}_i = c_i x_i \left( 1 + \frac{1+r_i}{r_i + a_i x_1 + \cdots + a_i x_N} - 1 \right), \quad i \in I_N,
\end{equation}

(4)

Ricker models where each $F_i$ has the form $F_i(y) = c_i \left( e^{r_i(1-y)} - 1 \right)$,

\begin{equation}
\dot{x}_i = c_i x_i \left( \exp\left[ r_i(1 - a_i x_1 - \cdots - a_i x_N) \right] - 1 \right), \quad i \in I_N.
\end{equation}

(5)

There is an extensive literature in population ecology and dynamical systems on (1) and its various particular cases. To name a few, Hirsch [2, 3, 4] investigated the dynamics of competitive and cooperative systems and Zeeman [14] studied bifurcations in competitive Lotka-Volterra systems focusing on three-dimensional cases. Hirsch [4] showed that competitive dissipative systems with the origin as a repellor has a global attractor $\Sigma$ on $\mathbb{R}_N^+ \setminus \{0\}$, where $\Sigma$ is homeomorphic to the standard $(N-1)$-simplex $\Delta^{N-1}$ by radial projection. Zeeman [14] called $\Sigma$ carrying simplex and used geometric analysis of nullclines of Lotka-Volterra systems to classify three-dimensional systems with stable nullclines into 33 classes, the dynamics of each class has a clear description on $\Sigma$. Similar to [14], Jiang, Niu and Zhu [11] did a complete classification of nullcline stable competitive three-dimensional Gompertz models. Jiang and Niu [10] further extended such classification to three-dimensional competitive systems with linearly determined nullclines including (2)–(5) and more. For a wider survey, see the above articles and the references cited therein.

Here we are concerned with the asymptotic behaviour of (1) when there is an equilibrium point $p \in \mathbb{R}_N^+$ that is globally attracting or repelling. For Lotka-Volterra systems, a criteron by Lyapunov function method is known for global asymptotic stability of a boundary or interior equilibrium point (see Theorem 3.2.1 in [12]). For competitive Lotka-Volterra systems, Zeeman and Zeeman [15] developed the split Lyapunov function method and provided sufficient conditions for an interior equilibrium point to be a global attractor or a global repellor. Hou and Baigent [5, 1] further developed the split Lyapunov function method and extended the above results for global attraction or repulsion to a boundary as well as interior equilibrium point of Lotka-Volterra systems that may not be competitive. In [6], the authors applied the Lyapunov function method and the split Lyapunov function method to dissipative systems (1) with both 0 and $\infty$ as repellors in $\mathbb{R}_N^+$, and obtained criteria for global asymptotic stability or global repulsion of an equilibrium point. These results can be viewed as further extension of [15, 5, 1] from Lotka-Volterra systems to Kolmogorov systems (1). Yu, Wang and Lu [13] obtained sufficient conditions for global stability of three-dimensional competitive Gompertz models. For Lotka-Volterra systems, there are also results for global repulsion or attraction by methods that are not using Lyapunov functions. For example, Hou used geometric method for global attraction [7, 8] and global repulsion [9].
In this paper, we are going to deal with a class of competitive dissipative systems (1) that has a carrying simplex $\Sigma$ and each nullcline is a concave or convex surface. By using geometric analysis of such nullclines, we provide sufficient conditions for an equilibrium point $p \in \mathbb{R}_+^N \setminus \{0\}$ to be globally asymptotically stable or for $p$ to be globally repelling.

The rest of the paper is organised as follows: 2. System description and notation. 3. Geometric method for global stability. 4. Proof of Theorem 3.1. 5. Geometric method for global repulsion. 6. Proof of Theorem 5.1. 7. Conclusion.

2. System description and notation

For convenience, we rewrite system (1) as

$$\dot{x} = f(x) \equiv D[x]F(x), \quad x \in \mathbb{R}_+^N,$$

where $D[x] = \text{diag}[x_1, \ldots, x_N]$ and $F \in C^1(\mathbb{R}_+^N, \mathbb{R}^N)$. Let $\text{int}\mathbb{R}_+^N$ denote the interior of $\mathbb{R}_+^N$. For any $x, y \in \mathbb{R}_+^N$, we write $x \ll y$ or $y \gg x$ if $y - x \in \text{int}\mathbb{R}_+^N$, $x \leq y$ or $y \geq x$ if $y - x \in \mathbb{R}_+^N$, and $x < y$ or $y > x$ if $x \leq y$ but $x \neq y$. We view each $x \in \mathbb{R}_+^N$ as a column vector and use $x^T$ as the transpose of $x$. With a slight abuse of notation, we shall use 0 for scalar and vector zero as well as the origin in $\mathbb{R}_+^N$.

Throughout the paper we assume that (6) meets the following assumptions:

(A1) $F(0) \gg 0$ so that the origin 0 is a repellor.

(A2) The system is dissipative: there is a compact invariant set that attracts uniformly each compact set of initial points.

(A3) The system is competitive: $\frac{\partial F_i}{\partial x_j} \leq 0$ for all $i, j \in I_N$ with $i \neq j$.

(A4) $\frac{\partial F_i}{\partial x_j}(p) < 0$ for every fixed point $p \in \mathbb{R}_+^N \setminus \{0\}$ and all $i, j \in I_N$.

Then the basin of repulsion of 0 in $\mathbb{R}_+^N$, $\text{Br}(0) = \{x \in \mathbb{R}_+^N : \alpha(x) = \{0\}\}$, is a bounded open set of $\mathbb{R}_+^N$ and $\Sigma = \partial \text{Br}(0) \setminus \text{Br}(0)$ is known as the carrying simplex. The theorem below (see Theorem 1.7 in [4] or Theorem 2.1 in [14]) describes the dynamics of (6) in terms of $\Sigma$.

Theorem 2.1. Under the assumptions (A1)–(A4), every trajectory in $\mathbb{R}_+^N \setminus \{0\}$ is asymptotic to one in $\Sigma$, and $\Sigma$ is a Lipschitz submanifold homeomorphic to the unit simplex in $\mathbb{R}_+^N$ by radial projection.

Now we explain some concepts that will be used later. Let $G \in C^1(\mathbb{R}_+^N, \mathbb{R})$ such that, for some $\alpha$ in the range of $G$,

$$\Gamma = \{x \in \mathbb{R}_+^N : G(x) = \alpha\}$$

is a connected $(N - 1)$-dimensional surface restricted to $\mathbb{R}_+^N$. Suppose that $\mathbb{R}_+^N$ is divided into three mutually exclusive connected subsets $\Gamma^-$, $\Gamma$ and $\Gamma^+$ with $0 \in \Gamma^-$. Then a point
$x \in \mathbb{R}^N_+$ is said to be below (on or above) $\Gamma$ if $x \in \Gamma^-$ ($x \in \Gamma$ or $x \in \Gamma^+$); a nonempty set $S \subset \mathbb{R}^N_+$ is said to be below (on or above) $\Gamma$ if $S \subset \Gamma^- \cup \Gamma$ ($\Gamma \subset S$ or $S \subset \Gamma^+ \cup \Gamma$); $S \subset \mathbb{R}^N_+$ is said to be strictly below (above) $\Gamma$ if $S \subset \Gamma^-$ ($S \subset \Gamma^+$).

The function $G$ is said to be convex if $G(sx + (1-s)y) \geq sG(x) + (1-s)G(y)$ holds for any two points $x, y$ in its domain and all $s \in [0, 1]$. For a surface $\Gamma$ with the division of $\mathbb{R}^N_+$ into $\Gamma^-$, $\Gamma$ and $\Gamma^+$, $\Gamma$ is said to be convex (concave) if for any distinct points $x, y \in \Gamma$, the line segment $\overline{xy} = \{tx + (1-t)y : 0 \leq t \leq 1\}$ is contained in $\Gamma^- \cup \Gamma$ ($\Gamma^+ \cup \Gamma$). Recall that a nonempty set $S \subset \mathbb{R}^N_+$ is called convex if $\overline{xy} \subset S$ for all $x, y \in S$. From these concepts we obtain some observations summarised in the following proposition.

**Proposition 2.2.** Assume that $\Gamma$ defined by (7) divides $\mathbb{R}^N_+$ into $\Gamma^-$, $\Gamma$ and $\Gamma^+$ as described above. Then the following statements are true.

(i) If $\Gamma$ is a plane in $\mathbb{R}^N_+ \setminus \{0\}$ then it is both convex and concave.

(ii) The surface $\Gamma$ is convex if and only if the set $\Gamma^- \cup \Gamma$ is convex; $\Gamma$ if concave if and only if $\Gamma \cup \Gamma^+$ is convex.

(iii) If the function $G$ is convex with $G(0) = \max_{x \in \mathbb{R}^N_+} G(x)$, then $\Gamma$ is also convex for any $\alpha < G(0)$ in the range of $G$.

(iv) If the function $-G$ is convex with $G(0) = \min_{x \in \mathbb{R}^N_+} G(x)$, then $\Gamma$ is also convex for any $\alpha > G(0)$ in the range of $G$.

(v) If the function $G$ is convex with $G(0) = \min_{x \in \mathbb{R}^N_+} G(x)$, then $\Gamma$ is concave for any $\alpha > G(0)$ in the range of $G$.

(vi) If the function $-G$ is convex with $G(0) = \max_{x \in \mathbb{R}^N_+} G(x)$, then $\Gamma$ is concave for any $\alpha < G(0)$ in the range of $G$.

The proof of Proposition 2.2 can be found in the Appendix at the end of this paper.

For any point $u \in \Gamma$, the tangent plane of $\Gamma$ at $u$ is

$$T_u(\Gamma) = \{x \in \mathbb{R}^N_+ : \nabla G(u)(x - u) = 0\} \tag{8}$$

if $\nabla G(u) \neq 0$, where $\nabla G(u) = (\frac{\partial G}{\partial u_1}, \ldots, \frac{\partial G}{\partial u_N})$ is viewed as a row vector. Denote the positive half $x_i$-axis by $J_i$ for all $i \in I_N$. Next, we assume that $\Gamma$ intersects at least one of the half axes $J_i$. If $\Gamma \cap J_i \neq \emptyset$, we assume that $R_i$ is the unique intersection point, i.e. $\Gamma \cap J_i = \{R_i\}$. If $\Gamma \cap J_i = \emptyset$, we say that the point $R_i$ does not exist. Now let $L(\Gamma)$ be the $(N-1)$-dimensional plane in $\mathbb{R}^N_+$ determined by these intersection points: If $R_i$ exists then $R_i \in L(\Gamma)$, if $R_i$ does not exist then $L(\Gamma)$ is parallel to the half axis $J_i$. Then the relative positions of $T_u(\Gamma), \Gamma$ and $L(\Gamma)$ are clear from the proposition below when $\Gamma$ is convex or concave.

**Proposition 2.3.** (a) Suppose $\Gamma$ given by (7) is convex. Then $\Gamma$ is above $L(\Gamma)$ but below $T_u(\Gamma)$ for any $u \in \Gamma$. (b) Suppose $\Gamma$ is concave and, if $\Gamma \cap J_j = \emptyset$ for some $j \in I_N$, for
any point \( w \in \Gamma \), the half line \( L(w)_j \) passing through \( w \) and parallel to \( J_j \) is contained in \( \Gamma \cap \Gamma^+ \). Then \( \Gamma \) is below \( L(\Gamma) \) and, for any \( u \in \Gamma \) with \( \nabla G(u)u \neq 0 \), \( \Gamma \) is above \( T_u(\Gamma) \).

The proof of Proposition 2.3 is also left to the Appendix.

For each \( i \in I_N \), the ith nullcline surface of (6) is defined by

\[
\Gamma_i = \{ x \in \mathbb{R}^N : F_i(x) = 0 \}.
\]

If \( \mathbb{R}_+^N \) is divided into three mutually exclusive connected subsets \( \Gamma_i^-, \Gamma_i, \Gamma_i^+ \) then the assumptions (A1)-(A3) imply that \( \dot{x}_i > 0 \) for \( x \in \Gamma_i^- \) and \( \dot{x}_i < 0 \) for \( x \in \Gamma_i^+ \). The ith coordinate plane is denoted by

\[
\pi_i = \{ x \in \mathbb{R}^N_+ : x_i = 0 \}.
\]

For any \( u, v \in \mathbb{R}^N_+ \) with \( u \leq v, i \in I_N \), and \( I \subset I_N \), define

\[
\{ u, v \} = \{ x \in \mathbb{R}^N_+ : u \leq x \leq v \},
\]

\[
\mathbb{R}^N_+(u) = \{ x \in \mathbb{R}^N_+ : x \geq u \},
\]

\[
\pi_i(u) = \{ x \in \mathbb{R}^N_+(u) : x_i = u_i \},
\]

\[
S(u, v_i) = \{ x \in \mathbb{R}^N_+(u) : x_i \geq v_i \},
\]

\[
S^0(u, v_i) = \{ x \in \mathbb{R}^N_+(u) : x_i > v_i \},
\]

\[
\mathbb{C}_I^0 = \{ x \in \mathbb{R}^N_+ : \forall i \in I, x_i = 0; \forall j \in I_N \setminus I, x_j > 0 \},
\]

\[
\mathbb{R}_I = \{ x \in \mathbb{R}^N_+ : \forall j \in I_N \setminus I, x_j > 0 \}.
\]

Then \( \mathbb{R}^N_+(0) = \mathbb{R}^N_+ \), \( \pi_i(0) = \pi_i \), \( \mathbb{C}_I^0 = \{ 0 \}, \mathbb{C}_0^0 = \text{int} \mathbb{R}^N_+ = \mathbb{R}_0 \) and \( \mathbb{R}_I = \mathbb{R}^N_+ \). For any nonempty set \( A \subset \mathbb{R}^N \) and \( \varepsilon > 0 \), the \( \varepsilon \)-neighbourhood of \( A \) is denoted by

\[
\mathcal{B}(A, \varepsilon) = \{ x \in \mathbb{R}^N : \| x - a \| < \varepsilon \text{ for some } a \in A \}.
\]

Suppose \( p \in \mathbb{C}_I^0 \) with \( I \neq I_N \) is an equilibrium point. Then \( p \in \Sigma \). We say that \( p \) is globally attracting if \( \lim_{t \to +\infty} x(x_0, t) = p \) for all \( x_0 \in \mathbb{R}_I \); \( p \) is globally repelling if for all \( x_0 \in (\Sigma \setminus \{ p \}) \cap \mathbb{R}_I \), we have \( \omega(x_0) \subset (\cup_{j \in I_N \setminus I} \pi_j) \cap \Sigma \) and \( \alpha(x_0) = \{ p \} \); \( p \) is called globally asymptotically stable if \( p \) is globally attracting and \( p \) is locally asymptotically stable in \( \mathbb{R}^N_+ \). Note that since \( \Sigma \) is a global attractor of (6) in \( \mathbb{R}^N_+ \setminus \{ 0 \} \), if \( p \) is globally repelling, \( p \) is essentially repelling on \( \Sigma \cap \mathbb{R}_I \). So we also say that \( p \) is globally repelling on \( \Sigma \).

### 3. Geometric method for global stability

In this section, we assume that \( p \in \mathbb{R}^N_+ \setminus \{ 0 \} \) is a nontrivial equilibrium point of (6) with support \( J = \{ j \in I_N : p_j > 0 \} \), i.e. \( p \in \mathbb{C}_I^0 \) for \( I = I_N \setminus J \neq I_N \). Then \( p \) is an interior equilibrium if \( J = I_N \) or on the boundary \( \partial \mathbb{R}^N_+ \) if \( J \) is a proper subset of \( I_N \). We call \( p \) saturated if \( F_i(p) \leq 0 \) for all \( i \in I_N \). Then, from the fact that \( F_i(p) \) is an eigenvalue of the
Jacobian matrix $Df(p)$ if $i \in I_N \setminus J$, it follows that a necessary condition for $p$ to be stable is that $p$ must be saturated.

Now assume that $p$ is a saturated equilibrium point. For each $i \in I_N$, if $F_i(p) = 0$ then $p$ is on the $i$th nullcline surface

$$
(19) \quad \Gamma_i = \{x \in \mathbb{R}_+^N : F_i(x) = 0\}
$$

and $\Gamma_i$ at $p$ has a tangent plane

$$
(20) \quad L_i(p) = \{x \in \mathbb{R}_+^N : \nabla F_i(p)(x - p) = 0\}
$$
as $\nabla F_i(p) \neq 0$ by (A4). We assume that on each positive half $x_i$-axis, $J_i$, (6) has a unique equilibrium point $R_i$, i.e. $\Gamma_i \cap J_i = \{R_i\}$. Assume also that each $\Gamma_i$ has at most one intersection point $R_{ij}$ with $J_j$ for each $j \in I_N$ ($R_{ii} = R_i$). Let $\tilde{L}_i$ be the plane in $\mathbb{R}_+^N$ determined by the intersection points $R_{ij}$ of $\Gamma_i$ with the coordinate axes: If $\Gamma_i$ intersects $J_j$ at a point $R_{ij}$ then $R_{ij} \in \tilde{L}_i$; if $R_{ij}$ does not exist then $\tilde{L}_i$ is parallel to $J_j$. Note that (A4) implies that $\nabla F_i(p) \ll 0$ and $p > 0$ so $\nabla F_i(p)p < 0$. If $\Gamma_i \cap J_j = \emptyset$, then, for any point $w \in \Gamma_i$, since $F_i(w) = 0$ and $\frac{\partial F_i}{\partial x_j} \leq 0$ by (A3), $F_i(x)$ is nonincreasing on $L_{(w)j}$ so $F_i(x) \leq F_i(w) = 0$ for all $x \in L_{(w)j}$. Thus, $L_{(w)j} \subset \Gamma_i \cup \Gamma_i^+$. Then, if $\Gamma_i$ is convex or concave, from Proposition 2.3 we see that $\Gamma_i$ is between $L_i(p)$ and $\tilde{L}_i$: if $\Gamma_i$ is convex then it is below $L_i(p)$ but above $\tilde{L}_i$; if $\Gamma_i$ is concave then it is above $L_i(p)$ but below $\tilde{L}_i$.

Suppose $F_i(p) < 0$ for some $i \in I_N \setminus J$. Then $p$ is above $\Gamma_i$ so the plane $L_i(p)$ is not tangent to $\Gamma_i$. If $\Gamma_i$ is concave then it is below $\tilde{L}_i$. However, if $\Gamma_i$ is convex, we may further assume that $F_i$ is a convex function with $F_i(0) = \max_{x \in \mathbb{R}_+^N} F_i(x)$ so that, by Proposition 2.2 (iii), both $\Gamma_i$ and the surface $\{x \in \mathbb{R}_+^N : F_i(x) = F_i(p)\}$ are convex surfaces and the former is below the latter. Note that $L_i(p)$ is tangent to $\{x \in \mathbb{R}_+^N : F_i(x) = F_i(p)\}$ at $p$ so $\{x \in \mathbb{R}_+^N : F_i(x) = F_i(p)\}$ is below $L_i(p)$. Then $\Gamma_i$ is also below $L_i(p)$. Hence, we can always find a plane above $\Gamma_i$ if $p$ is above $\Gamma_i$.

Now for each $k \in I_N$, define an upper plane $L_k^u$ by $L_k^u = L_k$ if $\Gamma_k$ is concave or $L_k^u = L_k(p)$ if $\Gamma_k$ is convex and, for each $j \in J$, define a lower plane $L_j^l$ by $L_j^l = L_j(p)$ if $\Gamma_j$ is concave or $L_j^l = \tilde{L}_j$ if $\Gamma_j$ is convex. Then each convex or concave $\Gamma_i$ is below $L_k^u$ for all $i \in I_N$ but is above $L_j^l$ for all $i \in J$.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $N \times N$ matrices with real entries such that

$$
(21) \quad L_i^u = \{x \in \mathbb{R}_+^N : (Ax)_i = 1\}, \quad i \in I_N,
$$

$$
(22) \quad L_j^l = \{x \in \mathbb{R}_+^N : (Bx)_i = 1\}, \quad i \in J.
$$

Then the entries of $A$ and $B$ can be determined as follows. First, suppose $\Gamma_i$ is concave, so we have $L_i^u = \tilde{L}_i$ and $L_i^l = L_i(p)$. If $\Gamma_i$ intersects the half axis $J_j$ at the point $R_{ij}$ with $r_{ij} > 0$ as its $j$th component, then $a_{ij}r_{ij} = 1$ so $a_{ij} = \frac{1}{r_{ij}}$; if $\Gamma_i$ does not intersect $J_j$ then
\( a_{ij} = 0 \). So \( a_{ij} \) is defined by

\[
(23) \quad a_{ij} = \begin{cases} 
0 & \text{if } \Gamma_i \text{ does not intersect } J_j, \\
\frac{1}{r_{ij}} & \text{if the } j\text{th component of } R_{ij} \text{ is } r_{ij}.
\end{cases}
\]

Since \( L_i(p) \) has the equation \( \nabla F_i(p)(x - p) = 0 \), we have \( \nabla F_i(p)x = \nabla F_i(p)p \) so \( (Bx)_i = 1 \) with

\[
(24) \quad (Bx)_i = (\nabla F_i(p)p)^{-1}\nabla F_i(p)x,
\]

i.e. \( (\nabla F_i(p)p)^{-1}\nabla F_i(p) \) is taken to be the \( i\)th row of \( B \). If \( \Gamma_i \) is convex then \( L_i^u = L_i(p) \) and \( L_i^l = \tilde{L}_i \). In this case, we have

\[
(25) \quad (Ax)_i = (\nabla F_i(p)p)^{-1}\nabla F_i(p)x,
\]

i.e. \( (\nabla F_i(p)p)^{-1}\nabla F_i(p) \) is taken to be the \( i\)th row of \( A \), and \( b_{ij} \) is given by

\[
(26) \quad b_{ij} = \begin{cases} 
0 & \text{if } \Gamma_i \text{ does not intersect } J_j, \\
\frac{1}{r_{ij}} & \text{if the } j\text{th component of } R_{ij} \text{ is } r_{ij}.
\end{cases}
\]

Note from (A4) that \( \frac{\partial F_i}{\partial x_j}(p) < 0 \) for all \( i,j \in I_N \) so that \( \nabla F_i(p)p = \sum_{j=1}^{N} \frac{\partial F_i}{\partial x_j}(p)p_j < 0 \). Thus, \( (\nabla F_i(p)p)^{-1}\nabla F_i(p) \gg 0 \). Then, from (A1)–(A4), (23)–(26) and the assumptions we see that

\[
(27) \quad \forall i, j \in I_N, a_{ii} > 0 \text{ and } a_{ij} \geq 0.
\]

Let

\[
(28) \quad Y = \left( \frac{1}{a_{11}}, \ldots, \frac{1}{a_{NN}} \right)^T.
\]

For any subset \( K \subset I_N \) and \( u \in \mathbb{R}^N \), the point \( u^K \in \mathbb{R}^N \) is defined by

\[
(29) \quad u^K_i = \begin{cases} 
u_i & \text{if } i \in K, \\
0 & \text{if } i \in I_N \setminus K.
\end{cases}
\]

We are now in a position to state the main result of this section in geometric terms.

**Theorem 3.1.** Assume that the following conditions hold.

(a) System (6) has a saturated equilibrium point \( p \in \mathbb{R}_+^N \setminus \{0\} \) with support \( J \subset I_N \).

(b) For each \( i \in I_N \), the nullcline surface \( \Gamma_i \) is either concave or convex, and if \( F_i(p) < 0 \) with \( \Gamma_i \) convex, the function \( F_i \) is also convex with \( F_i(0) = \max_{x \in \mathbb{R}_+^N} F_i(x) \).

(c) For each \( i \in J \), either the point \( Y^{I_N \setminus \{i\}} \) is below \( L_i^1 \) or the set \( L_i^1 \cap [0,Y^{I_N \setminus \{i\}}] \) is strictly above \( L_i^2 \) for all \( j \in I_N \setminus \{i\} \).

Then \( p \) is globally attracting. If, in addition, all eigenvalues of the Jacobian matrix \( Df(p) \) have negative real parts, then \( p \) is globally asymptotically stable.
Remark 1. If \( p \) is a boundary equilibrium point with \( F_i(p) < 0 \) for some \( i \in I_N \setminus J \) and \( \Gamma_i \) is convex, \( p \) is above \( \Gamma_i \). As \( F_i(0) > 0 \) and \( F_i \) is continuous, there is a number \( s_i \in (0, 1) \) such that \( F_i(s_i p) = 0 \). Since \( s_i p \in \Gamma_i \), \( L_i(s_i p) \) is a tangent plane of \( \Gamma_i \) at \( s_i p \). By the convexity of \( \Gamma_i \), \( \Gamma_i \) is below \( L_i(s_i p) \). Thus, as an alternative to the part of the condition (b) in Theorem 3.1, instead of requiring \( F_i \) to be a convex function, we may define \( L^u_i = L_i(s_i p) \) with
\[
(Ax)_i = (\nabla F_i(s_i p)s_i p)^{-1} \nabla F_i(s_i p)x
\]
and require the inequalities in (27) hold.

Remark 2. Since \( L^l_i \) is described by the equation \( (Ax)_i = 1 \) and \( L^u_j \) by the equation \( (Ax)_j = 1 \), condition (c) in Theorem 3.1 is ensured by the following inequalities: For each \( i \in J \), either
\[
(BY^{I_N \setminus \{i\}})_i < 1
\]
or
\[
\forall j \in I_N \setminus \{i\}, \max \left\{ 0, \frac{b_{ij}}{a_{jj}} (1 - (AY^{I_N \setminus \{i,j\}})_j) \right\} < 1 - (BY^{I_N \setminus \{i,j\}})_i.
\]
Indeed, it is obvious that (30) holds if and only if \( Y^{I_N \setminus \{i\}} \) is below \( L^l_i \). Since \( L^u_i \) and \( L^l_i \) have equations
\[
(Ax)_i \equiv a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{iN}x_N = 1,
(Bx)_i \equiv b_{i1}x_1 + b_{i2}x_2 + \cdots + b_{iN}x_N = 1
\]
respectively and \( L^l_i \) is below \( L^u_i \), we must have
\[
\forall i, j \in I_N, a_{ij} \leq b_{ij}.
\]
If \( Y^{I_N \setminus \{i\}} \) is not below \( L^l_i \), then \( Y^{I_N \setminus \{i\}} \) is on or above \( L^l_i \) so \( (BY^{I_N \setminus \{i\}})_i \geq 1 \). If (31) holds, then
\[
(AY^{I_N \setminus \{i,j\}})_i \leq (BY^{I_N \setminus \{i,j\}})_i < 1
\]
so \( Y^{I_N \setminus \{i,j\}} \) is below \( L^l_i \) for all \( j \in I_N \setminus \{i\} \). Thus, the line segment \( [Y^{I_N \setminus \{i,j\}}, Y^{I_N \setminus \{i\}}] \) and the plane \( L^l_i \) have a unique intersection point \( Q_j \) with \( \frac{1}{b_{ij}} (1 - (BY^{I_N \setminus \{i,j\}})_i) \) as its \( j \)th component. From (31) we obtain
\[
\frac{1}{a_{ij}} (1 - (AY^{I_N \setminus \{i,j\}})_j) < \frac{1}{b_{ij}} (1 - (BY^{I_N \setminus \{i,j\}})_i).
\]
If the expression on the left-hand side of the above inequality is negative, then \( [Y^{I_N \setminus \{i,j\}}, Y^{I_N \setminus \{i\}}] \) is strictly above \( L^u_j \) so \( Q_j \) is above \( L^u_j \). Otherwise, since the expression on the left-hand side of the above inequality is the \( j \)th component of the intersection point of the plane \( L^u_j \) with the line segment \( [Y^{I_N \setminus \{i,j\}}, Y^{I_N \setminus \{i\}}] \), the above inequality shows that \( Q_j \) is above \( L^u_j \) for all \( j \in I_N \setminus \{i\} \). In particular, \( Y^{I_N \setminus \{i\}} \) is above \( L^u_j \) for every \( j \in I_N \setminus \{i\} \). For each \( k \in I_N \setminus \{i, j\} \),
\[
(AY^{I_N \setminus \{i,j\}})_k \geq a_{kk} Y_k = 1
\]
Remark have negative real parts, then \( p \). Then \( p \)

\[ (32) \quad \text{or} \quad \forall (33) \]

Corollary 3.2. Assume that the following conditions hold.

a) System (6) has a saturated equilibrium point \( p \in \mathbb{R}_+^N \setminus \{0\} \) with support \( J \subset I_N \).

b) For each \( i \in I_N \), the nullcline surface \( \Gamma_i \) is a plane.

c) For each \( i \in J \), either the point \( Y^I_N \setminus \{i\} \) is below \( \Gamma_i \) or the set \( \Gamma_i \cap [0, Y^I_N \setminus \{i\}] \) is strictly above \( \Gamma_j \) for all \( j \in I_N \setminus \{i\} \).

Then \( p \) is globally attracting. If, in addition, all eigenvalues of the Jacobian matrix \( Df(p) \) have negative real parts, then \( p \) is globally asymptotically stable.

Remark 3. When each \( \Gamma_i \) is a plane in \( \mathbb{R}_+^N \) with equation \((Ax)_i = 1\), from Remark 2 we see that condition (c) in Corollary 3.2 is guaranteed by the following inequalities: For each \( i \in J \), either

\[ (32) \quad (AY^I_N \setminus \{i\})_i < 1 \]

or

\[ (33) \quad \forall j \in I_N \setminus \{i\}, \max \left\{ 0, \frac{a_{ij}}{a_{jj}} (1 - (AY^I_N \setminus \{i,j\})_j) \right\} < 1 - (AY^I_N \setminus \{i,j\})_i. \]

Example 3.3. Consider the Ricker model (5) with \( N = 3 \), \( r_i > 0 \), \( c_i > 0 \) and

\[ A = \begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 1 \end{pmatrix}. \]

It has an interior equilibrium \( p = (\frac{4}{7}, \frac{4}{7}, \frac{4}{7})^T \) and \( Y = (1, 1, 1)^T \). Since \((AY^{(2,3)})_1 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4} < 1\), \((AY^{(1,3)})_2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4} < 1\) and \((AY^{(1,2)})_3 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4} < 1\), (32) holds. Then, from Corollary 3.2 and Remark 3, \( p \) is globally attracting. In addition, if each eigenvalue of \( Df(p) \) has a negative real part, then \( p \) is globally asymptotically stable.

Note that the conditions (32) and (33) can be applied to any one of the systems (2)–(5). In particular, for Lotka-Volterra system (2), these are consistent with the conditions given in [7].

Example 3.4. Consider the system

\[ \dot{x}_i = x_i \left[ 1 - a_1 \ln(1 + x_i) - a_2 \ln(1 + x_{i+1}) - \cdots - a_N \ln(1 + x_{i+N-1}) \right] = x_i F_i(x), \]
for \( i \in I_N \) and \( x \in \mathbb{R}_+^N \), where the \( a_j \) are positive constants and \( x_j+ = x_j \). The system has an interior equilibrium point \( p = p_0(1, \ldots, 1)^T \) with
\[
p_0 = e^{1/(a_1 + \cdots + a_N)} - 1.
\]
Since \( \ln(1 + su + (1 - s)v) \geq s \ln(1 + u) + (1 - s) \ln(1 + v) \) for \( u \geq 0 \), \( v \geq 0 \) and \( 0 \leq s \leq 1 \), each \( F_i \) satisfies
\[
\forall x, y \in \mathbb{R}_+^N, \quad F_i(sx + (1 - s)y) \leq sF_i(x) + (1 - s)F_i(y).
\]
This shows that \( \Gamma_i = \{ x \in \mathbb{R}_+^N : F_i(x) = 0 \} \) is concave. Then
\[
\frac{\partial F}{\partial x}(p) = -\frac{1}{1 + p_0} \begin{pmatrix} a_1 & a_2 & \cdots & a_N \\ a_N & a_1 & \cdots & a_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix}.
\]
By (24),
\[
B = \left( D \left[ \frac{\partial F}{\partial x}(p) \right] \right)^{-1} \frac{\partial F}{\partial x}(p) = p_0^{-1} \left( \sum_{i=1}^N a_i \right)^{-1} \begin{pmatrix} a_1 & a_2 & \cdots & a_N \\ a_N & a_1 & \cdots & a_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix}.
\]
The intersection points of \( \Gamma_1 \) with the coordinate axes are
\[
(e^{1/a_1} - 1, 0, \ldots, 0)^T, (0, e^{1/a_2} - 1, 0, \ldots, 0)^T, \ldots, (0, \ldots, 0, e^{1/a_N} - 1)^T.
\]
Thus, from (23),
\[
A = \begin{pmatrix} a_1' & a_2' & \cdots & a_N' \\ a_N' & a_1' & \cdots & a_{N-1}' \\ \vdots & \vdots & \ddots & \vdots \\ a_2' & a_3' & \cdots & a_1' \end{pmatrix}, \quad a_i' = \frac{1}{e^{1/a_i} - 1} > 0.
\]
Clearly, \( a_{ij} > 0 \) for all \( i, j \in I_N \) so (27) holds. By (28), \( Y = (e^{1/a_1} - 1)(1, \ldots, 1)^T \). Note that \( A \) and \( B \) are both circulant matrices. Then (30) becomes
\[
\left( \sum_{i=2}^N a_i \right)(e^{1/a_1} - 1) < \left( \sum_{i=1}^N a_i \right)(e^{1/(a_1 + \cdots + a_N)} - 1)
\]
and (31) becomes
\[
\max\left\{ 0, p_0^{-1} \left( \sum_{i=1}^N a_i \right)^{-1} a_j(e^{1/a_1} - 1) \left[ 1 - (e^{1/a_1} - 1) \sum_{k \in I_N \setminus \{1, j\}} a_k' \right] \right\}
\]
\[
< 1 - (e^{1/a_1} - 1)p_0^{-1} \left( \sum_{i=1}^N a_i \right)^{-1} \sum_{k \in I_N \setminus \{1, j\}} a_k, \quad \forall j \in I_N \setminus \{1\}.
\]
By Remark 2 and Theorem 3.1, if either (34) or (35) holds, then \( p \) is globally attracting.

We observe that for fixed \( a_1 > 0 \), (34) holds when \( a_2, \ldots, a_N \) are small enough.
4. Proof of Theorem 3.1

The proof of Theorem 3.1 is divided into three steps.

**Proof of Theorem 3.1. Step 1.** We first show that \( \omega(x_0) \subset [0, Y] \) for all \( x_0 \in \mathbb{R}^N_+ \). For each \( i \in I_N \) and every \( x \in \mathbb{R}^N_+ \) with \( x_i > Y_i \), \( x \) is above \( L^u_i \). Since \( L^u_i \) is above \( \Gamma_i \), we have \( x \in \Gamma_i^+ \) so \( \dot{x}_i = x_i F_i(x) < 0 \) due to \( F_i(0) > 0 \) by (A1). Thus, for any \( \delta > 0 \), the flow of the system will be transversal to the plane \( x_i = Y_i + \delta \) downwardly, so \( \omega(x_0) \) is strictly below the plane \( x_i = Y_i + \delta \) for all \( x_0 \in \mathbb{R}^N_+ \). Therefore, \( \omega(x_0) \subset [0, Y] \) for all \( x_0 \in \mathbb{R}^N_+ \).

**Step 2.** Assume that \( \omega(x_0) \subset [u, v] \subset [0, Y] \) for all \( x_0 \in \mathbb{R} \). If for \( v' \) with \( v'_i = u_i \) for some \( i \in J \) and \( v'_j = v_j \) for all \( j \in I_N \setminus \{i\} \), either \( v' \) is below \( L^i \) or \([u, v'] \cap L^i \) is strictly above \( L^j \) for all \( j \in I_N \setminus \{i\} \), we show the existence of \( \delta > 0 \) such that \( \omega(x_0) \subset [u, v] \) for all \( x_0 \in \mathbb{R} \), where \( \bar{u}_i = u_i + \delta \leq v_i \) and \( \bar{u}_j = u_j \) for all \( j \in I_N \setminus \{i\} \).

If \( v' \) is below \( L^i \), then \([u, v'] \) is strictly below \( L^i \). By the compactness of \([u, v'] \), there is a \( \delta > 0 \) such that the set \( B([u, v'], 2\delta) \cap \mathbb{R}^N_+ \) is strictly below \( L^i \). As \( L^i \) is below \( \Gamma_i \), \( B([u, v'], 2\delta) \cap \mathbb{R}^N_+ \) is strictly below \( \Gamma_i \) so any solution in \( B([u, v'], 2\delta) \cap \mathbb{R}^N_+ \setminus \pi_i \) satisfies \( x_i(t) = x_i(t) F_i(x(t)) > 0 \), i.e. \( x_i(t) \uparrow \). We show that, for \( \bar{u} \) with \( \bar{u}_i = u_i + \delta \) and \( \bar{u}_j = u_j \) for all \( j \in I_N \setminus \{i\} \),

\[
\omega(x_0) \subset [\bar{u}, v], \quad \forall x_0 \in \mathbb{R}.
\]

Suppose (36) is not true so \( \omega(x_0) \cap [u, v'] \neq \emptyset \) for some \( x_0 \in \mathbb{R} \), where \( v''_i = v'_i + \delta = u_i + \delta \) and \( v''_j = v_j \) for all \( j \in I_N \setminus \{i\} \). As \( \omega(x_0) \subset [u, v] \) and \( \omega(x_0) \) is compact, there is a \( y^0 \in \omega(x_0) \) such that \( y_i \geq y^0_i \) for all \( y \in \omega(x_0) \). If \( y^0_i > 0 \) then \( y^0 \in B([u, v'], 2\delta) \cap \mathbb{R}^N_+ \setminus \pi_i \) so \( x_i(y^0, t) \) is increasing for \( |t| \) small enough. Thus, \( x_i(y^0, t) < y^0_i \) for \( t \) close to 0 from left. As the whole orbit \( \gamma(y^0) \) is contained in \( \omega(x_0) \), this contradicts \( y_i \geq y^0_i \) for all \( y \in \omega(x_0) \).

Hence, we must have \( y_i^0 = 0 \), so \( u_i = 0 \) and \( y^0 \in \omega(x_0) \cap [u, v'] \subset \pi_i \). But \( x_i(t) \uparrow \) in \( B([u, v'], 2\delta) \cap \mathbb{R}^N_+ \setminus \pi_i \) means that \([u, v'] \) repels the solutions away so \([u, v'] \cap \omega(x_0) = \emptyset \), a contradiction to \( \omega(x_0) \cap [u, v'] \). Therefore, we must have (36).

Now suppose \([u, v'] \cap L^i \) is strictly above \( L^u_i \) for all \( j \in I_N \setminus \{i\} \) (see Figure 1 (a) for illustration). Consider the plane

\[
L^i = \{ x \in \mathbb{R}^N_+ : \sum_{j \in I_N \setminus \{i\}} b_{ij} x_j + b_{ii}(u_i + \delta) = 1 \}.
\]

Then \( L^i \) is parallel to the \( x_i \)-axis and passes through \( L^i \cap \{ x \in \mathbb{R}^N_+ : x_i = u_i + \delta \} \). Note that for each \( x \in L^i \cap [u, v'] \),

\[
\sum_{j \in I_N \setminus \{i\}} b_{ij} x_j + b_{ii}(u_i + \delta) \geq (Bx)_i = 1,
\]
We show that (36) holds. Suppose (36) is not true. Then \( \omega \) is also strictly below \( x \) above \( \delta \) is 0 small enough the set \( B(L_i^n \cap [u, v'], \varepsilon) \) is also strictly above \( L_j^n \) for all \( j \in I_N \setminus \{i\} \). Hence, \( L_i \cap [u, v'] \) is strictly above \( L_j^n \) for all \( j \in I_N \setminus \{i\} \). This shows that each solution \( x(t) \) in \( [u, v] \) satisfies \( x_j(t) \downarrow \) for all \( j \in I_N \setminus \{i\} \) as long as \( x(t) \) is on or above \( L_i \) and \( x_j(t) \neq 0 \). Therefore, for any solution \( x(t) \) staying in a very small vicinity of \([u, v]\), for \( t \geq T \), once it goes below \( L_i \) it will stay below \( L_i \) forever. Since \( \omega(x_0) \subset [u, v] \) for all \( x_0 \in \mathbb{R}_I \), \( \omega(x_0) \) must be strictly below \( L_i \). The subset \([u, v'] \subset \pi_i \) below \( L_i \) is also strictly below \( L_i \).

We show that (36) holds. Suppose (36) is not true. Then \( \omega(x_0) \cap [u, v'] \neq \emptyset \) for some \( x_0 \in \mathbb{R}_I \). Since \( \omega(x_0) \subset [u, v] \), there is \( y_0 \in \omega(x_0) \cap [u, v'] \) such that \( y_i \geq y_i^0 \) for all \( y \in \omega(x_0) \). If \( u_i = 0 \), as the subset of \([u, v'] \subset \pi_i \) below \( L_i \) is strictly below \( \Gamma_i \), this set

![Figure 1. (a) Illustration of condition (c) in theorem 3.1 for \( N = 3 \) and distinct \( i, j, k \) in \( \{1, 2, 3\} \). (b) Illustration of \( L_i \) and \( L_i' \) for \( N = 3 \).](image)

So \( x \) is on or above \( L_i' \). Thus, \( L_i' \cap [u, v'] \) is above \( L_i' \) (see Figure 1 (b) for illustration).

Also, we have

\[
\lim_{\delta \to 0} \sup_{x \in L_i' \cap [u, v']} \{ \inf_{y \in L_i' \cap [u, v']} \|y - x\| \} = 0.
\]

Hence, \( \forall \varepsilon > 0, \exists \delta_0 > 0 \) such that \( \varepsilon > 0 \) small enough the set \( B(L_i' \cap [u, v'], \varepsilon) \) is also strictly above \( L_j^n \) for all \( j \in I_N \setminus \{i\} \). Thus, for \( \delta \in (0, \delta_0] \), \( L_i' \cap [u, v'] \) is strictly above \( L_j^n \) for all \( j \in I_N \setminus \{i\} \). As \( L_i' \) is parallel to the \( x_i \)-axis and \( a_{ji} \geq 0 \) in the equation \( (Ax)_j = 1 \) for \( L_j^n \); if \( (Ax)_j > 1 \) then \( (Ax')_j > 1 \) for \( x' \) with \( x'_i \geq x_i \) and \( x'_k = x_k \) for \( k \in I_N \setminus \{i\} \). Hence, \( L_i' \cap [u, v] \) is strictly above \( L_j^n \) for all \( j \in I_N \setminus \{i\} \). This shows that each solution \( x(t) \) in \([u, v]\) satisfies \( x_j(t) \downarrow \) for all \( j \in I_N \setminus \{i\} \) as long as \( x(t) \) is on or above \( L_i' \) and \( x_j(t) \neq 0 \). Therefore, for any solution \( x(t) \) staying in a very small vicinity of \([u, v]\), for \( t \geq T \), once it goes below \( L_i' \) it will stay below \( L_i' \) forever. Since \( \omega(x_0) \subset [u, v] \) for all \( x_0 \in \mathbb{R}_I \), \( \omega(x_0) \) must be strictly below \( L_i' \). The subset \([u, v'] \subset \pi_i \) below \( L_i' \) is also strictly below \( L_i' \).
repels the solutions in \( \mathbb{R}_I \) away from \( \pi_i \). Since \( \omega(x_0) \) is strictly below \( L_i' \), we must have \( \omega(x_0) \cap \pi_i = \emptyset \), so \( y_i^0 > 0 \). Then, since \( y^0 \) is below \( \Gamma_i \), \( x_i(y^0, t) < y_i^0 \) for \( t < 0 \) close enough to 0. As \( \gamma(y^0) \subset \omega(x_0) \), this contradicts \( y_i \geq y_i^0 \) for all \( y \in \omega(x_0) \). Therefore, (36) holds.

**Step 3.** Let \( u(s) = sp \) and \( v(s) = sp + (1 - s)Y \) for \( s \in [0, 1] \). We show that \( \omega(x_0) \subset [u(s), v(s)] \) for all \( x_0 \in \mathbb{R}_I \) and all \( s \in [0, 1] \). Thus, \( \omega(x_0) = [u(1), v(1)] = \{p\} \) for all \( x_0 \in \mathbb{R}_I \) and the conclusion of Theorem 3.1 holds.

From step 1 we know that \( \omega(x_0) \subset [0, Y] = [u(0), v(0)] \) for all \( x_0 \in \mathbb{R}_I^N \). By step 2 and condition (c), there is a \( \delta \in (0, 1) \) such that \( \omega(x_0) \subset [u(\delta), Y] \) for all \( x_0 \in \mathbb{R}_I \). Now define an affine map \( m_\delta : \mathbb{R}_I^N \to \mathbb{R}_I^N(u(\delta)) \) by \( m_\delta(x) = \delta p + (1 - \delta)x \). Then \( m_\delta(p) = p \) and \( m_\delta(x) - p = (1 - \delta)(x - p) \). Thus, \( m_\delta \) maps the line segment \( \overline{pp} \) to \( (1 - \delta)\overline{pp}, [0, Y] \) to \( [u(\delta), v(\delta)] \), and each \( L_i(p) \) to \( L_i(p) \cap \mathbb{R}_I^N(u(\delta)) \). Now consider the set

\[
C_j = \{sp + (1 - s)y : \forall y \in (\partial \mathbb{R}_I^N) \cap \tilde{L}_j, \forall s \in [0, 1]\}
\]

Then \( C_j \) is a cone surface with \( p \) as the vertex and \( \tilde{L}_j \) as its base. The map \( m_\delta \) maps \( C_j \) to \( C_j \cap \mathbb{R}_I^N(u(\delta)) \) and \( \tilde{L}_j \) in \( \mathbb{R}_I^N \) to a plane \( \tilde{L}_j \) in \( \mathbb{R}_I^N(u(\delta)) \). Note that \( C_j \cap \mathbb{R}_I^N(u(\delta)) \) is a cone with \( p \) as its vertex and \( \tilde{L}_j \) as its base (see Figure 2 for illustration). Thus, if \( \Gamma_j \) is concave (convex) then both \( \tilde{L}_j \) and \( \tilde{L}_j \) are above (below) \( \Gamma_j \). Hence, \( m_\delta \) maps \( L_i^u \) and \( L_j^u \) in \( \mathbb{R}_I^N \) to planes \( L_i^u(\delta) \) and \( L_j^u(\delta) \) in \( \mathbb{R}_I^N(u(\delta)) \) for each \( i \in J \) and all \( j \in I_N \setminus \{i\} \). Due to the nature of the map \( m_\delta \) projecting points along straight lines towards \( p \), the relative positions of \( L_i^u \) and \( L_j^u \) in \( \mathbb{R}_I^N \) are preserved for the planes \( L_i^u(\delta) \) and \( L_j^u(\delta) \) in \( \mathbb{R}_I^N(u(\delta)) \) for each \( i \in J \) and all \( j \in I_N \setminus \{i\} \).

For each \( i \in I_N \), \( \Gamma_i \) in \( \mathbb{R}_I^N(u(\delta)) \) is below \( L_i^u(\delta) \). The intersection point \( R_{ii} \) of \( L_i^u \) with the \( x_i \)-axis has \( x_i \) as its \( i \)-th component and 0 as other components. The point \( R_{ii} \) is
mapped to $m_\delta(R_{ii}) = \delta p + (1 - \delta) R_{ii}$, which has $\delta p_i + \frac{1 - \delta}{a_{ii}}$ as its $i$th component and $\delta p_j$ as its $j$th component for $j \neq i$. Since $L_i^u(\delta)$ is below the plane $x_i = \delta p_i + \frac{1 - \delta}{a_{ii}}$, by the reasoning similar to that in step 1 we see that $\omega(x_0)$ is below this plane for all $x_0 \in \mathbb{R}_I$. As $v(\delta) = m_\delta(Y) = \delta p + (1 - \delta)Y$ with $v_i(\delta) = \delta p_i + \frac{1 - \delta}{a_{ii}}$, we have $\omega(x_0) \subset [u(\delta), v(\delta)]$ for all $x_0 \in \mathbb{R}_I$.

For each $i \in J$, let $v_i(\delta)' = u_i(\delta)$ and $v_j(\delta)' = v_j(\delta)$ for $j \in I_N \setminus \{i\}$. Then condition (c) and the nature of $m_\delta$ imply that either $v(\delta)'$ is below $L_i^I(\delta)$ or $L_i^I(\delta) \cap [u(\delta), v(\delta)']$ is strictly above $L_i^u(\delta)$ for all $j \in I_N \setminus \{i\}$. From step 2 again, we can always replace $\delta \in (0, 1)$ by a larger one. Repetition of the above process shows that $\omega(x_0) \subset [u(\delta), v(\delta)]$ holds for all $x_0 \in \mathbb{R}_I$ and all $\delta \in (0, 1)$. Taking the limit $\delta \to 1^-$, we obtain $\omega(x_0) = \{p\}$. \qed

5. Geometric method for global repulsion

In this section, we assume that $p$ is an interior equilibrium point, so $p \in \text{int} \mathbb{R}^N_+$. When each nullcline surface $\Gamma_i$ is concave or convex, we define the planes $L_i^u$ and $L_i^I$ in the same way as in section 3 for $i \in I_N$. Then each $\Gamma_i$ is below $L_i^u$ but above $L_i^I$,

$$L_i^I = \{x \in \mathbb{R}^N_+ : (Bx)_i = 1\}, \quad L_i^u = \{x \in \mathbb{R}^N_+ : (Ax)_i = 1\}.$$ 

Thus,

$$\forall i, j \in I_N, a_{ij} \leq b_{ij}. \quad (37)$$

We also assume that for each $i \in I_N$, the intersection point of $L_i^I$ with the positive half $x_i$-axis is above $L_i^u$ for all $j \in I_N \setminus \{i\}$. Then $a_{ji}/b_{ii} > 1$ so

$$\forall i \in I_N, \forall j \in I_N \setminus \{i\}, a_{ji} > b_{ij} > 0. \quad (38)$$

From (37) and (38) we have

$$a_{jj} \leq b_{jj} < a_{ij} \leq b_{ij} \quad (i \neq j).$$

For any $k \in I_N \setminus \{i\}$, as $b_{ii} < a_{ji}$, if $b_{ik} > a_{jk}$ then the system of simultaneous equations

$$b_{ii}x_i + b_{ik}x_k = 1, \quad a_{ji}x_i + a_{jk}x_k = 1 \quad (39)$$

has a solution

$$x_i = \frac{a_{jk} - b_{ik}}{b_{ij}a_{jk} - b_{ik}a_{ji}} > 0, \quad x_j = \frac{b_{ii}a_{jk} - b_{ik}a_{ji}}{b_{ii}a_{jk} - b_{ik}a_{ji}} > 0.$$ 

This shows that $L_j^u$ and $L_i^I$ restricted to $\cap_{m \in I_N \setminus \{i, k\}} \pi_m$ has a unique intersection point. This is obviously true for $k = j$ as $b_{ij} > a_{jj}$. If $b_{ik} \leq a_{jk}$ then (39) has no solution with $x_i > 0$. Thus, the largest possible $i$th component of the points in $L_i^I \cap L_j^u$ is

$$\max \left\{ \frac{b_{ik} - a_{jk}}{a_{ji}b_{ik} - a_{jk}b_{ii}} : k \in I_N \setminus \{i\} \right\} \text{ if } b_{ik} > a_{jk}.$$
Then from (40) we see that conditions (a)–(c) guarantee the existence of $U$ and $W$ unstable manifold a saddle point with a one-dimensional stable manifold $\cup_{i} = \max_{i}$. Thus, for each $i \in I_{N}$, the intersection point of $L_{i}^{L}$ with the positive half $x_{i}$-axis is above $L_{i}^{L}$ for all $j \in I_{N} \setminus \{i\}$. For each $i \in I_{N}$, the intersection point of $L_{i}^{L}$ with the positive half $x_{i}$-axis is above $L_{i}^{L}$ for all $j \in I_{N} \setminus \{i\}$. Now define $U \gg 0$ by

\begin{equation}
U_{i} = \max \left\{ \frac{b_{ik} - a_{jk}}{a_{ij}b_{ik} - a_{jk}b_{ii}} : j, k \in I_{N} \setminus \{i\} \text{ if } b_{ik} > a_{jk} \right\}, i \in I_{N}.
\end{equation}

For a surface $\Gamma$ in $\mathbb{R}_{+}^{N}$, we call it strongly balanced if for all distinct points $u, v \in \Gamma$, neither $u - v$ nor $v - u$ is in $\mathbb{R}_{+}^{N}$.

**Theorem 5.1.** Assume that the following conditions hold.

(a) System (6) has an interior equilibrium point $p \in \text{int}\mathbb{R}_{+}^{N}$.

(b) For each $i \in I_{N}$, the nullcline surface $\Gamma_{i}$ is strongly balanced and either convex or concave.

(c) For each $i \in I_{N}$, the intersection point of $L_{i}^{L}$ with the positive half $x_{i}$-axis is above $L_{i}^{L}$ for all $j \in I_{N} \setminus \{i\}$.

(d) For each $i \in I_{N}$, either $\pi_{i} \cap [0, U]$ or $L_{i}^{L} \cap \pi_{i} \cap [0, U]$ is strictly below $L_{j}^{L}$ for all $j \in I_{N} \setminus \{i\}$, where $U$ is defined by (40).

Then $p$ is globally repelling.

**Remark 4.** Let $D \subset \mathbb{R}_{+}^{N}$ be a bounded region such that $\Gamma_{i} \subset D$ for all $i \in I_{N}$. Then, by Proposition 2.2 (iii) and (vi), the requirement of each $\Gamma_{i}$ to be concave or convex in part of condition (b) in Theorem 5.1 is guaranteed if each function $F_{i}$ or $-F_{i}$ is a convex function in $D$ with $F_{i}(0) = \max_{x \in \mathbb{R}_{+}^{N}} F_{i}(x)$. The requirement that each $\Gamma_{i}$ is strongly balanced is ensured by the following:

\begin{equation}
\forall i, j \in I_{N}, \forall u \in D, \frac{\partial F_{i}}{\partial x_{j}}(u) < 0.
\end{equation}

Indeed, (41) implies that $F_{i}(x)$ is strictly increasing in each $x_{j}$ for $x \in D$. Thus, for any $u, v \in D$ with $u < v$, we have $F_{i}(u) > F_{i}(v)$, so it is impossible to have both $u \in \Gamma_{i}$ and $v \in \Gamma_{i}$.

**Remark 5.** The algebraic condition equivalent to condition (c) in Theorem 5.1 is (38). Then from (40) we see that conditions (a)–(c) guarantee the existence of $U \gg 0$: $U_{i}$ is the maximum of the ith components of all the possible intersection points of $L_{i}^{L}$ with $\cup_{j \in I_{N} \setminus \{i\}} L_{j}^{u}$. Note that each set $L_{i}^{u} \cap \pi_{i} \cap [0, U]$ in condition (d), if not empty, is a convex hull which is determined by linear combinations of a finite number of vertices $v_{i1}, \ldots, v_{im}$. Thus, $L_{i}^{u} \cap \pi_{i} \cap [0, U]$ is strictly below $L_{j}^{L}$ for all $j \in I_{N} \setminus \{i\}$ if and only if each vertex $v_{ik}$ is below $L_{j}^{L}$ for all $j \in I_{N} \setminus \{i\}$. This will be clear from Figure 3 in Example 5.4 later.

**Remark 6.** Under the conditions of Theorem 5.1, from Theorem 2.1 we see that $p$ is a saddle point with a one-dimensional stable manifold $W^{s}(p)$ and $(N - 1)$-dimensional unstable manifold $W^{u}(p) = \text{int}\Sigma \setminus \{p\}$. Thus, for each $x_{0} \gg 0$, we have $\omega(x_{0}) = \{p\}$ if $x_{0} \in W^{s}(p)$ and $\omega(x_{0}) \subset \partial\Sigma$ if $x_{0} \not\in W^{s}(p)$. For each $x_{0} \in \text{int}\Sigma \setminus \{p\}$, we have $\omega(x_{0}) \subset \partial\Sigma$ and $\alpha(x_{0}) = \{p\}$. 

\(\text{GLOBAL STABILITY AND REPULSION 15}\)
For a particular class of systems (6) when each $\Gamma_i$ is a plane, so $\Gamma_i = L^u_i = L^l_i$, it is both concave and convex. Then condition (c) of Theorem 5.1 guarantees that each $\Gamma_i$ is strongly balanced. Thus, condition (b) is redundant and Theorem 5.1 is simplified as follows.

**Corollary 5.2.** Assume that the following conditions hold.

(a) System (6) has an equilibrium point $p \in \text{int}\mathbb{R}^N_+$.

(b) For each $i \in I_N$, the nullcline surface $\Gamma_i$ is a plane.

(c) Each axial equilibrium point $R_i$ is above $\Gamma_j$ for all $j \in I_N \setminus \{i\}$.

(d) For each $i \in I_N$, either $\pi_i \cap [0, U]$ or $\Gamma_i \cap [0, U]$ is strictly below $\Gamma_j$ for all $j \in I_N \setminus \{i\}$, where $U$ is given by (40) with $L^l_i = L^u_i = \Gamma_i$.

Then $p$ is globally repelling.

**Example 5.3.** Consider the system (4) with $r_i > 0$, $c_i > 0$ and $a_{ij} > 0$ for all $i, j \in I_N$. Suppose $p \in \text{int}\mathbb{R}^N_+$ is an interior equilibrium point. Then each $\Gamma_i$ is a plane,

$$\forall i \in I_N, \Gamma_i = L^l_i = L^u_i = \{x \in \mathbb{R}^N_+ : a_{i1}x_1 + \cdots + a_{iN}x_N = 1\}.$$ 

Assume that each axial equilibrium $R_i$ is above $\Gamma_j$ for all $j \in I_N \setminus \{i\}$, i.e.

$$\forall i, j \in I_N (i \neq j), a_{ji} > a_{ii} > 0.$$ 

Define $U \gg 0$ by

$$\forall i \in I_N, U_i = \max \left\{ \frac{a_{ik} - a_{jk}}{a_{ji}a_{ik} - a_{jk}a_{ii}} : j, k \in I_N \setminus \{i\} \text{ if } a_{ik} > a_{jk} \right\}.$$ 

Then, by Corollary 5.2, $p$ is globally repelling if either $\pi_i \cap [0, U]$ or $\Gamma_i \cap [0, U]$ is strictly below $\Gamma_j$ for all $i \in I_N$ and $j \in I_N \setminus \{i\}$.

Note that the above result for (4) is also true for other systems in (2)–(5). In particular, for Lotka-Volterra system (2), this result is consistent with [9].

**Example 5.4.** Consider the system

$$\begin{align*}
\dot{x}_1 &= x_1(1 - 2ax_1 - ax_1^2 - x_2 - x_3) = x_1F_1(x), \\
\dot{x}_2 &= x_2(1 - x_1 - 2ax_2 - ax_2^2 - x_3) = x_2F_2(x), \\
\dot{x}_3 &= x_3(1 - x_1 - x_2 - 2ax_3 - ax_3^2) = x_3F_3(x),
\end{align*}$$

where $a > 0$ is a constant. The system has an interior equilibrium point $p = p_0(1, 1, 1)^T$ with $p_0$ satisfying $ap_0^2 + 2(a + 1)p_0 = 1$, so

$$p_0 = \frac{1}{a} \left[ \sqrt{a + (a + 1)^2} - (a + 1) \right] = \frac{1}{\sqrt{a^2 + 3a + 1} + a + 1}.$$
Then
\[
\begin{align*}
\Gamma_1 &= \{ x \in \mathbb{R}_+^3 : 2ax_1 + ax_2^2 + x_2 + x_3 = 1 \}, \\
\Gamma_2 &= \{ x \in \mathbb{R}_+^3 : x_1 + 2ax_2 + ax_2^2 + x_3 = 1 \}, \\
\Gamma_3 &= \{ x \in \mathbb{R}_+^3 : x_1 + x_2 + 2ax_3 + ax_3^2 = 1 \}, \\
L_1(p) &= 2a(1 + p_0)x_1 + x_2 + x_3 = 1 + ap_0^2, \\
L_2(p) &= x_1 + 2a(1 + p_0)x_2 + x_3 = 1 + ap_0^2, \\
L_3(p) &= x_1 + x_2 + 2a(1 + p_0)x_3 = 1 + ap_0^2.
\end{align*}
\]

As \( \Gamma_1 \) intersects the axes at \((\sqrt{1 + \frac{1}{a}} - 1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T\) respectively and \((\sqrt{1 + \frac{1}{a}} - 1)^{-1} = a(\sqrt{1 + \frac{1}{a}} + 1)\), we have
\[
\tilde{L}_1 = \{ x \in \mathbb{R}_+^3 : a(\sqrt{1 + \frac{1}{a}} + 1)x_1 + x_2 + x_3 = 1 \}.
\]

Similarly,
\[
\begin{align*}
\tilde{L}_2 &= \{ x \in \mathbb{R}_+^3 : x_1 + a(\sqrt{1 + \frac{1}{a}} + 1)x_2 + x_3 = 1 \}, \\
\tilde{L}_3 &= \{ x \in \mathbb{R}_+^3 : x_1 + x_2 + a(\sqrt{1 + \frac{1}{a}} + 1)x_3 = 1 \}.
\end{align*}
\]

Note that
\[
\forall i \in I_3, \forall x, y \in \mathbb{R}_+^3, F_i(sx + (1 - s)y) \geq sF_i(x) + (1 - s)F_i(y)
\]
so \( F_1, F_2 \) and \( F_3 \) are convex functions with \( F_i(0) = \max_{x \in \mathbb{R}_+^3} F_i(x) \) for \( i \in I_3 \). By Proposition 2.2 (iii), \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) are convex. Then \( L_1^i = \tilde{L}_i \) and \( L_2^i = L_i(p) \) for \( i \in I_3 \). Since \( \frac{\partial F_i}{\partial x_i} = -2a - 2ax_i < 0 \) and \( \frac{\partial F_i}{\partial x_j} = -1 < 0 \) for \( i, j \in I_3 (i \neq j) \), by Remark 4 each \( \Gamma_i \) is strongly balanced. Thus, conditions (a) and (b) of Theorem 5.1 are fulfilled.

If \( a \in (0, 0.3) \), then we have \( p_0 < \frac{1}{2} \) so \( p_0^2 < \frac{1}{2} \) and
\[
\sqrt{1 + \frac{1}{a}} - 1 \geq \sqrt{\frac{1.3}{0.3} - 1} > \frac{4.3}{4} \geq 1 + \frac{1}{4}a > 1 + a p_0^2.
\]

Thus, the equilibrium point \((\sqrt{1 + \frac{1}{a}} - 1, 0, 0)^T\), which is the intersection point of \( L_1^i \) with the positive half \( x_1 \)-axis, is above \( L_2^i \) and \( L_3^i \). By symmetry, condition (c) of Theorem 5.1 is met.

To check condition (d), we need to find \( U \gg 0 \) given by (40). The point in \( L_1^i \cap L_2^i \cap \pi_3 \) is given by the solution of
\[
a(\sqrt{1 + \frac{1}{a}} + 1)x_1 + x_2 = 1, \ x_1 + 2a(1 + p_0)x_2 = 1 + ap_0^2,
\]
which has the components

\[ x_1 = \frac{1 + ap_0^2 - 2a(1 + p_0)}{1 - 2a^2(\sqrt{1 + \frac{1}{a}} + 1)(1 + p_0)}, \quad x_2 = \frac{1 - a(\sqrt{1 + \frac{1}{a}} + 1)(1 + ap_0^2)}{1 - 2a^2(\sqrt{1 + \frac{1}{a}} + 1)(1 + p_0)}, \quad x_3 = 0. \]

The point in \( L_l^1 \cap L_u^3 \cap \pi_3 \) is given by the solution of

\[ a(\sqrt{1 + \frac{1}{a}} + 1)x_1 + x_2 = 1, \quad x_1 + x_2 = 1 + ap_0^2, \]

which has the components

\[ x_1 = \frac{ap_0^2}{1 - a(\sqrt{1 + \frac{1}{a}} + 1)}, \quad x_2 = \frac{1 - a(\sqrt{1 + \frac{1}{a}} + 1)(1 + ap_0^2)}{1 - a(\sqrt{1 + \frac{1}{a}} + 1)}, \quad x_3 = 0. \]

The point in \( L_l^1 \cap L_u^3 \cap \pi_2 \) is the same as that in \( L_l^1 \cap L_u^3 \cap \pi_3 \) with the swap of \( x_2 \) and \( x_3 \), and the point in \( L_l^1 \cap L_u^3 \cap \pi_2 \) is the same as that in \( L_l^1 \cap L_u^3 \cap \pi_3 \) with the swap of \( x_2 \) and \( x_3 \). We can easily check that the function

\[ f(s) = \frac{ap_0^2 + s}{1 - a(\sqrt{1 + \frac{1}{a}} + 1) + a(\sqrt{1 + \frac{1}{a}} + 1)s} \]

is increasing in \( s \), so

\[ \frac{1 + ap_0^2 - 2a(1 + p_0)}{1 - 2a^2(\sqrt{1 + \frac{1}{a}} + 1)(1 + p_0)} = f(1 - 2a(1 + p_0)) > f(0) = \frac{ap_0^2}{1 - a(\sqrt{1 + \frac{1}{a}} + 1)}. \]

Then, by (40), \( U_1 \) is the maximum of the first component of the points in \( L_l^1 \cap L_u^3 \cap \pi_3 \), \( L_l^1 \cap L_u^3 \cap \pi_2 \), \( L_l^1 \cap L_u^3 \cap \pi_3 \), \( L_l^1 \cap L_u^3 \cap \pi_2 \) and \( L_l^1 \cap L_u^3 \cap \pi_2 \). Thus,

\[ U_1 = \frac{1 + ap_0^2 - 2a(1 + p_0)}{1 - 2a^2(\sqrt{1 + \frac{1}{a}} + 1)(1 + p_0)} \]

and by symmetry, \( U_2 = U_3 = U_1 \).

Next, we derive a condition on \( a \) so that condition (d) of Theorem 5.1 is satisfied. The set \( L_l^u \cap \pi_1 \cap [0, U] \) is the line segment \( AB \) (see Figure 3) on the plane \( x_1 = 0 \) with \( A(u_0, U_3) \) and \( B(U_2, u_0) \), where

\[ u_0 = 1 + ap_0^2 - U_3 = \frac{2a(1 + p_0)[1 - a(\sqrt{1 + \frac{1}{a}} + 1)(1 + ap_0^2)]}{1 - 2a^2(\sqrt{1 + \frac{1}{a}} + 1)(1 + p_0)}. \]

The point in \( L_l^2 \cap L_l^1 \cap \pi_1 \) is given by the solution of

\[ a(\sqrt{1 + \frac{1}{a}} + 1)x_2 + x_3 = 1, \quad x_2 + x_3 = 1 + ap_0^2, \]
Figure 3. Illustration of $L_1^u \cap \pi_1 \cap [0,U], L_2^l \cap \pi_1$ and $L_3^l \cap \pi_1$.

which has the components

$$x_1 = 0, x_2 = \frac{ap_0^2}{1 - a(\sqrt{1 + \frac{1}{a}} + 1)}, \quad x_3 = \frac{1 - a(\sqrt{1 + \frac{1}{a}} + 1)(1 + ap_0^2)}{1 - a(\sqrt{1 + \frac{1}{a}} + 1)}.$$

As the axial fixed point $(0, \sqrt{1 + \frac{1}{a}} - 1, 0)^T$, which is the intersection point of $L_2^l$ with the positive half $x_2$-axis, is above $L_1^u$, any point in $L_2^l \cap \pi_1$ with

$$\frac{ap_0^2}{1 - a(\sqrt{1 + \frac{1}{a}} + 1)} < x_2 \leq \sqrt{1 + \frac{1}{a}} - 1$$

is above $L_1^u$. Thus, $L_1^u \cap \pi_1 \cap [0,U]$ is strictly below $L_2^l$ if $u_0 > \frac{ap_0^2}{1 - a(\sqrt{1 + \frac{1}{a}} + 1)}$ (see Figure 3).

Note that $p_0 < \frac{1}{2}$ so $\frac{1 + p_0}{p_0^2} > 6$. Then, if $a \in (0, 0.3]$ is small enough to satisfy

$$(43) \quad 12[1 - a(\sqrt{1 + \frac{1}{a}} + 1)(1 + ap_0^2)] \geq \frac{ap_0^2}{1 - a(\sqrt{1 + \frac{1}{a}} + 1)} + 2a^2(\sqrt{1 + \frac{1}{a}} + 1)(1 + p_0) \geq 1,$$

$L_1^u \cap \pi_1 \cap [0,U]$ is strictly below $L_2^l$. Similarly, (43) also ensures that $L_1^u \cap \pi_1 \cap [0,U]$ is strictly below $L_3^l$. By symmetry, (43) guarantees condition (d) of Theorem 5.1. Therefore, $p$ is a global repellor if $a \leq 0.3$ and satisfies (43).

6. Proof of Theorem 5.1

To prepare for the proof of Theorem 5.1, we present five lemmas below, of which the first three reveal some general properties of (6) under certain conditions and the last two are closely related to the conditions of Theorem 5.1.
Lemma 6.1. If the $i$th axial fixed point $R_i$ is above (below) $\Gamma_j$ for all $j \in I_N \setminus \{i\}$, then $R_i$ is an attractor in $\mathbb{R}^N_+$ (a repeller in $\Sigma$).

Proof. Note that $\nabla (x_i F_i(x))_{x=R_i} \ll 0$ by (A4),
\[
\frac{\partial x_j F_j(x)}{\partial x_k}|_{x=R_i} = 0, \quad \frac{\partial x_j F_j(x)}{\partial x_j}|_{x=R_i} = F_j(R_i), \quad k \neq j \neq i.
\]

Thus, the eigenvalue of $\frac{\partial D[x]F(x)}{\partial x}|_{x=R_i}$ with an eigenvector on $x_i$-axis is negative and the $F_j(R_i)$ are eigenvalues of $\frac{\partial D[x]F(x)}{\partial x}|_{x=R_i}$ with an eigenvector transversal to the $x_i$-axis. If $R_i$ is above (below) $\Gamma_j$ for all $j \in I_N \setminus \{i\}$, then $\frac{\partial D[x]F(x)}{\partial x}|_{x=R_i}$ has $N$ negative eigenvalues ($N-1$ positive eigenvalues with eigenvectors transversal to the $x_i$-axis) so $R_i$ is an attractor in $\mathbb{R}^N_+$ (a repeller in $\Sigma$).

Lemma 6.2. Assume that each $\Gamma_i$ is strongly balanced. For any $u \in \mathbb{R}^N_+ \setminus \{0\}$ with support $I \subset I_N$, if there is a nonempty $I_0 \subset I$ such that $u$ is below $\Gamma_j$ for all $j \in I_0$ but is on $\Gamma_k$ for all $k \in I \setminus I_0$, then $u \in Br(0)$.

Proof. If $I_0 = I$, then $x(u,t)$ is below $\Gamma_j$ for all $j \in I$ and sufficiently small $|t|$. Since each $\Gamma_j$ is strongly balanced, by the monotone property of competitive systems, we have $x(u,t_2) < x(u,t_1) < u$ for all $t_2 < t_1 < 0$. Then there is a $q \in \mathbb{R}^N_+$ with $q < u$ such that $\alpha(u) = \{q\}$ so $q$ is an equilibrium point. We show that $q = 0$ so that $u \in Br(0)$.

For each $i \in I$, since $u_i > 0$, $F_i(u) > 0$, $F_i(0) > 0$ by (A1), and $F_i(v) < 0$ for sufficiently large $|v|$ by (A3), if $F_i(q) \leq 0$, then the continuity of $F_i$ ensures the existence of $q', u' \in \Gamma_i$ satisfying $q \leq q' < u < u'$. This contradicts the assumption that $\Gamma_i$ is strongly balanced. Therefore, we must have $F_i(q) > 0$. Since $q$ is an equilibrium point, we have $D[q]F(q) = 0$ so $q = 0$ and $u \in Br(0)$.

If $I_0 \neq I$, let $u(\varepsilon) > 0$ be defined by
\[
u_j(\varepsilon) = u_j \text{ for } j \not\in I \setminus I_0, u_k(\varepsilon) = u_k - \varepsilon \text{ for } k \in I \setminus I_0
\]
for sufficiently small $\varepsilon > 0$. Then $u(\varepsilon) < u$. As each $\Gamma_i$ is strongly balanced, we have $F_i(u(\varepsilon)) > 0$ so $u(\varepsilon)$ is below $\Gamma_i$ for all $i \in I$. Thus, from the case of $I_0 = I$, we have
\[u(\varepsilon) \in Br(0) \text{ and } F_i(x(u(\varepsilon), t)) > 0, \quad \forall t < 0, \forall i \in I\]
for each sufficiently small $\varepsilon > 0$. Then, by continuous dependence on initial values, we have $F_i(x(u, t)) \geq 0$ for all $t < 0$ and $i \in I$. But $\dot{x}_j(u, 0) = u_j F_j(u) > 0$ for $j \in I_0$, so $x_j(u, t) < u_j$ for all $t < 0$ and $j \in I_0$. Then $x(u, t) < u$ for all $t < 0$. Since each $\Gamma_i$ is strongly balanced, we have $F_i(x(u, t)) > 0$ for all $t < 0$ and $i \in I$. Hence, $x(u, t) \in Br(0)$ for all $t < 0$ so $u \in Br(0)$.

Lemma 6.3. Assume that each $\Gamma_i$ is strongly balanced. For any $u \in \mathbb{R}^N_+ \setminus \{0\}$ with support $I \subset I_N$, if there is a nonempty $I_0 \subset I$ such that $u$ is above $\Gamma_j$ for all $j \in I_0$ but is on $\Gamma_k$ for all $k \in I \setminus I_0$, then $u \in Br(\infty)$. 
The proof of Lemma 6.3 is similar to that of Lemma 6.2 so we omit it here.

**Lemma 6.4.** Assume that each $\Gamma_i$ is strongly balanced. Assume the existence of $u \in \mathbb{R}^N_+$, $i \in I_N$ and $v_i > u_i$ such that either $S^0(u, v_i)$ or $\Gamma_i \cap S^0(u, v_i)$ is strictly above $\Gamma_j$ for all $j \in I_N \setminus \{i\}$. Then, for each $x_0 \in \mathbb{R}^N_+ \setminus \{0\}$, if $\omega(x_0) \subset \mathbb{R}^N_+(u)$ then either $\omega(x_0) \subset \mathbb{R}^N_+(u) \setminus S^0(u, v_i)$ or $\omega(x_0) = \{R_i\}$ (so $R_i \in \mathbb{R}^N_+(u)$). Moreover, if the whole trajectory $\gamma(x_0)$ is in $\Sigma \cap \mathbb{R}^N_+(u)$ for some $x_0 \in \text{int}\Sigma$, then either $\omega(x_0) = \{R_i\}$ and $\alpha(x_0) \subset \Sigma \cap (\mathbb{R}^N_+(u) \setminus S^0(u, v_i))$ or $\gamma(x_0) \subset \Sigma \cap (\mathbb{R}^N_+(u) \setminus S^0(u, v_i))$.

**Proof.** If $S^0(u, v_i)$ is strictly above $\Gamma_i$ then it is strictly above $\Gamma_j$ for all $j \in I_N$. By Lemma 6.3, $S^0(u, v_i) \subset Br(\infty)$ so $\Sigma \cap S^0(u, v_i) = \emptyset$. If $\omega(x_0) \subset \mathbb{R}^N_+(u)$ for some $x_0 \in \mathbb{R}^N_+ \setminus \{0\}$, as $\omega(x_0) \subset \Sigma$ by Theorem 2.1, we have $\omega(x_0) \cap S^0(u, v_i) = \emptyset$ so $\omega(x_0) \subset \mathbb{R}^N_+(u) \setminus S^0(u, v_i)$.

Now suppose $\Gamma_i \cap S^0(u, v_i) \neq \emptyset$. Since this set is strictly above $\Gamma_j$ for all $j \in I_N \setminus \{i\}$, either $S^0(u, v_i)$ contains no equilibrium point or, if $R_i \in S^0(u, v_i)$, the axial equilibrium point $R_i$ is the unique equilibrium point in $S^0(u, v_i)$ and, by Lemma 6.1, $R_i$ is an attractor. By Lemma 6.3, any point on or above $\Gamma_i$ in $S^0(u, v_i) \setminus \{R_i\}$ belongs to $Br(\infty)$, so it can be neither an $\omega$-limit point nor an $\alpha$-limit point.

For any $x_0 \in \mathbb{R}^N_+ \setminus \{0\}$, if $\omega(x_0) \subset \mathbb{R}^N_+(u)$ with $\omega(x_0) \cap S^0(u, v_i) \neq \emptyset$, we show that $\omega(x_0) = \{R_i\}$. Indeed, if $R_i \in \omega(x_0) \cap S^0(u, v_i)$, then $R_i$ is the unique $\omega$-limit point in $Ba(R_i)$. As $\omega(x_0)$ is connected, we must have $\omega(x_0) = \{R_i\}$. Now suppose $R_i \notin \omega(x_0) \cap S^0(u, v_i)$. Then $\omega(x_0) \cap S^0(u, v_i)$ is strictly below $\Gamma_i$ so it contains no equilibrium point. For any point $q \in \omega(x_0) \cap S^0(u, v_i)$, $x(q, t) \in \omega(x_0)$ for all $t \in \mathbb{R}$ and $x_i(q, t)$ is increasing as long as $x(q, t) \in \omega(x_0) \cap S^0(u, v_i)$. Thus,

$$x(q, t) \in \omega(x_0) \cap S(u, q_i) \subset \omega(x_0) \cap S^0(u, v_i)$$

and $x_i(q, t)$ is increasing for all $t \geq 0$. Since $\omega(x_0) \cap S(u, q_i)$ is compact and strictly below $\Gamma_i$, we have

$$\delta_0 = \min \{F_i(x) : x \in \omega(x_0) \cap S(u, q_i)\} > 0$$

so

$$x_i(q, t) = q_i \exp \left( \int_0^t F_i(x(q, s)) ds \right) \geq q_i e^{\delta_0 t}, \forall t \geq 0.$$  

This leads to the unboundedness of $x(q, t)$ for $t \geq 0$, a contradiction to $x(q, t) \in \omega(x_0) \cap S(u, q_i)$. Hence, we have shown that the case $R_i \notin \omega(x_0) \cap S^0(u, v_i) \neq \emptyset$ does not exist. Therefore, for any $x_0 \in \mathbb{R}^N_+ \setminus \{0\}$ with $\omega(x_0) \subset \mathbb{R}^N_+(u)$, we have either $\omega(x_0) \subset \mathbb{R}^N_+(u) \setminus S^0(u, v_i)$ or $\omega(x_0) = \{R_i\}$ so $R_i \in \mathbb{R}^N_+(u)$.

Next, we suppose $\gamma(x_0) \subset \Sigma \cap \mathbb{R}^N_+(u)$ for some $x_0 \in \text{int}\Sigma$. If $\gamma(x_0) \cap \Sigma \cap S^0(u, v_i) \neq \emptyset$, as any point on or above $\Gamma_i$ (except $R_i$) belongs to $Br(\infty)$ and $R_i \notin \gamma(x_0)$, $\gamma(x_0) \cap \Sigma \cap S^0(u, v_i)$ is strictly below $\Gamma_i$. Thus, $x_i(x_0, t)$ is increasing as long as $x(x_0, t) \in S^0(u, v_i)$. By $\gamma(x_0) \subset \Sigma \cap \mathbb{R}^N_+(u)$, there is a $t_1 \in \mathbb{R}$ such that $x(x_0, t) \in S^0(u, v_i)$ for $t > t_1$ but $x(x_0, t) \in \mathbb{R}^N_+(u) \setminus S^0(u, v_i)$ for $t \leq t_1$. Thus, $\alpha(x_0) \subset \Sigma \cap (\mathbb{R}^N_+(u) \setminus S^0(u, v_i))$ and, as $x_i(x_0, t)$
is increasing for \( t > t_1 \), \( \omega(x_0) \subset S^0(u,v_i) \). It then follows from the previous paragraph that \( \omega(x_0) = \{R_i\} \). If \( \gamma(x_0) \cap \Sigma \cap S^0(u,v_i) = \emptyset \) then \( \gamma(x_0) \subset \Sigma \cap (\mathbb{R}^N_+ \setminus S^0(u,v_i)) \). \( \square \)

**Lemma 6.5.** Under the conditions of Theorem 5.1, assume the existence of \( u \in \mathbb{R}^N_+ \), \( i \in I_N \) and \( v > u \) with \( v_i > u_i \) such that \( u \) is below \( \Gamma_i \) and either \( \pi_i(u) \cap [u,v] \) or \( \Gamma_i \cap \pi_i(u) \cap [u,v] \) is below \( \Gamma_j \) for all \( j \in I_N \setminus \{i\} \) and \( R_i \not\subset [u,v] \). Then there is a \( \delta \in (0,v_i - u_i) \) such that if \( \omega(x_0) \subset [u,v] \) for some \( x_0 \neq 0 \), then either \( \omega(x_0) \subset \pi_i \) (and \( u_i = 0 \)) or \( \omega(x_0) \subset [u',v] \), where \( u_i' = u_i + \delta \) and \( u_j' = u_j \) for all \( j \in I_N \setminus \{i\} \). Moreover, if \( \gamma(x_0) \subset \Sigma \cap [u,v] \) for some \( x_0 \in \text{int} \Sigma \), then either \( \omega(x_0) \subset \pi_i \) and \( \alpha(x_0) \subset [u',v] \) or \( \gamma(x_0) \subset \Sigma \cap [u',v] \).

**Proof.** Let \( D = \{x \in \pi_i(u) \cap [u,v] : F_i(x) \geq 0\} \). Since \( u \) is below \( \Gamma_i \), we have \( F_i(u) > 0 \) so \( u \in D \) and \( D \neq \emptyset \). By the assumption, \( D \) is strictly below \( \Gamma_j \) for all \( j \in I_N \setminus \{i\} \), so \( R_i \) is the only possible nontrivial equilibrium in \( D \). But \( D \subset [u,v] \) and \( R_i \not\subset [u,v] \). Hence, \( R_i \not\subset D \) so \( D \) contains no nontrivial equilibrium point. Since each point in \( D \) is on or below \( \Gamma_i \), by Lemma 6.2 \( D \subset Br(0) \). Since \( Br(0) \) is open in \( \mathbb{R}^N_+ \), \( D \) is compact, and \( F_i \) is continuous, there is a small \( \delta \in (0,v_i - u_i) \) such that the set \( S = \{x \in [u,v] : x_i \leq u_i + \delta, F_i(x) \geq -\delta\} \) is a subset of \( Br(0) \). Thus, any nontrivial point in \( S \) is neither an \( \omega \)-limit point nor an \( \alpha \)-limit point. So \( \omega(x_0) \cap S = \emptyset \) if \( x_0 \neq 0 \). Now suppose \( \omega(x_0) \subset [u,v] \) and \( \omega(x_0) \cap ([u,v] \setminus [u',v]) \neq \emptyset \) for some \( x_0 \in \mathbb{R}^N_+ \setminus \{0\} \). We show that \( \omega(x_0) \subset \pi_i \) so \( u_i = 0 \). Since \( y \in \omega(x_0) \cap ([u,v] \setminus [u',v]) \) implies \( y \not\in S \) so \( F_i(y) < -\delta \), by the compactness of \( \omega(x_0) \) and the continuity of \( F_i \), there is an \( \varepsilon > 0 \) such that

\[
\forall z \in B(\omega(x_0),\varepsilon) \text{ with } z_i \leq u_i + \delta + \varepsilon, F_i(z) \leq -\delta/2.
\]

By definition of \( \omega(x_0) \) and the assumption \( \omega(x_0) \cap ([u,v] \setminus [u',v]) \neq \emptyset \), there is a \( T > 0 \) such that \( x_i(x_0,T) < u_i + \delta + \varepsilon \) and \( x(x_0,t) \in B(\omega(x_0),\varepsilon) \) for all \( t \geq T \). Then, by (46),

\[
x_i(x_0,t) = x_i(x_0,T) \exp \left( \int_T^t F_i(x(x_0,s))ds \right) \leq x_i(x_0,T)e^{-\delta(t-T)/2}
\]

for \( t > T \) as long as \( x_i(x_0,t) < u_i + \delta + \varepsilon \). This shows that \( \lim_{t \to +\infty} x_i(x_0,t) = 0 \), so \( \omega(x_0) \subset \pi_i \) and \( u_i = 0 \).

Finally, suppose \( \gamma(x_0) \subset \Sigma \cap [u,v] \) for some \( x_0 \in \text{int} \Sigma \). If \( \gamma(x_0) \cap \Sigma \cap [u,v] \setminus [u',v] \neq \emptyset \), then, as \( \Sigma \cap Br(0) = \emptyset \) so \( \gamma(x_0) \cap S = \emptyset \), from the definition of \( S \) we see that \( F_i(x(x_0,t)) < -\delta \) so \( x_i(x_0,t) \) is decreasing as long as \( x_i(x_0,t) \leq u_i' \). This shows the existence of \( T \in \mathbb{R} \) such that \( x_i(x_0,t) \leq u_i' \) and \( F_i(x(x_0,t)) < -\delta \) for all \( t \geq T \) but \( x_i(x_0,t) > u_i' \) for \( t < T \). Therefore, \( \alpha(x_0) \subset \Sigma \cap [u',v] \) and, from (47), \( \omega(x_0) \subset \pi_i \) and \( u_i = 0 \). If \( \gamma(x_0) \cap \Sigma \cap [u,v] \setminus [u',v] = \emptyset \), then \( \gamma(x_0) \subset \Sigma \cap [u',v] \). \( \square \)

With the help of Lemmas 6.1–6.5, we are now in a position to prove Theorem 5.1.

**Proof of Theorem 5.1.** For each \( i \in I_N \), by condition (c) and (40) we see that \( L^i_1 \cap S^0(0,U_i) \) is strictly above \( L^j_2 \) for all \( j \in I_N \setminus \{i\} \). From condition (b) we know that \( \Gamma_i \) is above \( L^i_1 \).
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and $\Gamma_j$ is below $L^u_j$. So $\Gamma_i \cap S^0(0, U_i)$ is strictly above $\Gamma_j$ for all $j \in I_N \setminus \{i\}$. By Lemma 6.4 with $u = 0$ and $v_i = U_i$, for each $x_0 \in \mathbb{R}^N_+ \setminus \{0\}$ we have either $\omega(x_0) \subset [0, U]$ or $\omega(x_0) = \{R_i\}$ for some $i \in I_N$. Moreover, for each $x_0 \in \text{int}\Sigma$, we have either $\omega(x_0) = \{R_i\}$ and $\alpha(x_0) \subset \Sigma \cap [0, U]$ for some $i \in I_N$ or $\gamma(x_0) \subset [0, U]$. By condition (d), for each $i \in I_N$, either $\pi_i \cap [0, U]$ or $L^u_i \cap \pi_i \cap [0, U]$ is strictly below $L^u_j$ for all $j \in I_N \setminus \{i\}$ so either $\pi_i \cap [0, U]$ or $\Gamma_i \cap \pi_i \cap [0, U]$ is strictly below $\Gamma_j$ for all $j \in I_N \setminus \{i\}$. Note that 0 is below $\Gamma_1$ by (A1). From (40) and condition (c) we know that the intersection point $R_{ii}$ of $L^1_i$ with the positive half $x_2$-axis satisfies $R_{ii} \not\in [0, U]$. As the $i$th axial equilibrium $R_i$ is on or above $L^1_i$ whereas $R_{ii} \subset L^1_i$, we must have $R_i \not\in [0, U]$.

Then, by Lemma 6.5 with $[u, v] = [0, U]$, there is a $\delta \in (0, 1)$ such that for all $x_0 \in \mathbb{R}^N_+ \setminus \{0\}$ we have either $\omega(x_0) \subset \pi_i$ for some $i \in I_N$ or $\omega(x_0) \subset [\delta p, U]$. Further, for all $x_0 \in \text{int}\Sigma$, we have either $\omega(x_0) \subset \pi_i \supset \Sigma$ and $\alpha(x_0) \subset [\delta p, U] \supset \Sigma$ for some $i \in I_N$ or $\gamma(x_0) \subset [\delta p, U] \supset \Sigma$.

Now define an affine map $m_\delta : \mathbb{R}^N_+ \to \mathbb{R}^N_+([\delta p])$ by

$$m_\delta(x) = \delta p + (1 - \delta)x.$$ 

Then $m_\delta(0) = \delta p$, $m_\delta(p) = p$, $m_\delta(U) = \delta p + (1 - \delta)U$. Let $[u(\delta), v(\delta)] = [m_\delta(0), m_\delta(U)]$. Then $m_\delta$ maps $[0, U]$ to $[u(\delta), v(\delta)]$, each $L_i(p)$ to $L_i(p) \cap \mathbb{R}^N_+([u(\delta)])$, and each $L_j$ to a plane $\tilde{L}_j$ in $\mathbb{R}^N_+(u(\delta))$. Note that $\tilde{L}_j$ is the convex hull of the vertex set $\{V_{j1}, \ldots, V_{jN}\}$, i.e.

$$\tilde{L}_j = \{s_1V_{j1} + \cdots + s_NV_{jN} : \forall i \in I_N, s_i \geq 0, s_1 + \cdots + s_N = 1\},$$

and $\tilde{L}_j$ is the convex hull of the vertex set $\{m_\delta(V_{j1}), \ldots, m_\delta(V_{jN})\}$. Since each $\Gamma_i$ is concave or convex, $\Gamma_i$ is between $L_i(p)$ and $\tilde{L}_i$, so $\Gamma_i \cap \mathbb{R}^N_+(u(\delta))$ is between $L_i(p) \cap \mathbb{R}^N_+(u(\delta))$ and $\tilde{L}_i$, the one above $\Gamma_i \cap \mathbb{R}^N_+(u(\delta))$ is denoted by $L^u_i(\delta)$ and the one below $\Gamma_i \cap \mathbb{R}^N_+(u(\delta))$ is denoted by $L^l_i(\delta)$. Then it follows from the radial projection feature of $m_\delta$ (centred at $p$) that the relationship between the positions of the $L^u_i(\delta)$, $L^l_i(\delta)$, $p$ and $[u(\delta), v(\delta)]$ in $\mathbb{R}^N_+(u(\delta))$ is exactly the same as that of the $L^u_i$, $L^l_i$, $p$ and $[0, U]$ in $\mathbb{R}^N_+$. Thus, for each $i \in I_N$, $L^u_i(\delta) \cap S^0(u(\delta), v_i(\delta))$ is strictly above $L^u_j(\delta)$ for all $j \in I_N \setminus \{i\}$ so $\Gamma_i \cap \mathbb{R}^N_+(u(\delta), v_i(\delta))$ is strictly above $\Gamma_j$ for all $j \in I_N \setminus \{i\}$. Following the conclusion from the previous paragraph and by Lemma 6.4, for each $x_0 \in \mathbb{R}^N_+ \setminus \{0\}$ we have either $\omega(x_0) \subset [u(\delta), v(\delta)]$ or $\omega(x_0) \subset \pi_k$ for some $k \in I_N$. Furthermore, for each $x_0 \in \text{int}\Sigma$, we have either $\omega(x_0) \subset \Sigma \cap \pi_k$ and $\alpha(x_0) \subset \Sigma \cap [u(\delta), v(\delta)]$ for some $k \in I_N$ or $\gamma(x_0) \subset \Sigma \cap [u(\delta), v(\delta)]$.

From condition (d) and the feature of $m_\delta$ we see that for each $i \in I_N$, $L^u_i(\delta) \cap \pi_i(u(\delta)) \cap [u(\delta), v(\delta)]$ is strictly below $L^u_j(\delta)$ for all $j \in I_N \setminus \{i\}$. By Lemma 6.5 again and repeating the above process, we obtain $\delta_1 \in (\delta, 1)$ so that $[u(\delta), v(\delta)]$ can be replaced by $[u(\delta_1), v(\delta_1)]$ in the above conclusion. Since this process can be repeated as long as $\delta_1 < 1$, by taking the supremum of such $\delta_1$, we obtain the conclusion with $[u(1), v(1)] = \{p\}$. Therefore, for each $x_0 \in \mathbb{R}^N_+ \setminus \{0\}$, we have either $\omega(x_0) \subset \pi_i$ for some $i \in I_N$ or $\omega(x_0) = \{p\}$; for each $x_0 \in \text{int}\Sigma$, we have either $\omega(x_0) \subset \Sigma \cap \pi_i$ for some $i \in I_N$ and $\alpha(x_0) = \{p\}$ or $\gamma(x_0) = \{p\}$ so $x_0 = p$. \qed
7. Conclusion

So far by using geometric analysis, we have obtained a sufficient condition (Theorem 3.1) for a boundary or an interior equilibrium point \( p \) to be globally asymptotically stable. We have also derived a sufficient condition (Theorem 5.1) for an interior equilibrium point to be globally repelling on \( \Sigma \). These results can be applied to a class of systems (6) when each nullcline surface \( \Gamma_i \) is concave or convex, so that an upper plane \( L^u_i \) above \( \Gamma_i \) and a lower plane \( L^l_i \) below \( \Gamma_i \) can be defined. Then, geometric conditions of the theorems are formed by using the relative positions of the \( L^u_i \) and the \( L^l_j \) on the boundary \( \partial \mathbb{R}^N_+ \) within a set \([0,V]\).

Note that Theorem 5.1 for global repulsion cannot be applied to a boundary equilibrium point \( p \in \mathbb{R}^N_+ \setminus \{0\} \) with support \( J \) a proper subset of \( I_N \). However, it can be applied to the \(|J|\)-dimensional subsystem

\[
\dot{x}_i = x_i F_i(x), \quad i \in J, \quad x \in \cap_{k \in I_N \setminus J} \pi_k \tag{48}
\]
as \( p \) is an interior equilibrium of (48). If \( p \) is globally repelling for the \(|J|\)-dimensional subsystem (48) and there is a saturated boundary equilibrium point \( p_0 \) that is globally attracting for system (6), then it might be possible for \( p \) to be globally repelling on \( \Sigma \).

**Theorem 7.1.** Assume that the following conditions hold:

(a) The \( k \)th axial equilibrium point \( R_k \) of (6) is saturated for some \( k \in I_N \).

(b) For each \( i \in I_N \), the nullcline surface \( \Gamma_i \) is either concave or convex. If \( \Gamma_i \) is convex with \( F_i(R_k) < 0 \) then the function \( F_i \) is also convex with \( F_i(0) = \max_{x \in \mathbb{R}^N_+} F_i(x) \).

(c) For all \( i,j \in I_N \setminus \{k\} \), the intersection point \( R_{ki} \) of \( L^l_k \) with the positive half \( x_i \)-axis is above \( L^u_j \).

(d) System (6) has an equilibrium \( p \in \mathbb{R}^N_+ \) with support \( J = I_N \setminus \{k\} \) and \( p \) as an interior equilibrium point of the subsystem

\[
\dot{x}_i = x_i F_i(x), \quad i \in J, \quad x \in \pi_k \tag{49}
\]
is globally repelling on \( \Sigma \cap \pi_k \).

(e) Any \( \alpha \) limit set \( \alpha(x_0) \) consists of a single equilibrium point if \( \alpha(x_0) \subset \Sigma \cap \pi_k \cap (\cup_{j \in J} \pi_j) \).

(f) The unstable manifold \( \mathcal{W}^u(q) \) for each equilibrium \( q \) in \( \Sigma \cap \pi_k \cap (\cup_{j \in J} \pi_j) \) is a subset of \( \cup_{j \in J} \pi_j \).

Then \( p \) is globally repelling on \( \Sigma \) and \( R_k \) is globally attracting. Moreover, if \( R_k \) is above \( \Gamma_i \) for all \( i \in J \), then \( R_k \) is globally asymptotically stable.

**Proof.** From condition (c) we know that either \( Y^J \) is below \( L^l_k \) or \( L^l_k \cap [0,Y] \cap \pi_k \) is strictly above \( L^u_j \) for all \( j \in I_N \setminus \{k\} \), where \( Y \) is defined by (28). By conditions (a)–(c) and
Theorem 3.1, $R_k$ is globally attracting. Thus, we have $\omega(x_0) = \{R_k\}$ for any $x_0 \in \Sigma \cap \mathbb{R}^J$. In particular, $R_k$ attracts the compact set $\Sigma_\delta = \{x \in \Sigma : x_k = \delta\}$ for sufficiently small $\delta > 0$. Condition (c) and Lemma 6.3 imply that $\Sigma \cap \pi_k$ is strictly below $\Gamma_k$. Thus, $\Sigma_\delta$ is strictly below $\Gamma_k$ for sufficiently small $\delta > 0$. Since $x_k(t)$ is increasing as long as $x(t)$ is below $\Gamma_k$, we have shown that $\alpha(x_0) \subset \Sigma \cap \pi_k$ for $x_0 \in \Sigma_\delta$ and, hence, for all $x_0 \in \Sigma \cap \mathbb{R}^J \setminus \{R_k\}$. As $p$ repels on $\Sigma \cap \pi_k$ by condition (d) and $p$ is below $\Gamma_k$, $p$ is a repeller on $\Sigma$. Thus, for any $\alpha(x_0) \subset \Sigma \cap \pi_k$, we have either $\alpha(x_0) = \{p\}$ or $\alpha(x_0) \subset \Sigma \cap \pi_k \cap (\cup_{j \in J} \pi_j)$. By condition (e) we know that, as $t \to -\infty$, $x(x_0, t)$ converges to $p$ or an equilibrium point in $\Sigma \cap \pi_k \cap (\cup_{j \in J} \pi_j)$ for $x_0 \in \Sigma \cap \mathbb{R}^J \setminus \{R_k\}$. Now we claim that $\alpha(x_0) = \{p\}$ for all $x_0 \in \text{int} \Sigma$. Indeed, $x_0 \gg 0$ so $x_0 \not\in \pi_i$ for all $i \in I_N$. By condition (f), $x_0 \not\in W^u(q)$ for any equilibrium point $q \in \Sigma \cap \pi_k \cap (\cup_{j \in J} \pi_j)$. Thus, $\alpha(x_0) \not\subset \Sigma \cap \pi_k \cap (\cup_{j \in J} \pi_j)$ so $\alpha(x_0) = \{p\}$. Therefore, $p$ is globally repelling on $\Sigma$. Finally, if $R_k$ is above $\Gamma_i$ for all $i \in J$, then the Jacobian matrix $Df(R_k)$ has $N$ negative eigenvalues, so $R_k$ is globally asymptotically stable. \hfill $\square$

**Example 7.2.** Consider the system

$$
\begin{align*}
\dot{x}_1 &= x_1(1 - 2ax_1 - ax_1^2 - x_2 - x_3 - x_4) = x_1F_1(x), \\
\dot{x}_2 &= x_2(1 - x_1 - 2ax_2 - ax_2^2 - x_3 - x_4) = x_2F_2(x), \\
\dot{x}_3 &= x_3(1 - x_1 - x_2 - 2ax_3 - ax_3^2 - x_4) = x_3F_3(x), \\
\dot{x}_4 &= x_4(2 - 3ax_1 - 3ax_2 - 3ax_3 - x_4) = x_4F_4(x),
\end{align*}
$$

(50)

where $a \in (0, 0.3]$ is a constant satisfying (43). The 3-dimensional subsystem on $\pi_4$ is the system (42) in Example 5.4. So $p = (p_0, p_0, p_0, 0)^T$ is globally repelling on $\Sigma \cap \pi_4$. This shows that system (50) satisfies condition (d) of Theorem 7.1. The axial equilibrium point $R_4 = (0, 0, 0, 2)^T$ is above $\Gamma_1, \Gamma_2$ and $\Gamma_3$ so it is saturated. Clearly, $\Gamma_4$ is a plane and $F_1, F_2, F_3$ are convex (see Example 5.4 in section 5). Thus, (50) meets conditions (a) and (b) of Theorem 7.1. The intersection points of $L_1^u, L_2^u, L_3^u$ and $L_4^u$ with the positive half $x_1$-axis are $\left(\frac{1 + ap_0^2}{2a(1 + p_0)}, 0, 0, 0\right)^T, (1, 0, 0, 0)^T, (1, 0, 0, 0)^T$ and $R_{41} = (\frac{2}{3a}, 0, 0, 0)^T$ respectively. As $p_0 < \frac{1}{2}$ and $a \leq 0.3$, we have

$$1 < \frac{1 + ap_0^2}{0.9} \leq \frac{1 + ap_0^2}{2a(1 + p_0)} < \frac{1}{2a} + \frac{1}{2\left[\left(\frac{1}{p_0}\right)^2 + \frac{1}{p_0}\right]} < \frac{1}{2a} + \frac{1}{12} = \frac{6 + a}{12a} < \frac{2}{3a}.$$ 

Thus, $R_{41}$ is above $L_1^u, L_2^u$ and $L_3^u$. By symmetry, $R_{42}$ and $R_{43}$ are also above $L_1^u, L_2^u$ and $L_3^u$. This shows that (50) satisfies condition (c) of Theorem 7.1. To check conditions (e) and (f), we note that the phase portrait on $\Sigma \cap \pi_4$ is given by Figure 4. From the flow on $\Sigma \cap \pi_4$ we see that any $\alpha(x_0) \subset \Sigma \cap \pi_4$ must consist of a single equilibrium point. Thus, condition (e) of Theorem 7.1 holds for (50). Since $\Sigma \cap \pi_4$ is strictly below $\Gamma_4$, for any equilibrium point $q \in \Sigma \cap \pi_4 \cap (\pi_1 \cup \pi_2 \cup \pi_3)$, $Df(q)$ has an eigenvector in $\pi_1 \cup \pi_2 \cup \pi_3$ transverse to $\pi_4$ corresponding to the positive eigenvalue $F_4(q)$. By the invariance of each $\pi_i$, we have $W^u(q) \subset (\pi_1 \cup \pi_2 \cup \pi_3)$. Thus, (50) satisfies condition (f) of Theorem 7.1. Then, by Theorem 7.1, $R_4$ is globally asymptotically stable and $p$ is globally repelling on $\Sigma$. 
Figure 4. Phase portrait for system (50) on $\Sigma \cap \pi_4$.

**Discussion.** For any equilibrium $p \in \Sigma$ with support $J \subset I_N$, we call $p$ saturated in reversed time if $F_i(p) \geq 0$ for all $i \in I_N$. As $F_i(p)$ is an eigenvalue of the Jacobian matrix $Df(p)$ if $i \in I_N \setminus J$, it follows that a necessary condition for $p$ to be a repellor on $\Sigma$ is that $p$ must be saturated in reversed time. Combining Theorems 5.1 and 7.1, we have obtained sufficient conditions for an equilibrium $p$ saturated in reversed time to be globally repelling on $\Sigma$ if $p$ has at most one zero component (i.e. $|J| \geq N - 1$). However, if $p$ has more than one zero components, our theorems for global repulsion are not applicable. Does the geometric method used here still have the power to deal with the problem of global repulsion when $|J| < N - 1$? This is left as an open problem.

**Appendix**

The proofs of Propositions 2.2 and 2.3 are given below.

**Proof of Proposition 2.2.** (i) and (ii) are straightforward from the definitions of convexity and concavity of a surface and convexity of a set.

(iii) Taking any $x, y \in \Gamma$ with $\alpha < G(0)$ in the range of $G$, by the convexity of $G$ we have

$$\forall s \in [0, 1], G(sx + (1-s)y) \geq sG(x) + (1-s)G(y) = s\alpha + (1-s)\alpha = \alpha. \tag{51}$$

Since $0 \in \Gamma^-$ and $G(0) > \alpha$, we must have $G(w) > \alpha$ for all $w \in \Gamma^-$. So $sx+(1-s)y \in \Gamma^- \cup \Gamma$ for all $s \in [0, 1]$. This shows that $\Gamma$ is convex.

(iv) Since $-G$ is convex, for any $\alpha > G(0)$ in the range of $G$ and $x, y \in \Gamma$, we have

$$\forall s \in [0, 1], -G(sx + (1-s)y) \geq -sG(x) - (1-s)G(y) = -s\alpha - (1-s)\alpha = -\alpha. \tag{52}$$
So \( G(sx + (1 - s)y) \leq \alpha \) for all \( s \in [0, 1] \). Then \( G(0) < \alpha \) and \( 0 \in \Gamma^- \) imply \( G(w) < \alpha \) for all \( w \in \Gamma^- \). Thus, \( sx + (1 - s)y \in \Gamma^- \cup \Gamma \) for all \( s \in [0, 1] \), so \( \Gamma \) is convex.

(v) Since \( G \) is convex, (51) holds for all \( x, y \in \Gamma \) with \( \alpha > G(0) \) in the range of \( G \). Since \( 0 \in \Gamma^- \) and \( G(0) < \alpha \), we must have \( G(w) < \alpha \) for all \( w \in \Gamma^- \) and \( G(w) \geq \alpha \) for all \( w \in \Gamma \cup \Gamma^+ \). It then follows from (51) that \( \overline{xy} \subset \Gamma \cup \Gamma^+ \), so \( \Gamma \) is concave.

(vi) Since \( -G \) is convex, (52) holds for all \( x, y \in \Gamma \) with \( \alpha < G(0) \) in the range of \( G \). Since \( 0 \in \Gamma^- \) and \( G(0) > \alpha \), we must have \( G(w) > \alpha \) for all \( w \in \Gamma^- \) and \( G(w) \leq \alpha \) for all \( w \in \Gamma \cup \Gamma^+ \). It then follows from (52) that \( \overline{xy} \subset \Gamma \cup \Gamma^+ \). This shows the concavity of \( \Gamma \).  

**Proof of Proposition 2.3.** Note that the sign of \( G(x) - \alpha \) on \( \Gamma^- \) is opposite to that on \( \Gamma^+ \). We first assume \( G(x) - \alpha < 0 \) for \( x \in \Gamma^- \) and \( G(x) - \alpha > 0 \) for \( x \in \Gamma^+ \).

(a) From the convexity of \( \Gamma \) and Proposition 2.2 (ii), \( \Gamma^- \cup \Gamma \) is a convex set. So, for each \( x \in \Gamma^- \cup \Gamma \setminus \{u\} \), \( \overline{ux} \subset \Gamma^- \cup \Gamma \). Thus,

\[
\forall s \in [0, 1], G(sx + (1 - s)y) - G(u) = G(u + s(x - u)) - \alpha \leq 0.
\]

From this it follows that

\[
D_{ux} G(u) = \lim_{s \to 0^+} \frac{1}{s} [G(u + s(x - u)) - G(u)] \leq 0.
\]

Since the directional derivative of \( G \) satisfies

\[
D_{ux} G(u) = \nabla G(u) \frac{(x - u)}{\|x - u\|} = \frac{1}{\|x - u\|} \nabla G(u)(x - u),
\]

we obtain \( \nabla G(u)(x - u) \leq 0 \) for all \( x \in \Gamma^- \cup \Gamma \). This shows that \( \Gamma^- \cup \Gamma \) is on one side of \( T_u(\Gamma) \). As \( 0 \in \Gamma^- \) and \( 0 \) is below \( T_u(\Gamma) \), the set \( \Gamma^- \cup \Gamma \) is below \( T_u(\Gamma) \) and so is \( \Gamma \).

To show that \( \Gamma \) is above \( L(\Gamma) \), we need only show that \( L(\Gamma) \) is below \( \Gamma \), i.e. \( L(\Gamma) \subset \Gamma^- \cup \Gamma \). If \( R_i, R_j \) exist for some distinct \( i, j \in I_N \), as \( R_i, R_j \in \Gamma \), by the convexity of \( \Gamma \) and Proposition 2.2 (i), \( R_i R_j \subset (\Gamma^- \cup \Gamma) \cap L(\Gamma) \). If \( R_i \) exists but \( R_j \) does not exist, then \( J_j \subset \Gamma^- \). Let \( Q_j \in J_j \) with \( v_j \) as its \( j \)th component. Then \( R_i Q_j \subset \Gamma^- \cup \Gamma \). As the half line \( L(R_i) \) passing through \( R_i \) and parallel to \( J_j \) lies in \( L(\Gamma) \), by the definition of \( L(\Gamma) \), and is the limit of \( R_i Q_j \) as \( v_j \to +\infty \), we also have \( L(R_i) \subset (\Gamma^- \cup \Gamma) \cap L(\Gamma) \). This shows that each one-dimensional edge of \( L(\Gamma) \) is contained in \( \Gamma^- \cup \Gamma \). Since \( \Gamma^- \cup \Gamma \) is convex and \( L(\Gamma) \) is both convex and concave, for any \( x, y \in (\Gamma^- \cup \Gamma) \cap L(\Gamma) \), we must have \( \overline{xy} \subset (\Gamma^- \cup \Gamma) \cap L(\Gamma) \). As each two-dimensional face of \( L(\Gamma) \) consists of \( \overline{xy} \) with \( x, y \) taking all the points in two one-dimensional edges, all two-dimensional faces of \( L(\Gamma) \) are contained in \( \Gamma^- \cup \Gamma \). Repeating this process a finite number of times, we obtain \( L(\Gamma) \subset \Gamma^- \cup \Gamma \). Hence, \( L(\Gamma) \) is below \( \Gamma \).

(b) By the concavity of \( \Gamma \) and Proposition 2.2 (ii), \( \Gamma \cup \Gamma^+ \) is convex. So, for any \( x \in \Gamma \cup \Gamma^+ \setminus \{u\} \), we have \( \overline{ux} \subset \Gamma \cup \Gamma^+ \). Thus,

\[
\forall s \in [0, 1], G(sx + (1 - s)y) - G(u) = G(u + s(x - u)) - \alpha \geq 0,
\]
from which follows $D_{x} G(u) \geq 0$. As $D_{x} G(u) = \frac{1}{\|x-u\|} \nabla G(u)(x-u)$, we obtain $\nabla G(u)(x-u) \geq 0$ for all $x \in \Gamma \cup \Gamma^+$. Thus, $\Gamma \cup \Gamma^+$ is on one side of $T_u(\Gamma)$. We shall see that
\begin{equation}
\forall w \in \Gamma, \mathbb{R}^N_+(w) = \{ x \in \mathbb{R}^N_+ : x \geq w \} \subset \Gamma \cup \Gamma^+.
\end{equation}
So, from this follows $\mathbb{R}^N_+(u) \subset \Gamma \cup \Gamma^+$ since $u \in \Gamma$. As $2u \in \mathbb{R}^N_+(u)$ so $2u \in \Gamma \cup \Gamma^+$, we have $\nabla G(u)(2u - u) = \nabla G(u)u \geq 0$. This together with $\nabla G(u)u \neq 0$ implies that $\nabla G(u)u > 0$ and $\nabla G(u)(0-u) < 0$. Thus, $\Gamma \cup \Gamma^+$ is on one side of $T_u(\Gamma)$ but 0 is on the other side of $T_u(\Gamma)$. Since 0 is below $T_u(\Gamma)$ by definition, $\Gamma \cup \Gamma^+$ is above $T_u(\Gamma)$ and so is $\Gamma$.

To show that $\Gamma$ is below $L(\Gamma)$, we need only show that $L(\Gamma)$ is above $\Gamma$, i.e. $L(\Gamma) \subset \Gamma \cup \Gamma^+$. For this purpose, we first show (55). We claim that $L(\Gamma) \subset \Gamma \cup \Gamma^+$ for all $i \in I_N$. Indeed, if $R_i$ does not exist, then the half line $L(i)$ lies in $\Gamma \cup \Gamma^+$ by assumption. If $R_i$ exists, then, for any $Q_i \in \Gamma$ with $v_i$ as its $i$th component and $Q_i$ by the convexity of $\Gamma \cup \Gamma^+$ and $Q_i$, $w \in \Gamma \cup \Gamma^+$, we have $wQ_i \subset \Gamma \cup \Gamma^+$. Since $L(i)$ is the limit of $wQ_i$, as $v_i \to +\infty$, we also have $L(i) \subset \Gamma \cup \Gamma^+$. Then it follows from the convexity of $\Gamma \cup \Gamma^+$ that
\begin{equation}
L(i) \times L(j) = \{ sx + (1-s)y : x \in L(i), y \in L(j), s \in [0,1] \} \subset \Gamma \cup \Gamma^+
\end{equation}
for all $i, j \in I_N$. Since $\mathbb{R}^N_+(w) = L(i) \times L(j) \times \cdots \times L(N)$, repeating the above process a finite number of times, we have shown (55).

Now if $R_i, R_j$ exist for some distinct $i, j \in I_N$, $R_i R_j$ is a one-dimensional edge of $L(\Gamma)$ and $R_i R_j \subset \Gamma \cup \Gamma^+$ by the convexity of $\Gamma \cup \Gamma^+$. If $R_i$ exists but $R_j$ does not exist, then $L(R_i j)$ is a one-dimensional edge of $L(\Gamma)$ and $L(R_i j) \subset \Gamma \cup \Gamma^+$ by assumption. Thus, every one-dimensional edge of $L(\Gamma)$ is contained in $\Gamma \cup \Gamma^+$. Then, following the same reasoning as we did in part (a), we obtain $L(\Gamma) \subset \Gamma \cup \Gamma^+$, so $L(\Gamma)$ is above $\Gamma$.

The proof is complete under the assumption $G(x) - \alpha < 0$ for $x \in \Gamma$ and $G(x) - \alpha > 0$ for $x \in \Gamma^+$. If $G(x) - \alpha > 0$ for $x \in \Gamma$ and $G(x) - \alpha < 0$ for $x \in \Gamma^+$, the above proof is still valid after swapping “≤” and “≥” in (53), (54) and some related inequalities. \hfill \Box

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**References**


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