TITLE A CLASS OF FUNCTIONAL DIFFERENTIAL EQUATIONS OF MIXED TYPE

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Zhanyuan Hou

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## ABSTRACT

This thesis is concerned with a class of linear functional differential equations of mixed type having the form

$$\frac{d}{dt}\{x(t) - D(t, x_t)\} = L_1(t, x_t) + L_2(t, x^t) + f(t), \qquad (1)$$

where, for each $t \geq \tau$, $D(t, .)$ and $L_1(t, .)$ from $C \equiv C([-r, 0], \mathbb{C}^N)$ to $\mathbb{C}^N$ and $L_2(t, .)$ from $C([0, \sigma], \mathbb{C}^N)$ to $\mathbb{C}^N$ are bounded linear operators. The thesis consists of three parts.

As a special case of (1), equations with piecewise constant arguments having the form

$$\frac{d}{dt}\{x(t) - D(x_t)\} = L_1(x_t) + L_2(x_{[t+\beta]}) + f(t) \qquad (2)$$

are investigated in Part I, where $D$ and $L_k (k = 1, 2)$ are from $C$ to $\mathbb{C}^N$, $\beta \geq 0$ is a constant and $[.]$ is the integer part function. Results on existence and uniqueness, representation of solutions, exponential estimates and asymptotic behaviour are obtained.

For the general equation (1), initial value problems are discussed and some simple properties of solutions given in Part II. If (2) is viewed as an instance of (1), $L_2(t, \psi)$ does not always involve the values of $\psi$ in the vicinity of $\sigma$. In Chapter 7, another case when (1) has such a property is discussed, namely when there is a sequence $\{t_n\}$ of Zero Advanced Points (ZAPs: for all $t \in [\tau, t_n]$, $L_2(t, x^t)$ does not involve any value of $x$ on $(t_n, \infty)$).

With existence and uniqueness guaranteed by a sequence of ZAPs and other suitable conditions, asymptotic solutions and asymptotic representation of solutions of the equation

$$\frac{d}{dt}\{x(t) - D(t, x_t)\} = A(t)x(t) + L_1(t, x_t) + L_2(t, x^t) \qquad (3)$$

are studied in Part III when the $N \times N$ matrix $A(t)$ is either bounded or diagonal. When $\|D(t, .)\|$ and $\|L_k(t, .)\|$ $(k = 1, 2)$ are small in a suitable sense as $t \to \infty$, the equation has $N$ solutions $x_1, x_2, \cdots, x_N$ on $[T - r, \infty)$, for sufficiently large $T \geq \tau$, such that every solution of the equation has the asymptotic representation

$$x(t) = X(t)c + o(e^{\delta(t) - \beta t}),$$

where $X = (x_1, x_2, \cdots, x_N)$, $c \in \mathbb{C}^N$ depends on $x$, $\delta(t)$ is an exponential bound for $X(t)$, and $\beta \geq 0$ is arbitrary. Asymptotic forms for $x_1, x_2, \cdots, x_N$ are also given when $A(t)$ is either diagonal or constant.

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# CHAPTER 0 INTRODUCTION

## § 0.1 TERMINOLOGY

We start by explaining some terms that are commonly used in the literature and that appear throughout the thesis.

A functional differential equation is a differential equation involving the values of the unknown functions at present, as well as at past or future times. The word time here stands for the independent variable. The concept of a functional differential equation is in this thesis confined to ordinary differential equations although it suits partial ones as well.

A differential difference equation is a functional differential equation in which the functionals, or operators, are represented by series or finite sums, each term of which involves the unknown function at only one time. General functional differential equations involve the values of the unknown functions in a time interval $[\tau_1(t), \tau_2(t)]$, which includes the present time $t$. In general, the deviations from the present, $\tau_1(t) - t$ and $\tau_2(t) - t$, may be bounded or not. Only equations with bounded deviations are considered in this thesis.

Functional differential equations can be classified into four types according to their deviations: retarded, advanced, neutral and mixed.

If the derivative at present is given in terms of past and present values of the functions in an equation, the equation is called retarded or delayed.

A neutral equation is one in which derivatives of functionals of the past history and the present state are involved but no future states occur in the equation.

An advanced equation involves only the present derivative, and future and present values of the functions. Although an advanced equation can be transformed to retarded one by replacing $t$ by $-t$, this is not helpful if we are interested only in equations on a half line $[t_0, \infty)$.

If an equation involves the derivative of functions at present, or derivatives of functionals of the past history, and past as well as future states, we say that the equation is of <u>mixed</u> type.

The type of an equation may depend on the time interval. If the equation belongs to one type on a sequence of intervals within $[t_0, \infty)$ and another type on the remaining intervals, it is also said to be of <u>alternating</u> type on $[t_0, \infty)$.

Any other type can be viewed as a special case of mixed type. However, the involvement of future states in a mixed equation is emphasized in this thesis.

## § 0.2 RECENT DEVELOPMENT OF MIXED EQUATIONS

It is well known that the topic of functional differential equations as a whole has developed as a branch of differential equations only in the recent few decades. However, a complete basic theory of retarded and neutral equations was formed in late seventies. We refer to Bellman and Cooke [3], Myshkis [26], Driver [11], and Hale [17] for such a theory. Since then there have appeared a large number of research results reflecting a variety of specialised interests. For instance, Győri and Ladas' book [14] includes most of the recent results in the oscillation theory of functional differential equations.

For advanced and mixed functional differential equations, miscellaneous problems have been posed for some specific equations found in both physical and theoretical contexts. Here some examples are mentioned.

Slater and Wilf [34] studied the equation

$$\frac{d}{dt}x(t) = k(t+1)\,x(t+1) - k(t)\,x(t),$$

which describes the slowing down of neutrons in a nuclear reactor. In the classic work of Pontryagin et al. [29], it is indicated that an optimal control problem involves an equation of mixed type. Chi et al. [5] studied a biological model by numerical methods.

In the last decade, oscillatory solutions have been studied for some special advanced and mixed equations, such as

$$\frac{d}{dt}y(t) = \sum_{i=1}^{n} p_i y(t+\tau_i), \quad p_i > 0, \ \tau_i > 0, \ i = 1, 2, \cdots, n$$

by Ladas et al. [22],

$$\frac{d^n}{dt^n}y(t) = p(t)\, y(g(t)) + q(t)\, y(h(t)), \quad g(t) < t, \ h(t) > t$$

by Kusano [21], and

$$\frac{d}{dt}x_i(t) + \sum_{j=1}^{n} \{a_{ij}(t)x_j(t - \tau_{ij}(t)) + b_{ij}(t)x_j(t + \sigma_{ij}(t))\} = 0,$$

$$\tau_{ij}(t) \geq 0, \ \sigma_{ij}(t) \geq 0, \ i, j = 1, 2, \cdots, n$$

by Gopalsamy [13].

As a special class, equations with piecewise constant arguments have been broadly studied since the beginning work by Cooke et al. in the early eighties ( see Shah and Wiener [33] ) and basic problems, such as the initial value problem, stability, oscillation etc., have been solved for some particular equations. We refer to Cooke and Wiener [9] and the references therein for more details.

Rustichini [31, 32] studied a class of autonomous equations of mixed type. The dynamics of the linear case was investigated by starting from the infinitesimal generator rather than from the semigroup of the solution map. Based on this analysis Rustichini investigated Hopf bifurcations for the nonlinear case.

Lin [24] studied the bounded solutions of a class of advanced linear differential difference equations on $[t_0, \infty)$.

For a class of mixed equations with the form

$$\frac{d}{dt}x(t) = \int_{\alpha}^{\beta} x(t + \tau)\, dr(\tau) + \int_{\gamma}^{\delta} x(t + \tau)\, d_{\tau}s(t, \tau),$$

Murovtsev and Myshkis [25] studied the existence of bilateral solutions (i.e., solutions on $(-\infty, \infty)$).

Although many scholars have engaged in the study of advanced and mixed equations and a large number of results have been obtained so far, there is no systematic general theory available for such equations.

## § 0.3 MOTIVATION

In both theory and practice, there is an increasing need to investigate equations of mixed type. In contrast to the theory of retarded and neutral equations, that of mixed equations is far from complete due to the difficulty coming from the involvement of future states. But if we restrict ourselves to some particular problems for special equations, various results can be found as mentioned above. However, any effort is worth making to investigate a broader class of equations and to establish a theory for some basic problems such as the Cauchy problem, asymptotic behaviour at infinity, and some qualitative properties.

This thesis aims to study some basic problems for a class of linear functional differential equations of mixed type.

## § 0.4 OUTLINE

The equations to be studied have the form

$$\frac{d}{dt}\{x(t) - D(t, x_t)\} = L_1(t, x_t) + L_2(t, x^t) + f(t), \tag{0.1}$$

where $f$ is a locally integrable function from $[t_0, \infty)$ to $\mathbb{C}^N$ (complex n-dimensional space), the functions $x_t$ and $x^t$ are defined by

$$x_t(\theta) = x(t + \theta) \text{ for } \theta \in [-r, 0]$$

and

$$x^t(\alpha) = x(t + \alpha) \text{ for } \alpha \in [0, \sigma],$$

$r$ and $\sigma$ are positive constants, and $D(t, .)$, $L_1(t, .)$ and $L_2(t, .)$ are linear mappings whose detailed description is given later. Some basic problems for (0.1) in general will be discussed in Part II.

As a special case of (0.1), equations with piecewise constant arguments having the form

$$\frac{d}{dt}\{x(t) - D(x_t)\} = L_1(x_t) + L_2(x_{[t+\beta]}) + f(t) \tag{0.2}$$

are to be discussed in Part I. Here $\beta > 0$ is a constant, $[.]$ is the integer part function, and $D$, $L_1$ and $L_2$ are bounded linear mappings from $C([-r, 0], \mathbb{C}^N)$ to $\mathbb{C}^N$.

In Part III, asymptotic solutions are to be studied for some special cases of (0.1). The equations to be dealt with have the form

$$\frac{d}{dt}\{x(t) - D(t, x_t)\} = A(t)\, x(t) + L_1(t, x_t) + L_2(t, x^t), \tag{0.3}$$

where $A(t)$ is an $N \times N$ matrix function having some special forms, and $D(t, .)$, $L_1(t, .)$ and $L_2(t, .)$ are small at infinity in some sense.

# PART ONE

# LINEAR AUTONOMOUS EQUATIONS WITH PIECEWISE CONSTANT ARGUMENTS

# CHAPTER 1 PRELIMINARIES

## § 1.1 INTRODUCTION

In the article of Myshkis [27], it was observed that a substantial theory did not exist for functional differential equations with piecewise continuous arguments. Since then some special equations with piecewise constant arguments (EPCA in short) have been investigated by a number of authors. For instance, equations of the form

$$\frac{d}{dt}x(t) = f(t, x(t), x(h(t)))$$

with $h(t) = [t]$, $[t - n]$, and $t - n[t]$ were studied by Cooke and Wiener [8], a second order equation

$$\frac{d^2}{dt^2}x(t) + p\,x(2[(t+1)/2]) = 0$$

was studied by Ladas et al. [23], a neutral equation

$$\frac{d}{dt}\{y(t) + p\,y(t + 1/2)\} = q\,y([t + 1/2])$$

by Huang [18], and a system

$$\frac{d}{dt}x(t) = Ax(t) + \sum_{j=-K}^{K} A_j x([t+j]) + f(t), \quad x(j) = c_j, \; -K \le j \le K-1$$

by Cooke and Wiener [9].

In summary, most of the previous studies in this direction have had the following features: (a) they dealt with special classes of differential difference equations, (b) they solved the problems by fundamental methods in differential equations. Moreover, only a few (such as [9]) investigated equations of mixed type which are essentially different from retarded and neutral equations for the Cauchy problem.

Clearly, there is still a lot of work to do for general functional differential EPCAs of mixed type. In this part, we study a class of linear functional differential EPCAs which we believe have not been studied by anyone else so far.

Throughout this thesis, $C \equiv C([-r, 0], \mathbb{C}^N)$ stands for the space of continuous functions from $[-r, 0]$ to $\mathbb{C}^N$ with the uniform norm $\|\varphi\| = \sup\{|\varphi(\theta)|: \theta \in [-r, 0]\}$.

The equations to be considered here have the form

$$\frac{d}{dt}\{x(t) - D(x_t)\} = L_1(x_t) + L_2(x_{[t+\beta]}) + f(t) \tag{1.1}$$

for $t \geq 0$, where $f$, from $[0, \infty)$ to $\mathbb{C}^N$, is locally integrable, $D$, $L_1$ and $L_2$ are bounded linear mappings from $C$ to $\mathbb{C}^N$ represented by

$$D(\varphi) = \int_{-r}^{0} \{d\xi(\theta)\}\varphi(\theta), \quad L_i(\varphi) = \int_{-r}^{0} \{d\eta_i(\theta)\}\varphi(\theta)$$

for $i = 1, 2$ and $\varphi \in C$, and $r > 0$ and $\beta \geq 0$ are constants. Here $\xi(\theta)$, $\eta_1(\theta)$ and $\eta_2(\theta)$ are $N \times N$ matrix functions of bounded variation and $\xi(\theta)$ satisfies

$$\lim_{\alpha \to 0} \mathrm{Var}_{[\alpha, 0]}(\xi) = 0.$$

We say that $x: [-r, \infty) \to \mathbb{C}^N$ satisfies (1.1) for $t \geq 0$ if $x(t)$ is continuous on $[-r, \infty)$ and $x(t) - D(x_t)$ is locally absolutely continuous on $[0, \infty)$, and (1.1) holds almost everywhere on $[0, \infty)$. The homogeneous equation corresponding to (1.1) is

$$\frac{d}{dt}\{x(t) - D(x_t)\} = L_1(x_t) + L_2(x_{[t+\beta]}). \tag{1.2}$$

We say that (1.1) and (1.2) are autonomous if the mappings $D$, $L_1$ and $L_2$ are independent of $t$. This is slightly different from the normal sense since the deviations vary with $t$.

The initial value problem will be discussed and existence and uniqueness theorems given in Chapter 2. Some other problems such as exponential estimates, the representation of solutions, the relationship between solutions of (1.1) and those of (1.2), asymptotic behaviour etc. will be investigated in Chapters 3-5.

## § 1.2 PRELIMINARIES

For convenience of later use, we state some results on linear autonomous neutral functional differential equations with the forms

$$\frac{d}{dt}\{x(t) - D(x_t)\} = L_1(x_t) \tag{1.3}$$

and

$$\frac{d}{dt}\{x(t) - D(x_t)\} = L_1(x_t) + f(t). \tag{1.4}$$

All of the results in this section before Lemma 1.1 can be found in Chapter 12 of [17].

For any $t_0 \in [0, \infty)$, $\varphi_0 \in C$ and any locally integrable $f$ on $[0, \infty)$, there is a unique solution $x: [t_0 - r, \infty) \to \mathbb{C}^N$ of (1.4) such that $x_{t_0} = \varphi_0$, i.e., $x$ is continuous on $[t_0 - r, \infty)$, $x(t) - D(x_t)$ is locally absolutely continuous on $[t_0, \infty)$, (1.4) holds almost everywhere, and $x(t_0 + \theta) = \varphi_0(\theta)$ for $\theta \in [-r, 0]$.

Since (1.3) is autonomous, its solution satisfies

$$x(t_0, \varphi_0)(t) = x(0, \varphi_0)(t - t_0)$$

for $t \geq t_0$. Also, for any $t \geq 0$, $x(0, \varphi_0)(t)$ is linear and continuous in $\varphi_0 \in C$. Thus we can define a bounded linear operator $T(t)$ on $C$ by

$$x_t(0, \varphi_0) = T(t)\varphi_0, \ t \geq 0. \tag{1.5}$$

By the definition of the symbol $x_t$ in § 0.4, we have

$$x(0, \varphi_0)(t + \theta) = x_t(0, \varphi_0)(\theta) = T(t)\varphi_0(\theta)$$

for $\theta \in [-r, 0]$. Hence, the solution of (1.3) with $x_0 = \varphi_0$ can be represented by either (1.5) or

$$x(0, \varphi_0)(t) = T(t)\varphi_0(0), \ t \geq 0. \tag{1.6}$$

The operator $T(t)$ possesses the property

$$T(t')T(t'') = T(t' + t'')$$

for $t', t'' > 0$. Moreover, there are real constants $b \geq 1$ and $a$ such that

$$\|T(t)\| \le b\,e^{at},\ t \ge 0,$$

where $\|T(t)\| = \sup\{\|T(t)\varphi\|:\ \varphi \in C,\ \|\varphi\| = 1\}$. By (1.5) the above inequality is equivalent to the exponential estimate

$$\|x_t(0, \varphi_0)\| \le b\,e^{at}\,|\varphi_0|,\ t \ge 0. \tag{1.7}$$

It is shown in [17] that the trivial solution of (1.3) is asymptotically stable if and only if the number a can be taken negative in (1.7).

Let $Y(t)$ be the matrix solution of the equation

$$Y(t) - D(Y_t) = I + \int_0^t L_1(Y_s)\,ds,\ t \ge 0, \tag{1.8}$$

with $Y_0$ given by $Y(0) = I$, the unit $N \times N$ matrix, and $Y(t) = 0$ for $t < 0$. Here $Y(t)$ is continuous from the right and is of bounded variation on any compact set. Then, for $t \ge t_0$, the solution of (1.4) can be represented by the variation of parameters formula

$$x(t_0, \varphi_0, f)(t) = T(t - t_0)\varphi_0(0) + \int_{t_0}^t Y(t - s)f(s)\,ds, \tag{1.9}$$

or alternatively by

$$x_t(t_0, \varphi_0, f) = T(t - t_0)\varphi_0 + \int_{t_0}^t Y_{t-s}f(s)\,ds. \tag{1.10}$$

From (1.10) it follows that

$$T(\sigma)\left\{\int_{t_0}^t Y_{t-s}f(s)\,ds\right\} = \int_{t_0}^t Y_{\sigma+t-s}f(s)\,ds \tag{1.11}$$

for $t \ge t_0$ and $\sigma \ge 0$.

It is also given in [17] that the solution of (1.4) satisfies

$$\|x_t(t_0, \varphi_0, f)\| \le b\,e^{a(t-t_0)}\left(\|\varphi_0\| + \int_{t_0}^t |f(s)|\,ds\right) \tag{1.12}$$

for some constants $b \geq 1$ and $a \geq 0$ and all $t \geq t_0$. Clearly, these constants may be different from those in (1.7) since $a$ in (1.7) may take negative values. In order to discuss the asymptotic behaviour of solutions of (1.1) in the later chapters, we need another estimate for the solution of (1.4) in which the constants are the same as those in (1.7).

**LEMMA 1.1** The solution of (1.8) satisfies

$$\|Y_t\| \leq b\, e^{at}, \; t \geq 0, \tag{1.13}$$

if the solution of (1.3) with $x_0 = \varphi_0$ satisfies (1.7) for every $\varphi_0$.

**Proof.** Let $\{\Phi_m\}$ be a sequence in $C([-r, 0], \mathbb{C}^{N \times N})$ satisfying $\|\Phi_m\| = 1$ and $\Phi_m(0) = I$ for $m \geq 1$ and $\lim_{m \to \infty} \Phi_m(\theta) = 0$ for $\theta \in [-r, 0)$. Then, for each $m \geq 1$, there is a unique matrix solution of (1.3), denoted by $X_m(t)$, such that $(X_m)_0 = \Phi_m$. For each $t \geq 0$, the solution $x(0, \varphi_0)(t)$ of (1.3) can be represented by an integral

$$x(0,\varphi_0)(t) = \int_{-r}^{0} \{d_\theta \eta(t, \theta)\} \varphi_0(\theta)$$

as (1.6) and (1.7) imply that $x(0, .)(t) = T(t).(0)$ is a bounded linear operator from $C$ to $\mathbb{C}^N$. Thus

$$X_m(t) = x(0, \Phi_m)(t) = \int_{-r}^{0} \{d_\theta \eta(t, \theta)\} \Phi_m(\theta)$$

for $t \geq 0$ and $m \geq 1$. Since the solution $Y(t)$ of (1.8) exists and $\lim_{m \to \infty} \Phi_m(\theta) = Y_0(\theta)$ holds for each $\theta \in [-r, 0]$, we have

$$\lim_{m \to \infty} X_m(t) = x(0, Y_0)(t) = Y(t)$$

for $t \geq 0$. From (1.7) and the condition of $\{\Phi_m\}$, the inequality

$$\|(X_m)_t\| \leq b\, e^{at}, \; t \geq 0,$$

holds for each $m \geq 1$. Then, by letting $m \to \infty$ for each $\theta \in [-r, 0]$ with $t + \theta \geq 0$, we obtain (1.13). #

**Remark**. From (1.13) and (1.10), it follows that the solution of (1.4) with $x_{t_0} = \varphi_0$ satisfies

$$\|x_t(t_0, \varphi_0, f)\| \leq b\, e^{a(t - t_0)}\left( \|\varphi_0\| + \int_{t_0}^{t} e^{a(t_0 - s)}|f(s)|\, ds \right) \tag{1.14}$$

for $t \geq t_0$ as long as the solution of (1.3) satisfies (1.7). Since $Y(t) = 0$ for $t < 0$, we also have

$$\left\| \int_{t_0}^{t} Y_{t+\sigma-s} f(s)\, ds \right\| \leq b \int_{t_0}^{t} e^{a(t+\sigma-s)}|f(s)|\, ds \tag{1.15}$$

for $t \geq t_0$ and any real $\sigma$.

Let $X(t)$ be the matrix solution of the equation

$$\frac{d}{dt}\{X(t) - D(X_t)\} = I + L_1(X_t), \quad t \geq 0 \tag{1.16}$$

with $X_0 = 0$, i.e. $X(t) = 0$ for $t \in [-r, 0]$. By applying (1.9) to (1.16), we obtain

$$X(t) = \int_0^t Y(t-s)\, ds = \int_0^t Y(s)\, ds \tag{1.17}$$

for $t \geq 0$.

# CHAPTER 2 EXISTENCE AND UNIQUENESS

The equation (1.1) is of neutral type when $\beta = 0$ and this is effectively covered by the standard theory of Hale [17]. If $\beta > 0$, however, (1.1) is of mixed type. For $\beta \geq 1$, (1.1) involves past and future states for any $t$ as $[t + \beta] > t$ holds for $t \geq 0$. But for $\beta \in (0, 1)$, (1.1) is of alternating type because for any $k = 1, 2, \cdots$, we have $[t + \beta] \leq t$ for $t \in [k, k + 1 - \beta)$ and $[t + \beta] > t$ for $t \in [k + 1 - \beta, k + 1)$.

When $\beta > 0$, it is natural to pose the initial value problem for (1.1) in the same way as for retarded and neutral equations. If, for any given $\varphi_0 \in C$, there is a continuous function $x(t)$ on $[-r, \infty)$ that satisfies (1.1) a.e. for $t \geq 0$ and

$$x_0 = \varphi_0, \text{ i.e., } x(\theta) = \varphi_0(\theta) \text{ for } \theta \in [-r, 0], \tag{2.1}$$

then $x(t)$ is said to be a solution of (1.1) with (2.1).

It will be shown that, under certain conditions, there exists a unique solution to the initial value problem (1.1) with (2.1) if $\beta \in (0, 1]$. However, the initial condition (2.1) is not enough to guarantee the uniqueness if $\beta > 1$.

## § 2.1 THE CASE OF $0 < \beta \leq 1$

**THEOREM** 2.1 Assume $\beta \in (0, 1]$. Then there is a unique solution of (1.1) with (2.1) if the matrix $L_2(X_\beta) - I$ is nonsingular.

**Proof**. By the step method, we only need to show that (1.1) with (2.1) has exactly one solution on $[0, 1]$ under the stated condition. Since (1.1) is neutral for $t \in [0, 1 - \beta)$ if $\beta < 1$, there is a unique solution of (1.1) with (2.1) on $[0, 1 - \beta]$. Thus we need to show that there exists a unique solution of (1.1) on $[1 - \beta, 1]$ for given $x_{1-\beta}$. This is equivalent to verifying that there is a unique solution of the equation

$$\frac{d}{dt}\{y(t) - D(y_t)\} = L_1(y_t) + L_2(y_\beta) + f(t + 1 - \beta), \ t \in [0, \ \beta], \tag{2.2}$$

with the initial condition $y_0 = \psi_0 \in C$.

Let $v$ be a constant in $\mathbb{C}^N$. Then, by (1.9) and (1.17), the solution of the equation

$$\frac{d}{dt}\{y(t) - D(y_t)\} = L_1(y_t) + v + f(t + 1 - \beta)$$

with $y_0 = \psi_0$ has the representation

$$y(t) = T(t)\psi_0(0) + X(t)v + \int_0^t Y(t - s)f(s + 1 - \beta)\, ds, \ t \geq 0. \tag{2.3}$$

We notice that $X(t) = Y(t) = 0$ for $t < 0$. Denote the function

$$\int_0^{\beta+\theta} Y(\beta + \theta - s)f(s + 1 - \beta)\, ds$$

by $F_\beta(\theta)$ for $\theta \in [-r, 0]$. Then it follows from (2.3) that

$$L_2(y_\beta) = L_2(T(\beta)\psi_0) + L_2(X_\beta)v + L_2(F_\beta).$$

Since $L_2(X_\beta) - I$ is nonsingular, the equation

$$v = L_2(T(\beta)\psi_0) + L_2(X_\beta)v + L_2(F_\beta) \tag{2.4}$$

has a unique solution

$$v = (I - L_2(X_\beta))^{-1}\{L_2(T(\beta)\psi_0) + L_2(F_\beta)\}. \tag{2.5}$$

By substituting (2.5) into (2.3), we obtain the unique solution of (2.2) with $y_0 = \psi_0$. #

Theorem 2.1 says that, for any locally integrable $f$ on $[0, \infty)$, (1.1) with (2.1) possesses a unique solution on $[-r, \infty)$ if $\det(L_2(X_\beta) - I) \neq 0$. So does (1.2) with (2.1). In fact, the converse of Theorem 2.1 is also true.

**THEOREM 2.2** Assume $\beta \in (0, 1]$. If there is a unique solution of (1.1) with (2.1) for every locally integrable function $f$ on $[0, \infty)$, then the matrix $L_2(X_\beta) - I$ is nonsingular.

**Proof.** From the proof of Theorem 2.1 we know that (1.1) with (2.1) has solutions (a unique solution) for $t$ in $[0, 1]$ if and only if (2.4) has solutions (a unique solution).

Suppose that $L_2(X_\beta) - I$ is singular. It is to be shown that there exists an $f$ such that (2.4) with this $f$ has no solution at all in $\mathbb{C}^N$, which is a contradiction to the assumption of the theorem.

Without loss of generality, we assume that $L_2(X_\beta)$ is in a Jordan form with possible 1's under the main diagonal and that the entries in the first row of $L_2(X_\beta) - I$ are zero. Then $L_2(X_\beta)$ has a generalised eigenvector $v = (p, 0, \cdots, 0)^T \in \mathbb{C}^N$ such that $L_2(X_\beta)v = v'$, where either $v' = v$ or $v' = v + (0, p, \cdots, 0)^T$. Now choose $p$ such that the first element of $L_2(T(\beta)\psi_0) + v'$ is not zero, and let $f(t + 1 - \beta) = v$ for $t \in [0, \beta]$. From the definition of $F_\beta$, we obtain

$$L_2(F_\beta) = L_2(X_\beta v) = L_2(X_\beta)v = v'.$$

Thus (2.4) becomes

$$(I - L_2(X_\beta))v = L_2(T(\beta)\psi_0) + v'. \tag{2.6}$$

By the choice of $p$, the first component equation in (2.6) is

$$0 = \text{nonzero constant},$$

which means that (2.6) has no solution at all in $\mathbb{C}^N$. #

**Remark.** The proof of Theorem 2.2 actually shows that, under the condition of $\det(I - L_2(X_\beta)) = 0$, there is an $f$ for given $\varphi_0 \in C$ such that (1.1) with (2.1) has no solution on $[0, 1]$ and so no solution on $[0, \infty)$. Also, for $\psi_0 = 0$ and $f(t + 1 - \beta) = 0$ for $t \in [0, \beta]$, (2.4) has infinitely many solutions, and so does (2.2) with $y_0 = 0 \in C$. Therefore, the homogeneous equation (1.2) with $x_0 = 0 \in C$ has infinitely many solutions on $[-r, 1]$ if $I - L_2(X_\beta)$ is singular.

**Remark**. Since $X(t)$ is continuous and $X_0 = 0$, the matrix $I - L_2(X_\beta)$ is nonsingular if $\beta$ is small enough (including $\beta = 0$). Thus existence and uniqueness always hold for small $\beta$.

## § 2.2 THE CASE OF $\beta > 1$

Suppose that $\beta$ satisfies $m < \beta \le m + 1$ for some integer $m \ge 1$ and let

$$t_0 = 0, \; t_k = k - \beta + [\beta], \; k = 1, 2, \cdots. \tag{2.7}$$

Since $0 \le \beta - [\beta] < 1$, $t_k$ satisfies $k - 1 < t_k \le k$ for each $k \ge 1$.

From (1.9) and (1.17) we know that the solution of the equation

$$\frac{d}{dt}\{y(t) - D(y_t)\} = L_1(y_t) + v + f(t)$$

for $t \ge t'$ is represented by

$$y(t) = T(t - t')y_{t'}(0) + X(t - t')v + \int_{t'}^{t} Y(t - s)f(s)\,ds.$$

Hence (1.1) is equivalent to the equation

$$x(t) = T(t - t_k)x_{t_k}(0) + X(t - t_k)L_2(x_{[\beta] + k}) + \int_{t_k}^{t} Y(t - s)f(s)\,ds \tag{2.8}$$

for $t \in [t_k, t_{k+1})$ and $k = 0, 1, 2, \cdots$.

**THEOREM 2.3** Suppose that $L_2(X_{\beta - m})$ is nonsingular. Then (1.1) with (2.1) has infinitely many solutions on $[-r, \infty)$.

**Proof**. Let

$$F_k(t) = \int_{t_k}^{t} Y(t-s)f(s)\,ds$$

for $t \in [-r, t_{k+1}]$ and $k = 0, 1, 2, \cdots$. Then (2.8) is equivalent to the system

$$L_2(x_{[\beta]+k}) = h_k, \; k = 0, 1, 2, \cdots, \tag{2.9a}$$

$$x(t) = T(t-t_k)x_{t_k}(0) + X(t-t_k)h_k + F_k(t), \; t \in [t_k, t_{k+1}), \; k = 0, 1, 2, \cdots. \tag{2.9b}$$

Since $x(t)$ is continuous, (2.9b) can be extended to $[t_k, t_{k+1}]$ for each $k \geq 0$ if necessary.

Suppose $\beta > [\beta] = m$. If $x(t)$ satisfies (2.9) on $[0, \infty)$, it follows from (2.9b) and (2.7) that

$$\begin{aligned} x_k(\theta) = x(k+\theta) &= T(k+\theta-t_k)x_{t_k}(0) + X(k+\theta-t_k)h_k + F_k(k+\theta) \\ &= T(\beta-[\beta])x_{t_k}(\theta) + X_{\beta-[\beta]}(\theta)h_k + F_k(k+\theta) \end{aligned} \tag{2.10}$$

for $\theta \in [-\min\{r, \beta-[\beta]\}, 0]$ and $k = 1, 2, \cdots$. For $\theta \in [-r, [\beta]-\beta]$ if $r > \beta - [\beta]$, since

$$k + \theta = (k - t_k) + \theta + t_k = \beta - [\beta] + \theta + t_k \leq t_k,$$

we have $F_k(k+\theta) = 0$, $X_{\beta-[\beta]}(\theta) = X(\beta - [\beta] + \theta) = 0$ and

$$T(\beta-[\beta])x_{t_k}(\theta) = x_{t_k}(\beta-[\beta]+\theta) = x(t_k+\beta-[\beta]+\theta) = x(k+\theta).$$

Hence, (2.10) holds for $\theta \in [-r, 0]$ and $k = 1, 2, \cdots$, i.e.,

$$x_k = T(\beta-m)x_{t_k} + X_{\beta-m}h_k + F_k(k+.), \; k = 1, 2, \cdots. \tag{2.11}$$

By substituting (2.11) with $k$ replaced by $m+k$ into (2.9a), we obtain

$$h_k = L_2(T(\beta-m)x_{t_{m+k}} + F_{m+k}(m+k+.)) + L_2(X_{\beta-m})h_{m+k}, \quad k = 0, 1, 2, \cdots. \tag{2.12}$$

Conversely, if $x(t)$ satisfies (2.9b) and (2.12) on $[0, \infty)$, then (2.11) and (2.12) imply that $x(t)$ also satisfy (2.9a). Therefore, (2.9) is equivalent to (2.9b) with (2.12).

Now we show that (2.9b) with (2.12) and the initial data (2.1) has infinitely many solutions on $[-r, \infty)$. In fact, for any given

$$h_0, h_1, \cdots, h_{m-1} \in \mathbb{C}^N,$$

$x(t)$ on $[-r, t_m)$ is uniquely determined by (2.9b) and (2.1). Since $L_2(X_{\beta-m})$ is nonsingular, $h_m$ is uniquely determined by (2.12) for $k = 0$. By employing (2.9b) and (2.12) alternately, we obtain $\{h_k: k \geq m\}$ and $x(t)$ on $[-r, \infty)$, which is the unique solution of (2.9b) with (2.12) and (2.1) for given $\{h_0, h_1, \cdots, h_{m-1}\}$. Hence the arbitrariness of $\{h_0, h_1, \cdots, h_{m-1}\}$ implies that (2.9b) with (2.12) and (2.1) has infinitely many solutions on $[-r, \infty)$.

Suppose $\beta = [\beta] = m + 1$. Then (2.7) implies $t_k = k$ for $k = 0, 1, 2, \cdots$. If $x(t)$ satisfies (2.9) on $[0, \infty)$, it follows from (2.9b) that

$$\begin{aligned} x(k + \theta) &= T(1 + \theta)x_{k-1}(0) + X(1 + \theta)h_{k-1} + F_{k-1}(k + \theta) \\ &= T(1)x_{k-1}(\theta) + X_{\beta-m}(\theta)h_{k-1} + F_{k-1}(k + \theta) \qquad (2.13) \end{aligned}$$

for $\theta \in [-\min\{r, 1\}, 0]$ and $k = 1, 2, \cdots$. Since (2.13) can be extended to $\theta \in [-r, 0]$ if $r > 1$, we obtain

$$x_k = T(1)x_{k-1} + X_1 h_{k-1} + F_{k-1}(k + .), \quad k = 1, 2, \cdots. \qquad (2.14)$$

By substitution, (2.14) and (2.9a) lead to the equation

$$h_k = L_2(T(1)x_{m+k} + F_{m+k}(m + k + 1 + .)) + L_2(X_1)h_{m+k}, \quad k = 0, 1, 2, \cdots. \quad (2.15)$$

Thus (2.9) is equivalent to (2.9b) with (2.15). Since $L_2(X_1)$ is nonsingular, by the same reasoning as above we can show that (2.9b) with (2.15) and (2.1) has infinitely many solutions on $[-r, \infty)$.

In either case, by the equivalence of (2.9) to (2.9b) with (2.12) or (2.9b) with (2.15), we have shown that (1.1) with (2.1) has infinitely many solutions on

$[-r, \infty)$ and, moreover, that each solution is uniquely determined by $\{h_0, h_1, \cdots, h_{m-1}\}$. #

Theorem 2.3 indicates that the initial condition (2.1) is not enough to determine a unique solution of (1.1). From the above proof we notice that extra data should be added to (2.1) and that the initial data should be given as

$$x_0 = \varphi_0, \ L_2(x_{|\beta|+k}) = h_k, \ k = 0, 1, 2, \cdots, m-1 \tag{2.16}$$

for any $\varphi_0 \in C$ and $h_0, h_1, \cdots, h_{m-1} \in \mathbb{C}^N$. Then the proof of Theorem 2.3 also proves the following result.

**THEOREM 2.4** Assume that $L_2(X_{\beta-m})$ is nonsingular. Then (1.1) with (2.16) possesses one and only one solution on $[-r, \infty)$.

We now show that, as an alternative to (2.16), $\varphi_0$ and m values of $x(t)$ in $[0, t_m]$ can be specified. Since $Y(t)$, the special solution of (1.8), is continuous from the right and $Y(0) = I$, (1.17) implies that $X(t)$ is nonsingular if $t > 0$ is small enough. Suppose $\{t_1', t_2', \cdots, t_m'\}$ satisfies

$$t_{k-1} < t_k' \le t_k, \ \det X(t_k' - t_{k-1}) \ne 0, \ k = 1, 2, \cdots, m. \tag{2.17}$$

Then alternative initial data to (2.16) can be set as

$$x_0 = \varphi_0, \ x(t_k') = u_k, \ k = 1, 2, \cdots, m. \tag{2.18}$$

for any $\varphi_0 \in C$ and $u_1, u_2, \cdots, u_m \in \mathbb{C}^N$.

**THEOREM 2.5** Assume that $L_2(X_{\beta-m})$ is nonsingular. Then (1.1) with (2.18) possesses a unique solution on $[-r, \infty)$.

**Proof.** By linking (2.9b) with (2.18), we obtain

$$u_k = T(t_k' - t_{k-1})x_{t_{k-1}}(0) + X(t_k' - t_{k-1})h_{k-1} + F_{k-1}(t_k')$$

for $k = 1, 2, \cdots, m$. From (2.17) we see that $\{h_0, h_1, \cdots, h_{m-1}\}$ is uniquely determined by $\{u_1, u_2, \cdots, u_m\}$ and vice versa. Then the conclusion follows from Theorem 2.4. #

Theorem 2.4 (Theorem 2.5) states that

$$\det L_2(X_{\beta - m}) \neq 0 \qquad (2.19)$$

is a sufficient condition for (1.1) with (2.16) ((2.18)) to have a unique solution on $[-r, \infty)$. Indeed, (2.19) is also a necessary condition in the following sense.

**THEOREM** 2.6 Suppose that (1.1) with (2.16) or (2.18) has a unique solution for every locally integrable $f$ on $[0, \infty)$. Then (2.19) holds.

The proof of Theorem 2.6 is similar to that of Theorem 2.2. Also, the remark after Theorem 2.2 adapts to the condition of Theorem 2.6.

## § 2.3 EXAMPLES AND REMARKS

**Example** 2.1 Wiener and Cooke [37] studied the equation

$$\frac{d}{dt}x(t) = Ax(t) + Bx([t + 1/2]) \qquad (2.20)$$

with the initial data $x(0) = c_0$, where $A$ and $B$ are constant matrices and $c_0$ is an N-dimensional vector, and obtained the following result:

Equation (2.20) with $x(0) = c_0$ has a unique solution on $[0, \infty)$ if $A$ and $e^{(-1/2)A} + (e^{(-1/2)A} - I)A^{-1}B$ are nonsingular.

We now use Theorem 2.1 to derive a uniqueness result for (2.20) which slightly differs from the above. Here $\beta = 1/2 < 1$ and $D = 0$, $L_1(\varphi) = A\varphi(0)$ and

$L_2(\varphi) = B\varphi(0)$ for $\varphi \in C([-r, 0], \mathbb{C}^N)$ with any $r > 0$. Since the mappings $D$ and $L_k$ $(k = 1, 2)$ only involve the value $\varphi(0)$ of the function $\varphi \in C$, the initial data space $C$ can be viewed as degenerating to $\mathbb{C}^N$. It is clear that

$$X(t) = \int_0^t e^{A(t-s)}\,ds = \int_0^t e^{As}\,ds, \quad t \geq 0,$$

is the solution of the equation

$$X'(t) = I + AX(t), \quad t \geq 0, \quad X_0 = 0$$

corresponding to (1.16) with $X_0 = 0$. By Theorem 2.1, (2.20) with $x(0) = c_0$ has a unique solution on $[0, \infty)$ if

$$L_2(X_{1/2}) - I = BX(1/2) - I = B\int_0^{1/2} e^{As}\,ds - I$$

is nonsingular. This condition differs from that in [37] even if $A^{-1}$ exists. However, if either $\{4k\pi i:\ k = 0, \pm 1, \pm 2, \cdots\}$ does not contain any eigenvalue of $A$ or $AB = BA$ and $\det A \neq 0$, this condition coincides with that in [37].

**Example** 2.2 Consider the equation

$$\frac{d}{dt}x(t) = Ax(t) + \sum_{j=-K}^{K} A_j x([t+j]) \tag{2.21}$$

with the initial data

$$x(j) = c_j, \quad -K \leq j \leq K-1. \tag{2.22}$$

Cooke and Wiener [9] defined a solution of (2.21) with (2.22) on $[0, \infty)$ as a continuous function $x$ on $[0, \infty)$ and $K$ values $x(-K), x(-K+1), \cdots, X(-1)$ which satisfy (2.22) and (2.21) for $t \geq 0$. They found that, in this sense, (2.21) with (2.22) has a unique solution on $[0, \infty)$ if $A$ and $(e^A - I)A^{-1}A_K$ are nonsingular.

We now show that the initial value problem (2.21) with (2.22) can be dealt with by Theorem 2.5 and the same uniqueness condition is obtained. Here $\beta = K > 1$, $r = 2K$ and

$$L_2(\varphi) = \sum_{j=-K}^{K} A_j \varphi(j - K)$$

for $\varphi \in C$. Since $X(t)$ is the same as in Example 2.1, we have

$$L_2(X_{\beta - m}) = L_2(X_1) = \sum_{j=-K}^{K} A_j X(1 + j - K) = A_K X(1) = A_K \int_0^1 e^{As}\, ds .$$

Clearly, $\det L_2(X_{\beta - m}) \neq 0$ if and only if $\det A_K \neq 0$ and $\det X(1) \neq 0$. Assume $\det L_2(X_{\beta - m}) \neq 0$. Then we can take $t_k' = t_k = k$ for $k = 1, 2, \cdots, K - 1$ and give the initial data as

$$x_0 = \varphi_0, \quad x(k) = c_k, \quad k = 1, 2, \cdots, K - 1. \tag{2.23}$$

By Theorem 2.5, (2.21) with (2.23) has a unique solution on $[-r, \infty)$. Notice that the values of a solution for $t \geq 0$ are actually determined by (2.21) and (2.22). Thus (2.21) with (2.22) has a unique solution on $[0, \infty)$ if $L_2(X_1)$ is nonsingular. If $A^{-1}$ exists, our uniqueness condition obviously coincides with the one in [9].

**Example** 2.3 Consider the scalar equation

$$\frac{d}{dt}\left\{ x(t) - a \int_{-r}^{0} \theta x_t(\theta)\, d\theta \right\} = b\, x(t) + c \int_{-r}^{0} x_{[t + m + 1/2]}(\theta)\, d\theta , \tag{2.24}$$

where $a \neq 0$, $b^2 \neq 4a$, $c \neq 0$, $m \geq 1$ and $r \geq 1/2$. Let

$$\lambda_1 = \frac{1}{2}\left(b - \sqrt{b^2 - 4a}\right) \text{ and } \lambda_2 = \frac{1}{2}\left(b + \sqrt{b^2 - 4a}\right).$$

Then, for $t \in [0, r]$,

$$X(t) = (b^2 - 4a)^{-1/2}(e^{\lambda_2 t} - e^{\lambda_1 t}) \tag{2.25}$$

is the solution of the equation

$$\frac{d}{dt}\left\{X(t) - a\int_{-r}^{0} \theta X_t(\theta)\,d\theta\right\} = 1 + b\,X(t) \tag{2.26}$$

with $X_0 = 0$. Since $[\beta] = m$ and

$$L_2(\varphi) = c\int_{-r}^{0} \varphi(\theta)\,d\theta$$

for $\varphi \in C$, we give the initial data as

$$x_0 = \varphi_0, \; c\int_{-r}^{0} x_{m+j}(\theta)\,d\theta = h_j, \; j = 0, 1, \cdots, m-1. \tag{2.27}$$

By Theorem 2.4, (2.24) with (2.27) has a unique solution on $[-r, \infty)$ if

$$\int_{-r}^{0} X_{1/2}(\theta)\,d\theta \neq 0,$$

that is,

$$(\lambda_2)^{-1}\left(e^{(1/2)\lambda_2} - 1\right) - (\lambda_1)^{-1}\left(e^{(1/2)\lambda_1} - 1\right) \neq 0.$$

We have seen through the examples that the efficient application of the theorems depends on finding the solution $X(t)$ of (1.16) with $X_0 = 0$ for $t \in [0, 1]$ and evaluating $L_2(X_1)$ or $L_2(X_{\beta - [\beta]})$ (if $\beta > [\beta]$). We end this chapter with some remarks on these issues.

(I). For a special class of equations, $X(t)$ can be represented explicitly in terms of the coefficients involved in (1.16). For instance, the solution of (2.26) with $X_0 = 0$ is given by (2.25) for $t \in [0, r]$. Indeed, (2.26) with $X_0 = 0$ can be solved by the following method. Let

$$y(t) = \int_0^t ds \int_0^s X(u)\,du.$$

Then, for $t \in [-r, 0]$, we have

$$y(t) = \frac{d}{dt}y(t) = \frac{d^2}{dt^2}y(t) = X(t) = 0.$$

Thus, for $t \in [0, r]$,

$$\int_{-r}^{0} \theta X_t(\theta)\, d\theta = \int_{-r}^{0} \theta \frac{d^2}{dt^2} y(t+\theta)\, d\theta = -\int_{-r}^{0} \frac{d}{dt} y(t+\theta)\, d\theta = -y(t).$$

By integrating (2.26) from 0 to t, we obtain

$$\frac{d^2}{dt^2}y(t) - b\frac{d}{dt}y(t) + ay(t) = t. \tag{2.28}$$

Therefore, $X(t)$ on $[0, r]$ is the solution of (2.26) with $X_0 = 0$ if and only if $y(t)$ on $[0, r]$ is the solution of (2.28) with $y(0) = dy(0)/dt = 0$. By solving (2.28) with $y(0) = dy(0)/dt = 0$, we can easily obtain $X(t) = d^2y(t)/dt^2$ as given by (2.25). This method can be extended to (1.16) if D and $L_1$ have the forms

$$D(\varphi) = \int_{-r}^{0} P_1(\theta)\varphi(\theta)\, d\theta + \sum_{j=0}^{m_1} A_j\varphi(-r_j)$$

and

$$L_1(\varphi) = \int_{-r}^{0} P_2(\theta)\varphi(\theta)\, d\theta + \sum_{j=0}^{m_2} B_j\varphi(-\sigma_j),$$

where $P_1(\theta)$ and $P_2(\theta)$ are polynomials in $\theta$ with matrix coefficients, and $r_j \in (0, r]$ and $\sigma_j \in [0, r]$.

(II). For $\beta \in (0, 1]$, we can obtain sufficient conditions on $\|D\|$, $\|L_1\|$, $\|L_2\|$ and $\beta$ for $L_2(X_\beta) - I$ to be nonsingular. By Theorem 2.1, these conditions guarantee that (1.1) with (2.1) has a unique solution on $[-r, \infty)$. For this purpose, we first define functions $\|D\|_t$, $\|L_1\|_t$, $\|L_2\|_t$ and $\|X\|_t$ as follows. We recall that D is represented by $\xi(\theta)$ on $[-r, 0]$. After extending the domain of $\xi$ to $(-\infty, 0]$ by letting $\xi(\theta) = \xi(-r)$ for $\theta \le -r$, we define $\|D\|_t$ by

$$\|D\|_t = \mathrm{Var}_{[-t,\, 0]}(\xi),\ t \ge 0.$$

Clearly, $\|D\|_t$ is nondecreasing, $\|D\|_r = \|D\|$, the usual definition of the norm, and $\|D\|_t = \|D\|_r$ for $t \geq r$. Similarly, $\|L_1\|_t$ and $\|L_2\|_t$ are defined by

$$\|L_i\|_t = \mathrm{Var}_{[-t,\,0]}(\eta_i), \quad t \geq 0, \quad i = 1, 2.$$

Let $|X(t)|$ be the norm of the matrix $X(t)$ and define $\|X\|_t$ by

$$\|X\|_t = \max\{\, |X(s)|:\ 0 \leq s \leq t\}, \quad t \geq 0.$$

**LEMMA 2.7** The solution of (1.16) with $X_0 = 0$ satisfies

$$\|X\|_t \leq (1 - \|D\|_t)^{-1} t \exp\{(1 - \|D\|_t)^{-1} t \|L_1\|_t\} \tag{2.29}$$

for $t \geq 0$ as long as $\|D\|_t < 1$.

**Proof**. For any $t' > 0$, by integrating (1.16) from 0 to t and then estimating it, we obtain

$$|X(t)| \leq \|D\|_t \|X\|_t + t + \int_0^t \|L_1\|_s \|X\|_s \, ds$$

for $t \in [0, t']$, which implies that

$$\|X\|_t \leq \|D\|_{t'} \|X\|_t + t' + \|L_1\|_{t'} \int_0^t \|X\|_s \, ds, \quad t \in [0, t'].$$

If $\|D\|_{t'} < 1$, then it follows that

$$\|X\|_t \leq (1 - \|D\|_{t'})^{-1} \left( t' + \|L_1\|_{t'} \int_0^t \|X\|_s \, ds \right)$$

for $t \in [0, t']$. By the Gronwall inequality, we have

$$\|X\|_t \leq (1 - \|D\|_{t'})^{-1} t' \exp\{(1 - \|D\|_{t'})^{-1} t \|L_1\|_{t'}\}$$

for $t \in [0, t']$. Thus (2.29) holds for $t = t'$. Since $t' > 0$ is arbitrary, (2.29) holds for $t \geq 0$. #

With the aid of Lemma 2.7, we obtain the following result.

**THEOREM 2.8** Equation (1.1) with (2.1) possesses a unique solution on $[-r, \infty)$ if

$$\|D\|_\beta + \beta\|L_2\|_\beta \exp\left\{(1 - \|D\|_\beta)^{-1}\beta\|L_1\|_\beta\right\} < 1. \tag{2.30}$$

**Proof.** If (2.30) holds, then (2.29) implies

$$|L_2(X_\beta)| \leq \|L_2\|_\beta\|X\|_\beta \leq (1 - \|D\|_\beta)^{-1}\beta\|L_2\|_\beta \exp\left\{(1 - \|D\|_\beta)^{-1}\beta\|L_1\|_\beta\right\} < 1.$$

Let $\lambda$ be an eigenvalue of $L_2(X_\beta)$ and $\alpha \in \mathbb{C}^N$ be a corresponding eigenvector satisfying $|\alpha| = 1$. Then it follows that

$$|\lambda| = |\lambda|.|\alpha| = |\lambda\alpha| = |L_2(X_\beta)\alpha| \leq |L_2(X_\beta)|.|\alpha| = |L_2(X_\beta)| < 1,$$

which implies that $L_2(X_\beta) - I$ is nonsingular. By Theorem 2.1, (1.1) with (2.1) possesses a unique solution on $[-r, \infty)$. #

**Example 2.4** Consider the equation

$$\frac{d}{dt}\left\{x(t) - \sum_{i=1}^{\infty} 2^{-(i+1)}x(t - 2^{-i})\right\}$$

$$= 2\int_{-1}^{0} x(t + \theta)\, d\theta + \frac{1}{4}x([t + 1/2]) - 5x([t + 1/2] - 1). \tag{2.31}$$

Here $\beta = \frac{1}{2}$, $\|D\|_\beta = \sum_{i=1}^{\infty} 2^{-(i+1)} = \frac{1}{2}$, $\|L_1\|_\beta = 1$, and $\|L_2\|_\beta = \frac{1}{4}$. Since

$$\|D\|_\beta + \beta\|L_2\|_\beta \exp\{(1 - \|D\|_\beta)^{-1}\beta\|L_1\|_\beta\} = 1/2 + (1/8)e < 1,$$

by Theorem 2.8 (2.31) with (2.1) has a unique solution on $[-1, \infty)$.

# CHAPTER 3
# REPRESENTATION OF SOLUTIONS

From § 1.2 we know that the solution $x(t_0, \varphi_0, f)(t)$ of the neutral equation (1.4) has the representation (1.9) for $t \geq t_0$, which shows the linear dependence on $\varphi_0$ and $f$ explicitly. Together with the exponential estimate (1.13), (1.9) also shows that $x(t_0, \cdot, 0)(t)$ and $x(t_0, 0, \cdot)(t)$ are bounded linear mappings from $C$ and $L(t_0, t)$, respectively, to $\mathbb{C}^N$ for any $t \geq t_0$.

We will show that similar results hold for solutions of (1.1). Exponential estimates, which are based on the representation, will be given in Chapter 4 so that the boundedness of the solution mappings defined below is proved. In this chapter, however, we only give the representation of solutions of (1.1) in terms of those of (1.3) and (1.4).

## § 3.1 A LEMMA

In deriving the representation, the following lemma plays an important role.

**LEMMA** 3.1 Suppose that $\{A_i: 0 \leq i < \infty\}$ is a sequence of matrices and $\{d_i: 1 \leq i < \infty\}$ a sequence of vectors, with compatible dimensions. If $\{b_i: 1 \leq i < \infty\}$ satisfies the system of equations

$$A_0 b_k + A_1 b_{k-1} + \cdots + A_{k-1} b_1 + d_k = 0, \; k = 1, 2, \cdots \tag{3.1}$$

and $(A_0)^{-1}$ exists, then we have the expression

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We will show that similar results hold for solutions of (1.1). Exponential estimates, which are based on the representation, will be given in Chapter 4 so that the boundedness of the solution mappings defined below is proved. In this chapter, however, we only give the representation of solutions of (1.1) in terms of those of (1.3) and (1.4).

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and $(A_0)^{-1}$ exists, then we have the expression

$$b_k = \sum_{j=1}^{k} (-1)^j \sum_{s(k,\,j)} \left((A_0)^{-1}A_{p_1}\right)\left((A_0)^{-1}A_{p_2}\right)\cdots\left((A_0)^{-1}A_{p_{j-1}}\right)\left((A_0)^{-1}d_{p_j}\right) \tag{3.2}$$

for $k \geq 1$, where

$$s(k, j) = \{(p_1, p_2, \ldots, p_j);\ p_i \geq 1 \text{ integer}, \sum_{i=1}^{j} p_i = k\}. \tag{3.3}$$

**Proof**. The expression (3.2) is clearly true for $k = 1$. Suppose that (3.2) holds for $k = 1, 2, \cdots, K$. Then, for $k = K + 1$, we have

$$\begin{aligned}
b_{K+1} &= -\sum_{i=1}^{K} \left((A_0)^{-1}A_i\right) b_{K+1-i} - (A_0)^{-1}d_{K+1} \\
&= -\sum_{i=1}^{K} \left((A_0)^{-1}A_i\right) \sum_{j=1}^{K+1-i} (-1)^j \times \\
&\qquad \sum_{s(K+1-i,\,j)} \left((A_0)^{-1}A_{p_1}\right)\cdots\left((A_0)^{-1}A_{p_{j-1}}\right)\left((A_0)^{-1}d_{p_j}\right) - (A_0)^{-1}d_{K+1} \\
&= \sum_{j=1}^{K} (-1)^{j+1} \sum_{i=1}^{K+1-j} \left((A_0)^{-1}A_i\right) \times \\
&\qquad \sum_{s(K+1-i,\,j)} \left((A_0)^{-1}A_{p_1}\right)\cdots\left((A_0)^{-1}A_{p_{j-1}}\right)\left((A_0)^{-1}d_{p_j}\right) - (A_0)^{-1}d_{K+1} \\
&= \sum_{j=1}^{K} (-1)^{j+1} \sum_{s(K+1,\,j+1)} \left((A_0)^{-1}A_{p_1}\right)\cdots\left((A_0)^{-1}A_{p_j}\right)\left((A_0)^{-1}d_{p_{j+1}}\right) \\
&\qquad - \sum_{s(K+1,\,1)} \left((A_0)^{-1}d_{p_1}\right) \\
&= \sum_{j=1}^{K+1} (-1)^j \sum_{s(K+1,\,j)} \left((A_0)^{-1}A_{p_1}\right)\cdots\left((A_0)^{-1}A_{p_{j-1}}\right)\left((A_0)^{-1}d_{p_j}\right).
\end{aligned}$$

By induction, (3.2) holds for every integer $k \geq 1$. #

## § 3.2 REPRESENTATION WHEN $\beta \in [0, 1]$

Suppose that $\beta$ satisfies $0 \le \beta \le 1$ and

$$\det(L_2(X_\beta) - I) \neq 0, \tag{3.4}$$

where $X(t)$ is the solution of (1.16) with $X_0 = 0$. Then, by Theorem 2.1, the solution of (1.1) with (2.1) uniquely exists on $[-r, \infty)$. As usual, we denote this solution by $x(\varphi_0, f)(t)$. By the linearity of (1.1) and (1.2), it follows that

$$x(\varphi_0, f)(t) = x(\varphi_0, 0)(t) + x(0, f)(t) \tag{3.5}$$

for $t \ge 0$. Moreover, the equations (1.1) and (1.2) imply the linearity of $x(\varphi_0, 0)(t)$ and $x(0, f)(t)$ in $\varphi_0$ and $f$, respectively, for each $t \ge 0$. Indeed, $x(., 0)(t)$ and $x(0, .)(t)$ are bounded linear mappings from $C$ and $L(0, t)$, respectively, to $\mathbb{C}^N$. Thus they have their own representations. We first summarize the above analysis as follows.

**THEOREM** 3.2 Assume $\beta \in [0, 1]$ and (3.4) holds. Then the solution of (1.1) has the decomposition (3.5). Moreover, for each $t \ge 0$, $x(., 0)(t)$ and $x(0, .)(t)$ are bounded linear mappings from $C$ and $L(0, t)$, respectively, to $\mathbb{C}^N$.

To represent the solution $x(\varphi_0, f)(t)$, we need the following notation. Let

$$B_0 = L_2\left(\int_0^1 Y_{\beta-s}\, ds\right) - I, \quad B_j = L_2\left(\int_0^1 Y_{j+\beta-s}\, ds\right),$$

$$F_j(\varphi_0) = L_2(T(j)\varphi_0) + L_2\left(\int_0^{1-\beta} Y_{j-s} ds\right) L_2(\varphi_0),$$

and

$$G_j(f) = L_2\left(\int_0^1 Y_{j-s} f(s)\, ds\right)$$

for $j = 1, 2, \cdots$. Here $T(t)$ is the solution operator of (1.3) defined by (1.5) and $Y(t)$ is the special solution of (1.8). Since

$$\left(\int_0^1 Y_{\beta - s}\, ds\right)(\theta) = \int_0^1 Y(\beta - s + \theta)\, ds = \int_{\beta+\theta-1}^{\beta+\theta} Y(s)\, ds = X(\beta + \theta) = X_\beta(\theta)$$

for $\theta \in [-r, 0]$, by the definition of $B_0$ and (3.4), $(B_0)^{-1}$ exists. Applying Lemma 3.1 to the system of equations

$$B_0 b_k + B_1 b_{k-1} + \cdots + B_{k-1} b_1 + F_k(\varphi_0) + G_k(f) = 0, \quad k = 1, 2, \cdots, \tag{3.6}$$

we see that each $b_k$ can be represented in terms of $\{B_j\}$, $\{F_j(\varphi_0)\}$ and $\{G_j(f)\}$ according to (3.2). We denote this representation for $b_k$ by

$$b_k = \widehat{F}_k(\varphi_0) + \widehat{G}_k(f), \quad k = 1, 2, \cdots. \tag{3.7}$$

Now we can represent $x(\varphi_0, f)(t)$ as follows.

**THEOREM** 3.3 Assume that $\beta \in [0, 1]$ and (3.4) holds. Then the solution of (1.1) is given by

$$x(\varphi_0, f)(t) = T(t)\varphi_0(0) + \int_0^{1-\beta} Y(t - s)\, dsL_2(\varphi_0) + \int_0^t Y(t - s)f(s)\, ds$$

$$+ \sum_{j=1}^{k} \int_{j-\beta}^{j+1-\beta} Y(t - s)\, ds\left\{\widehat{F}_j(\varphi_0) + \widehat{G}_j(f)\right\} \tag{3.8}$$

for $t \in [0, 1 - \beta)$ with $k = 0$ and $t \in [k - \beta, k + 1 - \beta)$ with $k \geq 1$.

**Proof**. Viewing the equation (1.1) as (1.4) and applying (1.9) and (1.10) to (1.1), we see that the solution $x(\varphi_0, f)(t)$ of (1.1) satisfies both

$$x(t) = T(t)\varphi_0(0) + \int_0^t Y(t - s)\{f(s) + L_2(x_{[\beta + s]})\}\, ds, \quad t \geq 0, \tag{3.9}$$

and

$$x_t = T(t)\varphi_0 + \int_0^t Y_{t-s}\{f(s) + L_2(x_{[\beta+s]})\}\, ds\,, \quad t \ge 0. \tag{3.10}$$

Then, for $t \in [0, 1 - \beta]$ (if $\beta < 1$), it follows from (3.9) that

$$x(\varphi_0, f)(t) = T(t)\varphi_0(0) + \int_0^t Y(t - s)\{f(s) + L_2(\varphi_0)\}\, ds$$

$$= T(t)\varphi_0(0) + \int_0^{1-\beta} Y(t - s)\, ds L_2(\varphi_0) + \int_0^t Y(t - s) f(s)\, ds\,,$$

which coincides with (3.8). For $t \in [k - \beta, k + 1 - \beta)$ with $k \ge 1$, (3.9) implies

$$x(\varphi_0, f)(t) = T(t)\varphi_0(0) + \int_0^{1-\beta} Y(t - s)\, ds L_2(\varphi_0)$$

$$+ \int_0^t Y(t - s) f(s)\, ds + \sum_{j=1}^{k} \int_{j-\beta}^{j+1-\beta} Y(t - s)\, ds L_2(x_j).$$

Comparing this with (3.8), we only need show that

$$L_2(x_j) = \widehat{F}_j(\varphi_0) + \widehat{G}_j(f), \ j = 1, 2, \cdots.$$

Equivalently, by (3.7) and (3.6), we only need show that $\{L_2(x_j) \colon j = 1, 2, \cdots\}$ satisfies

$$B_0 L_2(x_k) + B_1 L_2(x_{k-1}) + \cdots + B_{k-1} L_2(x_1)$$

$$+ F_k(\varphi_0) + G_k(f) = 0, \ k = 1, 2, \cdots. \tag{3.11}$$

From (3.10) we have

$$x_k = T(k)\varphi_0 + \int_0^k Y_{k-s}\{f(s) + L_2(x_{[\beta+s]})\}\, ds$$

$$= T(k)\varphi_0 + \int_0^k Y_{k-s} f(s)\, ds + \int_0^{k+1-\beta} Y_{k-s} L_2(x_{[\beta+s]})\, ds$$

$$= T(k)\varphi_0 + \int_0^{1-\beta} Y_{k-s}\, ds L_2(\varphi_0) + \int_0^{\beta} Y_{k-s} f(s)\, ds + \sum_{j=0}^{k-1} \int_0^1 Y_{j+\beta-s} ds\, L_2(x_{k-j})$$

for $k \geq 1$. Then the definitions of $B_j$, $F_j(\varphi_0)$ and $G_j(f)$ yield

$$L_2(x_k) = F_k(\varphi_0) + G_k(f) + L_2\left(\int_0^1 Y_{\beta-s}\, ds\right) L_2(x_k) + \sum_{j=1}^{k-1} B_j L_2(x_{k-j})$$

for $k \geq 1$, i.e., $\{L_2(x_j): j = 1, 2, \cdots\}$ satisfies (3.11). Therefore, (3.8) holds for $t \geq 0$. #

From (3.8) we immediately obtain

$$x(\varphi_0, 0)(t) = T(t)\varphi_0(0) + \int_0^{1-\beta} Y(t-s)\, ds L_2(\varphi_0)$$

$$+ \sum_{j=1}^{k} \int_{j-\beta}^{j+1-\beta} Y(t-s)\, ds\, \widehat{F}_j(\varphi_0) \tag{3.12}$$

and

$$x(0, f)(t) = \int_0^t Y(t-s) f(s)\, ds + \sum_{j=1}^{k} \int_{j-\beta}^{j+1-\beta} Y(t-s)\, ds\, \widehat{G}_j(f) \tag{3.13}$$

for $t \in [0, 1-\beta)$ with $k = 0$ and $t \in [k-\beta, k+1-\beta)$ with $k \geq 1$. Thus, for each $t \geq 0$, (3.12) and (3.13) can be viewed as representations of $x(., 0)(t)$ and $x(0, .)(t)$ respectively.

## § 3.3 REPRESENTATION WHEN $\beta > 1$

We postulate that $\beta$ satisfies $m < \beta \leq m + 1$ for some integer $m \geq 1$ and that

$$\det L_2(X_{\beta - m}) \neq 0. \tag{3.14}$$

Then, for any locally integrable $f$ on $[0, \infty)$, any vector

$$v_0 = (h_{m-1}{}^T, h_{m-2}{}^T, \cdots, h_0{}^T)^T \in \mathbb{C}^N \times \mathbb{C}^N \times \cdots \times \mathbb{C}^N = \mathbb{C}^{mN}$$

and any $\varphi_0 \in C$, Theorem 2.4 guarantees that (1.1) with (2.16) possesses a unique solution on $[-r, \infty)$. We denote this solution by $x(\varphi_0, v_0, f)(t)$. Then, from the equation itself, we obtain the following result analogous to Theorem 3.2.

**THEOREM** 3.4 Assume that $\beta \in (m, m+1]$ for some integer $m \geq 1$ and that (3.14) holds. Then the solution of (1.1) has the decomposition

$$x(\varphi_0, v_0, f)(t) = x(\varphi_0, 0, 0)(t) + x(0, v_0, 0)(t) + x(0, 0, f)(t) \tag{3.15}$$

for $t \geq 0$. Furthermore, for each $t \geq 0$, $x(0, ., 0)(t)$ is a linear transformation from $\mathbb{C}^{mN}$ to $\mathbb{C}^N$ and $x(., 0, 0)(t)$ and $x(0, 0, .)(t)$ are bounded linear mappings from $C$ and $L(0, t)$, respectively, to $\mathbb{C}^N$.

**Remark**. The boundedness of $x(., 0, 0)(t)$ and $x(0, 0, .)(t)$ will be shown in Chapter 4.

The method of obtaining the representation of the solution $x(\varphi_0, v_0, f)(t)$ is similar to that for the case $\beta \in [0, 1]$ but rather more complicated. We first define $\{P_j: j = 0, 1, 2, \cdots\}$ and $\{P_j': j = m, m+1, \cdots\}$ by

$$P_m = L_2\left(\int_0^1 Y_{\beta - s}\, ds\right) - I, \tag{3.16}$$

$$P_m' = L_2\left(\int_0^{\cdot 1} Y_{m + [\beta - m] - s}\, ds\right) - I, \tag{3.17}$$

$$P_j = L_2\left(\int_0^1 Y_{j + \beta - m - s}\, ds\right) \tag{3.18}$$

for $j = 0, 1, \cdots, m-1, m+1, \cdots$ and

$$P_j' \;=\; L_2\left(\int_0^{1} Y_{j+[\beta-m]-s}\,ds\right) \tag{3.19}$$

for $j \geq m+1$, where $\{t_k\}$ is given by (2.7). Clearly, $P_j' = P_j$ for $j \geq m$ if $\beta = m+1$. Let

$$u_j(\varphi_0) \;=\; L_2(T(j+[\beta-m])\varphi_0), \tag{3.20}$$

$$w_j(f) \;=\; L_2\left(\int_0^{j+[\beta-m]} Y_{j+[\beta-m]-s}f(s)\,ds\right) \tag{3.21}$$

for $j \geq m$ and

$$U_k(\varphi_0) \;=\; (u_{km+m-1}(\varphi_0)^T,\, u_{km+m-2}(\varphi_0)^T,\, \cdots,\, u_{km}(\varphi_0)^T)^T, \tag{3.22}$$

$$W_k(f) \;=\; (w_{km+m-1}(f)^T,\, w_{km+m-2}(f)^T,\, \cdots,\, w_{km}(f)^T)^T \tag{3.23}$$

for $k \geq 1$. We also define $\{\Phi_k:\ k = 0, 1, 2, \cdots\}$ by

$$\Phi_k \;=\; \begin{bmatrix} P_{km} & P_{km+1} & \cdots & P_{km+m-1} \\ P_{km-1} & P_{km} & \cdots & P_{km+m-2} \\ \cdots & \cdots & \cdots & \cdots \\ P_{km-m+1} & P_{km-m+2} & \cdots & P_{km} \end{bmatrix} \tag{3.24}$$

for $k \geq 0$, where $P_j$ is given by (3.16) and (3.18) for $j \geq 0$ and $P_j = 0$ for $j < 0$. Replacing $P_j$ by $P_j'$ in the last column of $\Phi_k$ for each $k \geq 1$, we also define a sequence $\{\Phi_k':\ k \geq 1\}$.

Since $Y(t) = 0$ for $t < 0$, by (1.17) we have

$$\left(\int_0^{1} Y_{\beta-m-s}\,ds\right)(\theta) \;=\; \int_0^{1} Y(\beta-m-s+\theta)\,ds \;=\; \int_0^{\beta-m+\theta} Y(s)\,ds$$

$$=\; X(\beta-m+\theta) \;=\; X_{\beta-m}(\theta)$$

for $\theta \in [-r,\ 0]$. Then, by (3.18), (3.14) and (3.24), both $(P_0)^{-1}$ and $(\Phi_0)^{-1}$ exist. From Lemma 3.1 we know that the solution $\{v_k:\ k \geq 1\}$ of the system of equations

$$\Phi_0 v_k + \Phi_1 v_{k-1} + \cdots + \Phi_{k-1} v_1$$

$$+ \{U_k(\varphi_0) + \Phi_k' v_0 + W_k(f)\} = 0, \ k = 1, 2, \cdots \tag{3.25}$$

can be represented in terms of $\{\Phi_k\}$ and $\{U_k(\varphi_0) + \Phi_k' v_0 + W_k(f)\}$ according to (3.2). We denote this representation for $v_k$ by $\widehat{U}_k(\varphi_0) + \widehat{\Phi}_k v_0 + \widehat{W}_k(f)$ $(k \geq 1)$. We also let $\widehat{U}_0(\varphi_0) = 0$, $\widehat{\Phi}_0 = I$, and $\widehat{W}_0(f) = 0$ and put

$$E_k(t) = \left( \int_{t_{km+m-1}}^{t_{km+m}} Y(t-s)\, ds, \int_{t_{km+m-2}}^{t_{km+m-1}} Y(t-s)\, ds, \cdots, \int_{t_{km}}^{t_{km+1}} Y(t-s)\, ds \right) \tag{3.26}$$

for $t \geq 0$ and $k = 0, 1, 2, \cdots$. We can now give the representation of the solution of (1.1) with (2.16).

**THEOREM** 3.5 Suppose that $\beta$ satisfies $m < \beta \leq m + 1$ for some integer $m \geq 1$ and that (3.14) holds. Then the solution of (1.1) with (2.16) has the representation

$$x(\varphi_0, v_0, f)(t) = T(t)\varphi_0(0) + \int_0^t Y(t-s) f(s)\, ds$$

$$+ \sum_{j=0}^{k} E_j(t) \left\{ \widehat{U}_j(\varphi_0) + \widehat{\Phi}_j v_0 + \widehat{W}_j(f) \right\} \tag{3.27}$$

for $t \in [t_{km}, t_{(k+1)m})$ with $k \geq 0$.

**Proof.** From the variation of parameters formulae (1.9) and (1.10), we deduce that any solution of (1.1) with $x_0 = \varphi_0$ must satisfy the equations (3.9) and (3.10). Moreover, $x(\varphi_0, v_0, f)(t)$ is the unique solution of both (3.9) and (3.10) with (2.16).

For $t \in [0, t_m]$, (3.9) with (2.16) results in

$$x(\varphi_0, v_0, f)(t) = T(t)\varphi_0(0) + \int_0^t Y(t-s) f(s)\, ds + \int_0^{t_m} Y(t-s) L_2(x_{[\beta+s]})\, ds$$

$$= T(t)\varphi_0(0) + \int_0^t Y(t - s)f(s)\,ds + \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} Y(t - s)\,ds\, h_j$$

$$= T(t)\varphi_0(0) + \int_0^t Y(t - s)f(s)\,ds + E_0(t)v_0.$$

Thus (3.27) holds for $t \in [t_{km}, t_{(k+1)m})$ with $k = 0$.

To show that (3.27) also holds for $k > 0$, we let

$$L_2(x_{[\beta]+i}) = h_i, \quad i = m, m + 1, \cdots \tag{3.28}$$

and

$$v_j = (h_{jm+m-1}{}^T, h_{jm+m-2}{}^T, \cdots, h_{jm}{}^T)^T, \quad j = 1, 2, \cdots. \tag{3.29}$$

By (2.16) and the definition of $v_0$, (3.28) can be extended to $i \geq 0$ and (3.29) to $j \geq 0$. Then, for $i \in \{0, 1, \cdots, m - 1\}$, $k \geq 1$, and $t \in [t_{km+i}, t_{km+i+1}) \subset [t_{km}, t_{(k+1)m})$, it follows from (3.9) that

$$x(\varphi_0, v_0, f)(t) = T(t)\varphi_0(0) + \int_0^t Y(t - s)f(s)\,ds + \int_0^{t_{(k+1)m}} Y(t - s)L_2(x_{[\beta+s]})\,ds$$

$$= T(t)\varphi_0(0) + \int_0^t Y(t - s)f(s)\,ds + \sum_{j=0}^{k} E_j(t)v_j\,. \tag{3.30}$$

If we could show that $\{v_j\}$ satisfies (3.25), then

$$v_k = \hat{U}_k(\varphi_0) + \hat{\Phi}_k v_0 + \hat{W}_k(f)$$

for $k \geq 0$ and (3.27) is obtained by substituting this into (3.30).

Now we verify that $\{v_j\}$ satisfies (3.25).

(i). Suppose that $\beta \in (m, m + 1)$. Then $[\beta] = m$, $[\beta - m] = 0$ and $t_{km+i} < km + i < t_{km+i+1}$. Since $x(\varphi_0, v_0, f)(t)$ satisfies (3.10), the substitution of $t = km + i$ into (3.10) leads to

$$x_{km+i} = T(km + i)\varphi_0 + \int_0^{km+i} Y_{km+i-s}f(s)\,ds + \int_0^{km+i} Y_{km+i-s}L_2(x_{[\beta+s]})\,ds\,.$$

From (3.28) and (2.7) we have

$$\int_0^{km+i} Y_{km+i-s} L_2(x_{[\beta+s]})\, ds = \int_0^{t_{km+i+1}} Y_{km+i-s} L_2(x_{[\beta+s]})\, ds$$

$$= \int_0^{t_1} Y_{km+i-s}\, ds\, h_0 + \sum_{j=1}^{km+i} \int_{t_j}^{t_{j+1}} Y_{km+i-s}\, ds\, h_j$$

$$= \int_0^{t_1} Y_{km+i+[\beta-m]-s}\, ds\, h_0 + \sum_{j=0}^{km+i-1} \int_0^{1} Y_{j+\beta-m-s}\, ds\, h_{km+i-j}.$$

(ii). Suppose that $\beta = m + 1$. Then $[\beta] = m + 1$, $[\beta - m] = 1$, and $[t_{km+i}, t_{km+i+1}) = [km + i, km + i + 1)$. By (3.10) again, we obtain

$$x_{km+i+1} = T(km+i+1)\varphi_0 + \int_0^{km+i+1} Y_{km+i+1-s} f(s)\, ds$$

$$+ \int_0^{km+i+1} Y_{km+i+1-s} L_2(x_{[\beta+s]})\, ds.$$

In this case,

$$\int_0^{km+i+1} Y_{km+i+1-s} L_2(x_{[\beta+s]})\, ds = \sum_{j=0}^{km+i} \int_{j}^{j+1} Y_{km+i+1-s}\, ds\, h_j$$

$$= \int_0^{t_1} Y_{km+i+[\beta-m]-s}\, ds\, h_0 + \sum_{j=0}^{km+i-1} \int_0^{1} Y_{j+\beta-m-s}\, ds\, h_{km+i-j}.$$

By letting $L_2(.)$ operate on the above equalities and taking (3.28) and (3.16)-(3.21) into account, we obtain

$$h_{(k-1)m+i} = L_2(x_{[\beta]+(k-1)m+i}) = L_2(x_{km+i+[\beta-m]})$$

$$= u_{km+i}(\varphi_0) + w_{km+i}(f) + P_{km+i}' h_0 + \sum_{j=0}^{km+i-1} P_j h_{km+i-j} +$$

$$+ L_2\left(\int_0^1 Y_{\beta - s}\, ds\right)h_{(k-1)m+i} - P_m h_{(k-1)m+i}$$

if $(k - 1)m + i \neq 0$, and

$$h_0 = u_m(\varphi_0) + w_m(f) + L_2\left(\int_0^1 Y_{m+[\beta - m]-s}\, ds\right)h_0 + \sum_{j=0}^{m-1} P_j h_{m-j}$$

if $(k - 1)m + i = 0$. By the definition of $P_m$ and $P_m'$, the above equalities can be written as

$$P_0 h_{km+i} + P_1 h_{km+i-1} + \cdots + P_{km+i-1} h_1 + P_{km+i}' h_0 + u_{km+i}(\varphi_0)$$

$$+ w_{km+i}(f) = 0, \quad k \geq 1, \ 0 \leq i \leq m - 1. \tag{3.31}$$

Then the definitions (3.22)-(3.24) and (3.29) lead to the equivalence of (3.31) and (3.25). Therefore, $\{v_k\}$ satisfies (3.25) and the proof is complete. #

**Remark**. From (3.27) it follows that

$$x(\varphi_0, 0, 0)(t) = T(t)\varphi_0(0) + \sum_{j=1}^{k} E_j(t)\widehat{U}_j(\varphi_0),$$

$$x(0, v_0, 0)(t) = \sum_{j=0}^{k} E_j(t)\widehat{\Phi}_j v_0,$$

and

$$x(0, 0, f)(t) = \int_0^t Y(t - s)f(s)\, ds + \sum_{j=1}^{k} E_j(t)\widehat{W}_j(f)$$

for $t \in [t_{km}, t_{(k+1)m})$ with $k \geq 0$. We can view these as representations of the mappings $x(., 0, 0)(t)$ and $x(0, 0, .)(t)$ and the transformation $x(0, ., 0)(t)$. Based on these representations, exponential estimates will be given in the next chapter.

# CHAPTER 4 EXPONENTIAL ESTIMATES

We mentioned in § 1.2 that solutions of the neutral equation (1.3) are exponentially bounded and that solutions of (1.4) satisfy the exponential estimate (1.13). In this chapter, similar estimates are obtained for solutions of (1.1) and (1.2).

For any finite set S, we denote the number of elements in S by N(S).

**LEMMA** 4.1 The number of elements in s(k, j) is

$$N(s(k,j)) = \binom{k-1}{j-1}, \tag{4.1}$$

where s(k, j) is given by (3.3) in Lemma 3.1.

**Proof.** There is only one element in s(1, 1). So (4.1) is true for k = 1.

Suppose that (4.1) holds for k = 1, 2, ⋯, K. We show that (4.1) also holds for k = K + 1. In fact,

$$N(s(K+1,1)) = 1 = \binom{K}{0},$$

i.e., (4.1) holds for (k, j) = (K + 1, 1). For (K + 1, j) with $1 < j \le K + 1$,

$$\begin{aligned} N(s(K+1,j)) &= N\left(\left\{(p_1, p_2, \cdots, p_j): \sum_{i=1}^{j} p_i = K+1, p_i \ge 1\right\}\right) \\ &= N\left(\left\{(p_1, p_2, \cdots, p_{j-1}, 1): \sum_{i=1}^{j-1} p_i = K, p_i \ge 1\right\}\right) \\ &\quad + N\left(\left\{(p_1, p_2, \cdots, p_{j-1}, 2): \sum_{i=1}^{j-1} p_i = K-1, p_i \ge 1\right\}\right) \\ &\quad + \cdots + N\left(\left\{(p_1, p_2, \cdots, p_{j-1}, K+2-j): \sum_{i=1}^{j-1} p_i = j-1, p_i \ge 1\right\}\right) \\ &= N(s(K, j-1)) + N(s(K-1, j-1)) + \cdots + N(s(j-1, j-1)) \end{aligned}$$

$$= \binom{K-1}{j-2} + \binom{K-2}{j-2} + \dots + \binom{j-2}{j-2} = \binom{K}{j-1} = \binom{K+1-1}{j-1}.$$

Therefore, (4.1) holds for $k = K + 1$. By induction, (4.1) holds for every integer $k \geq 1$. #

## § 4.1 ESTIMATES FOR THE CASE OF $\beta \in [0, 1]$

From Theorem 3.3 we know that the solution of (1.2) with (2.1) has the representation

$$x(\varphi_0, 0)(t) = T(t)\varphi_0(0) + \int_0^{1-\beta} Y(t-s)\, ds L_2(\varphi_0)$$

$$+ \sum_{j=1}^{k} \int_{j-\beta}^{j+1-\beta} Y(t-s)\, ds\, \widehat{F}_j(\varphi_0) \tag{4.2}$$

for $t \in [0, 1-\beta)$ with $k = 0$ and $t \in [k-\beta, k+1-\beta)$ with $k \geq 1$, where $\{\widehat{F}_k(\varphi_0)\colon k \geq 1\}$ is the solution of (3.6) with $f = 0$. Then (1.7) and (1.15) imply the estimates

$$|x(\varphi_0, 0)(t)| \leq \|T(t)\varphi_0\| + \left|\int_0^{1-\beta} Y(t-s)\, ds L_2(\varphi_0)\right|$$

$$\leq b\left(1 + \|L_2\| \int_0^{1-\beta} e^{-as}\, ds\right) e^{at} \|\varphi_0\| \tag{4.3}$$

for $t \in [0, 1-\beta]$ and

$$|x(\varphi_0, 0)(t)| \leq \|T(t)\varphi_0\| + \left|\int_0^{1-\beta} Y(t-s)\, ds L_2(\varphi_0)\right| + \sum_{j=1}^{k} \left|\int_{j-\beta}^{j+1-\beta} Y(t-s)\, ds\, \widehat{F}_j(\varphi_0)\right|$$

$$\leq b\left(1 + \|L_2\|\int_0^{1-\beta} e^{-as}\,ds\right)e^{at}\|\varphi_0\| + \sum_{j=1}^{k} b\int_{j-\beta}^{j+1-\beta} e^{a(t-s)}\,ds\,|\widehat{F}_j(\varphi_0)| \tag{4.4}$$

for $t \in [k-\beta, k+1-\beta)$ with $k \geq 1$. From Lemma 3.1 and (3.6), we have

$$\widehat{F}_k(\varphi_0) = \sum_{j=1}^{k}(-1)^j\sum_{s(k,j)}\left((B_0)^{-1}B_{p_1}\right)\left((B_0)^{-1}B_{p_2}\right)\cdots\left((B_0)^{-1}B_{p_{j-1}}\right)\left((B_0)^{-1}F_{p_j}(\varphi_0)\right) \tag{4.5}$$

for $k \geq 1$. Now we use (1.7) and (1.15) again to estimate $|\widehat{F}_k(\varphi_0)|$. By the definitions of $B_j$ and $F_j(\varphi_0)$ given after Theorem 3.2, we obtain

$$|B_j| = \left|L_2\left(\int_0^1 Y_{j+\beta-s}\,ds\right)\right| \leq b\,\|L_2\|\int_0^1 e^{a(j+\beta-s)}\,ds,$$

$$|F_j(\varphi_0)| \leq |L_2(T(j)\varphi_0)| + \left|L_2\left(\int_0^{1-\beta} Y_{j-s}\,ds\right)L_2(\varphi_0)\right|$$

$$\leq b\|L_2\|\left(1 + \|L_2\|\int_0^{1-\beta} e^{-as}\,ds\right)e^{aj}\|\varphi_0\|$$

for $j \geq 1$. For the time being, we denote

$$b\,|(B_0)^{-1}|.\|L_2\|\int_0^1 e^{a(\beta-s)}\,ds \quad\text{and}\quad b\,|(B_0)^{-1}|.\|L_2\|\left(1 + \|L_2\|\int_0^{1-\beta} e^{-as}\,ds\right)$$

by $M_0$ and $M_1$ respectively. Then, from (4.5) and Lemma 4.1,

$$\left|\widehat{F}_k(\varphi_0)\right| \leq \sum_{j=1}^{k}\sum_{s(k,j)}\left|(B_0)^{-1}\right|^j|B_{p_1}|.|B_{p_2}|\cdots|B_{p_{j-1}}|.|F_{p_j}(\varphi_0)|$$

$$\leq \sum_{j=1}^{k}\binom{k-1}{j-1}(M_0)^{j-1}M_1e^{ak}\|\varphi_0\| = M_1(1+M_0)^{k-1}e^{ak}\|\varphi_0\|$$

for $k \geq 1$. Substituting this into (4.4), we obtain

$$|x(\varphi_0,0)(t)| \leq b\,e^{at}\left\{1 + \|L_2\|\int_0^{1-\beta} e^{-as}\,ds + \sum_{j=1}^{k}\int_{j-\beta}^{j+1-\beta} e^{a(j-s)}\,ds\,M_1(1+M_0)^{j-1}\right\}\|\varphi_0\|$$

$$\leq b\left(1 + \|L_2\| \int_0^{1-\beta} e^{-as}\, ds\right)\left\{1 + M_0 \sum_{j=1}^{k} (1 + M_0)^{j-1}\right\} e^{at} \|\varphi_0\|$$

$$= b\left(1 + \|L_2\| \int_0^{1-\beta} e^{-as}\, ds\right)(1 + M_0)^k e^{at} \|\varphi_0\|$$

$$\leq b' \|\varphi_0\| \exp\{a + \ln(1 + M_0)\} t$$

for $t \in [k - \beta, k + 1 - \beta)$ with $k \geq 1$ and some constant $b' \geq 1$.

Summarizing the above discussion, we come to the following conclusion.

**THEOREM** 4.2 Assume that $\beta \in [0, 1]$ and (3.4) holds. Then there are constants $a_1$ and $b_1$ with

$$b_1 \geq 1, \; a_1 \leq a + \ln\left(1 + b\left|(B_0)^{-1}\right| \|L_2\| \int_0^1 e^{a(\beta - s)}\, ds\right) \tag{4.6}$$

such that the solution of (1.2) with (2.1) satisfies

$$\|x_t(\varphi_0, 0)\| \leq b_1 e^{a_1 t} \|\varphi_0\| \tag{4.7}$$

for $t \geq 0$.

To establish an exponential estimate for the solution of (1.1) with $x_0 = 0$, we can imitate the above reasoning and obtain an inequality

$$\|x_t(0, f)\| \leq b_1' \int_0^{\alpha(t)} e^{a_1'(t - s)} |f(s)|\, ds, \; t \geq 0,$$

where the constants $a_1'$ and $b_1'$, after replacing $a_1$ and $b_1$, satisfy (4.6) and $\alpha(t)$ is to be defined later. But the most important conclusion $a_1' = a_1$, which will be used in the next chapter, can not be drawn by this method. However, our goal can be reached by employing a variation of parameters formula and the estimate (4.7).

**THEOREM** 4.3 Assume that $\beta \in [0,\ 1]$ and (3.4) holds. Then the solution of (1.1) with $x_0 = 0$ satisfies

$$\|x_t(0, f)\| \le b_2 \int_0^{\alpha(t)} e^{a_1(t-s)} |f(s)|\, ds \tag{4.8}$$

for $t \ge 0$ and some constant $b_2 \ge 1$, where $a_1$ is the same as in (4.7) and $\alpha(t)$ is defined by

$$\alpha(t) = \max\{[t + \beta],\ t\},\ t \ge 0. \tag{4.9}$$

**Proof.** We first define $\{f_n\}$ by

$$f_n(t) = f(t + n),\ t \ge 0,\ n = 0, 1, 2, \cdots. \tag{4.10}$$

Notice that the homogeneous equation (1.2) is autonomous and that the function $[t + \beta] - t$ is 1-periodic. Then, since $[t + \beta] = t + \{[t + \beta] - t\}$ for $t \ge 0$, by uniqueness and the decomposition (3.5), the solution of (1.1) with $x_0 = 0$ satisfies

$$x(0, f)(t) = x(0, f_0)(t)$$

for $t \ge 0$ and

$$x(0, f)(t) = x(x_1(0, f_0), f_1)(t - 1) = x(x_1(0, f_0), 0)(t - 1) + x(0, f_1)(t - 1)$$

for $t \ge 1$. Repeating the above procedure $n$ times, we have

$$x(0, f)(t) = x(x_1(0, f_0), 0)(t - 1) + x(x_1(0, f_1), 0)(t - 2) + \cdots$$
$$+ x(x_1(0, f_{n-1}), 0)(t - n) + x(0, f_n)(t - n) \tag{4.11}$$

for $t \ge n \ge 1$. Hence we need to estimate $x(0, f_n)(t - n)$ for $t \in [n, n + 1)$ and $n \ge 0$. For each $n \ge 0$, it follows from (3.13) that

$$x(0, f_n)(t) = \int_0^t Y(t - s) f_n(s)\, ds$$

for $t \in [0, 1 - \beta)$ and

$$x(0, f_n)(t) = \int_0^t Y(t-s) f_n(s)\, ds + \int_{1-\beta}^t Y(t-s)\, ds\, \widehat{G}_1(f_n)$$

for $t \in [1-\beta, 1)$. Since $\{\widehat{G}_j(f)\}$ is the solution of (3.6) with $\varphi_0 = 0$, we have

$$\widehat{G}_1(f_n) = -(B_0)^{-1} G_1(f_n) = -(B_0)^{-1} L_2\left(\int_0^1 Y_{1-s} f_n(s)\, ds\right), \quad n = 0, 1, 2, \cdots.$$

Then, from the representation of $x(0, f_n)(t)$ and (1.15), we obtain

$$|x(0, f_n)(t)| \le b \int_0^t e^{a(t-s)} |f_n(s)|\, ds$$

for $t \in [0, 1-\beta)$ and

$$|x(0, f_n)(t)| \le b \int_0^t e^{a(t-s)} |f_n(s)|\, ds + b \int_{1-\beta}^t e^{a(t-s)}\, ds \left| (B_0)^{-1} L_2\left(\int_0^1 Y_{1-u} f_n(u)\, du\right)\right|$$

$$\le b \int_0^t e^{a(t-s)} |f_n(s)|\, ds + b \int_{1-\beta}^t e^{a(t-s)}\, ds \left|(B_0)^{-1}\right| . \|L_2\|\, b \int_0^1 e^{a(1-u)} |f_n(u)|\, du$$

$$\le b\left(1 + b\left|(B_0)^{-1}\right| . \|L_2\| \int_{1-\beta}^1 e^{a(1-s)}\, ds\right) \int_0^t e^{a(t-s)} |f_n(s)|\, ds$$

for $t \in [1-\beta, 1)$. Thus there is a $\rho \ge 1$ such that

$$|x(0, f_n)(t)| \le \rho \int_0^{n+1} e^{a_1(t-s)} |f_n(s)|\, ds, \quad n \ge 0, \tag{4.12}$$

for $t \in [0, 1)$ and

$$\|x_1(0, f_n)\| \le \rho \int_0^1 e^{a_1(1-s)} |f_n(s)|\, ds, \quad n \ge 0. \tag{4.13}$$

Then, for $t \in [n, n+1)$ with $n \ge 1$, Theorem 4.2 and the relations (4.10)-(4.13) lead to the estimate

$$
\begin{aligned}
|x(0, f)(t)| &\le \sum_{i=1}^{n} |x(x_1(0, f_{i-1}), 0)(t-i)| + |x(0, f_n)(t-n)| \\
&\le \sum_{i=1}^{n} b_1 e^{a_1(t-i)} \|x_1(0, f_{i-1})\| + \rho \int_0^{\omega(t-n)} e^{a_1(t-n-s)} |f_n(s)|\, ds \\
&\le b_1 \rho e^{a_1 t} \sum_{i=1}^{n} \int_0^{1} e^{a_1(1-s-i)} |f_{i-1}(s)|\, ds + \rho \int_0^{\omega(t)} e^{a_1(t-s)} |f(s)|\, ds \\
&\le b_1 \rho \int_0^{\omega(t)} e^{a_1(t-s)} |f(s)|\, ds .
\end{aligned}
$$

This, together with (4.12), implies

$$
|x(0, f)(t)| \le b_1 \rho \int_0^{\omega(t)} e^{a_1(t-s)} |f(s)|\, ds
$$

for $t \ge 0$. Then (4.8) follows from letting $b_2 = b_1 \rho \|e^{a_1 \cdot}\|$ #

Combining Theorem 4.2 with Theorem 4.3, we see that there are constants $a_1$ and $b_1$ satisfying (4.6) such that the solution of (1.1) with (2.1) satisfies

$$
\|x_t(\varphi_0, f)\| \le b_1 e^{a_1 t} \left( \|\varphi_0\| + \int_0^{\omega(t)} e^{-a_1 s} |f(s)|\, ds \right) \tag{4.14}
$$

for $t \ge 0$.

**Remark.** Theorems 4.2 and 4.3 also show that the linear mappings $x(., 0)(t)$ and $x(0, .)(t)$ are bounded for each $t \ge 0$.

## § 4.2 ESTIMATES FOR THE CASE OF $\beta > 1$

We suppose that $\beta$ satisfies $m < \beta \le m + 1$ for some integer $m \ge 1$ and that (3.14) holds. Then, by Theorem 3.5, the solution of (1.2) with (2.16) has the representation

$$x(\varphi_0, v_0, 0)(t) = T(t)\varphi_0(0) + \sum_{j=0}^{k} E_j(t)\left\{\hat{U}_j(\varphi_0) + \hat{\Phi}_j v_0\right\} \tag{4.15}$$

for $t \in [t_{km}, t_{(k+1)m})$ with $k \ge 0$.

To obtain an estimate of $|x(\varphi_0, v_0, 0)(t)|$, we specify the norm of the nm-vector $v_0$ by

$$|v_0| = |(h_{m-1}{}^T, h_{m-2}{}^T, \cdots, h_0{}^T)^T| = \max\{|h_0|, |h_1|, \cdots, |h_{m-1}|\} \tag{4.16}$$

and deduce the norm for matrices $E_j(t)$ and $\Phi_j$. Let

$$M_2 = 1 + b\|L_2\|\max(1, e^{a(m-1)})\max\left\{e^{a|\beta|}, \int_0^1 e^{a(\beta - m - s)}\,ds \sum_{i=1}^{m} e^{ai}\right\}, \tag{4.17}$$

where $a$ and $b$ are defined by (1.7).

**THEOREM 4.4** Assume that $\beta \in (m, m+1]$ for some integer $m \ge 1$ and that (3.14) holds. Then there are constants $a^*$ and $b^*$ satisfying

$$b^* \ge 1, \quad a^* \le a + (1/m)\ln\{1 + |(\Phi_0)^{-1}|\, M_2 e^{-am}\} \tag{4.18}$$

such that the solution of (1.2) with (2.16) satisfies

$$\|x_t(\varphi_0, v_0, 0)\| \le b^* e^{a^* t}(\|\varphi_0\| + |v_0|) \tag{4.19}$$

for $t \ge 0$.

**Proof.** The representation (4.15) indicates that $|E_j(t)|$ and $\left|\hat{U}_j(\varphi_0) + \hat{\Phi}_j v_0\right|$ should be estimated first. By (3.26) and (1.15),

$$|E_j(t)| = \sum_{i=0}^{m-1}\left|\int_{t_{jm+i}}^{t_{jm+i+1}} Y(t-s)\,ds\right| \le b\int_{t_{jm}}^{t_{(j+1)m}} e^{a(t-s)}\,ds$$

for $t \geq 0$ and $j = 0, 1, 2, \cdots$. Since $\{\hat{U}_k(\varphi_0) + \hat{\Phi}_k v_0\}$ is the solution of (3.25) with $f = 0$, by Lemma 3.1 we have the representation

$$\hat{U}_k(\varphi_0) + \hat{\Phi}_k v_0 = \sum_{j=1}^{k} (-1)^j \sum_{s(k,\, j)} \left((\Phi_0)^{-1}\Phi_{p_1}\right) \cdots \left((\Phi_0)^{-1}\Phi_{p_{j-1}}\right)(\Phi_0)^{-1}\left(U_{p_j}(\varphi_0) + \Phi_{p_j}' v_0\right),$$

$$k = 1, 2, \cdots. \tag{4.20}$$

Notice that (3.16)-(3.20) and (1.15) result in the inequalities

$$|P_m| \leq |I| + \left|L_2\left(\int_0^t Y_{\beta - s}\, ds\right)\right| \leq 1 + b\|L_2\| \int_0^t e^{a(\beta - s)}\, ds,$$

$$|P_m'| \leq |I| + \left|L_2\left(\int_0^{t'} Y_{[\beta] - s}\, ds\right)\right| \leq 1 + b\|L_2\| \int_0^t e^{a(\beta - s)}\, ds,$$

$$|P_n| \leq b\|L_2\| \int_0^t e^{a(n + \beta - m - s)}\, ds \quad (n \geq 0,\ n \neq m),$$

$$|P_n'| \leq b\|L_2\| \int_0^{t'} e^{a(n + [\beta] - m - s)}\, ds \leq b\|L_2\| \int_0^t e^{a(n + \beta - m - s)}\, ds \quad (n > m),$$

and

$$|u_n(\varphi_0)| \leq b\|L_2\| e^{a(n + [\beta] - m)} \|\varphi_0\| \quad (n \geq m).$$

Then, from (3.24), (3.22) and (4.17), we obtain

$$|\Phi_1| = \max\left\{\sum_{n=1}^{m} |P_{i+n}| :\ i = 0, 1, \cdots, m-1\right\}$$

$$\leq \max\left\{1 + b\|L_2\| \int_0^1 e^{a(i + \beta - m - s)}\, ds \sum_{n=1}^{m} e^{an} :\ i = 0, 1, \cdots, m-1\right\} \leq M_2,$$

$$|\Phi_1'| = \max\left\{\sum_{n=1}^{m-1} |P_{i+n}| + |P_{i+m}'| :\ i = 0, 1, \cdots, m-1\right\} \leq M_2,$$

$$|\Phi_j| = \max\left\{\sum_{n=1}^{m} |P_{(j-1)m+i+n}| :\ i = 0, 1, \cdots, m-1\right\} \leq M_2 e^{a(j-1)m},$$

$$|\Phi_j'| = \max\left\{\sum_{n=1}^{m-1} |P_{(j-1)m+i+n}| + |P_{jm+i}'|:\ i = 0, 1, \cdots, m-1\right\} \le M_2 e^{a(j-1)m}$$

for $j > 1$ and

$$\begin{aligned}|U_j(\varphi_0)| &= \max\{\, |u_{jm+i}(\varphi_0)|:\ i = 0, 1, \cdots, m-1\} \\ &\le b\,\|L_2\|\, e^{a(jm+|\beta|-m)} \max(1, e^{a(m-1)})\, \|\varphi_0\| \\ &\le M_2\, e^{a(j-1)m}\, \|\varphi_0\|\end{aligned}$$

for $j \ge 1$. Hence, from (4.20) and Lemma 4.1,

$$\begin{aligned}\left|\hat{U}_k(\varphi_0) + \hat{\Phi}_k v_0\right| &\le \sum_{j=1}^{k} \sum_{s(k,j)} \left|(\Phi_0)^{-1}\right| \left|\Phi_{p_1}\right| \cdot \left|\Phi_{p_2}\right| \cdots \left|\Phi_{p_{j-1}}\right| \left(|U_{p_j}(\varphi_0)| + \left|\Phi_{p_j}'\right| \cdot |v_0|\right) \\ &\le \sum_{j=1}^{k} \binom{k-1}{j-1} \left(\left|(\Phi_0)^{-1}\right| M_2 e^{-am}\right)^{j} e^{akm} (\|\varphi_0\| + |v_0|) \\ &= \left|(\Phi_0)^{-1}\right| M_2 e^{a(k-1)m} \left(1 + \left|(\Phi_0)^{-1}\right| M_2 e^{-am}\right)^{k-1} (\|\varphi_0\| + |v_0|)\end{aligned}$$

for $k \ge 1$. As for $k = 0$, we have $\hat{U}_0(\varphi_0) + \hat{\Phi}_0 v_0 = v_0$ by definition.

Now, with the above preparation, we can derive the estimate (4.19). From (4.15) we obtain, for $t \in [0, t_m)$,

$$\begin{aligned}|x(\varphi_0, v_0, 0)(t)| &\le |T(t)\varphi_0(0)| + |E_0(t)v_0| \\ &\le b\,e^{at}\|\varphi_0\| + b\int_0^{t_m} e^{a(t-s)}\,ds\,|v_0| \le b' e^{at}(\|\varphi_0\| + |v_0|),\end{aligned}$$

where

$$b' = b \max\left(1, \int_{(N-\beta)}^{t_m} e^{-as}\,ds\right).$$

For $t \in [t_{km}, t_{(k+1)m})$ with $k \ge 1$,

$$|x(\varphi_0, v_0, 0)(t)| \le |T(t)\varphi_0(0)| + |E_0(t)v_0| + \sum_{j=1}^{k} |E_j(t)| \cdot \left|\hat{U}_j(\varphi_0) + \hat{\Phi}_j v_0\right|$$

$$\leq b' e^{at}(\|\varphi_0\| + |v_0|)$$

$$+ \sum_{j=1}^{k} b \int_{t_{jm}}^{t_{(j+1)m}} e^{a(t-s)} ds\, M_2 \left|(\Phi_0)^{-1}\right| e^{a(j-1)m} \left(1 + M_2 \left|(\Phi_0)^{-1}\right| e^{-am}\right)^{j-1} (\|\varphi_0\| + |v_0|)$$

$$\leq b' e^{at}(\|\varphi_0\| + |v_0|)\left\{1 + M_2 \left|(\Phi_0)^{-1}\right| e^{-am} \sum_{j=1}^{k} \left(1 + M_2 \left|(\Phi_0)^{-1}\right| e^{-am}\right)^{j-1}\right\}$$

$$= b'\left(1 + M_2 \left|(\Phi_0)^{-1}\right| e^{-am}\right)^{k} e^{at}(\|\varphi_0\| + |v_0|).$$

For $t \in [t_{km}, t_{(k+1)m})$ with $k \geq 0$, since

$$\left(1 + \left|(\Phi_0)^{-1}\right| M_2 e^{-am}\right)^{k} \leq b'' \exp\left\{\frac{t_{km}}{m} \ln\left(1 + \left|(\Phi_0)^{-1}\right| M_2 e^{-am}\right)\right\}$$

$$\leq b'' \exp\left\{\frac{t}{m} \ln\left(1 + \left|(\Phi_0)^{-1}\right| M_2 e^{-am}\right)\right\}$$

for some $b''$ independent of $k$, it follows that

$$|x(\varphi_0, v_0, 0)(t)| \leq b'b'' (\|\varphi_0\| + |v_0|) \exp\left\{a + \frac{1}{m}\ln\left(1 + \left|(\Phi_0)^{-1}\right| M_2 e^{-am}\right)\right\}t$$

for all $t \geq 0$. Therefore, there are constant $a^*$ and $b^*$ satisfying (4.18) such that (4.19) holds for $t \geq 0$. #

**Remark.** Notice that

$$E_0(t) = \left(0, \cdots, 0, \int_{t_i}^{t} Y(t-s)\, ds, \int_{t_{i-1}}^{t_i} Y(t-s)\, ds, \cdots, \int_{0}^{t_1} Y(t-s)\, ds\right)$$

for $t \in [t_i, t_{i+1}) \subset [0, t_m)$ with $i \in \{0, 1, \cdots, m-1\}$. If we define $|v_0|_i$ by

$$|v_0|_i = \max\{|h_0|, |h_1|, \cdots, |h_i|\}, \tag{4.21}$$

then the above proof enables us to derive

$$|x(\varphi_0, v_0, 0)(t)| \leq b^* e^{a^* m} (\|\varphi_0\| + |v_0|_i) \tag{4.22}$$

for $t \in [t_i, t_{i+1}) \subset [0, t_m)$, since

$$|E_0(t)v_0| \leq |E_0(t)|.|v_0|_i$$

holds for $t$ in this interval. Soon we will use (4.19) and (4.22) to obtain an exponential estimate for $x(0, 0, f)(t)$.

Let $\{f_k\}$ be defined by (4.10). To indicate the dependence on $(\varphi_0, v_0, f)$, we denote $v_k$, defined by (3.28) and (3.29), by $v_k(\varphi_0, v_0, f)$ for $k \geq 1$. Then, for the same reason as in § 4.1, the solution of (1.1) with $(\varphi_0, v_0) = (0, 0)$ satisfies

$$x(0, 0, f)(t) = x(0, 0, f_0)(t)$$

for $t \geq 0$ and

$$\begin{aligned} x(0, 0, f)(t) &= x(x_m(0, 0, f_0), v_1(0, 0, f_0), f_m)(t - m) \\ &= x(x_m(0, 0, f_0), v_1(0, 0, f_0), 0)(t - m) + x(0, 0, f_m)(t - m) \end{aligned}$$

for $t \geq m$. Repeating the above procedure $k$ times, we obtain

$$\begin{aligned} x(0, 0, f)(t) &= x(0, 0, f_{km})(t - km) \\ &\quad + \sum_{i=1}^{k} x(x_m(0, 0, f_{(i-1)m}), v_1(0, 0, f_{(i-1)m}), 0)(t - im) \end{aligned} \tag{4.23}$$

for $t \geq km$. From (4.19), (4.22) and (4.23), we shall show the following result.

**THEOREM 4.5** Assume that $\beta \in (m, m + 1]$ for some integer $m \geq 1$ and that (3.14) holds. Then there is a constant $c^* \geq 1$ such that the solution of (1.1) with $(\varphi_0, v_0) = (0, 0)$ satisfies

$$\|x_t(0, 0, f)\| \leq c^* \int_0^{\alpha(t)} e^{a^*(t - s)} |f(s)| \, ds \tag{4.24}$$

for $t \geq 0$, where $a^*$ is as in (4.9) and $\alpha(t)$ is defined by $\alpha(t) = t$ for $t \in [0, m + [\beta] - \beta)$ and $\alpha(t) = \max\{t, [n + 1 - \beta]\}$ for $t \in [n - \beta, n + 1 - \beta)$ with $n \geq m + [\beta]$.

**Proof.** We first estimate $|x(0, 0, f)(t)|$ for $t \in [0, m)$, $\|x_m(0, 0, f)\|$, $|v_1(0, 0, f)|_i$ for $i = 0, 1, \cdots, m - 1$, and $|v_1(0, 0, f)|$. By Theorem 3.5, we have the representation

$$x(0, 0, f)(t) = \int_0^t Y(t - s)f(s)\,ds$$

for $t \in [0, t_m)$ and

$$x(0, 0, f)(t) = \int_0^t Y(t - s)f(s)\,ds + E_1(t)\widehat{W}_1(f)$$

for $t \in [t_m, m)$. From the proof of Theorem 3.5 we know that

$$v_k(0, 0, f) = \widehat{W}_k(f) \quad (k = 1, 2, \cdots)$$

satisfies (3.25) with $(\varphi_0, v_0) = (0, 0)$. Thus

$$v_1(0, 0, f) = \widehat{W}_1(f) = -(\Phi_0)^{-1}W_1(f).$$

Since $\Phi_0$, by (3.24), has the triangular form

$$\Phi_0 = \begin{bmatrix} P_0 & P_1 & \cdots & P_{m-1} \\ 0 & P_0 & \cdots & P_{m-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & P_0 \end{bmatrix},$$

$(\Phi_0)^{-1}$ is also triangular. Then, From (4.21), (3.23) and (3.21), it follows that

$$\begin{aligned} |v_1(0, 0, f)|_i &\le |(\Phi_0)^{-1}|.|W_1(f)|_i \\ &= |(\Phi_0)^{-1}| \max\{|w_m(f)|, |w_{m+1}(f)|, \cdots, |w_{m+i}(f)|\} \\ &\le b\,\|L_2\|.\left|(\Phi_0)^{-1}\right| \max(1, e^{a(m-1)}) \int_0^{[\beta]+1} e^{a([\beta]-s)}|f(s)|\,ds \\ &\le b_1 \int_0^{[\beta]+1} e^{a^*(m-s)}|f(s)|\,ds \end{aligned} \tag{4.25}$$

for $i = 0, 1, \cdots, m - 1$ and some constant $b_1 > 0$. In particular,

$$|v_1(0, 0, f)| = |v_1(0, 0, f)|_{m-1} \le b_1 \int_0^{[\beta]+m-1} e^{a^*(m-s)}|f(s)|\, ds\,. \tag{4.26}$$

Suppose that $t \in [t_m, m)$. Since

$$E_1(t) = \left(0, \cdots, 0, \int_{t_m}^{t} Y(t-s)\, ds\right),$$

we have

$$\left|E_1(t)\widehat{W}_1(f)\right| \le \left|\int_{t_m}^{t} Y(t-s)\, ds\right| . \left|\widehat{W}_1(f)\right|_0 \le b\, b_1 \int_{t_m}^{t} e^{a(t-s)}\, ds \int_0^{[\beta]} e^{a^*(m-s)}|f(s)|\, ds.$$

Then, by (1.15), there is a $b_2 \ge 1$ such that

$$|x(0, 0, f)(t)| \le b \int_0^t e^{a(t-s)}|f(s)|\, ds \le b_2 \int_0^t e^{a^*(t-s)}|f(s)|\, ds \tag{4.27}$$

for $t \in [0, t_m)$,

$$|x(0, 0, f)(t)| \le \left|\int_0^t Y(t-s) f(s)\, ds\right| + \left|E_1(t)\widehat{W}_1(f)\right| \le b_2 \int_0^m e^{a^*(t-s)}|f(s)|\, ds \tag{4.28}$$

for $t \in [t_m, m)$ (as $[\beta] = m$ if $t_m < m$), and

$$\|x_m(0, 0, f)\| \le b_2 \int_0^m e^{a^*(m-s)}|f(s)|\, ds\,. \tag{4.29}$$

By the definition of $\alpha(t)$, (4.27) and (4.28) are equivalent to

$$|x(0, 0, f)(t)| \le b_2 \int_0^{\alpha(t)} e^{a^*(t-s)}|f(s)|\, ds \tag{4.30}$$

for $t \in [0, m)$.

We now show that (4.30), with an increased value of $b_2$, actually holds for $t \ge 0$. For any $t \ge m$, there is a $k \ge 1$ such that either $t \in [km + t_i, km + t_{i+1}) \subset [km, (k+1)m)$ for some $i \in \{0, 1, \cdots, m-1\}$ or $t \in [km + t_m, (k+1)m)$. In the first case, we see from (4.23), (4.19), (4.22), (4.25)-(4.27), and (4.29) that

$$|x(0, 0, f)(t)| \le \sum_{n=1}^{k-1} b^* e^{a^*(t-nm)}(\|x_m(0, 0, f_{(n-1)m})\| + |v_1(0, 0, f_{(n-1)m})|)$$

$$+ b^* e^{a^*(t-km)} (\|x_m(0, 0, f_{(k-1)m})\| + |v_1(0, 0, f_{(k-1)m})|_i)$$

$$+ |x(0, 0, f_{km})(t - km)|$$

$$\le \sum_{n=1}^{k-1} b^* e^{a^*(t-nm)} \times$$

$$\left\{ b_2 \int_0^m e^{a^*(m-s)} |f(s + (n-1)m)|\, ds + b_1 \int_0^{[\beta]+m-1} e^{a^*(m-s)} |f(s + (n-1)m)|\, ds \right\}$$

$$+ b^* e^{a^*(t-km)} \left\{ b_2 \int_0^m e^{a^*(m-s)} |f(s + (k-1)m)| ds + b_1 \int_0^{[\beta]+i} e^{a^*(m-s)} |f(s + (k-1)m)| ds \right\}$$

$$+ b_2 \int_0^{t-km} e^{a^*(t-km-s)} |f(s + km)|\, ds$$

$$\le b^* b_2 \int_0^t e^{a^*(t-s)} |f(s)|\, ds + b_1^* \int_0^{(k-1)m+[\beta]+i} e^{a^*(t-s)} |f(s)|\, ds$$

for some positive constant $b_1^*$ independent of $k$ and $i$. Since

$$\max\{t,\ (k-1)m + [\beta] + i\} = \max\{t,\ [km + t_{i+1}]\} = \alpha(t),$$

(4.30) holds for $t$ in the first case if we replace $b_2$ by $b^* b_2 + b_1^*$. For $t \in [km + t_m, (k+1)m)$, (4.26) and (4.28) indicate that we should replace $i$ by $m - 1$ and the upper bound $t - km$ of the integral by $m$ in the above inequality. We then obtain

$$|x(0, 0, f)(t)| \le b_2^* \int_0^{(k+1)m} e^{a^*(t-s)} |f(s)|\, ds$$

for some constant $b_2^*$ ($\ge 1$) independent of $k$. Since $t \in [km + t_m, (k+1)m)$ implies $t_m = m - \beta + [\beta] < m$ and the relation

$$[km + t_m, (k + 1)m) \subset [km + t_m, (k + 1)m + t_1)$$

holds, we have

$$\alpha(t) = \max\{t, [(k + 1)m + t_1]\} = \max\{t, (k + 1)m\} = (k + 1)m.$$

Thus (4.30) holds for $t \in [km + t_m, (k + 1)m)$ if we replace $b_2$ by $b_2^*$. Therefore, we have shown (4.30) for some constant $b_2 \geq 1$ and all $t \geq 0$. Thus (4.24) holds for some constant $c^* \geq 1$ and all $t \geq 0$. #

From Theorems 4.4 and 4.5 we see that the solution of (1.1) with (2.16) satisfies

$$\|x_t(\varphi_0, v_0, f)\| \leq b^* e^{a^* t}\left\{\|\varphi_0\| + |v_0| + \int_0^{\alpha(t)} e^{-a^* s}|f(s)|\, ds\right\} \tag{4.31}$$

for $t \geq 0$, where $a^*$ and $b^*$ are constants satisfying (4.18).

The estimates (4.19) and (4.24) also indicate that the linear mappings $x(., 0, 0)(t)$ and $x(0, 0, .)(t)$ are bounded for each $t \geq 0$.

# CHAPTER 5 ASYMPTOTIC BEHAVIOUR

In this chapter, we employ the previous results to discuss the asymptotic behaviour of solutions of (1.1) and (1.2). We assume that the conditions for existence and uniqueness are satisfied.

## § 5.1 ASYMPTOTIC PROPERTIES OF SOLUTIONS

We first define a space $C_\beta$ in order to unify the discussion for the two cases of $\beta \in [0, 1]$ and $\beta > 1$. If $\beta \in [0, 1]$, we simply define

$$C_\beta = C = C([-r, 0], \mathbb{C}^N).$$

If $\beta \in (m, m + 1]$ for some integer $m \geq 1$, we define

$$C_\beta = C \times \mathbb{C}^{mN}$$

with a norm $\|.\|$ given by

$$\|\varphi_\beta\| = \|(\varphi_0, v_0)\| = \|\varphi_0\| + |v_0|,$$

where $\varphi_0 \in C$, $v_0 \in \mathbb{C}^{mN}$ and $\varphi_\beta = (\varphi_0, v_0) \in C_\beta$. Then, for any $\beta \geq 0$, $C_\beta$ is a Banach space. In this notation, we can simply denote the solution $x(\varphi_0, v_0, f)(t)$ or $x(\varphi_0, f)(t)$ of (1.1) by $x(\varphi_\beta, f)(t)$.

**Definition** 5.1 The equation (1.2) is said to be stable if, for any $\varepsilon > 0$, there is a $\delta > 0$ such that the solution of (1.2) satisfies

$$\|x_t(\varphi_\beta, 0)\| < \varepsilon \tag{5.0}$$

for $t \geq 0$ provided that $\|\varphi_\beta\| < \delta$. We say that (1.2) is uniformly asymptotically stable (UAS in short) if it is stable and if, for any $\varepsilon > 0$ and $M > 0$, there is a $T \geq 0$ such that each solution of (1.2) with $\|\varphi_\beta\| \leq M$ satisfies (5.0) for $t \geq T$.

**Definition** 5.2 The solutions of (1.1) are said to be uniformly bounded if, for any $\delta > 0$, there is an $M > 0$ such that each solution of (1.1) with $\|\varphi_\beta\| \leq \delta$ satisfies

$$\|x_t(\varphi_\beta, f)\| \leq M$$

for $t \geq 0$. The solutions of (1.1) are said to be uniformly asymptotic to zero (UAZ in short) if they are uniformly bounded and if, for any $\delta > 0$ and $\varepsilon > 0$, there is a $T \geq 0$ such that each solution of (1.1) with $\|\varphi_\beta\| \leq \delta$ satisfies

$$\|x_t(\varphi_\beta, f)\| < \varepsilon$$

for $t \geq T$.

**Remarks**. (i). The homogeneous equation (1.2) is stable if and only if its solutions are uniformly bounded. Also, (1.2) is UAS if and only if its solutions are UAZ. These conclusions are derived immediately from the definitions and the linearity of (1.2).

(ii). The uniformity in Definitions 5.1 and 5.2 refers only to the initial data $\varphi_\beta$ in $C_\beta$. For autonomous equations the choice of initial point $t_0$ is immaterial. Thus the definitions of stability, uniform asymptotic stability and uniform boundedness are equivalent to those for autonomous delay and neutral equations (see [17]). Although the term UAZ is not standard, it describes an important asymptotic feature of solutions.

In the following, we give some results concerning asymptotic properties of (1.1) and (1.2). The first one reveals the equivalence of UAS to negative exponential boundedness of (1.2).

**THEOREM 5.1** The equation (1.2) is UAS if and only if there are constants $a^* < 0$ and $b^* \geq 1$ such that each solution of (1.2) satisfies

$$\|x_t(\varphi_\beta, 0)\| \leq b^* e^{a^* t} \|\varphi_\beta\| \tag{5.1}$$

for $t \geq 0$.

**Proof.** The "if" part of the assertion follows immediately from Definition 5.1. Thus we only need show that, if (1.2) is UAS, then (5.1) holds for some constants $a^* < 0$ and $b^* \geq 1$. If $\beta \in [0, 1]$, the proof is the same as for the neutral equation (1.3) (see Ch.12 in [17]) and is also covered by the proof for $\beta > 1$.

Assume that $\beta \in (m, m+1]$ for some integer $m \geq 1$ and that (1.2) is UAS. Then, for

$$\varepsilon = [2(1 + \|L_2\|)]^{-1},$$

there is an integer $K \geq 0$ such that each solution of (1.2) with $\|\varphi_\beta\| = 1$ satisfies $\|x_t(\varphi_\beta, 0)\| < \varepsilon$ for $t \geq Km$. From the definition of $v_j$ given by (3.28) and (3.29),

$$v_K(\varphi_\beta, 0) = (L_2(x_{[\beta]+Km+m-1}(\varphi_\beta, 0))^T, L_2(x_{[\beta]+Km+m-2}(\varphi_\beta, 0))^T, \cdots, L_2(x_{[\beta]+Km}(\varphi_\beta, 0))^T)^T.$$

It follows that

$$|v_K(\varphi_\beta, 0)| \leq \|L_2\| \max\{ \|x_{[\beta]+Km+i}(\varphi_\beta, 0)\|: i = 0, 1, \cdots, m-1\} \leq \varepsilon \|L_2\|$$

and

$$\|(x_{Km}(\varphi_\beta, 0), v_K(\varphi_\beta, 0))\| = \|x_{Km}(\varphi_\beta, 0)\| + |v_K(\varphi_\beta, 0)| \leq \varepsilon(1 + \|L_2\|) = 2^{-1}$$

for $\|\varphi_\beta\| = 1$. This, together with the linearity of (1.2), implies that the solution of (1.2) satisfies

$$\|(x_{Km}(\varphi_\beta, 0), v_K(\varphi_\beta, 0))\| \leq 2^{-1} \|\varphi_\beta\| \tag{5.2}$$

for any $\varphi_\beta \in C_\beta$. Since $[t + \beta] - t$ is 1-periodic, by uniqueness the solution of (1.2) satisfies

$$x(\varphi_\beta, 0)(t) = x(\varphi_0, v_0, 0)(t) = x(x_{km}(\varphi_\beta, 0), v_k(\varphi_\beta, 0), 0)(t - km)$$

for $t \geq km - r$ and $k = 0, 1, 2, \cdots$. Thus

$$x_{2Km}(\varphi_\beta, 0) = x_{Km}(x_{Km}(\varphi_\beta, 0), v_K(\varphi_\beta, 0), 0)$$

and

$$\begin{aligned} v_{2K}(\varphi_\beta, 0) &= (L_2(x_{[\beta]+2Km+m-1}(\varphi_\beta, 0))^T, L_2(x_{[\beta]+2Km+m-2}(\varphi_\beta, 0))^T, \\ &\qquad \cdots, L_2(x_{[\beta]+2Km}(\varphi_\beta, 0))^T)^T \\ &= (L_2(x_{[\beta]+Km+m-1}(x_{Km}(\varphi_\beta, 0), v_K(\varphi_\beta, 0), 0))^T, \\ &\qquad L_2(x_{[\beta]+Km+m-2}(x_{Km}(\varphi_\beta, 0), v_K(\varphi_\beta, 0), 0))^T, \\ &\qquad \cdots, L_2(x_{[\beta]+Km}(x_{Km}(\varphi_\beta, 0), v_K(\varphi_\beta, 0), 0))^T)^T \\ &= v_K(x_{Km}(\varphi_\beta, 0), v_K(\varphi_\beta, 0), 0). \end{aligned}$$

By (5.2) we have

$$\|(x_{2Km}(\varphi_\beta, 0), v_{2K}(\varphi_\beta, 0))\| \leq 2^{-1}\|(x_{Km}(\varphi_\beta, 0), v_K(\varphi_\beta, 0))\| \leq 2^{-2}\|\varphi_\beta\|.$$

Inductively, we obtain

$$\|(x_{kKm}(\varphi_\beta, 0), v_{kK}(\varphi_\beta, 0))\| \leq 2^{-k}\|\varphi_\beta\| \tag{5.3}$$

for $k \geq 1$. In fact, (5.3) also holds for $k = 0$ as

$$(x_0(\varphi_\beta, 0), v_0(\varphi_\beta, 0)) = (\varphi_0, v_0) = \varphi_\beta.$$

Since (1.2) is stable, by remark (i) and the linearity of (1.2) there is an $M \geq 1$ such that each solution of (1.2) satisfies

$$\|x_t(\varphi_\beta, 0)\| \leq M\|\varphi_\beta\| \tag{5.4}$$

for $t \geq 0$ and $\varphi_\beta \in C_\beta$.

Now for any $t \geq 0$, there is an integer $k \geq 0$ such that $t \in [kKm, (k+1)Km)$. Then, by (5.3) and (5.4),

$$\|x_t(\varphi_\beta, 0)\| = \|x_{t-kKm}(x_{kKm}(\varphi_\beta, 0), v_{kK}(\varphi_\beta, 0), 0)\|$$

$$\leq M \|(x_{kKm}(\varphi_\beta, 0), v_{kK}(\varphi_\beta, 0))\|$$

$$\leq M\, 2^{-k} \|\varphi_\beta\| \leq 2M \exp\{-t\,(Km)^{-1} \ln 2\} \|\varphi_\beta\|.$$

Therefore, the solution of (1.2) satisfies (5.1) with some constants $a^* < 0$ and $b^* \geq 1$. #

The next result reveals the relationship between the asymptotic behaviour of (1.2) and of (1.3).

**THEOREM** 5.2 Assume that the neutral equation (1.3) is UAS. Then, for $\beta \in [0, 1]$ and sufficiently small $\|L_2\|$, (1.2) is also UAS.

**Proof.** Since (1.3) is UAS, there are constants $a < 0$ and $b \geq 1$ such that the solution of (1.3) satisfies

$$\|T(t)\varphi_0\| \leq b\, e^{at} \|\varphi_0\|$$

for $t \geq 0$ and $\varphi_0 \in C$ (refer to § 1.2). By Theorem 4.2, there exist constants $b^*$ and $a^*$ satisfying

$$b^* \geq 1 \text{ and } a^* \leq a + \ln\left\{1 + b\left|(B_0)^{-1}\right| . \|L_2\| \int_0^1 e^{a(\beta - s)}\, ds\right\}$$

such that each solution of (1.2) satisfies (5.1). From the definition of $B_0$ in § 3.2, we have

$$B_0 = L_2\left(\int_0^1 Y_{\beta - s}\, ds\right) - I.$$

Thus $|(B_0)^{-1}| \to 1$ as $\|L_2\| \to 0$. Therefore, (5.1) holds for some $a^* < 0$ and $b^* \geq 1$ if $\|L_2\|$ is small enough. By Theorem 5.1, (1.2) is UAS if $\|L_2\|$ is small enough. #

**Remark.** Theorem 5.2 cannot be extended to the case of $\beta > 1$. This will be shown by Example 5.2 in § 5.4.

The next two results are for the boundedness of solutions of (1.1).

**THEOREM** 5.3 The solutions of (1.1) are uniformly bounded if one of the following conditions is met:

(A). (1.2) is stable and $f \in L(0, \infty)$;

(B). (1.2) is UAS and $\int_t^{t+1} |f(s)|\, ds$ is bounded on $[0, \infty)$.

**Proof.** By Theorems 4.3 and 4.5, the solution of (1.1) with $\varphi_\beta = 0$ satisfies

$$\|x_t(0, f)\| \le c^* \int_0^{t} e^{a^*(t-s)} |f(s)|\, ds \tag{5.5}$$

for $t \ge 0$ as long as each solution of (1.2) satisfies (5.1).

Suppose (A) holds. Then, by (5.4), the stability of (1.2) implies that we can take $a^* = 0$ in (5.1). Thus each solution of (1.1) satisfies

$$\|x_t(\varphi_\beta, f)\| \le \|x_t(\varphi_\beta, 0)\| + \|x_t(0, f)\| \le b^*\|\varphi_\beta\| + c^* \int_0^{\infty} |f(s)|\, ds$$

for $t \ge 0$. By Definition 5.2, solutions of (1.1) are uniformly bounded.

Assume that (B) holds. Then Theorem 5.1 implies that each solution of (1.2) satisfies (5.1) with $a^* < 0$. Let

$$M_0 = \sup\left\{ \int_t^{t+1} |f(s)|\, ds : \; t \in [0, \infty) \right\}. \tag{5.6}$$

From (5.5) we obtain

$$\|x_t(0, f)\| \le c^* e^{a^*[t]} \int_0^{t-[t]} |f(s)|\, ds + \sum_{n=-1}^{[t]-1} c^* e^{a^* n} \int_{t-n-1}^{t-n} |f(s)|\, ds$$

$$\le c^* M_0 e^{-a^*} \sum_{n=0}^{[t]+1} e^{a^* n} \le c^* M_0 e^{-a^*}(1 - e^{a^*})^{-1},$$

and so

$$\|x_t(\varphi_\beta, f)\| \le b^*\|\varphi_\beta\| + M_0c^*e^{-a^*}(1 - e^{a^*})^{-1},$$

which implies the uniform boundedness of solutions of (1.1). #

**THEOREM** 5.4 The solutions of (1.1) are UAZ if (1.2) is UAS and f satisfies

$$\lim_{t \to \infty}\int_t^{t+1} |f(s)|\, ds = 0. \tag{5.7}$$

**Proof.** By Theorem 5.1, (5.1) holds for some constant $a^* < 0$ as (1.2) is UAS. Since (5.7) implies the boundedness of the integral in (5.6), by Theorem 5.3 solutions of (1.1) are uniformly bounded. Then, for any $\varepsilon > 0$ and $\delta > 0$, we only need to find a $T \ge 0$ such that each solution of (1.1) with $\|\varphi_\beta\| \le \delta$ satisfies $\|x_t(\varphi_\beta, f)\| < \varepsilon$ for $t \ge T$.

By (5.7) again, there is a $T' \ge 0$ such that

$$\int_t^{t+1} |f(s)|\, ds < \varepsilon(2c^*)^{-1}e^{a^*}(1 - e^{a^*}) =_{def} \varepsilon'$$

for $t \ge T'$. Since $e^{a^*t} \to 0$ as $t \to \infty$, there is a $T \ge T' + 1$ such that

$$e^{a^*(t-T')} < \frac{1}{2}\varepsilon\left\{b^*e^{a^*T'}\delta + c^*\int_0^{T'} |f(s)|\, ds\right\}^{-1}$$

holds for $t \ge T$. Then, for $\|\varphi_\beta\| \le \delta$ and $t \ge T$, we have

$$\|x_t(\varphi_\beta, f)\| \le \|x_t(\varphi_\beta, 0)\| + \|x_t(0, f)\|$$

$$\le b^*e^{a^*t}\|\varphi_\beta\| + c^*\int_0^{t+1} e^{a^*(t-s)}|f(s)|\, ds$$

$$\le b^*e^{a^*t}\|\varphi_\beta\| + c^*\int_0^{T'} e^{a^*(t-s)}|f(s)|\, ds + c^*\{e^{-a^*} + 1 + e^{a^*} + \cdots\}\varepsilon'$$

$$\le e^{a^*(t-T')}\left\{b^*e^{a^*T'}\delta + c^*\int_0^{T'}|f(s)|\,ds\right\} + \frac{1}{2}\varepsilon < \varepsilon.$$

Thus solutions of (1.1) are UAZ. #

## § 5.2 THE SPECTRUM

From the discussion in § 5.1, we realize that the asymptotic behaviour of solutions of (1.2) plays an important role in that of solutions of (1.1). For the neutral equation (1.3), the asymptotic behaviour of its solutions can be studied by analysing the spectrum of the solution operator $T(t)$ for $t \ge 0$ (see Ch.12 in [17]). This idea can also be applied to our equation (1.2). In the following, we first define a "solution operator" for (1.2) and then investigate the relationship between its spectrum and the asymptotic behaviour of solutions of (1.2).

We define such an operator $T_\beta$ on $C_\beta$ as follows. If $\beta \in [0, 1]$ and $x(\varphi_0, 0)(t)$ is the solution of (1.2) with (2.1), we let

$$T_\beta\varphi_\beta = x_1(\varphi_\beta, 0) \tag{5.8}$$

for $\varphi_\beta \in C_\beta$ (where, in fact, $C_\beta = C$ and $\varphi_\beta = \varphi_0$). If $\beta$ satisfies $m < \beta \le m + 1$ for some integer $m \ge 1$, we let

$$T_\beta\varphi_\beta = (x_m(\varphi_\beta, 0), v_1(\varphi_\beta, 0)) \tag{5.9}$$

for $\varphi_\beta \in C_\beta$ ( $C_\beta = C \times \mathbb{C}^{mN}$ and $\varphi_\beta = (\varphi_0, v_0)$), where $v_1(\varphi_\beta, 0)$ is defined by (3.28) and (3.29). We denote the spectrum of $T_\beta$ by $\sigma(T_\beta)$ and the spectral radius by $\rho(T_\beta)$.

**THEOREM** 5.5 The equation (1.2) is UAS if and only if there is an integer $k \ge 1$ such that

$$\rho(T_\beta{}^k) < 1. \tag{5.10}$$

If

$$\rho(T_\beta{}^k) > 1 \tag{5.11}$$

for some $k \geq 1$, then (1.2) is unstable.

**Proof**. (A). Assume $\beta \in [0, 1]$. Then, by (5.8), we have

$$T_\beta{}^j \varphi_\beta = x_j(\varphi_\beta, 0), \; j = 0, 1, 2, \cdots.$$

Since $C_\beta$ is a Banach space, by a spectral radius theorem (refer to § 19.5 in [28]), the relation

$$\rho(T_\beta{}^k) = \lim_{j \to \infty} (\|T_\beta{}^{kj}\|)^{1/j} = \lim_{j \to \infty} (\|x_{kj}(\cdot, 0)\|)^{1/j} \tag{5.12}$$

holds for any integer $k \geq 1$.

If (1.2) is UAS, then Theorem 5.1 implies the inequality (5.1) with $a^* < 0$. Thus we have

$$(\|T_\beta{}^j\|)^{1/j} \leq (b^*)^{1/j} e^{a^*}, \; j = 1, 2, \cdots. \tag{5.13}$$

Then (5.10) with $k = 1$ follows from (5.12) and (5.13). On the other hand, if (5.10) holds for some $k \geq 1$, by (5.12) again, there is an integer $J \geq 1$ such that

$$(\|x_{kj}(\cdot, 0)\|)^{1/j} \leq (1/2)(1 + \rho(T_\beta{}^k)) =_{def} \varepsilon$$

for $j \geq J$. Hence

$$\|x_{kj}(\varphi_\beta, 0)\| \leq \varepsilon^j \|\varphi_\beta\|$$

for $j \geq J$. By Theorem 4.2, there is an $M \geq 1$ such that

$$\|x_t(\varphi_\beta, 0)\| \leq M \|\varphi_\beta\|$$

for $t \in [0, kJ]$. Then, for $t \in [kj, k(j+1))$ with $j \geq J$, we have

$$\|x_t(\varphi_\beta, 0)\| = \|x_{t-kj}(x_{kj}(\varphi_\beta, 0), 0)\| \leq M \|x_{kj}(\varphi_\beta, 0)\| \leq M \varepsilon^j \|\varphi_\beta\|. \tag{5.14}$$

Since $\varepsilon < 1$, (5.14) implies (5.1) with $a^* < 0$. Therefore, (1.2) is UAS.

If (5.11) holds for some $k \geq 1$, then, by (5.12) and the definition of $\|x_t(., 0)\|$, there is a sequence $\{\varphi_j: \varphi_j \in C = C_\beta, \|\varphi_j\| = 1, j = 1, 2, \cdots\}$ such that

$$\lim_{j \to \infty} \|x_{kj}(\varphi_j, 0)\| = \infty.$$

By Definition 5.1, (1.2) is unstable.

(B). Assume that $\beta \in (m, m+1]$ for some integer $m \geq 1$. From (5.9) we obtain

$$T_\beta^2\varphi_\beta = T_\beta(T_\beta\varphi_\beta) = (x_m(T_\beta\varphi_\beta, 0),\ v_1(T_\beta\varphi_\beta, 0))$$

$$= (x_m(x_m(\varphi_\beta, 0), v_1(\varphi_\beta, 0), 0),\ v_1(x_m(\varphi_\beta, 0), v_1(\varphi_\beta, 0), 0)).$$

For the same reason as in the proof of (5.3), it follows that

$$T_\beta^2\varphi_\beta = (x_{2m}(\varphi_\beta, 0), v_2(\varphi_\beta, 0))$$

and, by induction,

$$T_\beta^j\varphi_\beta = (x_{jm}(\varphi_\beta, 0), v_j(\varphi_\beta, 0)).$$

Thus, instead of (5.12), we have

$$\lim_{j \to \infty}(\|x_{kjm}(., 0)\|)^{1/j} \leq \rho(T_\beta^k)$$

$$\leq \lim_{j \to \infty}(\|x_{kjm}(., 0)\| + \|v_{kj}(., 0)\|)^{1/j} \tag{5.15}$$

for $k \geq 1$. If (1.2) is UAS, by Theorem 5.1 (5.1) holds with $a^* < 0$. Since the definition of $v_j$, (3.28) and (3.29), implies

$$|v_j(\varphi_\beta, 0)| \leq \|L_2\| \max\{\ \|x_{[\beta]+jm+i}(\varphi_\beta, 0)\|:\ i = 0, 1, \cdots, m-1\}$$

$$\leq b^*\|L_2\|\, e^{a^*([\beta]+jm)}\|\varphi_\beta\|,$$

from (5.15) we obtain

$$\rho(T_\beta) \leq \lim_{j \to \infty}(b^* e^{a^*jm} + b^*\|L_2\|\, e^{a^*([\beta]+jm)})^{1/j} = e^{a^*m} < 1.$$

Conversely, we assume that (5.10) holds for some $k \geq 1$. Then, by (5.15), we have

$$\|x_{kjm}(\cdot, 0)\| \le \|T_\beta^{kj}\| \le \varepsilon^j \tag{5.16}$$

for some $\varepsilon \in (0, 1)$, some integer $J \ge 1$ and all $j \ge J$. Since the solution of (1.2) satisfies

$$x_t(\varphi_\beta, 0) = x_{t-kjm}(x_{kjm}(\varphi_\beta, 0), v_{kj}(\varphi_\beta, 0), 0) = x_{t-kjm}(T_\beta^j \varphi_\beta, 0)$$

for $t \ge jkm$, Theorem 4.4 and (5.16) imply (5.1) with $a^* < 0$. Thus (1.2) is UAS.

If (5.11) holds for some $k \ge 1$, then there is a sequence $\{\varphi_j: \varphi_j \in C_\beta, \|\varphi_j\| = 1, j = 1, 2, \cdots\}$ such that

$$\lim_{j \to \infty} \{\|x_{kjm}(\varphi_j, 0)\| + |v_{kj}(\varphi_j, 0)|\} = \infty. \tag{5.17}$$

Since the stability of (1.2) implies (5.4) which, together with the definition of $v_j$, yields the boundedness of

$$\{\|x_{kjm}(\cdot, 0)\| + \|v_{kj}(\cdot, 0)\|: j = 1, 2, \cdots\},$$

(5.17) shows that (1.2) is unstable. #

**Remark.** Since further investigation on the spectrum of $T_\beta^k$ is beyond the scope of this thesis, we only give Theorem 5.5 for the general case of (1.2). For some special cases of (1.2), however, some further results can be easily obtained. In § 5.3 we shall deal with one of the special cases when the functional differential equation (1.3) degenerates to an ordinary differential equation.

## § 5.3 THE CHARACTERISTIC EQUATION

The equation considered here has the form

$$\frac{d}{dt}x(t) = A\,x(t) + L_2(x_{[t+\beta]}) \tag{5.18}$$

and the equation corresponding to (1.3) for this special case of (1.2) is

$$\frac{d}{dt}x(t) = A\,x(t), \tag{5.19}$$

where $A$ is an $N \times N$ constant matrix.

**THEOREM 5.6** There is an integer $k \geq 1$ and a matrix $\Psi$ such that the spectrum of $T_\beta{}^k$ for (5.18) satisfies

$$\sigma(T_\beta{}^k) = \{0\} \cup \{\lambda:\ \lambda \in \sigma(\Psi)\}. \tag{5.20}$$

Moreover, $\sigma(\Psi)$ determines the stability of (5.18). In other words, we have

(A) (5.18) is UAS if and only if $|\lambda| < 1$ for $\lambda \in \sigma(\Psi)$;

(B) (5.18) is stable if and only if $|\lambda| \leq 1$ for $\lambda \in \sigma(\Psi)$ and, for $|\lambda| = 1$, the multiplicity of $\lambda$ as an eigenvalue is equal to its number of independent eigenvectors.

**Proof.** Since the fundamental matrix solution of (5.19) is $e^{At}$, by viewing (5.19) as (1.3), we have

$$T(t)\varphi_0(\theta) = e^{A(t+\theta)}\varphi_0(0),\ \theta \in [-r,\ 0]$$

for any $\varphi_0 \in C$ and $t \geq r$. Clearly, the range of $T(t)$ in $C$ is finite dimensional as long as $t \geq r$. Thus, for any $\beta \geq 0$, there is an integer $k \geq 1$ such that the range of $T(k)$ when $\beta \in [0,\ 1]$, or of $T(km)$ when $\beta \in (m, m+1]$ for some $m \geq 1$, is finite dimensional in $C$. We note that the matrix solution of (1.8) is

$$Y(t) = e^{At} \text{ for } t \geq 0;\ Y(t) = 0 \text{ for } t < 0.$$

Then, by the definition of $T_\beta$ and the representations (3.12) and (3.27), the range of $T_\beta{}^k$ in $C_\beta$ is finite dimensional. Thus $T_\beta{}^k$ is compact. Since $C_\beta$ is infinite dimensional, $\sigma(T_\beta{}^k)$ consists of zero and eigenvalues.

For any $\lambda \in \sigma(T_\beta{}^k)$ with $\lambda \neq 0$, the corresponding eigenvectors of $T_\beta{}^k$ must be in $T_\beta{}^k C_\beta$. Since $T_\beta{}^k C_\beta$ is finite dimensional, by choosing a basis of $T_\beta{}^k C_\beta$ denoted by $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_q)$, $\alpha_i \in T_\beta{}^k C_\beta$, $1 \leq i \leq q$, we obtain a matrix $\Psi$ such that

$$T_\beta{}^k\alpha = \alpha\Psi \text{ and } T_\beta{}^k\varphi_\beta = \alpha\Psi c$$

for $\varphi_\beta = \alpha c \in T_\beta{}^k C_\beta$. If $\varphi_\beta$ is an eigenvector of $T_\beta{}^k$ corresponding to $\lambda$, then it follows that

$$\alpha\Psi c = T_\beta{}^k\varphi_\beta = \lambda\varphi_\beta = \lambda\alpha c,$$

which is equivalent to

$$\alpha(\Psi - \lambda I)c = 0.$$

Since $\alpha_1, \alpha_2, \cdots, \alpha_q$ are linearly independent vectors in $C_\beta$, the above equation is also equivalent to

$$(\Psi - \lambda I)c = 0 \text{ or } \det(\Psi - \lambda I) = 0. \tag{5.21}$$

Equation (5.21) means that $\lambda \in \sigma(\Psi)$. By the equivalence of the above equations, we also have $\sigma(\Psi) \subseteq \sigma(T_\beta{}^k)$. Thus (5.20) holds.

By (5.20) the spectral radius of $T_\beta{}^k$ is given by

$$\rho(T_\beta{}^k) = \rho(\Psi) = \max\{\,|\lambda|: \lambda \in \sigma(\Psi)\}.$$

Then the conclusion (A) follows from Theorem 5.5.

In order to prove the conclusion (B), we first show that (5.18) is stable if and only if $\{T_\beta{}^{kj}\alpha: j \geq 0\}$ is bounded. The "only if" part follows immediately from Remark (i) in § 5.1 and the definition of $T_\beta$. Suppose then that $\{T_\beta{}^{kj}\alpha: j \geq 0\}$ is bounded. By Theorems 4.2 and 4.4, there is an $M \geq 1$ such that each solution of (5.18) satisfies

$$\|x_t(\varphi_\beta, 0)\| \leq M \tag{5.22}$$

for $\varphi_\beta \in C_\beta$ with $\|\varphi_\beta\| = 1$ and $t \in [0, k([\beta] + 1)]$. Thus

$$\{T_\beta{}^k\varphi_\beta: \varphi_\beta \in C_\beta, \|\varphi_\beta\| = 1\}$$

is bounded in $T_\beta{}^k C_\beta$. Since $\alpha$ is a basis of $T_\beta{}^k C_\beta$ and $\{T_\beta{}^{kj}\alpha: j \geq 0\}$ is bounded, then

$$\{T_\beta{}^{kj}\varphi_\beta: \varphi_\beta \in C_\beta, \|\varphi_\beta\| = 1, j = 1, 2, \cdots\}$$

is also bounded. From the definition of $T_\beta$,

$$\{\|x_{jk}(\varphi_\beta, 0)\|: \ \varphi_\beta \in C_\beta, \ \|\varphi_\beta\| = 1, \ j = 1, 2, \cdots\} \ (\beta \in [0, \ 1])$$

or

$$\{\|x_{jkm}(\varphi_\beta, 0)\| + |v_{jk}(\varphi_\beta, 0)|: \ \varphi_\beta \in C_\beta, \|\varphi_\beta\| = 1, j = 1, 2, \cdots\} \ (\beta \in (m, m+1], m \geq 1)$$

is bounded. Hence, from the proof of Theorem 5.5, (5.22) holds for all $t \geq 0$ and $\varphi_\beta \in C_\beta$ with $\|\varphi_\beta\| = 1$, i.e., solutions of (5.18) are uniformly bounded. By Remark (i) in § 5.1, (5.18) is stable.

Since $T_\beta^{kj} \alpha = \alpha \Psi^j$ for $j \geq 1$, the boundedness of $\{T_\beta^{kj} \alpha: j \geq 0\}$ is equivalent to that of $\{\Psi^j\}$, which is also equivalent to the condition on $\sigma(\Psi)$ as stated in (B). Thus (B) holds. #

Theorem 5.6 indicates that the behaviour of solutions of (5.18) is completely determined by the eigenvalues of the matrix $\Psi$. Therefore, we can call $\Psi$, (5.21) and its solutions the characteristic matrix, characteristic equation and eigenvalues, respectively, of the equation (5.18).

To analyse the asymptotic behaviour for a given equation, we only need to determine an integer $k \geq 1$, the matrix $\Psi$ for $T_\beta^k$ and its eigenvalues. Some examples will now be given to show the application of the theorems in the next section.

## § 5.4 EXAMPLES

We give three examples in this section. The first one deals with a differential difference equation which was studied by Wiener and Cooke [37] and also in our Example 2.1. This shows that our general theory covers some previous work in this direction. The second example verifies the remark after Theorem 5.2. While the first two examples involve degenerated equations in which $r = 0$, i.e., the space C degenerates to $\mathbb{C}^N$, the third example deals with a differential integral equation with $r > 0$.

**Example** 5.1 Consider the equation

$$\frac{d}{dt}x(t) = A\,x(t) + B\,x([t + 1/2]) \tag{2.20}$$

with $x(0) = c_0$. In [37] the following result is given:

Assume that $A^{-1}$ exists and let

$$M_0 = M(-1/2)^{-1}M(1/2),\ M(\pm 1/2) = e^{\pm(1/2)A} + (e^{\pm(1/2)A} - I)A^{-1}B.$$

Then (2.20) is UAS if and only if the eigenvalues of $M_0$ satisfy $|\lambda_j| < 1$, $j = 1, 2, \cdots, N$.

Recall that the sufficient condition

$$\det\left(B\int_0^{1/2} e^{As}\,ds - I\right) \neq 0$$

for (2.20) with $x(0) = c_0$ to have a unique solution on $[0, \infty)$, obtained in Example 2.1, is different from the condition

$$\det A \neq 0 \text{ and } \det M(-1/2) \neq 0$$

given in [37]. We will derive from Theorem 5.6 a necessary and sufficient condition for (2.20) to be UAS, which also differs slightly from that in [37].

Here $\beta = 1/2$ and $L_2(\varphi) = B\varphi(0)$ for $\varphi \in C$. Thus we have $r = 0$. The solution operator $T(t)$ of (5.19) is compact and satisfies

$$T(t)\varphi_0 = e^{At}\varphi(0)$$

for any $t \geq 0$ and $\varphi_0 \in C = \mathbb{C}^N$. Hence we take $k = 1$. By the definition of $T_\beta$ and the representation (3.12),

$$T_\beta\varphi_0 = x_1(\varphi_0, 0) = x(\varphi_0, 0)(1)$$

$$= T(1)\varphi_0(0) + \int_0^{1/2} Y(1 - s)\,ds L_2(\varphi_0) + \int_{1/2}^{1} Y(1 - s)ds\,\widehat{F}_1(\varphi_0)$$

$$= e^{A}\varphi_0(0) + \int_0^{1/2} e^{A(1-s)}\, ds B\varphi_0(0) + \int_{1/2}^{1} e^{A(1-s)}\, ds\, \widehat{F}_1(\varphi_0).$$

From the definitions in § 3.2 it follows that

$$\widehat{F}_1(\varphi_0) = -(B_0)^{-1}F_1(\varphi_0)$$

$$= -\left\{L_2\left(\int_0^1 Y_{\beta-s}\, ds\right) - I\right\}^{-1}\left\{L_2(T(1)\varphi_0) + L_2\left(\int_0^{1-\beta} Y_{1-s}\, ds\right)L_2(\varphi_0)\right\}$$

$$= -\left\{B\left(\int_0^{1/2} e^{A(1/2-s)}\, ds\right) - I\right\}^{-1}\left\{Be^{A}\varphi_0(0) + B\left(\int_0^{1/2} e^{A(1-s)}\, ds\right)B\varphi_0(0)\right\}$$

$$=_{def}\ B^*\varphi_0(0).$$

Thus, by substitution, we have

$$T_\beta\varphi_0 = \left\{e^{A} + \int_0^{1/2} e^{A(1-s)}\, ds\, B + \int_{1/2}^{1} e^{A(1-s)}\, ds\, B^*\right\}\varphi_0(0)$$

$$= \left\{I - \int_0^{1/2} e^{As}\, ds\left(B\int_0^{1/2} e^{As}\, ds - I\right)^{-1} B\right\}e^{(1/2)A}\left\{e^{(1/2)A} + \left(\int_0^{1/2} e^{As}\, ds\right)B\right\}\varphi_0(0)$$

$$=_{def}\ \Psi\varphi_0(0).$$

By Theorem 5.6, (2.20) is UAS if and only if the eigenvalues of $\Psi$ satisfy $|\lambda_j| < 1$, $j = 1, 2, \cdots, N$. This condition is different from that in [37]. However, if either $\{4k\pi i\colon k = 0, 1, 2, \cdots\}$ does not contain any eigenvalue of $A$ or $\det A \neq 0$ and $AB = BA$, we have $\Psi = M_0$, i.e., our condition agrees with that in [37].

**Example** 5.2 Consider the scalar equation

$$\frac{d}{dt}x(t) = -x(t) + \varepsilon x([t+2]), \tag{5.23}$$

where $\varepsilon \neq 0$ is a constant. Here $\beta = 2$ and $L_2(\varphi) = \varepsilon\varphi(0)$ for $\varphi \in C$. Thus we have $m = 1$, $r = 0$ and $C = \mathbb{C}$. As in Example 5.1, we take $k = 1$ and calculate $\Psi$ for $T_\beta$.

From the definition of $T_\beta$ we have

$$T_\beta \varphi_\beta = (x_1(\varphi_\beta, 0),\ v_1(\varphi_\beta, 0))$$

for $\varphi_\beta = (\varphi_0, v_0) \in C_\beta = C \times \mathbb{C} = \mathbb{C}^2$. Then it follows from (3.27) that

$$x_1(\varphi_\beta, 0) = x(\varphi_0, v_0, 0)(1) = T(1)\varphi_0(0) + E_0(1)v_0$$

$$= e^{-1}\varphi_0(0) + \int_0^1 e^{s-1}\,ds\, v_0 = (e^{-1}, 1 - e^{-1})\varphi_\beta^T.$$

Since $v_1(\varphi_\beta, 0)$ satisfies (3.25) with $f = 0$, from (3.25), (3.24), (3.22), (3.20) and (3.16)-(3.18), we obtain

$$v_1(\varphi_\beta, 0) = -(\Phi_0)^{-1}\{U_1(\varphi_0) + \Phi_1' v_0\} = -(P_0)^{-1}\{u_m(\varphi_0) + P_m v_0\}$$

$$= -\left(L_2\left(\int_0^1 Y_{\beta-m-s}\,ds\right)\right)^{-1}\left\{L_2\left(T(m + [\beta - m])\varphi_0\right) + \left(L_2\left(\int_0^1 Y_{\beta-s}\,ds\right) - I\right)v_0\right\}$$

$$= -\left(\varepsilon\int_0^1 Y(1-s)\,ds\right)^{-1}\left\{\varepsilon T(2)\varphi_0(0) + \left(\varepsilon\int_0^1 Y(2-s)\,ds - I\right)v_0\right\}$$

$$= -\left(\varepsilon\int_0^1 e^{s-1}\,ds\right)^{-1}\left\{\varepsilon e^{-2}\varphi_0(0) + \left(\varepsilon\int_0^1 e^{s-2}\,ds - 1\right)v_0\right\}$$

$$= \left(\frac{1}{e(1-e)}, \frac{e}{\varepsilon(e-1)} - \frac{1}{e}\right)\varphi_\beta^T.$$

Hence, by representing $T_\beta\varphi_\beta$ as $\varphi_\beta\Psi$, we have

$$\Psi = \begin{bmatrix} e^{-1} & e^{-1}(1-e)^{-1} \\ (e-1)e^{-1} & \varepsilon^{-1}(e-1)^{-1}e - e^{-1} \end{bmatrix}.$$

Since $\Psi$ has the eigenvalues

$$\lambda_{1,2} = \frac{1}{2\varepsilon}\left\{\frac{e}{e-1} \pm \sqrt{\left(\frac{e}{e-1}\right)^2 - \frac{4\varepsilon}{e-1}}\right\},$$

there is a $\delta > 0$ such that

$$\rho(\Psi) = \max\{|\lambda_1|, |\lambda_2|\} > 1$$

for $0 < |\varepsilon| \le \delta$. By Theorem 5.6, (5.23) is unstable if $\varepsilon$ satisfies $0 < |\varepsilon| \le \delta$. This example shows that (5.23) is always unstable for sufficiently small $|\varepsilon| \neq 0$ although the equation $x'(t) = - x(t)$ is UAS.

**Example** 5.3 Consider the equation

$$\frac{d}{dt}x(t) = Ax(t) + \int_{-1}^{0} H(\theta)x([t+2]+\theta)\, d\theta, \qquad (5.24)$$

where $A$ is a nonsingular $N \times N$ matrix. In this example, $\beta = 2$, $r = 1$ and

$$L_2(\varphi) = \int_{-1}^{0} H(\theta)\varphi(\theta)\, d\theta$$

for $\varphi \in C$. We assume that

$$\det(e^A - I) \neq 0 \text{ and } \det \int_{-1}^{0} H(\theta)\left(e^{A(1+\theta)} - I\right) d\theta \neq 0 .$$

Then, by taking $k = 1$ ($\ge r$), we claim that the eigenvalues of (5.24) are the solutions of the equation

$$\det\left\{\lambda^2 \int_{-1}^{0} H(\theta)\left(e^{A(1+\theta)} - I\right) d\theta - \lambda \int_{-1}^{0} H(\theta)\left(e^{A\theta} - I\right)e^{A} d\theta - \lambda A + Ae^{A}\right\} = 0. \quad (5.25)$$

In fact, by the same formulae as in Example 5.2, we have

$$T_\beta \varphi_\beta = (x_1(\varphi_\beta, 0),\ v_1(\varphi_\beta, 0)),$$

$$x_1(\varphi_\beta, 0)(\theta) = T(1)\varphi_0(0) + \int_0^1 Y(1+\theta-s)\, ds\, v_0$$

$$= e^{A(1+\theta)}\varphi_0(0) + \int_0^{1+\theta} e^{A(1+\theta-s)}\, ds\, v_0$$

$$= e^{A(1+\theta)}\varphi_0(0) + \left(e^{A(1+\theta)} - I\right)A^{-1} v_0$$

for $\theta \in [-1, 0]$, and

$$
\begin{aligned}
v_1(\varphi_\beta, 0) &= -\left(L_2\left(\int_0^1 Y_{\beta - m - s}\, ds\right)\right)^{-1}\left\{L_2(T(\beta)\varphi_0) + \left(L_2\left(\int_0^1 Y_{\beta - s}\, ds\right) - I\right) v_0\right\} \\
&= -\left(\int_{-1}^0 H(\theta)\int_0^{1+\theta} e^{A(1+\theta-s)} ds\, d\theta\right)^{-1} \times \\
&\quad \left\{\int_{-1}^0 H(\theta)e^{A(2+\theta)} d\theta\, \varphi_0(0) + \left(\int_{-1}^0 H(\theta)\int_0^1 e^{A(2+\theta-s)} ds\, d\theta - I\right) v_0\right\} \\
&= -A\left(\int_{-1}^0 H(\theta)\left(e^{A(1+\theta)} - I\right) d\theta\right)^{-1} \times \\
&\quad \left\{\int_{-1}^0 H(\theta)e^{A(2+\theta)} d\theta\, \varphi_0(0) + \left(\int_{-1}^0 H(\theta)e^{A(1+\theta)} d\theta\left(e^A - I\right)A^{-1} - I\right) v_0\right\} \\
&=_{def}\ A_1\varphi_0(0) + A_2 v_0.
\end{aligned}
$$

We denote the ith column of a matrix $M$ by $col_i(M)$ and let

$$\gamma_{1,i} = (col_i(e^{A(1+\theta)}), col_i(A_1)) \text{ and } \gamma_{2,i} = (col_i((e^{A(1+\theta)} - I)A^{-1}), col_i(A_2))$$

for $i = 1, 2, \cdots, N$. Then, if we view

$$\alpha = (\gamma_{1,1}, \gamma_{1,2}, \cdots, \gamma_{1,N}, \gamma_{2,1}, \gamma_{2,2}, \cdots, \gamma_{2,N})$$

as a basis of $T_\beta C_\beta$, we obtain

$$T_\beta\gamma_{1,i} = (e^{A(1+\theta)}col_i(e^A) + (e^{A(1+\theta)} - I)A^{-1}col_i(A_1),\ A_1col_i(e^A) + A_2col_i(A_1))$$

and

$$
\begin{aligned}
T_\beta\gamma_{2,i} = (e^{A(1+\theta)}&col_i((e^A - I)A^{-1}) + (e^{A(1+\theta)} - I)A^{-1}col_i(A_2), \\
&A_1col_i((e^A - I)A^{-1}) + A_2col_i(A_2))
\end{aligned}
$$

for $i = 1, 2, \cdots, N$. That is,

$$T_\beta\gamma_{1,i} = \alpha\begin{bmatrix} col_i(e^A) \\ col_i(A_1) \end{bmatrix} \text{ and } T_\beta\gamma_{2,i} = \alpha\begin{bmatrix} col_i((e^A - I)A^{-1}) \\ col_i(A_2) \end{bmatrix}$$

for $i = 1, 2, \cdots, N$. Thus $T_\beta \alpha = \alpha\Psi$, where

$$\Psi = \begin{bmatrix} e^A & (e^A - I)A^{-1} \\ A_1 & A_2 \end{bmatrix}.$$

Since $\det (e^A - I)A^{-1} \neq 0$, by putting

$$\Psi_1(\lambda) = \begin{bmatrix} e^A - \lambda I & I \\ A_1 & (A_2 - \lambda I)A(e^A - I)^{-1} \end{bmatrix}$$

and

$$\Psi_0 = \begin{bmatrix} I & 0 \\ 0 & (e^A - I)A^{-1} \end{bmatrix},$$

we obtain $\Psi - \lambda I = \Psi_1(\lambda)\Psi_0$. Since $\det \Psi_0 \neq 0$, the equation $\det (\Psi - \lambda I) = 0$ is equivalent to $\det \Psi_1(\lambda) = 0$. By changing the first $N$ rows in $\det \Psi_1(\lambda)$ to $(0, I)$, we have

$$\det \Psi_1(\lambda) = (-1)^{N^2} \det\{A_1 - (A_2 - \lambda I)A(e^A - I)^{-1}(e^A - \lambda I)\}.$$

From the definitions of $A_1$ and $A_2$,

$$A_1 - (A_2 - \lambda I)A(e^A - I)^{-1}(e^A - \lambda I)$$

$$= -A\left(\int_{-1}^{0} H(\theta)\left(e^{A(1+\theta)} - I\right)d\theta\right)^{-1} \times$$

$$\left\{\lambda^2\int_{-1}^{0} H(\theta)\left(e^{A(1+\theta)} - I\right)d\theta - \lambda\int_{-1}^{0} H(\theta)\left(e^{A\theta} - I\right)d\theta - \lambda A + Ae^A\right\}\left(e^A - I\right)^{-1}.$$

Therefore, $\lambda$ satisfies $\det (\Psi - \lambda I) = 0$ if and only if $\lambda$ satisfies (5.25).

**Remark**. For some special equations with $r = 0$, it is possible to use specific methods which may be simpler than our general method.

# PART TWO
# LINEAR DIFFERENTIAL EQUATIONS OF MIXED TYPE

In this part, we mainly discuss existence, as well as uniqueness, of solutions with given initial data, and some simple properties of solutions such as exponential estimates, boundedness, and finite limits in $\mathbb{C}^N$.

This part is composed of two chapters. The first one, Chapter 6, covers the discussion of some different ways of giving initial data. In Chapter 7, however, we concentrate on solutions on $[t_0 - r, \infty)$ with the initial condition $x_{t_0} = \varphi_0$ for $\varphi_0 \in C$. Conditions for uniqueness will be emphasized and some simple properties discussed. Chapter 7 also serve as preparation for Part III in which we are going to study asymptotic behaviour of solutions.

# CHAPTER 6 INITIAL VALUE PROBLEMS

The equations to be discussed in Chapters 6 and 7 have the forms

$$\frac{d}{dt}\{x(t) - D(t, x_t)\} = L_1(t, x_t) + L_2(t, x^t) + f(t) \tag{6.1}$$

and

$$\frac{d}{dt}\{x(t) - D(t, x_t)\} = L_1(t, x_t) + L_2(t, x^t), \tag{6.2}$$

where, for each $t \geq \tau$ and some constants $r > 0$ and $\sigma > 0$, $D(t, .)$ and $L_1(t, .)$ are bounded linear operators from $C(-r) \equiv C([-r, 0], \mathbb{C}^N)$ to $\mathbb{C}^N$, and $L_2(t, .)$ is a bounded linear operator from $C(\sigma) \equiv C([0, \sigma], \mathbb{C}^N)$ to $\mathbb{C}^N$. For each $t \geq \tau$ and $\varphi \in C(-r)$, $D(t, \varphi)$ is represented by

$$D(t, \varphi) = \int_{-r}^{0} \{d_\theta \xi(t, \theta)\}\varphi(\theta), \tag{6.3}$$

where $\xi(t, .)$ is of bounded variation on $[-r, 0]$ and satisfies

$$\lim_{\theta \to 0^-} \xi(t, \theta) = 0 = \xi(t, 0). \tag{6.4}$$

We assume that the operator $D(t, .)$ is continuous in $t$ in the sense that

$$\lim_{t \to t_0} \|D(t, .) - D(t_0, .)\| = 0 \tag{6.5}$$

for any $t_0 \geq \tau$. We also assume that, for each $\varphi \in C(-r)$ and $\psi \in C(\sigma)$, the functions $L_1(t, \varphi)$, $L_2(t, \psi)$ and $f(t)$ are Lebesgue measurable and that $\|L_1(t, .)\|$, $\|L_2(t, .)\|$ and $|f(t)|$ are locally integrable. As usual, $x_t \in C(-r)$ is defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$ and $x^t \in C(\sigma)$ by $x^t(\alpha) = x(t + \alpha)$ for $\alpha \in [0, \sigma]$.

For any $t_0 \geq \tau$ and any $t_1 > t_0$, we say that $x(t)$ is a solution of (6.1) on $[t_0 - r, t_1 + \sigma]$ if $x(t)$ is continuous on $[t_0 - r, t_1 + \sigma]$ and $x(t) - D(t, x_t)$ is locally absolutely continuous, and (6.1) holds almost everywhere, on $[t_0, t_1]$. If $t_1 = \infty$, then $[t_0 - r, t_1 + \sigma]$ and $[t_0, t_1]$ are replaced by $[t_0 - r, \infty)$ and $[t_0, \infty)$, respectively.

## § 6.1 INTRODUCTORY REMARKS

Before we start dealing with (6.1), let us recall some facts about neutral equations. For a neutral functional differential equation, not necessarily linear, we need to specify an initial function $\varphi_0 \in C(-r)$ and to find a solution $x(t)$ on $[t_0 - r, t_0 + \alpha]$, for some $\alpha > 0$, such that $x_{t_0} = \varphi_0$. Under general assumptions on the equation, such a solution uniquely exists provided that $\alpha$ is small enough. Then, by viewing $x$ on $[t_0 + \alpha - r, t_0 + \alpha]$ as the initial data and applying the same idea as above, we can extend the unique solution with $x_{t_0} = \varphi_0$ to a larger interval $[t_0 - r, t_0 + \alpha']$. If the equation is linear, the unique solution with $x_{t_0} = \varphi_0$ on a finite interval can be extended to $[t_0 - r, \infty)$. Since $C(-r)$ degenerates to $\mathbb{C}^N$ when $r = 0$, ordinary differential equations can be viewed as degenerate neutral equations. Thus the initial value problem for neutral equations, as well as the method of finding its solution theoretically, is a natural generalization of that for ordinary differential equations.

For functional differential equations of mixed type, although there have appeared some results in a variety of specific contexts (see the relevant references quoted in § 0.2), very few have dealt with the initial value problem in general. It is also natural to wonder, for this problem, if we can generalize the ideas from neutral equations to mixed equations.

The main purpose of this chapter is to discuss some different ways of posing the initial value problem for mixed equation (6.1)

## § 6.2 SOLUTIONS ON A FINITE INTERVAL

In this section, we consider the initial value problem for (6.1) on a finite interval $[t_0, t_1]$, for any $t_0 \geq \tau$ and any $t_1 > t_0$. If a function $x(t)$ satisfies (6.1) a.e. for $t \in [t_0, t_1]$, then the equation itself implies that the values of $x(t)$ on $[t_0 - r, t_1 + \sigma]$, rather than on $[t_0, t_1]$, are involved. It is reasonable to guess that such a solution on $[t_0 - r, t_1 + \sigma]$ is in some way determined by its values on $[t_0 - r, t_0]$ and $[t_1, t_1 + \sigma]$. Thus we pose the initial value problem for (6.1) as follows.

**IVP1**. For any given $\varphi_0 \in C(-r)$ and $\psi_0 \in C_0(\sigma)$, where

$$C_0(\sigma) = \{\psi \in C(\sigma):\ \psi(0) = 0\},$$

we look for a solution $x(t)$ of (6.1) on $[t_0 - r, t_1 + \sigma]$ satisfying

$$x_{t_0} = \varphi_0 \text{ and } x^{t_1} = \psi_0 + x(t_1). \tag{6.6}$$

First of all, we need to investigate existence and uniqueness of the solution to IVP1.

**THEOREM 6.1** For any $t_0 \ge \tau$ and any $t_1 > t_0$, (6.1) with (6.6) possesses a unique solution on $[t_0 - r, t_1 + \sigma]$ if $t_1 - t_0$ is small enough.

**Proof.** It is clear that (6.1) with (6.6) is equivalent to the integral equation

$$x(t) = \varphi_0(0) - D(t_0, \varphi_0) + D(t, x_t)$$

$$+ \int_{t_0}^{t} \{L_1(s, x_s) + L_2(s, x^s) + f(s)\}\, ds\,,\ t \in [t_0, t_1], \tag{6.7}$$

with (6.6). Let $\varphi^*(t)$, $\psi^*(t)$ and $x^*(t)$ be defined by

$$(\varphi^*)_{t_0} = \varphi_0,\ \varphi^*(t) = \varphi_0(0) \text{ for } t \in [t_0, t_1 + \sigma],$$

$$(\psi^*)^{t_1} = \psi_0,\ \psi^*(t) = \psi_0(0) = 0 \text{ for } t \in [t_0 - r, t_1],$$

and

$$(x^*)_{t_0} = 0,\ (x^*)^{t_1} = x^*(t_1),\ x^*(t) = x(t) - \varphi_0(0) \text{ for } t \in [t_0, t_1].$$

Then $x(t)$ is a solution of (6.7) with (6.6) on $[t_0 - r, t_1 + \sigma]$ if and only if $x^*(t)$ satisfies the equation

$$x^*(t) = h(t) + D(t, (x^*)_t) + \int_{t_0}^{t} \{L_1(s, (x^*)_s) + L_2(s, (x^*)^s)\}\, ds\,, \quad t \in [t_0, t_1], \tag{6.8}$$

with $x^*(t_0) = 0$, where

$$h(t) = D(t, (\varphi^*)_t) - D(t_0, \varphi_0) + \int_{t_0}^{t} \{L_1(s, (\varphi^*)_s) + L_2(s, (\psi^*)^s + \varphi_0(0)) + f(s)\}\, ds\,.$$

Since $x^*(t)$ is required to be continuous on $[t_0 - r, t_1 + \sigma]$ and constant on $[t_0 - r, t_0]$ and $[t_1, t_1 + \sigma]$, it is regarded as a function on $[t_0, t_1]$. For each $t$, we define $\|D(t, .)\|_s$ by

$$\|D(t, .)\|_s = \mathrm{Var}_{[-s, 0]}(\xi(t, .)) \quad (s \geq 0)$$

as in Remark II in § 2.3 (p.24). Then

$$|D(t, (x^*)_t)| \leq \|D(t, .)\|_{t - t_0} \|(x^*)_t\|$$

for $t \in [t_0, t_1]$ as $x^*(t) = 0$ for $t \leq t_0$. Since $\|D(t, .)\|_s$ is nondecreasing in $s$ and $\|D(t, .)\|_s = \|D(t, .)\|_r = \|D(t, .)\|$ for $s \geq r$, we have

$$\|D(t, .)\|_{t - t_0} \leq \|D(t, .) - D(t_0, .)\| + \|D(t_0, .)\|_{t - t_0}.$$

From (6.3)-(6.5) it follows that

$$\lim_{t \to t_0^+} \|D(t, .)\|_{t - t_0} = 0.$$

Therefore, when $t_1 - t_0$ is small enough, we have

$$\|D(t, .)\|_{t - t_0} + \int_{t_0}^{t} \{\|L_1(s, .)\| + \|L_2(s, .)\|\}\, ds \leq \delta < 1 \tag{6.9}$$

for $t \in [t_0, t_1]$ and some constant $\delta > 0$. Let $\aleph \subset C([t_0 - r, t_1 + \sigma], \mathbb{C}^N)$ such that each $u$ in $\aleph$ satisfies $u_{t_0}(\theta) = 0$ for $\theta \in [-r, 0]$ and $u^{t_1}(\alpha) = u(t_1)$ for $\alpha \in [0, \sigma]$. Then $\aleph$ with the usual norm in $C$ is a Banach space. Defining a map on $\aleph$ by (6.8) and applying the contraction mapping principle, we see that (6.8) has exactly one solution $x^*(t)$ in $\aleph$, and so (6.1) with (6.6) possesses a unique solution $x(t)$ on $[t_0 - r, t_1 + \sigma]$, if (6.9) holds. #

**Remarks.** (i). Theorem 6.1 indicates that, when $t_1 - t_0$ is small enough, a solution of (6.1) on $[t_0 - r, t_1 + \sigma]$ is uniquely determined by its values on $[t_0 - r, t_0]$ and $[t_1, t_1 + \sigma]$ in the sense of (6.6), but we can not specify

$$x_{t_0} = \varphi_0 \text{ and } x^{t_1} = \psi \tag{6.10}$$

for any $\varphi_0 \in C(-r)$ and any $\psi \in C(\sigma)$. Indeed, for any given $\varphi_0 \in C(-r)$ and $\psi_0 \in C_0(\sigma)$, let $\alpha_0 = x(t_1)$ and $\psi = \alpha + \psi_0$, where $x(t)$ is the unique solution of (6.1) with (6.6) on $[t_0 - r, t_1 + \sigma]$. Then, for $\alpha = \alpha_0$, $x(t)$ is also a solution of (6.1) with (6.10) on $[t_0 - r, t_1 + \sigma]$. For any $\alpha \in \mathbb{C}^N$ and $\alpha \neq \alpha_0$, if (6.1) with (6.10) has a solution $\underline{x}(t)$ on $[t_0 - r, t_1 + \sigma]$, then $\underline{x}_{t_0} = \varphi_0$, $\underline{x}(t_1) = \psi(0) = \alpha$, and $\underline{x}^{t_1} = \psi = \alpha + \psi_0 = \underline{x}(t_1) + \psi_0$. Thus $\underline{x}(t)$ is also a solution of (6.1) with (6.6) on $[t_0 - r, t_1 + \sigma]$. By the uniqueness of the solution of (6.1) with (6.6) on $[t_0 - r, t_1 + \sigma]$, we have $\underline{x}(t) \equiv x(t)$ for $t \in [t_0 - r, t_1 + \sigma]$, which implies $\alpha = \underline{x}(t_1) = x(t_1) = \alpha_0$. This contradicts the assumption $\alpha \neq \alpha_0$. Therefore, if $\alpha \neq \alpha_0$, (6.1) with (6.10) has no solution at all on $[t_0 - r, t_1 + \sigma]$.

(ii). In Theorem 6.1, the condition that $t_1 - t_0$ is small enough is essential. Indeed, for large $t_1 - t_0$, (6.1) with (6.6) may have no solution at all. Even if there is a solution, uniqueness may not hold. This is illustrated by the example below.

Consider the equation

$$\frac{d}{dt}x(t) = x(t-1) + x(t+1) \tag{6.11}$$

for $t \in [t_0, t_1] = [0, T]$ with $T \leq 1$, together with (6.6), where $r = \sigma = 1$. Clearly, $x(t)$ is a solution of (6.11) with (6.6) on $[-1, T+1]$ if and only if $x(t)$ satisfies

$$x(t) = \varphi_0(0) + x(T)\,t + \int_0^t (\varphi_0(s-1) + \psi_0(s+1-T))\,ds \tag{6.12}$$

for $t \in [0, T]$. Then (6.12) has a solution on $[0, T]$ if and only if

$$(1 - T)x(T) = \varphi_0(0) + \int_0^T (\varphi_0(s - 1) + \psi_0(s + 1 - T))\,ds \tag{6.13}$$

has a solution $x(T)$ in $\mathbb{C}^N$. For $T \in (0, 1)$, since (6.13) possesses a unique solution $x(T)$, (6.12) on $[0, T]$, and thus (6.11) with (6.6) on $[-1, T + 1]$, possesses a unique solution. If $T = 1$, however, either (6.13) has infinitely many solutions or it has no solution at all according as the relation

$$\varphi_0(0) + \int_0^1 (\varphi_0(s - 1) + \psi_0(s))\,ds = 0 \tag{6.14}$$

holds or not. Thus, if $T = 1$, (6.11) with (6.6) has either infinitely many solutions, or no solution at all, on $[-1, 2]$ depending on whether (6.14) holds or not.

(iii). Let $t_1 > t_0$ and $t_2 > t_1$. Suppose that $x_1(t)$ is a solution of (6.1) on $[t_0 - r, t_1 + \sigma]$ and $x_2(t)$ a solution on $[t_1 - r, t_2 + \sigma]$ satisfying $(x_2)_{t_1} = (x_1)_{t_1}$. If $x_3(t)$, defined by $x_3(t) = x_1(t)$ on $[t_0 - r, t_1]$ and by $x_3(t) = x_2(t)$ on $[t_1, t_2 + \sigma]$, is a solution of (6.1) on $[t_0 - r, t_2 + \sigma]$, we say that $x_3(t)$ is an extension of the solution $x_1(t)$ from $[t_0 - r, t_1 + \sigma]$ to $[t_0 - r, t_2 + \sigma]$.

If $x_2(t) = x_1(t)$ also holds for $t \in [t_1, t_1 + \sigma]$, then $x_3(t)$ must be an extension of $x_1(t)$ as $x_3(t)$ is continuous on $[t_0 - r, t_2 + \sigma]$ and satisfies (6.1) for $t \in [t_0, t_2]$. Otherwise, $x_3(t)$ may not be a solution of (6.1) on $[t_0 - r, t_1 + \sigma]$, let alone a solution on $[t_0 - r, t_2 + \sigma]$.

As an example, we consider (6.11) again. Let $x_1(t)$ be a solution of (6.11) on $[-1, 3/2]$ satisfying $(x_1)_0 = \varphi_0$ and $(x_1)^{1/2} = \omega + x_1(1/2)$, and $x_2(t)$ a solution on $[-1/2, 2]$ satisfying $(x_2)_{1/2} = (x_1)_{1/2}$ and $(x_2)^1 = \psi_0 + x_2(1)$, where $\psi_0$ and $\omega$ in $C_0(1)$ satisfy

$$\omega(1/2 + \alpha) - \psi_0(\alpha) \neq \text{constant for } \alpha \in [0, 1/2]. \tag{6.15}$$

If $x_3(t)$ is a solution of (6.11) on $[-1, 3/2]$, then, from the definition of $x_3(t)$ and (6.11), it follows that $x_2(t) = x_1(t)$ for $t \in [1, 3/2]$. On $[1, 3/2]$, since $x_1(t) = x_1(1/2) + \omega(t - 1/2)$ and $x_2(t) = x_2(1) + \psi_0(t - 1)$, we have

$\omega(t - 1 + 1/2) = \psi_0(t - 1) + x_2(1) - x_1(1/2)$, which contradicts (6.15). Thus $x_3(t)$ is not a solution of (6.11) on $[-1, 3/2]$.

This example also shows that we can not use Theorem 6.1 to extend a solution of (6.1) on $[t_0 - r, t_1 + \sigma]$ to a larger interval, not to mention extending it to $[t_0 - r, \infty)$.

## § 6.3 SOLUTIONS ON A HALF LINE

Inspired by the initial value problem for neutral differential equations, we expect that a solution of (6.1) on $[t_0 - r, \infty)$, for any $t_0 \geq \tau$, is determined by its values on $[t_0 - r, t_0]$.

**IVP2**. For any $t_0 \geq \tau$ and any $\varphi_0 \in C(-r)$, we look for a solution $x(t)$ of (6.1) on $[t_0 - r, \infty)$ satisfying $x_{t_0} = \varphi_0$.

We suppose that $\|D(t, .)\|$, $\|L_1(t, .)\|$ and $\|L_2(t, .)\|$ are bounded on $[\tau, \infty)$ and let $\|D\|$, $\|L_1\|$ and $\|L_2\|$ be their suprema.

**THEOREM** 6.2 Suppose that $\|D\|$, $\|L_1\|$ and $\|L_2\|$ are small enough that

$$\|D\| + \lambda^{-1}(\|L_1\| + \|L_2\|e^{\lambda\sigma}) < 1, \tag{6.16}$$

where $\lambda$ is a particular positive number, and that $\int_\tau^t f(s)\, ds\, e^{-\lambda t}$ is bounded. Then, for every $(t_0, \varphi_0) \in [\tau, \infty) \times C(-r)$, (6.1) with $x_{t_0} = \varphi_0$ has at least one solution, and a unique solution of order $e^{\lambda t}$, on $[t_0 - r, \infty)$.

**Proof**. On $[t_0, \infty)$, the equation (6.1) with $x_{t_0} = \varphi_0$ is equivalent to

$$x(t) = \varphi_0(0) - D(t_0, \varphi_0) + D(t, x_t) + \int_{t_0}^{t} \{L_1(s, x_s) + L_2(s, x^s) + f(s)\}\, ds$$

with $x_{t_0} = \varphi_0$ and this can be written as

$$y(t) = h_1(t) + D(t, y_t) + \int_{t_0}^{t} \{L_1(s, y_s) + L_2(s, y^s)\}\, ds \tag{6.17}$$

with $y_{t_0} = 0$, where $y_t = x_t - (\varphi^*)_t$ and $\varphi^*(t) = \varphi_0(0)$ for $t \geq t_0$, $(\varphi^*)_{t_0} = \varphi_0$, and

$$h_1(t) = D(t, (\varphi^*)_t) - D(t_0, \varphi_0) + \int_{t_0}^{t} \{L_1(s, (\varphi^*)_s) + L_2(s, \varphi_0(0)) + f(s)\}\, ds.$$

We show that (6.17) with $y_{t_0} = 0$ has, on $[t_0 - r, \infty)$, a unique solution of order $e^{\lambda t}$. Let

$$S = \{u \in C([t_0 - r, \infty), \mathbb{C}^N):\ u_{t_0} = 0,\ u(t)e^{-\lambda t} \text{ is bounded}\}.$$

Then $S$ with the norm

$$\|u\|_S = \sup\{|u(t)|e^{-\lambda t}:\ t \geq t_0\}$$

is a Banach space. By the condition of the theorem, $h_1 \in S$. Defining an operator $A$ on $S$ by the right hand side of (6.17), we have

$$|Au(t)|\, e^{-\lambda t} \leq \|h_1\|_S + \left( \|D\| + \int_{t_0}^{t} \{\|L_1\| + \|L_2\| e^{\lambda\sigma}\} e^{\lambda(\rho - t)}\, d\rho \right) \|u\|_S$$

for $u \in S$ and $t \geq t_0$. Thus $Au \in S$. Then, by (6.16), the contraction mapping principle leads to the conclusion that $A$ has a unique fixed point in $S$, i.e., (6.17) has a unique solution on $[t_0 - r, \infty)$ if restricted to the space $S$. Therefore, (6.1) with $x_{t_0} = \varphi_0$ has a unique solution of order $e^{\lambda t}$ and thus at least one solution on $[t_0 - r, \infty)$. #

The method of the above proof provides us with a by-product.

**COROLLARY** 6.3 Suppose that (6.16) holds for some $\lambda > 0$. Then, for any $t_0 \geq \tau$ and any $t_1 > t_0$, no matter how large $t_1 - t_0$ is, (6.1) with (6.6) possesses a unique solution on $[t_0 - r, t_1 + \sigma]$.

**Proof**. Since (6.1) with (6.6) is equivalent to (6.8), by applying the contraction mapping principle on

$$S^* = \{u \in C([t_0 - r, t_1 + \sigma], \mathbb{C}^N): u_{t_0} = 0, u^{t_1} = u(t_1)\}$$

with a norm $\|\cdot\|_{S^*}$ defined by

$$\|u\|_{S^*} = \max\{|u(t)| e^{-\lambda t}: t \in [t_0, t_1]\},$$

we see that (6.8) has a unique solution in $S^*$. #

**Remarks**. (i). Corollary 6.3 indicates that (6.1) with (6.6) always possesses a unique solution on any finite interval $[t_0 - r, t_1 + \sigma]$ provided that $\|D\|$, $\|L_1\|$ and $\|L_2\|$ are small enough to guarantee (6.16). However, in order for (6.1) with $x_{t_0} = \varphi_0$ to have a solution on $[t_0 - r, \infty)$, Theorem 6.2 also requires $\int_\tau^t f(s)\, ds\, e^{-\lambda t}$ be bounded on $[\tau, \infty)$. It is not clear whether this is an essential condition, but at least the proof of Theorem 6.2 fails without it.

(ii). Although the smallness of $\|D(t, .)\|$, $\|L_1(t, .)\|$ and $\|L_2(t, .)\|$ may be in some different sense, such as the alternative conditions given in Chapter 7, some such condition is essential for both Theorem 6.2 and Corollary 6.3. In other words, the required solution may not exist without such a condition. The following example will illustrate this point. Let $B_0$, $B_1$, $B_2$ and $B_3$ be $3 \times 3$ matrices given by

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0.5 & 0 \\ 1 & 0 & -0.5 \\ -3 & -2 & 0 \end{bmatrix}, \begin{bmatrix} -0.5 & -0.5 & 0 \\ -1 & -1 & -1 \\ 2 & 2 & 1.5 \end{bmatrix}, \text{ and } \begin{bmatrix} 0.5 & 0.5 & 0.25 \\ -1 & -1 & -0.5 \\ 1 & 1 & 0.5 \end{bmatrix}$$

respectively. We claim that, for some $\varphi_0 \in C(-r)$ with $N = 3$, the equation

$$\frac{d}{dt}\{x(t) - x(t - r)\} = \sum_{j=0}^{3} B_j x(t + 2j - r) - \sum_{j=0}^{3} B_j x(t + 2j) \tag{6.18}$$

with $x_0 = \varphi_0$ has no solution at all on $[-r, \infty)$. In fact, by letting

$$y(t) = x(-t) - x(-t-r), \tag{6.19}$$

(6.18) is transformed to

$$\frac{d}{dt}y(t) = \sum_{j=0}^{3} B_j y(t-2j). \tag{6.20}$$

Kappel [19] showed that (6.20) is degenerate with respect to $q = (0, 0, 1)^T$ at $t_1 = 6$, i.e., for any $\varphi \in C([-6, 0], \mathbb{C}^3)$, the solution of (6.20) with $y_0 = \varphi$ satisfies $q^T y(t) \equiv 0$ for $t \geq 6$. Thus, for any $y^* \in \mathbb{C}^3$ with $q^T y^* \neq 0$, (6.20) has no solution on $[-6, 6]$ satisfying $y(6) = y^*$. Then (6.19) implies that, for any $\varphi_0 \in C([-r, 0], \mathbb{C}^3)$ satisfying

$$q^T(\varphi_0(0) - \varphi_0(-r)) \neq 0, \tag{6.21}$$

(6.18) with $x_{-6} = \varphi_0$ has no solution at all on $[-6-r, 6]$. Since (6.18) is autonomous, (6.18) with $x_0 = \varphi_0$ has no solution on $[-r, 12]$, let alone on $[-r, \infty)$, as long as $\varphi_0$ satisfies (6.21).

(iii). According to Theorem 6.2, the smallness of $\|D(t, .)\|$, $\|L_1(t, .)\|$ and $\|L_2(t, .)\|$ implies existence. However, uniqueness (except for solutions of order $e^{\lambda t}$) is not guaranteed no matter how small $\|D\|$, $\|L_1\|$ and $\|L_2\|$ are. As an example, we consider the equation

$$\frac{d}{dt}x(t) = A\,x(t+1) + B\,x(t-1), \tag{6.22}$$

where $A$ and $B$ are $N \times N$ matrices satisfying $\det A \neq 0$ and

$$\lambda^{-1}(|B| + |A|e^{\lambda}) < 1. \tag{6.23}$$

From Theorem 6.2 we know that, for every $t_0 \in (-\infty, \infty)$ and every $\varphi_0 \in C(-1)$, (6.22) with $x_{t_0} = \varphi_0$ has, on $[t_0 - 1, \infty)$, a unique solution of order $e^{\lambda t}$. We will show that, without this order restriction, (6.22) with $x_0 = \varphi_0$ has at least two solutions on $[-1, \infty)$ for some specific $\varphi_0 \in C(-1)$.

For this purpose, we first analyse the distribution of the eigenvalues of (6.22), i.e., the solutions of the characteristic equation

$$\Delta(\mu) \equiv \det\{\mu I - Ae^{\mu} - Be^{-\mu}\} = 0. \qquad (6.24)$$

Clearly, $e^{N\mu}\Delta(\mu)$ and $\Delta(\mu)$ have the same zeros and $e^{N\mu}\Delta(\mu)$ is a polynomial in $\mu$ and $e^{\mu}$. Furthermore, the highest powers of $\mu$ and $e^{\mu}$ are N and 2N respectively. Let

$$e^{N\mu}\Delta(\mu) = \sum_{i=0}^{N}\sum_{j=0}^{2N} a_{i,j}\mu^i e^{j\mu}.$$

Since $A^{-1}$ exists and

$$e^{N\mu}\Delta(\mu) = \det\{\mu e^{\mu} I - e^{2\mu}A - B\} = \det(-A)\det\{e^{2\mu}I - \mu e^{\mu}A^{-1} + A^{-1}B\},$$

we have $a_{i,j} = 0$ for $i + j > 2N$, $i \in \{0, 1, \cdots, N\}$ and $j \in \{0, 1, \cdots, 2N\}$, and

$$a_{N,N} = 1, \; a_{0,2N} = \det(-A) \neq 0.$$

Because $e^{N\mu}\Delta(\mu)$ has at least two terms with the highest indices $i + j = 2N$ (i.e., $e^{N\mu}\Delta(\mu)$ has no principle term), by Theorem A1 in [17] (p338), $e^{N\mu}\Delta(\mu)$ has an infinity of zeros with arbitrarily large real parts. Thus (6.24) has infinitely many solutions whose real parts can be arbitrarily large.

Let $\mu$ be a solution of (6.24) with $\mathrm{Re}\mu > \lambda$. Then there is a $c_0 \in \mathbb{C}^N$ such that $x_1(t) = e^{\mu t}c_0$ is a solution of (6.22) on $(-\infty, \infty)$. Let $\varphi_0(\theta) = (x_1)_0(\theta) = e^{\mu\theta}c_0$ for $\theta \in [-1, 0]$. We claim that (6.22) with $x_0 = \varphi_0$ has at least two solutions on $[-1, \infty)$. Indeed, (6.22) has a unique solution $x_2(t)$ on $[-1, \infty)$ with $(x_2)_0 = \varphi_0$ such that $x_2(t)e^{-\lambda t}$ is bounded. On the other hand, $x_1(t)e^{-\lambda t}$ is unbounded on $[-1, \infty)$. Thus $x_1(t)$ and $x_2(t)$ are solutions of (6.22) with $(x_1)_0 = (x_2)_0 = \varphi_0$ and $x_1 \neq x_2$.

We have shown that, for some $\varphi_0 \in C(-r)$, (6.22) with $x_0 = \varphi_0$ has at least two solutions on $[-1, \infty)$ provided that (6.23) holds. Actually, for every $\varphi_0 \in C(-r)$, (6.22) with $x_0 = \varphi_0$ has infinitely many independent solutions on $[-1, \infty)$. This assertion will be shown in the next section.

## § 6.4 SOLUTIONS ON A HALF LINE WITH EXTRA INITIAL DATA

We expect there to be some ideal initial condition with which the solution of (6.1) on $[t_0 - r, \infty)$ is unique if one exists. Theorem 6.2 and its remarks suggest that the initial condition $x_{t_0} = \varphi_0$ is not enough for uniqueness. Although finding a way to give a kind of ideal initial condition for the general equation (6.1) is a significant problem, we do not tackle it in this thesis. For autonomous equations, however, it is possible to give some data on $[t_0, t_0 + \sigma]$ in addition to $x_{t_0} = \varphi_0$.

In this section, we study only the simple equation

$$\frac{d}{dt}x(t) = A\,x(t+1) + B\,x(t-1) \tag{6.22}$$

in order to show the possibility of giving a kind of ideal initial condition. We assume that $A^{-1}$ exists and that (6.23) holds throughout this section. We also denote the first (second or nth) order derivative of $x(t)$ by $x'(t)$ ($x''(t)$ or $x^{(n)}(t)$).

Our first task is to find a suitable space $C_1$ so that we can specify any $\varphi \in C_1$ as extra initial data on $[0, 1]$. Since $A^{-1}$ exists, we denote $-A^{-1}B$ by $G$. Let $C_1$ be a subset of $C^{\infty}([-1, 0], \mathbb{C}^N)$ such that each $\varphi \in C_1$ satisfies

$$\varphi(-1) = 0, \quad A^{-1}\varphi'(-1) = \varphi(0), \tag{6.25}$$

$$\psi_{k+1}(-1) = \psi_k(0), \quad k = 2, 3, \cdots \tag{6.26}$$

with

$$\begin{aligned}\psi_{2n}(\theta) = {} & A^{-(2n-1)}\varphi^{(2n-1)}(\theta) + \sum_{i, j \geq 0;\, i+j = 2n-3} A^{-i}GA^{-j}\varphi^{(2n-3)}(\theta) \\ & + \sum_{i, j, k \geq 0;\, i+j+k = 2n-5} A^{-i}GA^{-j}GA^{-k}\varphi^{(2n-5)}(\theta) \\ & + \cdots + \sum_{i, j \geq 0;\, i+j = n-1} G^{i}A^{-1}G^{j}\varphi(\theta)\end{aligned} \tag{6.27}$$

and

$$\psi_{2n+1}(\theta) = A^{-(2n)}\varphi^{(2n)}(\theta) + \sum_{i, j \geq 0;\, i+j = 2(n-1)} A^{-i}GA^{-j}\varphi^{(2n-2)}(\theta)$$

$$+ \sum_{i, j, k \geq 0;\, i+j+k = 2(n-2)} A^{-i}GA^{-j}GA^{-k}\varphi^{(2n-4)}(\theta)$$

$$+ \cdots + \sum_{i, j, k \geq 0;\, i+j+k = n-1} G^{i}A^{-1}G^{j}A^{-1}G^{k}\varphi''(\theta) + G^{n}\varphi(\theta) \tag{6.28}$$

for $\theta \in [-1, 0]$ and $n = 1, 2, \cdots$.

We need to show that $C_1$ is a nontrivial space.

**THEOREM** 6.4 If $x(t)$ on $[-1, \infty)$ is a solution of (6.22) with $x_0 = 0$, then $x_1 \in C_1$. Moreover, $C_1$ is of infinite dimension.

**Proof**. As $0 \in C_1$, we assume that $x(t)$ on $[-1, \infty)$ is a nontrivial solution of (6.22) with $x_0 = 0$ and show that $x_1 \in C_1$ and $x_1 \neq 0$.

Since $x(t)$ satisfies (6.22) for $t \geq 0$, then, for any integer $k \geq 0$ and $t \in [k, k+1]$, we have

$$x_{k+1}'(t-k-1) = A\, x_{k+2}(t-k-1) + B\, x_k(t-k-1).$$

As $A^{-1}$ exists, $x(t)$ satisfies

$$x_{k+2}(\theta) = A^{-1}x_{k+1}'(\theta) + G\, x_k(\theta) \tag{6.29}$$

for $\theta \in [-1, 0]$ and $k = 0, 1, 2, \cdots$. Letting $k = 0$ in (6.29), we have

$$x_2(\theta) = A^{-1}x_1'(\theta) + Gx_0(\theta) = A^{-1}x_1'(\theta),$$

which, together with $x_2 \in C^1([-1, 0], \mathbb{C}^N)$, implies $x_1 \in C^2([-1, 0], \mathbb{C}^N)$. Thus, by (6.29) and $x_3 \in C^1([-1, 0], \mathbb{C}^N)$, we obtain

$$x_3(\theta) = A^{-1}x_2'(\theta) + Gx_1(\theta) = A^{-2}x_1''(\theta) + Gx_1(\theta)$$

and $x_1 \in C^3([-1, 0], \mathbb{C}^N)$. Inductively, it follows from (6.29) and $x \in C^1([0, \infty), \mathbb{C}^N)$ that

$$x_{2n}(\theta) = A^{-(2n-1)}x_1^{(2n-1)}(\theta) + \sum_{i, j \geq 0;\, i+j = 2n-3} A^{-i}GA^{-j}x_1^{(2n-3)}(\theta) +$$

$$+ \sum_{i,j,k \ge 0;\, i+j+k = 2n-5} A^{-i}GA^{-j}GA^{-k}x_1^{(2n-5)}(\theta)$$

$$+ \cdots + \sum_{i,j \ge 0;\, i+j = n-1} G^iA^{-1}G^j x_1'(\theta) \tag{6.30}$$

and

$$x_{2n+1}(\theta) = A^{-(2n)}x_1^{(2n)}(\theta) + \sum_{i,j \ge 0;\, i+j = 2(n-1)} A^{-i}GA^{-j}x_1^{(2n-2)}(\theta)$$

$$+ \sum_{i,j,k \ge 0;\, i+j+k = 2(n-2)} A^{-i}GA^{-j}GA^{-k}x_1^{(2n-4)}(\theta)$$

$$+ \cdots + \sum_{i,j,k \ge 0;\, i+j+k = n-1} G^iA^{-1}G^jA^{-1}G^kx_1'(\theta) + G^nx_1(\theta) \tag{6.31}$$

for $n = 1, 2, \cdots$ and $x_1 \in C^\infty([-1, 0], \mathbb{C}^N)$. By the continuity of $x$,

$$x_{k+1}(-1) = x_k(0), \ k = 0, 1, 2, \cdots. \tag{6.32}$$

Then (6.30)-(6.32) with $x_0 = 0$ imply that $x_1$ satisfies (6.25)-(6.28). Thus $x_1 \in C_1$. Since $x$ is nontrivial, by (6.30) and (6.31), we must have $x_1 \neq 0$.

Recall that, for every eigenvalue $\mu$ of (6.22) with $\mathrm{Re}\mu > \lambda$, there is a $c_0 \in \mathbb{C}^N$ such that (6.22) with $x_0(\theta) = e^{\mu\theta}c_0$, $\theta \in [-1, 0]$, has at least two solutions on $[-1, \infty)$; one is $e^{\mu t}c_0$ and another is of order $e^{\lambda t}$. Then the difference of the two solutions is a nontrivial solution of (6.22) on $[-1, \infty)$ with $x_0 = 0$. Since (6.22) has infinitely many eigenvalues whose real parts can be arbitrarily large, (6.22) with $x_0 = 0$ has infinitely many independent solutions on $[-1, \infty)$. The infinity of such solutions implies that $C_1$ is of infinite dimension. #

Since (6.22) with $x_0 = \varphi_0$ has a unique solution of order $e^{\lambda t}$ on $[-1, \infty)$, we denote $x_1$ of this solution by $\varphi_{0\lambda}$. With the well defined space $C_1$, now we can pose the initial value problem for (6.22) as follows.

**IVP3.** For any $\varphi_0 \in C(-1)$ and any $\varphi \in C_1$, we look for a solution of (6.22) on $[-1, \infty)$ satisfying

$$x_0 = \varphi_0 \text{ and } x_1 = \varphi_{0\lambda} + \varphi. \tag{6.33}$$

**THEOREM 6.5** For any $\varphi_0 \in C(-1)$ and any $\varphi \in C_1$, (6.22) with (6.33) possesses a unique solution on $[-1, \infty)$.

*Proof*. We first show that, for any $\varphi \in C_1$, (6.22) with $x_0 = 0$ and $x_1 = \varphi$ possesses a unique solution on $[-1, \infty)$. By taking (6.25)-(6.28) into account, we define a continuous function $x$ on $[-1, \infty)$ by

$$x(t) = \begin{cases} 0 & \text{for } t \in [-1, 0], \\ \varphi(t-1) & \text{fot } t \in [0, 1], \\ \psi_k(t-k) \text{ for } t \in [k-1, k], & k = 2, 3, \cdots. \end{cases}$$

Since the definition of $x$ is equivalent to

$$x_0(\theta) = 0, \ x_1(\theta) = \varphi(\theta), \text{ and } x_k(\theta) = \psi_k(\theta), \ k = 2, 3, \cdots$$

for $\theta \in [-1, 0]$, by (6.27) and (6.28) again $x$ satisfies (6.30) and (6.31) for $n = 1, 2, \cdots$. Then we derive from (6.30) and (6.31) that

$$x_2(\theta) = A^{-1}x_1'(\theta) = A^{-1}x_1'(\theta) + Gx_0(\theta),$$

$$\begin{aligned} x_{2n+1}(\theta) - A^{-1}x_{2n}'(\theta) &= GA^{-(2n-2)}x_1^{(2n-2)}(\theta) \\ &+ G\Sigma_{i, j \geq 0;\, i+j = 2(n-2)} A^{-i}GA^{-j}x_1^{(2n-4)}(\theta) + \cdots \\ &+ G\Sigma_{i, j, k \geq 0;\, i+j+k = n-2} G^iA^{-1}G^jA^{-1}G^kx_1^{(2n-4)}(\theta) \\ &+ GG^{n-1}x_1(\theta) = Gx_{2n-1}(\theta), \end{aligned}$$

and

$$x_{2(n+1)}(\theta) - A^{-1}x_{2n+1}'(\theta) = Gx_{2n}(\theta)$$

for $\theta \in [-1, 0]$ and $n \geq 1$. Thus $x(t)$ satisfies (6.29), i.e., $x(t)$ on $[-1, \infty)$ is a solution of (6.22) with $x_0 = 0$ and $x_1 = \varphi$.

If there are two solutions of (6.22) with $x_0 = 0$ and $x_1 = \varphi$, then their difference is a solution of (6.22) with $x_0 = 0$ and $x_1 = 0$. This solution is trivial as it satisfies (6.30) and (6.31). Therefore, the function $x$ on $[-1, \infty)$ defined above is the unique solution of (6.22) with $x_0 = 0$ and $x_1 = \varphi$.

For any $\varphi_0 \in C(-1)$ and any $\varphi \in C_1$, let $x_1(t)$ stand for the unique solution of (6.22) with $(x_1)_0 = \varphi_0$ such that $x_1(t)e^{-\lambda t}$ is bounded on $[-1, \infty)$, and $x_2(t)$ for the unique solution of (6.22) on $[-1, \infty)$ with $(x_2)_0 = 0$ and $(x_2)_1 = \varphi$. Then $x(t) = x_1(t) + x_2(t)$ on $[-1, \infty)$ is a solution of (6.22) with (6.33). Since the solution of (6.22) with $x_1 = x_0 = 0$ is trivial, $x(t) = x_1(t) + x_2(t)$ is the unique solution of (6.22) with (6.33). #

From Theorems 6.4 and 6.5 we see that $C_1$ and the set of solutions of (6.22) with $x_0 = 0$ on $[-1, \infty)$ are in 1-1 correspondence in the following way.

**COROLLARY** 6.6 The initial data space $C_1$ satisfies

$$C_1 = \{\varphi \in C(-1) : \text{(6.22) with } x_0 = 0 \text{ has a solution } x \text{ on } [-1, \infty) \text{ such that } x_1 = \varphi\}.$$

For equation (6.22) with (6.23), Theorem 6.5 indicates that IVP3 is well posed. That is to say, existence and uniqueness always hold when $|A|$ and $|B|$ are small enough. Unfortunately, this virtue of IVP3 is achieved with the high price of a complicated description of the space $C_1$. Because of this it is difficult to find concrete examples from $C_1$.

For some special cases of (6.22), Corollary 6.6 can help us to find such functions in $C_1$. For instance, the advanced scalar equation

$$\frac{d}{dt}x(t) = a\,x(t+1), \tag{6.34}$$

where $a$ is a constant satisfying $0 < |a| < e^{-1}$, is a special case of (6.22) with $A = a$, $B = 0$ and $N = 1$, and (6.23) holds for $\lambda = 1$. Since there is no delay involved in the equation, the initial data $x_0 = \varphi_0$ is given in $\mathbb{C}$ instead of $C(-1)$. Then, for any pair of eigenvalues of (6.34), $\mu_1$ and $\mu_2$, the function

$$\varphi(\theta) = e^{\mu_1(1+\theta)} - e^{\mu_2(1+\theta)}, \quad \theta \in [-1, 0],$$

belongs to $C_1$ as $e^{\mu_1 t} - e^{\mu_2 t}$ is a solution of (6.34) with $x_0 = 0$ on $[0, \infty)$.

# CHAPTER 7
# SOLUTIONS TO IVP2 ON THE HALF LINE

Our main interest in this chapter and thereafter is in solutions of (6.1) and (6.2) on the half line $[t_0 - r, \infty)$. In Chapter 6, we discussed the three IVPs for (6.1) and mentioned the shortcomings of each. Since the theorem on IVP1 can not provide any information about solutions on $[t_0 - r, \infty)$, we will not discuss IVP1 any more. The overwhelming defect of IVP2 is that uniqueness is not guaranteed when existence holds. The purpose of IVP3 is to show the possibility of overcoming this flaw of IVP2 by providing extra initial data. But, in this thesis, we do not intend to go any further along this line. Instead, we will probe another way to improve IVP2, to find some condition on $L_2(t, .)$ rather than give extra initial data.

Such a condition on $L_2(t, .)$ is to be discussed in § 7.1 and its relation to existence and uniqueness in § 7.2. In § 7.3, we will prove a few simple properties of the solution of (6.1) with $x_{t_0} = \varphi_0$.

## § 7.1 ZERO ADVANCED POINTS

Suppose that $L_2(t, \psi)$ in (6.1) has the representation

$$L_2(t, \psi) = \int_0^\sigma d_\alpha \eta(t, \alpha)\psi(\alpha),$$

where, for each $t \geq \tau$, $\eta(t, .)$ is of bounded variation on $[0, \sigma]$. The following definition states a condition on $L_2$ which is important for the discussion of existence and uniqueness in the next section.

**DEFINITION** 7.1 A point $t^* \in (\tau, \infty)$ is called a zero advanced point of (6.1), shortened as ZAP, if, for all $t \in [\tau, t^*)$, $L_2(t, x^t)$ does not involve any value of $x(t)$ on $(t^*, \infty)$.

From the representation of $L_2$ we can see that $t^*$ is a ZAP of (6.1) if, for every $t \in [\tau, t^*)$ satisfying $t + \sigma > t^*$, $\eta(t, \alpha)$ is constant for $\alpha \in (t^* - t, \sigma]$.

It is obvious that, for a linear neutral differential equation on $[\tau, \infty)$, every point in $(\tau, \infty)$ is a ZAP as the equation can be viewed as (6.1) with $L_2(t, .) \equiv 0$.

**Example** 7.1 Suppose $\beta \in [0, 1]$ and consider the equation

$$\frac{d}{dt}\{x(t) - D(x_t)\} = L_1(x_t) + L_2(x_{[t+\beta]}), \tag{7.1}$$

which was studied in Part I. For every integer $n \geq 0$, each point in $[n, n + 1 - \beta]$ is a ZAP of (7.1).

**Example** 7.2 Suppose that $L_3(t, .)$ satisfies the same condition as $L_1(t, .)$ (see the beginning of Ch. 6) and that $\Delta(t)$ is a real continuous function of $t$. If $t^*$ satisfies $t + \Delta(t) \leq t^*$ for $t \leq t^*$, then $t^*$ is a ZAP of the equation

$$\frac{d}{dt}\{x(t) - D(t, x_t)\} = L_1(t, x_t) + L_3(t, x_{t+\Delta(t)}). \tag{7.2}$$

If $\Delta(t) > 0$ holds for all $t \geq \tau$ and $L_3(t, \varphi)$ always involves the value $\varphi(0)$, then (7.2) has no ZAP at all on $[\tau, \infty)$.

**Example** 7.3 The autonomous equation

$$\frac{d}{dt}\{x(t) - D(x_t)\} = L_1(x_t) + L_2(x^t) + f(t) \tag{7.3}$$

on $[0, \infty)$ has no ZAP at all if $L_2(\psi)$ involves the values of $\psi(\alpha)$ for $\alpha \in (0, \sigma]$.

For any $t_0 \geq \tau$, if $t^* > t_0$ is a ZAP and $x(t)$ satisfies (6.1) a.e. on $[t_0, t^*]$, we say that $x(t)$ is a solution of (6.1) on $[t_0 - r, t^*]$, instead of on $[t_0 - r, t^* + \sigma]$, as no values of $x(t)$ on $(t^*, t^* + \sigma]$ are involved.

**LEMMA** 7.1 Suppose that $t_0 \geq \tau$ and $t_2 > t_0$, and assume that $t_1 \in (t_0, t_2)$ is a ZAP. If $x_1(t)$ is a solution of (6.1) on $[t_0 - r, t_1]$ and $x_2(t)$ a solution on

$[t_1 - r, t_2 + \sigma]$ (or $[t_1 - r, \infty)$) with $(x_2)_{t_1} = (x_1)_{t_1}$, then $x_3(t)$, defined by $x_3(t) = x_1(t)$ on $[t_0 - r, t_1]$ and by $x_3(t) = x_2(t)$ on $[t_1, t_2 + \sigma]$ (or $[t_1\ \infty)$), is an extension of $x_1(t)$ from $[t_0 - r, t_1]$ to $[t_0 - r, t_2 + \sigma]$ (or $[t_0 - r, \infty)$).

**Proof.** It is clear that $x_3(t)$ is continuous on $[t_0 - r, t_2 + \sigma]$ (or $[t_0 - r, \infty)$) and satisfies (6.1) a.e. on $[t_0, t_2]$ (or $[t_0, \infty)$). #

The remarks in § 6.2 tell us that not every solution on a finite interval $[t_0 - r, t_1 + \sigma]$ can be extended to a larger interval. However, if $t_1$ is a ZAP, Theorem 6.1 and Lemma 7.1 ensure that every solution on $[t_0 - r, t_1]$ has an extension to a larger interval. This suggests that a solution of (6.1) on a finite interval might be extended to $[t_0 - r, \infty)$ by employing a sequence of ZAPs.

## § 7.2 EXISTENCE AND UNIQUENESS

We know from § 6.3 that, if $\int_t^t f(s)\, ds\, e^{-\lambda t}$ is bounded for some $\lambda > 0$, then the smallness of $\|D\|$, $\|L_1\|$ and $\|L_2\|$ implies existence of at least one solution of (6.1) with $x_{t_0} = \varphi_0$ on $[t_0 - r, \infty)$, but does not imply the uniqueness. With the help of a sequence of ZAPs, however, the smallness of $\|D\|$, $\|L_1\|$ and $\|L_2\|$ guarantees existence and uniqueness of such a solution on $[t_0 - r, \infty)$ without any restriction on the size of f. This will be verified soon.

**THEOREM** 7.2 Suppose that $\{\tau_n: n \geq 1\}$ is a sequence of ZAPs of (6.1) satisfying $\tau_n \uparrow \infty$ as $n \to \infty$. Then (6.1) with $x_{t_0} = \varphi_0$ possesses a unique solution on $[t_0 - r, \infty)$ for every $(t_0, \varphi_0) \in [\tau, \infty) \times C(-r)$ if and only if (6.1) with

$x_{t_0} = \varphi_0$ possesses a unique solution on $[t_0 - r, \tau_{n+1}]$ for each integer $n \geq 0$ and every $(t_0, \varphi_0) \in [\tau_n, \tau_{n+1}) \times C(-r)$, where $\tau_0 = \tau$.

**Proof.** For convenience, we symbolize the above conclusion as "(A) holds if and only if (B) holds".

Suppose that (A) holds. Then, for each $n \geq 0$ and every $(t_0, \varphi_0) \in [\tau_n, \tau_{n+1}) \times C(-r)$, (6.1) with $x_{t_0} = \varphi_0$ possesses a unique solution $x(t)$ on $[t_0 - r, \infty)$. Clearly, $x(t)$ is also a solution of (6.1) with $x_{t_0} = \varphi_0$ on $[t_0 - r, \tau_{n+1}]$ as $\tau_{n+1}$ is a ZAP. We show that this solution is unique. If (6.1) has another solution $x_1(t)$ on $[t_0 - r, \tau_{n+1}]$ satisfying $(x_1)_{t_0} = \varphi_0$ and $x_1 \neq x$, then, by (A) and Lemma 7.1, $x_1(t)$ can be extended to $[t_0 - r, \infty)$. Since the solution of (6.1) with $x_{t_0} = \varphi_0$ on $[t_0 - r, \infty)$ is unique, we must have $x_1(t) \equiv x(t)$ on $[t_0 - r, \infty)$, which is a contradiction. Hence (B) holds.

Assume that (B) holds. We verify that, for every $(t_0, \varphi_0) \in [\tau, \infty) \times C(-r)$, (6.1) with $x_{t_0} = \varphi_0$ possesses a unique solution on $[t_0 - r, \infty)$. Since $\tau_n \uparrow \infty$ as $n \to \infty$, there is an integer $m$ such that $t_0 \in [\tau_m, \tau_{m+1})$. Thus (B) implies that (6.1) has a unique solution $x_1(t)$ on $[t_0 - r, \tau_{m+1}]$ satisfying $(x_1)_{t_0} = \varphi_0$. By (B) again, (6.1) also has a unique solution $x_2(t)$ on $[\tau_{m+1} - r, \tau_{m+2}]$ such that $(x_2)_{\tau_{m+1}} = (x_1)_{\tau_{m+1}}$. The application of induction produces a sequence $\{x_n(t)\}$ such that, for each $n \geq 1$, $x_{n+1}(t)$ on $[\tau_{m+n} - r, \tau_{m+n+1}]$ is the unique solution of (6.1) with $(x_{n+1})_{\tau_{m+n}} = (x_n)_{\tau_{m+n}}$. Let $x(t) = x_1(t)$ for $t \in [t_0 - r, \tau_{m+1}]$ and $x(t) = x_{n+1}(t)$ for each $n \geq 1$ and all $t \in [\tau_{m+n}, \tau_{m+n+1}]$. Then, by applying Lemma 7.1 repeatedly, we see that $x(t)$ on $[t_0 - r, \infty)$ is a solution of (6.1) with $x_{t_0} = \varphi_0$. This solution is also unique since any solution of (6.1) with $x_{t_0} = \varphi_0$ on $[t_0 - r, \infty)$ can yield a sequence $\{x_n(t)\}$ and such a sequence is uniquely determined by $(x_1)_{t_0} = \varphi_0$. #

**Remark.** Theorem 7.2 converts the initial value problem on the half line $[t_0 - r, \infty)$ into that on finite intervals. The latter is much easier to cope with by applying fixed point theorems.

**THEOREM** 7.3 Assume that $\{\tau_n: n \geq 1\}$ is a sequence of ZAPs of (6.1) satisfying $\tau_n \uparrow \infty$ as $n \to \infty$, and that there is a number $\lambda \geq 0$ satisfying

$$\|D(t, .)\| + \int_{\tau_n}^{t} \{\|L_1(s, .)\| + \|L_2(s, .)\| e^{\lambda\sigma}\} e^{\lambda(s - t)}\, ds \; < \; 1 \tag{7.4}$$

for each $n \geq 0$ and all $t \in [\tau_n, \tau_{n+1}]$. Then (6.1) with $x_{t_0} = \varphi_0$ possesses a unique solution on $[t_0 - r, \infty)$ for every $(t_0, \varphi_0) \in [\tau, \infty) \times C(-r)$.

**Proof**. From Theorem 7.2 we need only show that, for each $n \geq 0$ and every $(t_0, \varphi_0) \in [\tau_n, \tau_{n+1}) \times C(-r)$, (6.1) with $x_{t_0} = \varphi_0$ possesses a unique solution on $[t_0 - r, \tau_{n+1}]$.

Let $(\varphi^*)_{t_0} = \varphi_0$, and $\varphi^*(t) = \varphi_0(0)$ and

$$h(t) = D(t, (\varphi^*)_t) - D(t_0, \varphi_0) + \int_{t_0}^{t} \{L_1(s, (\varphi^*)_s) + L_2(s, \varphi_0(0)) + f(s)\} ds$$

for $t \in [t_0, \tau_{n+1}]$. Then $h(t)$ is continuous and $h(t_0) = 0$. Since (6.1) with $x_{t_0} = \varphi_0$ is equivalent to the equation

$$y(t) = h(t) + D(t, y_t) + \int_{t_0}^{t} \{L_1(s, y_s) + L_2(s, y^s)\}\, ds \tag{7.5}$$

with $y_{t_0} = 0$ through the relation

$$x(t) = \varphi^*(t) + y(t), \; t \in [t_0 - r, \tau_{n+1}],$$

we show that (7.5) with $y_{t_0} = 0$ possesses a unique solution on $[t_0 - r, \tau_{n+1}]$.

It is known that the set

$$S = \{u \in C([t_0 - r, \tau_{n+1}], \mathbb{C}^N): u_{t_0} = 0\}$$

with the distance

$$\|u_1 - u_2\|_S = \sup\{|u_1(t) - u_2(t)|e^{-\lambda t}: t_0 \leq t \leq \tau_{n+1}\}$$

is a complete metric space. By viewing the right hand side of (7.5) (with $y_{t_0} = 0$) as the definition of a mapping $A$ on S, we have $A: S \to S$ and

$$\|Au_1 - Au_2\|_S = \sup\{|Au_1(t) - Au_2(t)|e^{-\lambda t}:\ t_0 \le t \le \tau_{n+1}\}$$

$$\le \sup\{\|D(t,.)\|.\|(u_1)_t - (u_2)_t\|e^{-\lambda t} + F(t, u_1, u_2):\ t_0 \le t \le \tau_{n+1}\},$$

where

$$F(t, u_1, u_2) = \int_{t_0}^{t} \{\|L_1(v,.)\|.\|(u_1)_v - (u_2)_v\| + |L_2(v, (u_1)^v - (u_2)^v)|\}e^{-\lambda t}\, dv\,.$$

As $\tau_{n+1}$ is a ZAP, it follows that, for $v < \tau_{n+1}$,

$$|L_2(v, (u_1)^v - (u_2)^v)| \le \|L_2(v,.)\| \sup\{|u_1(s) - u_2(s)|:\ v \le s \le \min(v + \sigma, \tau_{n+1})\}$$

$$\le \|L_2(v,.)\|e^{\lambda(v+\sigma)}\|u_1 - u_2\|_S.$$

Thus

$$\|Au_1 - Au_2\|_S \le \|u_1 - u_2\|_S \times$$

$$\sup\left\{\|D(t,.)\| + \int_{t_0}^{t} \{\|L_1(v,.)\| + \|L_2(v,.)\|e^{\lambda\sigma}\}e^{\lambda(v-t)}\, dv:\ t_0 \le t \le \tau_{n+1}\right\}.$$

Since (6.5) implies the continuity of $\|D(t,.)\|$ on $[\tau, \infty)$, by (7.4), $A$ is contractive. Therefore, (7.5) with $y_{t_0} = 0$ possesses a unique solution on $[t_0 - r, \tau_{n+1}]$. #

**Remark**. The proof of Theorem 7.3 is similar to that of Theorem 6.1. But now with the aid of a sequence of ZAPs, existence with uniqueness is obtained without any order restriction on x.

Theorem 7.3 is not convenient from the practical point of view for we have to check (7.4) on every interval $[\tau_n, \tau_{n+1}]$. However, this defect can be compensated by a little sacrifice, replacing (7.4) by a slightly stronger condition.

**COROLLARY** 7.4 Assume that $\{\tau_n: n \ge 1\}$ is a sequence of ZAPs of (6.1) satisfying $\tau_n \uparrow \infty$ as $n \to \infty$, and that there is a number $\lambda \ge 0$ satisfying

$$\|D(t,.)\| + \int_{\tau}^{t} \{\|L_1(s,.)\| + \|L_2(s,.)\|e^{\lambda\sigma}\}e^{\lambda(s-t)}\,ds < 1 \tag{7.6}$$

for all $t \geq \tau$. Then, for every $(t_0, \varphi_0) \in [\tau, \infty) \times C(-r)$, (6.1) with $x_{t_0} = \varphi_0$ possesses a unique solution on $[t_0 - r, \infty)$.

**Proof.** Since (7.6) implies (7.4) on every $[\tau_n, \tau_{n+1}]$, this is an immediate consequence of Theorem 7.3. #

**COROLLARY 7.5** Assume that $\{\tau_n: n \geq 1\}$ is a sequence of ZAPs of (6.1) satisfying $\tau_n \uparrow \infty$ as $n \to \infty$, and that there is a number $\lambda > 0$ satisfying

$$\|D\| + \lambda^{-1}(\|L_1\| + \|L_2\|e^{\lambda\sigma}) \leq 1, \tag{7.7}$$

where $\|D\|$ and $\|L_k\|$ are the suprema of $\|D(t,.)\|$ and $\|L_k(t,.)\|$ $(k = 1, 2)$ on $[\tau, \infty)$. Then, for every $(t_0, \varphi_0) \in [\tau, \infty) \times C(-r)$, (6.1) with $x_{t_0} = \varphi_0$ possesses a unique solution on $[t_0 - r, \infty)$.

**Proof.** This follows immediately from Corollary 7.4. #

Through Corollaries 7.4 and 7.5 we have seen the significance of $\{\tau_n\}$, which, with the smallness of $\|D(t,.)\|$ and $\|L_k(t,.)\|$ $(k = 1, 2)$, guarantees both existence and uniqueness of the solution of (6.1) with $x_{t_0} = \varphi_0$ on $[t_0 - r, \infty)$.

If $\|D(t,.)\|$ and $\|L_k(t,.)\|$ $(k = 1, 2)$ are bounded but not small enough for the above to hold, it is still possible to have existence with uniqueness provided that every $\tau_{n+1} - \tau_n$ is small enough. Let

$$\|D\|_\delta = \sup\{\|D(t,.)\|_\delta: \tau \leq t < \infty\},$$

where, for each $t \in [\tau, \infty)$, $\|D(t,.)\|_\delta$ is defined as in the proof of Theorem 6.1 (p.79). We claim that $\|D\|_\delta$ is nondecreasing in $\delta$. In fact, from (6.3) and the Riesz theorem (Theorem 13.1 in [28]), we have

$$\|D(t,.)\|_\delta = \mathrm{Var}_{[-\delta,0]}(\xi(t,.)) \tag{7.8}$$

for $t \geq \tau$ and $\delta \geq 0$. Thus, for each $t \geq \tau$, $\|D(t, .)\|_\delta$ is nondecreasing for $\delta \geq 0$. For any $\delta_1 \geq 0$ and $\delta_2 > \delta_1$, since

$$\|D\|_{\delta_2} = \sup\{\|D(t, .)\|_{\delta_2} : t \in [\tau, \infty)\}$$

and $\|D(t, .)\|_{\delta_2} \geq \|D(t, .)\|_{\delta_1}$ for every $t \geq \tau$, we have $\|D\|_{\delta_2} \geq \|D(t, .)\|_{\delta_1}$ for all $t \geq \tau$, which implies $\|D\|_{\delta_2} \geq \|D\|_{\delta_1}$.

**COROLLARY** 7.6 Suppose that $\{\tau_n\}$ is as in Theorem 7.2, that $\delta > 0$ satisfies

$$\|D\|_\delta + \delta(\|L_1\| + \|L_2\|) \leq 1, \tag{7.9}$$

and that

$$\tau_{n+1} - \tau_n < \delta \tag{7.10}$$

for every integer $n \geq 0$. Then, for every $(t_0, \varphi_0) \in [\tau, \infty) \times C(-r)$, (6.1) with $x_{t_0} = \varphi_0$ possesses a unique solution on $[t_0 - r, \infty)$.

**Proof**. We refer to the proof of Theorem 7.3 with $\lambda = 0$. Since $u_{t_0} = 0$ for every $u \in S$, we actually have

$$|D(t, (u_1)_t - (u_2)_t)| \leq \|D(t, .)\|_{t - t_0} \|u_1 - u_2\|_S$$

for $t \in [t_0, \tau_{n+1}]$. Then

$$\|Au_1 - Au_2\|_S \leq \|u_1 - u_2\|_S \times$$

$$\sup\left\{\|D(t, .)\|_{t - t_0} + \int_{t_0}^{t} \{\|L_1(v, .)\| + \|L_2(v, .)\|\}\, dv : t_0 \leq t \leq \tau_{n+1}\right\}$$

$$\leq \|u_1 - u_2\|_S \{\|D\|_{\tau_{n+1} - \tau_n} + (\tau_{n+1} - \tau_n)(\|L_1\| + \|L_2\|)\}.$$

By (7.9) and (7.10), $A$ is contractive on $S$. #

**Remark**. For each $t \geq \tau$, it follows from (7.8) and (6.4) that

$$\lim_{\delta \to 0^+} \|D(t, .)\|_\delta = 0. \tag{7.11}$$

Suppose that either $D(t, .) = D(.)$ does not depend on $t$, or (7.11) holds uniformly for $t \in [\tau, \infty)$. Then $\|D\|_\delta \to 0$ as $\delta \to 0^+$. Thus, provided that $\|L_1(t, .)\|$ and $\|L_2(t, .)\|$ are bounded, there must be a $\delta > 0$ satisfying (7.9). By Corollary 7.6, (6.1) with $x_{t_0} = \varphi_0$ always possesses a unique solution on $[t_0 - r, \infty)$ as long as every $\tau_{n+1} - \tau_n$ is small enough.

As a special case of (6.2), we consider conditions for existence and uniqueness that might be imposed on the equation (7.2). Here we assume that $\Delta(t)$ is piecewise continuous and satisfies $-r_1 \le \Delta(t) \le \sigma$ for $t \ge \tau$ and some $r_1 > 0$, and that $\{\tau_n\}$ is a sequence of ZAPs satisfying

(a). $\tau_n \to \infty$ as $n \to \infty$;

(b). $\tau_{2(k-1)} \le \tau_{2k-1} < \tau_{2k}$ for $k = 1, 2, \cdots$;

(c). each interval $[\tau_{2(k-1)}, \tau_{2k-1}]$ is composed of ZAPs;

(d). $\Delta(t) > 0$ a.e. on each interval $[\tau_{2k-1}, \tau_{2k}]$.

Then (7.2) is a neutral equation on every interval $[\tau_{2(k-1)}, \tau_{2k-1}]$ and a mixed equation on every $[\tau_{2k-1}, \tau_{2k}]$. Since, for each $t \ge \tau$, $L_3(t, x_{t+\Delta(t)})$ involves the values of $x$ on $[t - r + \Delta(t), t + \Delta(t)] \subset [t - r - r_1, t + \sigma]$, we should replace the initial data space $C(-r)$ by $C(-r - r_1)$ if we view (7.2) as (6.2). If (7.2) is restricted to $[\tau_{2k-1}, \tau_{2k}]$, however, (d) implies that $C(-r)$ can still serve as the initial data space. We add a subscript "-" to the elements in $C(-r - r_1)$ so as to distinguish them from elements in $C(-r)$. For instance, $x_t \in C(-r)$ but $x_{t-} \in C(-r - r_1)$.

From Theorem 7.2 and existence and uniqueness for neutral equations, we derive that (7.2) with $x_{t_0-} = \varphi_{0-}$ possesses a unique solution on $[t_0 - r - r_1, \infty)$ for any $(t_0, \varphi_{0-}) \in [\tau, \infty) \times C(-r - r_1)$ if and only if (7.2) with $x_{t_0} = \varphi_0$ possesses a unique solution on $[t_0 - r, \tau_{2k}]$ for each $k \ge 1$ and every $(t_0, \varphi_0) \in [\tau_{2k-1}, \tau_{2k}) \times C(-r)$.

If $\|D(t, .)\|$ is bounded on the intervals $[\tau_{2k-1}, \tau_{2k}]$, $k = 1, 2, \cdots$, we define

$$\|D\|_+ = \sup\{\|D(t, .)\|:\ t \in [\tau_{2k-1}, \tau_{2k}],\ k \geq 1\}$$

and

$$\|D\|_{+\delta} = \sup\{\|D(t, .)\|_\delta :\ t \in [\tau_{2k-1}, \tau_{2k}],\ k \geq 1\}$$

for $\delta \geq 0$. Similarly, we can define $\|L_1\|_+$ and $\|L_3\|_+$.

**THEOREM** 7.7 Equation (7.2) with $x_{t_0} = \varphi_0$ possesses a unique solution on $[t_0 - r - r_1, \infty)$ for every $(t_0, \varphi_0) \in [\tau, \infty) \times C(-r - r_1)$ if one of the following conditions holds.

(i). There is a $\lambda \geq 0$ satisfying

$$\|D(t, .)\| + \int_{\tau_{2k-1}}^{t} \{\|L_1(s, .)\| + \|L_3(s, .)\|e^{\lambda\sigma}\}e^{\lambda(s-t)}\, ds < 1 \tag{7.12}$$

for each $k \geq 1$ and all $t \in [\tau_{2k-1}, \tau_{2k}]$.

(ii). There is a $\lambda > 0$ satisfying

$$\|D\|_+ + \lambda^{-1}(\|L_1\|_+ + \|L_3\|_+ e^{\lambda\sigma}) \leq 1. \tag{7.13}$$

(iii). There is a $\delta > 0$ such that

$$\|D\|_{+\delta} + \delta(\|L_1\|_+ + \|L_3\|_+) \leq 1,\ \tau_{2k} - \tau_{2k-1} < \delta,\ k \geq 1. \tag{7.14}$$

**Proof**. For every $t \in [\tau_{2k-1}, \tau_{2k}]$ with $\Delta(t) > 0$, $L_3(t, .)$ can be decomposed as $L_3(t, .) = F_1(t, .) + F_2(t, .)$, where $F_1(t, x_{t+\Delta(t)})$ only involves the values of $x(s)$ for $s \leq t$ and $F_2(t, x_{t+\Delta(t)})$ the values of $x(s)$ for $s \geq t$. Then it follows from the Riesz theorem (Theorem 13.1 in [28]) that

$$\|L_3(t, .)\| = \|F_1(t, .)\| + \|F_2(t, .)\|.$$

For $\lambda \geq 0$, since

$$\|L_1(t, .) + F_1(t, .)\| + \|F_2(t, .)\|e^{\lambda\sigma} \leq \|L_1(t, .)\| + \|F_1(t, .)\| + \|F_2(t, .)\|e^{\lambda\sigma}$$

$$\leq \|L_1(t, .)\| + \|L_3(t, .)\|e^{\lambda\sigma},$$

(7.12) implies (7.4) on every $[\tau_{2k-1}, \tau_{2k}]$. Similarly, we can replace (7.7) by (7.13), and (7.9) and (7.10) by (7.14). Then the conclusion follows from Theorem 7.3 and Corollaries 7.5 and 7.6. #

Some examples are given below to show the application of our theorems.

**Example** 7.4 The equation

$$\frac{d}{dt}\left\{x(t) - \frac{1}{3}x(t-2)\right\} = \frac{1}{3}x(t) + \frac{1}{3e}\int_t^{t+\cos^2 t} x(s)\,ds \tag{7.15}$$

with $x_{t_0} = \varphi_0$ possesses a unique solution on $[t_0 - 2, \infty)$ for every $(t_0, \varphi_0) \in [\tau, \infty) \times C(-2)$ $(\tau \le 0)$. Indeed, by letting $\tau_n = (1/2 + 2n)\pi$, $n = 1, 2, \cdots$, and $\tau_0 = \tau$, we have $t + \cos^2 t \le \tau_n$ for each $n \ge 1$ and all $t \le \tau_n$ since $t + \cos^2 t$ is an increasing function. Thus each $\tau_n$ is a ZAP and $\tau_n \to \infty$ as $n \to \infty$. Obviously,

$$\|D\| + (\|L_1\| + \|L_2\|e^{\sigma}) \le 1/3 + 1/3 + e/3e = 1.$$

Then the conclusion follows from Corollary 7.5.

**Example** 7.5 Consider the equation

$$\frac{d}{dt}\{x(t) - A(t)x(t-r)\} = B_1(t)x(t) + B_2(t)x(t + |\sin t|) + f(t), \tag{7.16}$$

where $r > \pi$, $f \in C([0, \infty), \mathbb{C}^N)$, the matrices $A(t)$, $B_1(t)$ and $B_2(t)$ are continuous, and

$$|B_1(t)| + |B_2(t)| \le p < \pi^{-1}, \ t \ge 0.$$

Since $\|D\|_\delta = 0$ for $\delta \in [0, r)$ and $\pi p < 1$, there is a $\delta \in (\pi, r)$ such that

$$\|D\|_\delta + \delta(\|L_1\| + \|L_2\|) \le \delta p \le 1.$$

It is obvious that $\tau_n = n\pi$, $n \ge 1$, are ZAPs for $t + |\sin t|$ is increasing. By Corollary 7.6, (7.16) with $x_{t_0} = \varphi_0$ possesses a unique solution on $[t_0 - r, \infty)$ for every $(t_0, \varphi_0) \in [\tau, \infty) \times C(-r)$. We note that the size of $|A(t)|$ is not relevant to existence and uniqueness as long as $r > \pi$.

**Example** 7.6 Consider the equation

$$\frac{d}{dt}x(t) = x(t - a \sin t) \tag{7.17}$$

on $[0, \infty)$, where $a$ is a constant satisfying $0 < a \le e^{-1}$. We claim that (7.17) with $x_{t_0} = \varphi_0$ possesses a unique solution on $[t_0 - a, \infty)$ for every $(t_0, \varphi_0) \in [0, \infty) \times C(-a)$. Viewing (7.17) as (7.2), we have $D = L_1 = 0$, $L_3(\varphi) = \varphi(0)$, and $\Delta(t) = - a \sin t$. Thus $r = 0$ and (with the notation given before Theorem 7.7) $r_1 = \sigma = a$. Let $\tau_n = n\pi$ for $n = 0, 1, 2, \cdots$. Then, for each $k \ge 1$, $t + \Delta(t) \le t$ for $t \in [\tau_{2(k-1)}, \tau_{2k-1}]$ and $\Delta(t) > 0$ and $t + \Delta(t) < \tau_{2k}$ for $t \in (\tau_{2k-1}, \tau_{2k})$. Hence $\{\tau_n\}$ is a sequence of ZAPs satisfying the conditions (a)-(d). Let $\lambda = e$. we have

$$\|D\|_+ + \lambda^{-1}(\|L_1\|_+ + \|L_3\|_+ e^{\lambda\sigma}) \le \lambda^{-1}e = 1.$$

Then Theorem 7.7 (ii) implies the truth of our claim.

## § 7.3 SIMPLE PROPERTIES

In this section, we discuss a few simple properties of the solutions of (6.1), such as an exponential estimate, boundedness, and the existence of finite limits at infinity.

Suppose that, for $i = 1, 2$ and some $\lambda > 0$, $\int_\tau^t \|L_i(s, .)\| e^{\lambda(s-t)} ds$ is bounded for $t \ge \tau$, and denote its least upper bound by $M_i(\lambda)$. Then, for $\alpha \in (0, \lambda)$ and $t \ge \tau$, an integration by parts leads to

$$\int_\tau^t \|L_i(s, .)\| e^{\alpha(s-t)} ds = e^{-\alpha t}\int_\tau^t e^{(\alpha-\lambda)s} d\left(\int_\tau^s \|L_i(v, .)\| e^{\lambda v} dv\right)$$

$$= \int_\tau^t \|L_i(s, .)\| e^{\lambda(s-t)}\, ds + (\lambda - \alpha)\int_\tau^t e^{\alpha(s-t)} \int_\tau^s \|L_i(v, .)\| e^{\lambda(v-s)}\, dv ds$$

$$\leq M_i(\lambda)\left\{1 + (\lambda - \alpha)\int_\tau^t e^{\alpha(s-t)}\, ds\right\}.$$

Thus

$$\int_\tau^t \|L_i(s, .)\| e^{\alpha(s-t)}\, ds \leq \left\{\frac{\lambda}{\alpha} - \left(\frac{\lambda}{\alpha} - 1\right) e^{\alpha(\tau - t)}\right\} M_i(\lambda), \quad i = 1, 2. \tag{7.18}$$

For $\alpha \geq \lambda$ and $i = 1, 2,$ it is obvious that

$$\int_\tau^t \|L_i(s, .)\| e^{\alpha(s-t)}\, ds \leq M_i(\lambda)$$

for $t \geq \tau$. Therefore, for every $\alpha > 0$ and $i = 1, 2,$ $\int_\tau^t \|L_i(s, .)\| e^{\alpha(s-t)}\, ds$ is bounded for $t \geq \tau$.

Let $S(\lambda)$ stand for the set of continuous functions on $[T, \infty)$, where the value $T \geq \tau$ may vary according to the context, such that $u(t)e^{-\lambda t}$ is bounded for each $u \in S(\lambda)$. Assume that

$$\|D\| + M_1(\lambda) + M_2(\lambda)e^{\lambda\sigma} < 1 \tag{7.19}$$

for some $\lambda > 0$. If the integral of $f$ belongs to $S(\lambda_1)$ for some $\lambda_1 \in [0, \lambda]$, then, for each $t_1 \geq \tau$, there is a constant $F(t_1)$ such that

$$\left|\int_{t_1}^t f(s)\, ds\right| \leq F(t_1)e^{\lambda_1(t - t_1)}, \quad t \geq t_1. \tag{7.20}$$

In this case, we let

$$K = (1 + \|D\|)\{1 - (\|D\| + M_1(\lambda) + M_2(\lambda)e^{\lambda\sigma})\}^{-1} \tag{7.21}$$

and

$$a = \inf\{s: \lambda_1 \leq s \leq \lambda,\ \|D\| + K^{-1} + (\lambda/s)(M_1(\lambda) + M_2(\lambda)e^{s\sigma}) \leq 1\}. \tag{7.22}$$

If $f = 0$, however, we define $a$ by the unique solution of the equation

$$G(a) =_{\det} \|D\| + (\lambda/a)(M_1(\lambda) + M_2(\lambda)e^{a\sigma}) = 1 \tag{7.23}$$

in $(0, \lambda)$ and $K$ by

$$K = (1 + \|D\|)\{1 - \|D\| - M_1(\lambda) - M_2(\lambda)e^{a\sigma}\}^{-1} \tag{7.24}$$

instead of (7.21) and (7.22) since the values of $a$ and $K$ determined by (7.23) and (7.24) are smaller than those given by (7.21) and (7.22).

**THEOREM** 7.8 Assume that (7.19) holds for some $\lambda > 0$ and that $f$ satisfies (7.20) for some $\lambda_1 \in [0, \lambda]$. Then, for any given $(t_0, \varphi_0) \in [\tau, \infty) \times C(-r)$, (6.1) with $x_{t_0} = \varphi_0$ possesses a unique solution $x$ in the class $S(\lambda)$. Moreover, this solution satisfies

$$\|x_t\| \le K(\|\varphi_0\| + F(t_0))e^{a(t - t_0)} \tag{7.25}$$

for $t \ge t_0$, where $a$ and $K$ are determined by (7.21) and (7.22), or by (7.23) and (7.24) if $f = 0$. Furthermore, if there is a sequence $\{\tau_n\}$ of ZAPs as in Theorem 7.2, then this solution is unique in $C([t_0 - r, \infty), \mathbb{C}^N)$.

**Proof.** The proof of Theorem 6.2 can be adopted almost verbatim to show existence and uniqueness in $S(\lambda)$. Below, we employ the successive approximations method to establish the estimate (7.25). We define a sequence of functions $\{x_k(t)\}$ by

$$x_0(t) = \varphi_0(0) + \int_{t_0}^{t} f(s)\,ds, \quad (x_0)_{t_0} = \varphi_0$$

and

$$x_{k+1}(t) = \varphi_0(0) - D(t_0, \varphi_0) + D(t, (x_k)_t)$$

$$+ \int_{t_0}^{t} \{L_1(s, (x_k)_s) + L_2(s, (x_k)^s) + f(s)\}\,ds, \quad (x_{k+1})_{t_0} = \varphi_0, \ k = 0, 1, 2, \cdots$$

for $t \geq t_0$. Since (6.1) and the integral equation obtained by integrating (6.1) from $t_0$ to $t$, both with $x_{t_0} = \varphi_0$, are equivalent for $t \geq t_0$, it is sufficient to show that $\{x_k(t)\}$ satisfies

$$\|(x_k)_t\| \leq K(\|\varphi_0\| + F(t_0))e^{a(t-t_0)}, \; k = 0, 1, 2, \cdots \tag{7.26}$$

for $t \geq t_0$ and that

$$\lim_{k \to \infty} x_k(t) = x(t_0, \varphi_0, f)(t) \tag{7.27}$$

uniformly for $t$ on any finite interval within $[t_0 - r, \infty)$, where $x(t_0, \varphi_0, f)(t)$ stands for the unique solution of (6.1) with $x_{t_0} = \varphi_0$ in the class $S(\lambda)$. Then (7.25) follows from (7.26) and (7.27).

For $k = 0$, (7.26) is obvious from (7.20)-(7.22). Suppose that (7.26) holds for some $k \geq 0$. Then, from the definition of $x_{k+1}(t)$, we obtain the estimate

$$\begin{aligned}
|x_{k+1}(t)| &\leq (1 + \|D\|)\|\varphi_0\| + \|D\|K(\|\varphi_0\| + F(t_0))e^{a(t-t_0)} + F(t_0)e^{\lambda_1(t-t_0)} \\
&\quad + \int_{t_0}^{t} \{\|L_1(s, .)\| + \|L_2(s, .)\|e^{a\sigma}\}K(\|\varphi_0\| + F(t_0))e^{a(s-t_0)}\, ds \\
&\leq K(\|\varphi_0\| + F(t_0))e^{a(t-t_0)} \times \\
&\quad \left\{\|D\| + K^{-1}\max\left\{1, (1 + \|D\|)e^{a(t_0-t)}\right\} + \int_{t_0}^{t} \{\|L_1(s, .)\| + \|L_2(s, .)\|e^{a\sigma}\}e^{a(s-t)}\, ds\right\} \\
&=_{def} \omega(t)K(\|\varphi_0\| + F(t_0))e^{a(t-t_0)}
\end{aligned}$$

for $t \geq t_0$. If $(1 + \|D\|)e^{a(t_0-t)} \leq 1$, by (7.18) and (7.22) we have

$$\omega(t) \leq \|D\| + K^{-1} + (\lambda/a)(M_1(\lambda) + M_2(\lambda)e^{a\sigma}) =_{def} \delta \leq 1.$$

Otherwise, by (7.18), (7.21) and (7.22),

$$\begin{aligned}
\omega(t) &\leq \|D\| + K^{-1}(1 + \|D\|)e^{a(t_0-t)} + \{(\lambda/a) - ((\lambda/a) - 1)e^{a(t_0-t)}\}(M_1(\lambda) + M_2(\lambda)e^{a\sigma}) \\
&\leq \delta + \{K^{-1}\|D\| - ((\lambda/a) - 1)(M_1(\lambda) + M_2(\lambda)e^{a\sigma})\}e^{a(t_0-t)}
\end{aligned}$$

$$\leq \delta + \{K^{-1}(1 + \|D\|) + \|D\| + M_1(\lambda) + M_2(\lambda)e^{\lambda\sigma} - \delta\}e^{\alpha(t_0 - t)}$$

$$= \delta + (1 - \delta)e^{\alpha(t_0 - t)} \leq 1$$

for $t \geq t_0$. In either case, we have derived (7.26) with $k$ replaced by $k + 1$. By induction, (7.26) holds for $k = 0, 1, 2, \cdots$.

Let $x(t_0, \varphi_0, f)(t) = y(t_0, \varphi_0, f)(t)e^{\lambda t}$ for $t \geq t_0 - r$. Then, restricted to the space of bounded continuous functions on $[t_0 - r, \infty)$, $y(t_0, \varphi_0, f)(t)$ is the unique solution of the equation

$$y(t) = (\varphi_0(0) - D(t_0, \varphi_0))e^{-\lambda t} + D(t, e^{\lambda \cdot}y_t)$$

$$+ \int_{t_0}^{t} f(s)\,ds\,e^{-\lambda t} + \int_{t_0}^{t} \{L_1(s, e^{\lambda \cdot}y_s) + L_2(s, e^{\lambda \cdot}y^s)\}e^{\lambda(s - t)}\,ds, \quad t \geq t_0, \tag{7.28}$$

with $y_{t_0} = \varphi_0 e^{-\lambda(t_0 + \cdot)}$. For each $k \geq 0$, by putting $x_k(t) = y_k(t)e^{\lambda t}$ for $t \geq t_0 - r$ and substituting it into (7.26), we know that $y_k(t)$ is bounded by $K(\|\varphi_0\| + F(t_0))e^{-\lambda t_0}$ on $[t_0, \infty)$. From the definition of $\{x_k\}$, $\{y_k\}$ satisfies

$$y_0(t) = e^{-\lambda t}\left\{\varphi_0(0) + \int_{t_0}^{t} f(s)\,ds\right\}, \quad (y_0)_{t_0} = \varphi_0 e^{-\lambda(t_0 + \cdot)}$$

and

$$y_{k+1}(t) = e^{-\lambda t}\int_{t_0}^{t} f(s)\,ds + \int_{t_0}^{t} \left(L_1(s, e^{\lambda \cdot}(y_k)_s) + L_2(s, e^{\lambda \cdot}(y_k)^s)\right)e^{\lambda(s - t)}ds$$

$$+ (\varphi_0(0) - D(t_0, \varphi_0))e^{-\lambda t} + D(t, e^{\lambda \cdot}(y_k)_t),$$

$$(y_{k+1})_{t_0} = \varphi_0 e^{-\lambda(t_0 + \cdot)}, \quad k = 0, 1, 2, \cdots \tag{7.29}$$

for $t \geq t_0 - r$. Then, by (7.19) and the Cauchy convergence principle, there is a bounded continuous function $y^*$ on $[t_0 - r, \infty)$ such that

$$\|y_k - y^*\| =_{def} \sup\{|y_k(t) - y^*(t)|: t \geq t_0 - r\} \to 0 \tag{7.30}$$

as $k \to \infty$. Let $k \to \infty$ in (7.29). It follows from (7.30) that $y^*(t)$ is a bounded solution of (7.28) with $(y^*)_{t_0} = \varphi_0 e^{-\lambda(t_0 + \cdot)}$. Since such a solution of (7.28) is unique, we must have $y^*(t) = y(t_0, \varphi_0, f)(t)$ for $t \geq t_0 - r$. Thus (7.27) holds uniformly for $t$ on any finite interval within $[t_0 - r, \infty)$.

If $f = 0$ and $a$ and $K$ are determined by (7.23) and (7.24) rather than (7.21) and (7.22), then, in the induction step,

$$|x_{k+1}(t)| \leq K\|\varphi_0\|e^{a(t-t_0)} \times$$

$$\left\{\|D\| + K^{-1}\left(1 + \|D\|e^{a(t_0-t)}\right) + \int_{t_0}^{t} \{\|L_1(s, .)\| + \|L_2(s, .)\|e^{a\sigma}\}e^{a(s-t)}\, ds\right\}$$

$$=_{def}\ \omega(t)K\|\varphi_0\|e^{a(t-t_0)}$$

for $t \geq t_0$ and

$$\omega(t) \leq \|D\| + (\lambda/a)(M_1(\lambda) + M_2(\lambda)e^{a\sigma})$$

$$+ \{K^{-1}(1 + \|D\|) - ((\lambda/a) - 1)(M_1(\lambda) + M_2(\lambda)e^{a\sigma})\}e^{a(t_0-t)} = 1.$$

Thus (7.25), with $F(t_0) = 0$, holds again.

If there is a sequence $\{\tau_n\}$ of ZAPs as in Theorem 7.2, then the uniqueness of the solution in $C([t_0 - r, \infty), \mathbb{C}^N)$ follows from (7.19) and Theorem 7.3. #

The next result deals with boundedeness, as well as the existence of finite limits at infinity, of the solutions.

**THEOREM** 7.9 Assume that

$$\|D\| + \int_{\tau}^{\infty} \{\|L_1(s, .)\| + \|L_2(s, .)\|\}ds < 1 \tag{7.31}$$

and that the integral of $f$ from $\tau$ to $t$ is bounded for $t \geq \tau$. Then, for every $(t_0, \varphi_0) \in [\tau, \infty) \times C(-r)$, (6.1) with $x_{t_0} = \varphi_0$ possesses a unique solution in the space of bounded continuous functions on $[t_0 - r, \infty)$. If, in addition, the integral of $f$ on $[\tau, \infty)$ is convergent, $\|D\| < 1/3$, and there is a $D_0$ such that

$$\lim_{t \to \infty} \|D(t, .) - D_0\| = 0, \tag{7.32}$$

then every bounded solution of (6.1) on $[t_0 - r, \infty)$ has a finite limit in $\mathbb{C}^N$ as $t \to \infty$. Moreover, such limits exhaust $\mathbb{C}^N$. If there is a sequence $\{\tau_n\}$ of ZAPs as in Theorem 7.2, then uniqueness also holds in $C([t_0 - r, \infty), \mathbb{C}^N)$.

**Proof**. For any $(t_0, \varphi_0) \in [\tau, \infty) \times C(-r)$, the existence, as well as uniqueness, of the bounded solution of (6.1) on $[t_0 - r, \infty)$ follows from (7.31), the boundedness of $\int_\tau^t f(s)\,ds$, and the proof of Theorem 6.2 with the trivial modification needed for $\lambda = 0$.

Suppose that $\int_\tau^\infty f(s)\,ds$ is convergent and that $D(t, .)$ satisfies $\|D\| < 1/3$ and (7.32). For any $t \geq t_0 + r$ and any $\theta \in [-r, 0]$, the integration of (6.1) from $t + \theta$ to $t$ yields

$$x(t) - x(t + \theta)$$

$$= D(t, x_t) - D(t + \theta, x_{t+\theta}) + \int_{t+\theta}^{t} \{L_1(s, x_s) + L_2(s, x^s) + f(s)\}\,ds$$

$$= D(t, x_t - x(t)) + D(t + \theta, x(t + \theta) - x_{t+\theta}) + D(t, I)(x(t) - x(t + \theta))$$

$$+ \big(D(t, I) - D(t + \theta, I)\big)x(t + \theta) + \int_{t+\theta}^{t} \{L_1(s, x_s) + L_2(s, x^s) + f(s)\}ds. \tag{7.33}$$

We introduce the following functions:

$$U(t) = \max\{|x(t) - x(t + \theta)|: -r \leq \theta \leq 0\}, \quad W(t) = \max\{U(t + \theta): -r \leq \theta \leq 0\},$$

and

$$V(t) = \|x_t\| \sup\{|D(t, I) - D(t + \theta, I)|: -r \leq \theta \leq 0\}$$

$$+ \max\left\{\left|\int_{t+\theta}^{t} \{L_1(s, x_s) + L_2(s, x^s) + f(s)\}\,ds\right|: -r \leq \theta \leq 0\right\}.$$

in order to show that $x(t)$ has a finite limit in $\mathbb{C}^N$ as $t \to \infty$. If $x(t)$ is a bounded solution of (6.1), then $U(t)$ and $W(t)$ are bounded and, by (7.31), (7.32) and the convergence of the integral of $f$ on $[\tau, \infty)$, $V(t) \to 0$ as $t \to \infty$. By the definition of $W$, there is a $\gamma \geq 0$ such that

$$\overline{\lim}_{t \to \infty} U(t) = \overline{\lim}_{t \to \infty} W(t) = \gamma.$$

Moreover, from (7.33) we derive

$$U(t) \leq 3\|D\|W(t) + V(t), \ t \geq t_0 + r. \tag{7.34}$$

Then, by letting $t \to \infty$, it follows from (7.34) that $\gamma \leq 3\|D\|\gamma$. We must have $\gamma = 0$ as $\|D\| < 1/3$. Now for any $p > 0$, replacing $\theta$ by $p$ in (7.33), we obtain

$$(1 - \|D\|)|x(t) - x(t + p)| \leq \|D\|(U(t) + U(t + p)) + |D(t, I) - D(t + p, I)|.|x(t + p)|$$
$$+ \left| \int_t^{t+p} \{L_1(s, x_s) + L_2(s, x^s) + f(s)\}ds \right|,$$

which implies

$$\lim_{t \to \infty} |x(t) - x(t + p)| = 0$$

uniformly for $p \geq 0$. Thus $\lim_{t \to \infty} x(t)$ exists by the Cauchy convergence principle.

Since (7.32) and $\|D\| < 1/3$ imply $\|D_0\| \leq \|D\| < 1/3$, then $|D_0(I)| \leq \|D_0\|.\|I\| < 1/3$. If $\lambda$ is an eigenvalue of $D_0(I)$ and $c_0$ is an eigenvector with $|c_0| = 1$, we have

$$|\lambda| = |\lambda c_0| = |D_0(I)c_0| \leq |D_0(I)|.|c_0| < 1/3,$$

which implies $\det(I - D_0(I)) \neq 0$. For any $\alpha \in \mathbb{C}^N$, we denote $(I - D_0(I))\alpha$ by $\beta$ and consider the equation

$$x(t) = \beta + D(t, x_t) - \int_t^{\infty} \{L_1(s, x_s) + L_2(s, x^s) + f(s)\}ds \tag{7.35}$$

for $t \geq \tau$. Since (7.31) holds, by applying the contraction mapping principle we can show that, for every $(t_0, \varphi_0) \in [\tau, \infty) \times C(-r)$ with $\varphi_0(0) = 0$, (7.35) with $x_{t_0} = x(t_0) + \varphi_0$ has a unique bounded solution on $[t_0 - r, \infty)$. Since each bounded solution of (7.35) is also a solution of (6.1) and every bounded solution of (6.1) has a finite limit at infinity, every bounded solution of (7.35) has a finite limit at infinity. Suppose such a solution satisfies $x(t) \to \delta$ as $t \to \infty$. Then the equality $\delta = D_0(I)\delta + \beta$ follows from (7.35) by letting $t \to \infty$. Thus

$$\delta = (I - D_0(I))^{-1}\beta = \alpha.$$

Hence, for every $\alpha \in \mathbb{C}^N$, (6.1) has a solution tending to $\alpha$ as $t \to \infty$.

Suppose that $\{\tau_n\}$ is a sequence of ZAPs as in Theorem 7.2. Since (7.31) implies (7.6) with $\lambda = 0$, the uniqueness in $C([t_0 - r, \infty), \mathbb{C}^N)$ is an immediate consequence of Corollary 7.4. #

**Remark**. For any $M > 0$, Theorem 7.9 also implies that all of the bounded solutions of (6.1) with $x_{t_0} = \varphi_0$, for $(t_0, \varphi_0) \in [\tau, \infty) \times C(-r)$ with $\|\varphi_0\| \le M$, have a common bound. Indeed, denoting the unique bounded solution by $x(t_0, \varphi_0, f)(t)$, its bound by $\|x(t_0, \varphi_0, f)\|$, and the bound of the integral of $f$ from $\tau$ to $t$ by $F$, we see that

$$|x(t_0, \varphi_0, f)(t)| \le (1 + \|D\|)\|\varphi_0\| + \|D\|.\|x(t_0, \varphi_0, f)\| + \left|\int_{t_0}^{t} f(s)\,ds\right| + \int_{t_0}^{t} \{\|L_1(s, .)\| + \|L_2(s, .)\|\}\,ds\,\|x(t_0, \varphi_0, f)\|$$

$$\le (1 + \|D\|)M + 2F + \left\{\|D\| + \int_{\tau}^{\infty} \{\|L_1(s, .)\| + \|L_2(s, .)\|\}\,ds\right\}\|x(t_0, \varphi_0, f)\|$$

for $t \ge t_0$. Then

$$\|x(t_0, \varphi_0, f)\| \le \left\{1 - \|D\| - \int_{\tau}^{\infty} \{\|L_1(s, .)\| + \|L_2(s, .)\|\}\,ds\right\}^{-1} \{(1 + \|D\|)M + 2F\}$$

for all $(t_0, \varphi_0) \in [\tau, \infty) \times C(-r)$ with $\|\varphi_0\| \le M$. In this sense, we may say that the solutions of (6.1) are bounded uniformly in $(t_0, \varphi_0)$ if there is a sequence $\{\tau_n\}$ as in Theorem 7.2.

We notice that Theorem 7.9 applies to the homogeneous equation (6.2) when $f = 0$. From the above proof and the remark, we obtain a further conclusion for (6.2).

**COROLLARY** 7.10 Assume that (7.31) holds and that $\{\tau_n\}$ is as in Theorem 7.2. Then (6.2) is uniformly stable but not asymptotically stable.

**Proof.** Since the uniform stability of (6.2) is equivalent to the uniform boundedness of the solutions of (6.2) in the above-mentioned sense, by Theorem 7.9 and its remark (6.2) is uniformly stable. For any $\beta \neq 0$ and $(t_0, \varphi_0) \in [\tau, \infty) \times C(-r)$ with $\varphi_0(0) = 0$, (7.31) implies that the equation

$$x(t) = \beta + D(t, x_t) - \int_t^\infty \{L_1(s, x_s) + L_2(s, x^s)\}ds \tag{7.36}$$

with $x_{t_0} = x(t_0) + \varphi_0$ has a unique bounded solution on $[t_0 - r, \infty)$, which is also a solution of (6.2). Since $\beta \neq 0$ implies that the solution does not tend to zero as $t \to \infty$, (6.2) is not asymptotically stable. #

# PART THREE
# ASYMPTOTIC SOLUTIONS AND ASYMPTOTIC REPRESENTATION OF SOLUTIONS

In the theory of asymptotic solutions of linear differential equations, there are two classes of equations which are fundamental and much studied.

One is the asymptotically diagonal equation

$$\frac{d}{dt}x(t) = \{\Lambda(t) + R(t)\}\,x(t), \tag{1*}$$

where $\Lambda = dg[\lambda_1, \lambda_2, \cdots, \lambda_N]$ is a diagonal matrix and $R$ an $N \times N$ matrix of locally integrable functions from $[\tau, \infty)$ to $\mathbb{C}$. In the various available results for asymptotic solutions of (1*) (see, for example, Chapter 1 in Eastham [12]), it is commonly assumed that $R(t)$ is in some sense small as $t \to \infty$, and it is expected that there exist $N$ independent solutions of (1*) with asymptotic forms

$$x_n(t) = \{e_n + o(1)\}\exp\left(\int_\tau^t \tilde{\lambda}_n(s)\,ds\right), \quad n = 1, 2, \cdots, N, \tag{2*}$$

where $e_n$ is the $n$th coordinate vector and $\tilde{\lambda}_n - \lambda_n$ is small as $t \to \infty$.

Another is the asymptotically autonomous retarded functional differential equation

$$\frac{d}{dt}x(t) = L(x_t) + F(t, x_t), \tag{3*}$$

where $L$ and, for each $t \geq \tau$, $F(t, .)$ are bounded linear operators from $C([-r, 0], \mathbb{C}^N)$ to $\mathbb{C}^N$, and $\|F(t, .)\|$ is small as $t \to \infty$. A typical result says that, if $\mu$ is a root of the characteristic equation

$$\det \Delta(\lambda) = 0 \text{ with } \Delta(\lambda) = \lambda I - L(e^{\lambda \cdot}) \tag{4*}$$

satisfying certain conditions (see the details in Hale [16]), then (3*) has a solution with the form

$$x(t) = \{v + o(1)\}\, e^{\mu t + s(t)} \qquad (5^*)$$

as $t \to \infty$, where $v \in \mathbb{C}^N$ is a constant vector depending on $x$ and $s(t)$ is small and can be expressed in some way. Cooke [7] extended this result to a broader class of equations including a term $A(t)\,(x(t - r(t)) - x(t - r'))$, where $A(t)$ is a bounded $N \times N$ matrix, $r(t) \in C([\tau, \infty), [0, r])$, $r' \in [0, r]$, and $r(t) - r'$ is small as $t \to \infty$.

One of the special cases of (3*) is the equation

$$\frac{d}{dt}x(t) = \Lambda\, x(t) + F(t, x_t), \qquad (6^*)$$

where $\Lambda = dg[\lambda_1, \lambda_2, \cdots, \lambda_N]$ is a constant diagonal matrix and $\|F(t, .)\|$ is small as $t \to \infty$. Then (6*) has N solutions of the form (5*) which, in this case, are consistent with the forms (2*). We refer to Haddock and Sacker [15], Arino and Györi [2], and Ai [1] for detailed representations of such solutions.

A common feature of the results for the two classes of equations is that, when the equation is in some sense asymptotic to a simpler equation, it has solutions asymptotic to those of the simpler one. On the other hand, the essential difference due to the infinite dimension of the solution space of (3*) seems not clearly emphasized in the above references. Since the solution space of (1*) is N dimensional, it is well represented by the N solutions in (2*). However, (6*) has infinitely many solutions independent of those N solutions of the form (2*) and their asymptotic behaviour draws our attention.

In a recent paper, Cassell and Hou [4] considered the equation

$$\frac{d}{dt}x(t) = \Lambda(t)x(t) + F(t, x_t), \qquad (7^*)$$

which can be viewed as a generalisation of both (1*) and (6*). Under appropriate conditions, they not only proved the existence of the N solutions in (2*) but also pointed out the order of smallness of the solutions independent of those in (2*) as

$t \to \infty$. Moreover, successive approximations of $\tilde{\lambda}_n$, $n = 1, 2, \cdots, N$, in (2*) were also given when $\|F(t, .)\| \in L^p(\tau, \infty)$ for some $p \in [1, \infty)$.

In this part of the thesis, we shall investigate the asymptotic behaviour of solutions of the equation

$$\frac{d}{dt}\{x(t) - D(t, x_t)\} = A(t)\,x(t) + L_1(t, x_t) + L_2(t, x^t), \tag{8*}$$

where $A(t)$ is an $N \times N$ matrix of locally integrable functions and the mappings $D(t, .)$, $L_1(t, .)$ and $L_2(t, .)$ are as in Part II (p.76). We assume throughout the whole part that existence and uniqueness of the solution of (8*) with $x_T = \varphi_0$ on $[T\text{-}r, \infty)$ hold for every $T \geq \tau$ and $\varphi_0 \in C$. We also assume that $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$, is a sequence of ZAPs of (8*).

Under some smallness conditions on $D(t, .)$, $L_1(t, .)$ and $L_2(t, .)$, we shall show that (8*) has N independent solutions $x_1(t)$, $x_2(t)$, $\cdots$, $x_N(t)$ which asymptotically approach, or behave like, those of the equation

$$\frac{d}{dt}x(t) = A(t)\,x(t) \tag{9*}$$

as $t \to \infty$. Further, we shall indicate the order of smallness of the solutions independent of the N special solutions.

Three chapters ( 8 - 10) are included in this part. In Chapter 8, we study the equation

$$\frac{d}{dt}x(t) = A\,x(t - r(t)) \tag{10*}$$

as a special case of (8*), where A is an $N \times N$ constant matrix and $r(t)$, not necessarily non-negative as some previous work has required (refer to §8.1), is small at infinity. We obtain a representation of the N special solutions of (10*) in terms of A and $r(t)$.

Chapter 9 deals with the special case of (8*) when $A(t) = \Lambda(t)$ is diagonal. Results parallel to those in [4] will be obtained.

In Chapter 10, we assume that $A(t)$ is bounded. We shall prove that a matrix solution

$$X(t, t_0) = (x_1(t, t_0), x_2(t, t_0), \cdots, x_N(t, t_0))$$

composed of N independent solutions $x_1, x_2, \cdots, x_N$ exists and behaves like that of (9*), though we can not give an explicit representation.

Since (10*) can be viewed as (8*) with

$$L_1(t, x_t) + L_2(t, x^t) = A(x(t - r(t)) - x(t))$$

and the equations in Chapters 9 and 10 do not cover such a case, we shall point out the possibility of extending the results of Chapters 8-10 to a broader class of equations in the conclusion of this thesis.

# CHAPTER 8
# ASYMPTOTIC SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH A SMALL DEVIATION

## § 8.1 INTRODUCTION

In this chapter, we investigate the asymptotic behaviour of solutions of the equation

$$\frac{d}{dt}x(t) = A\, x(t - r(t)), \tag{8.1}$$

where $A$ is an $N \times N$ constant matrix, $r(t) \in C([\tau, \infty), [-\sigma, \rho])$ for some constants $\sigma > 0$ and $\rho > 0$, and $r(t) \to 0$ as $t \to \infty$. We stress that the condition $r(t) \geq 0$ in the references quoted below is not required here. The results for (8.1) can be extended to the scalar equation

$$\frac{d}{dt}x(t) = a(t)\, x(t) + b(t)\,(x(t - r(t)) - x(t)), \tag{8.2}$$

where $a(t)$ and $b(t)$ are locally integrable functions, as well as to some other vector equations.

Cooke [6] studied the particular case (8.2) when $a(t) = b(t) = a$ is a constant, and showed that every solution of (8.2) has the asymptotic form

$$x(t) = \left\{ \exp\left( - at + a^2 \int_0^t r(s)\, ds \right) \right\} (c + o(1)) \tag{8.3}$$

as $t \to \infty$ if $r(t) \in [0, \rho]$, $r(t) \to 0$ as $t \to \infty$, and either $r(t) \in L^p(\tau, \infty)$ for some $p \in (1, 2)$ with $r'(t)$ bounded or $r(t) \in L(\tau, \infty)$. Kato [20] generalised the result to (8.1) when $r(t) \in [0, \rho]$, and either $r(t) \in L(\tau, \infty)$ or $r(t) \in L^2(\tau, \infty)$ and, in the latter case, $r(t)$ is either Lipschitzian or monotone for large $t$. This is not entirely satisfactory as it does not cover the case of $p > 2$. Moreover, when $c = 0$ in (8.3), the order of smallness of $x(t)$ is not clear enough. Further, the restriction $r(t) \geq 0$ can be eliminated. Our purpose is to try to sort out these problems for (8.1) and (8.2).

In this chapter, we replace $r$ by $\rho$ in the usual definition of the space $C$, i.e. $C \equiv C([-\rho, 0], \mathbb{C}^N)$, in order to avoid any confusion with $r(t)$.

## § 8.2 MAIN RESULTS

The usual way to prove that solutions of an equation have a certain asymptotic form is to change the equation to one whose solutions have finite limits in $\mathbb{C}^N$ as $t \to \infty$. As we shall see later, (8.1) can be transformed to an equation of the form

$$\frac{d}{dt}x(t) = F(t)\,x(t) + G(t)\,(x(t - r(t)) - x(t)), \tag{8.4}$$

where $F(t) \in L(\tau, \infty)$, $G(t) \in L^\infty(\tau, \infty)$, and both are $N \times N$ matrices. Accordingly, we give a result for (8.4) before we tackle (8.1).

For any $p \in [1, \infty]$ and any $f \in L^p(\tau, \infty)$, we denote the norm of $f$ by $\|f\|_p$ or, to mark the relevance of $[T, \infty)$, by $\|f\|_p(T, \infty)$. Since $r(t) \to 0$ as $t \to \infty$, for any $T \geq \tau$ the function

$$\varepsilon(T) = \sup\{\, |r(t)|: \ t \geq T \,\} \tag{8.5}$$

tends to zero as $T \to \infty$.

**THEOREM** 8.1 Assume that $F(t) \in L(\tau, \infty)$, $G(t) \in L^\infty(\tau, \infty)$, and $r(t) \to 0$ as $t \to \infty$. Then, for $T \geq \tau$ satisfying

$$\|F\|_1(T, \infty) + 2\,\varepsilon(T)\,\|G\|_\infty(T, \infty) < 1, \tag{8.6}$$

(8.4) has $N$ solutions on $[T\text{-}r, \infty)$ with the forms

$$x_n(t) = e_n + o(1), \quad n = 1, 2, \cdots, N, \tag{8.7}$$

as $t \to \infty$, where $e_n$ is the $n$th coordinate vector. Furthermore, every solution of (8.4) has the asymptotic representation

$$x(t) = X(t)c + o(e^{-\beta t}), \tag{8.8}$$

where $X(t) = (x_1(t), x_2(t), \cdots, x_N(t))$, $c \in \mathbb{C}^N$ is a constant depending on $x$, and $\beta \geq 0$ is arbitrary.

The proof of Theorem 8.1 is left to § 8.3. With the aid of Theorem 8.1, we can prove the following result for (8.1). Let

$$\eta_0(t) = A, \quad \eta_{k+1}(t) = A \exp\left(\int_t^{t-\omega} \eta_k(s)\, ds\right), \quad k = 0, 1, 2, \cdots \tag{8.9}$$

for $t \geq \tau$ and let all of them vanish for $t < \tau$.

**THEOREM** 8.2 Assume that $r(t) \to 0$ as $t \to \infty$ and $\|r_t\| \in L^p(\tau, \infty)$ for some $p \in [1, \infty)$. Then, for sufficiently large $T \geq \tau$, (8.1) has $N$ solutions on $[T - r, \infty)$ with the forms

$$x_n(t) = \left\{\exp\left(\int_T^t \eta_M(s)\, ds\right)\right\}(e_n + o(1)), \quad n = 1, 2, \cdots, N, \tag{8.10}$$

as $t \to \infty$, where $M$ satisfies $M < p \leq M + 1$ and $\eta_M(t)$ is determined by (8.9). Moreover, every solution of (8.1) has the form (8.8) as $t \to \infty$, where $X(t)$ is composed of the $N$ solutions in (8.10).

**Proof.** Since each $\eta_k(t)$ in (8.9) can be expanded as a power series in $A$ with scalar functions as coefficients, they all commute with $A$ and with each other. It is clear from (8.9) that each $\eta_k(t)$ is bounded. Let us first show that

$$\|(\eta_k - \eta_{k-1})_t\| \in L^{p(k)}(\tau, \infty) \tag{8.11}$$

for $k = 1, 2, \cdots$, where $p(k) = \max(1, pk^{-1})$ and as usual, $(\eta_k - \eta_{k-1})_t(\theta) = (\eta_k - \eta_{k-1})(t + \theta)$ for $\theta \in [-\rho, 0]$ and $\|.\|$ is the norm in $C$. From (8.9) we have

$$|\eta_1(t) - \eta_0(t)| = |A\,(e^{-Ar(t)} - I)| \le c_1\,|r(t)|$$

for some $c_1 > 0$. Then, by the assumption on $r(t)$, $\|(\eta_1 - \eta_0)_t\| \in L^p(\tau, \infty)$, i.e., (8.11) holds for $k = 1$. If (8.11) holds for $k$, then

$$|\eta_{k+1}(t) - \eta_k(t)| = \left| A\left[\exp\left(\int_t^{t-r(t)} \eta_k(s)\,ds\right) - \exp\left(\int_t^{t-r(t)} \eta_{k-1}(s)\,ds\right)\right]\right|$$

$$\le c_2\left|\int_t^{t-r(t)} (\eta_k(s) - \eta_{k-1}(s))\,ds\right| \le c_2|r(t)|.\|\eta_k - \eta_{k-1}\|_{[t-\rho,\,t+\sigma]}$$

for some $c_2 > 0$, where

$$\|\eta_k - \eta_{k-1}\|_{[t-\rho,\,t+\sigma]} = \max\{|(\eta_k - \eta_{k-1})(s)| : s \in [t-\rho, t+\sigma]\}.$$

Since $\|(\eta_k - \eta_{k-1})_t\| \in L^{p(k)}(\tau, \infty)$ implies $\|\eta_k - \eta_{k-1}\|_{[t-r',\,t+r'']} \in L^{p(k)}(\tau, \infty)$ for any $r' > 0$ and $r'' > 0$, it follows that $\|(\eta_{k+1} - \eta_k)_t\| \in L^{p(k+1)}(\tau, \infty)$. Then we have proved (8.11) for $k = 1, 2, \cdots$.

By transforming (8.1) with

$$x(t) = \left\{\exp\left(\int_\tau^t \eta_M(s)\,ds\right)\right\} y(t), \tag{8.12}$$

we obtain

$$\frac{d}{dt}y(t) = (\eta_{M+1}(t) - \eta_M(t))\,y(t) + \eta_{M+1}(t)\,(y(t - r(t)) - y(t)). \tag{8.13}$$

Since $(\eta_{M+1} - \eta_M) \in L(\tau, \infty)$ and $\eta_{M+1}(t)$ is bounded, by Theorem 8.1 and (8.12) the proof of the theorem is complete. #

**Remark** (i). From Theorem 8.1 and the proof of Theorem 8.2, we can give a condition in terms of the value $T \ge \tau$ for which the $N$ solutions of (8.1) in (8.10) exist on $[T-r, \infty)$. In fact, for any $T \ge \tau + M\rho$, we let $c_0(T) = |A|$ and

$$c_{k+1}(T) = |A|\exp\{\varepsilon(T - M\rho)c_k(T)\},\ k = 0, 1, \cdots, M. \tag{8.14}$$

Then, from (8.9) and (8.14), $\|\eta_0\|_\infty = c_0(T)$ and

$$\|\eta_{k+1}\|_{\infty}(T-(M-k)\rho, \infty) = \sup\{|\eta_{k+1}(t)| : t \ge T-(M-k)\rho\} \le c_{k+1}(T)$$

for $k = 0, 1, \cdots, M$, where $\varepsilon(t)$ is given by (8.5). Since (8.14) implies

$$c_0(T) \le c_1(T) \le \cdots \le c_{M+1}(T)$$

and for any $\alpha, \beta \in \mathbb{C}$,

$$|e^{\alpha} - e^{\beta}| \le e^{\max(|\alpha|, |\beta|)}|\alpha - \beta|,$$

for $t \ge T$ we have

$$\begin{aligned}
|\eta_{M+1}(t) - \eta_M(t)| &\le |A|\, e^{\varepsilon(T)c_M(T)} \left| \int_{t}^{t+\sigma} (\eta_M(s) - \eta_{M-1}(s))\, ds \right| \\
&\le c_{M+1}(T)\,|r(t)|.\|\eta_M - \eta_{M-1}\|_{[t-\rho, t+\sigma]} \\
&\le c_{M+1}(T)\,|r(t)|\, c_M(T)\, \|r\|_{[t-\rho, t+\sigma]}\|\eta_{M-1} - \eta_{M-2}\|_{[t-2\rho, t+2\sigma]} \\
&\le \left(c_{M+1}(T)\, \|r\|_{[t-\rho, t+\sigma]}\right)^2 \|\eta_{M-1} - \eta_{M-2}\|_{[t-2\rho, t+2\sigma]} \\
&\le \left(c_{M+1}(T)\, \|r\|_{[t-(M-1)\rho, t+(M-1)\sigma]}\right)^M \|\eta_1 - \eta_0\|_{[t-M\rho, t+M\sigma]} \\
&\le \left(c_{M+1}(T)\, \|r\|_{[t-M\rho, t+M\sigma]}\right)^{M+1}. \qquad (8.15)
\end{aligned}$$

By (8.14), (8.15) and (8.6), $T \ge \tau + Mr$ is a suitable value if it satisfies

$$(c_{M+1}(T))^{M+1} \int_T^{\infty} \left( \|r\|_{[t-M\rho, t+M\sigma]} \right)^{M+1} dt + 2\varepsilon(T)\, c_{M+1}(T) < 1. \qquad (8.16)$$

**Remark** (ii). For any particular number $p \in [1, \infty)$, we can further simplify the expressions in (8.10). If $p = 1$, then $M = 0$ and

$$\exp\left( \int_0^t \eta_M(s)\, ds \right) = e^{At}.$$

If $p \in (1, 2]$, then $M = 1$ and $\eta_1(t) = Ae^{-Ar(t)} = A - A^2 r(t) + L$, where we let $L$ stand for any function in $L(T, \infty)$, not necessarily the same at each appearance. Hence

$$\left\{\exp\left(\int_0^t \eta_1(s)\,ds\right)\right\}(I + o(1)) = \left\{\exp\left(At - A^2\int_0^t r(s)\,ds\right)\right\}(I + o(1))\,B$$

for some constant matrix B. In this case, we can replace (8.10) by

$$x_n(t) = \left\{\exp\left(At - A^2\int_0^t r(s)\,ds\right)\right\}(e_n + o(1)), \quad n = 1, 2, \cdots, N. \tag{8.17}$$

If $p \in (2, 3]$, then $M = 2$ and

$$\begin{aligned}\eta_2(t) &= A\exp\left(\int_t^{t+\omega} Ae^{-Ar(s)}\,ds\right)\\ &= A + A^2\int_t^{t+\omega} e^{-Ar(s)}\,ds + \frac{1}{2}A^3\left(\int_t^{t+\omega} e^{-Ar(s)}\,ds\right)^2 + L\\ &= A - A^2 r(t) - A^3\int_t^{t+\omega} r(s)\,ds + \frac{1}{2}A^3 r^2(t) + L.\end{aligned}$$

Thus we can replace (8.10) by

$$x_n(t) = \left\{\exp\left[At - A^2\int_0^t ds\left(r(s) - \frac{1}{2}Ar^2(s) + A\int_s^{s+\omega} r(u)\,du\right)\right]\right\}(e_n + o(1)),$$
$$n = 1, 2, \cdots, N. \tag{8.18}$$

Step by step we can simplify (8.10) for $M = 3, 4, \cdots$.

**Remark** (iii). Kato [20] requires $0 \le r(t) \le \rho$ while we require that $-\sigma \le r(t) \le \rho$, $r(t) \to 0$ as $t \to \infty$, and (8.1) has a sequence $\{t_n\}$ of ZAPs. Setting aside this difference, we compare our condition and result with Kato's (see § 8.1) when p satisfies $1 \le p \le 2$. From the proof of Theorem 8.2 we can see that the condition $\|r_t\| \in L(\tau, \infty)$ can be replaced by $r(t) \in L(\tau, \infty)$ if $p = 1$. In this case, our condition coincides with Kato's. Since $r(t)$ is bounded, the condition $\|r_t\| \in L^p(\tau, \infty)$ for some p in $(1, 2]$ implies $\|r_t\| \in L^2(\tau, \infty)$. For this case, our conditions and Kato's are independent: the condition $\|r_t\| \in L^2(\tau, \infty)$ always implies $r(t) \in L^2(\tau, \infty)$ (and is equivalent to it when $r(t)$ is monotone), but it is unrelated to any Lipschitz condition. When $p$ is in $[1, 2]$, Kato's result states that every solution of (8.1) has the form

$$x(t) = \left\{\exp\left(At - A^2\int_0^t r(s)\,ds\right)\right\}(c + o(1)) \tag{8.19}$$

as $t$ tends to infinity. When $c = 0$, (8.19) does not give the order of smallness of $x(t)$ unlike our result.

## § 8.3 THE PROOF OF THEOREM 8.1

Consider the integral equation

$$x(t) = e_k - \int_t^\infty \{F(s)x(s) + G(s)(x(s - r(s)) - x(s))\}\,ds \tag{8.20}$$

for $t \geq \tau$ and each $k \in \{1, 2, \cdots, N\}$.

**LEMMA 8.3** Assume that $F(t)$, $G(t)$ and $r(t)$ satisfy the conditions of Theorem 8.1 and that $T \geq \tau$ satisfies (8.6). Then, for each $k$ in $\{1, 2, \cdots, N\}$, (8.20) has a solution on $[T - \rho, \infty)$ satisfying $x_k(t) = x_k(T)$ for $t \in [T - \rho, T]$ and $x_k(t) = e_k + o(1)$ as $t \to \infty$.

**Proof.** Let $S \subset C([T - \rho, \infty), \mathbb{C}^N)$ such that each $x \in S$ satisfies $x(t) = x(T)$ for $t \in [T - \rho, T]$, $x \in L^\infty(T, \infty)$, and $x(t - r(t)) - x(t) \in L(T, \infty)$. Define

$$\|x\|_S = \max\left\{\|x\|_\infty, \|x\|_0 = \int_T^\infty |x(t - r(t)) - x(t)|\,dt\right\}.$$

Then $S$ with the norm $\|.\|_S$ is a Banach space. Let $k$ in $\{1, 2, \cdots, N\}$ be fixed.

For any x in S, we define $Ax(t)$ by the right hand side of (8.20) for $t \geq T$ and $Ax(t) = Ax(T)$ for $t \in [T - \rho, T]$. Then $Ax(t)$ is continuous and bounded. Put $t^* = \max(t, T)$. Then, for $t \geq T$, we have

$$|Ax(t - r(t)) - Ax(t)| = \left| \int_t^{(t - r(t))^*} \{F(s)x(s) + G(s)(x(s - r(s)) - x(s))\} \, ds \right|$$

$$\leq \int_{(t - \varepsilon(t))^*}^{t + \varepsilon(t)} |F(s)x(s) + G(s)(x(s - r(s)) - x(s))| \, ds \in L(T, \infty).$$

Thus $A$ maps S into itself.

Let $u = x_1 - x_2$ and $\chi = Ax_1 - Ax_2$ for $x_1, x_2 \in S$. Then, for $t \geq T$,

$$|\chi(t)| = \left| \int_t^{\infty} \{F(s)u(s) + G(s)(u(s - r(s)) - u(s))\} \, ds \right|$$

$$\leq \|F\|_1(T, \infty) \, \|u\|_\infty + \|G\|_\infty(T, \infty) \, \|u\|_0 \tag{8.21}$$

and

$$|\chi(t - r(t)) - \chi(t)| = \left| \int_t^{(t - r(t))^*} \{F(s)u(s) + G(s)(u(s - r(s)) - u(s))\} \, ds \right|$$

$$\leq \int_{(t - \varepsilon(t))^*}^{t + \varepsilon(t)} |F(s)u(s) + G(s)(u(s - r(s)) - u(s))| \, ds .$$

Hence, by defining the above integrand to be zero for $s < T$, we obtain

$$\|\chi\|_0 \leq \int_T^{\infty} dt \int_{-\varepsilon(T)}^{\varepsilon(T)} |F(s + t)u(s + t) + G(s + t)(u(s + t - r(s + t)) - u(s + t))| \, ds$$

$$\leq 2\varepsilon(T) \int_T^{\infty} |F(s) \, u(s) + G(s) \, (u(s - r(s)) - u(s))| \, ds$$

$$\leq 2\varepsilon(T) \left\{ \|F\|_1(T, \infty) \, \|u\|_\infty + \|G\|_\infty(T, \infty) \, \|u\|_0 \right\}. \tag{8.22}$$

Thus

$$\|Ax_1 - Ax_2\|_\infty \le \left(\|F\|_1(T,\infty), \|G\|_\infty(T,\infty)\right)\begin{bmatrix} \|x_1 - x_2\|_\infty \\ \|x_1 - x_2\|_0 \end{bmatrix} \tag{8.21*}$$

and

$$\|Ax_1 - Ax_2\|_0 \le 2\varepsilon(T)\left(\|F\|_1(T,\infty), \|G\|_\infty(T,\infty)\right)\begin{bmatrix} \|x_1 - x_2\|_\infty \\ \|x_1 - x_2\|_0 \end{bmatrix}. \tag{8.22*}$$

Define U and V as follows:

$$U = \begin{bmatrix} \|F\|_1(T,\infty) & \|G\|_\infty(T,\infty) \\ 2\varepsilon(T)\,\|F\|_1(T,\infty) & 2\varepsilon(T)\,\|G\|_\infty(T,\infty) \end{bmatrix}, \quad V = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then, from (8.21*) and (8.22*), we derive

$$\|A^n x_1 - A^n x_2\|_\infty \le \left(\|F\|_1(T,\infty), \|G\|_\infty(T,\infty)\right) U^{n-1} V\, \|x_1 - x_2\|_S, \tag{8.23a}$$

$$\|A^n x_1 - A^n x_2\|_0 \le 2\varepsilon(T)\left(\|F\|_1(T,\infty), \|G\|_\infty(T,\infty)\right) U^{n-1} V\, \|x_1 - x_2\|_S \tag{8.23b}$$

for $n = 1, 2, \cdots$. Since (8.6) ensures that the eigenvalues of $U$ satisfy $|\mu_i| < 1$, $i = 1, 2$, which implies $U^n \to 0$ as $n \to \infty$, there is an $m \ge 1$ such that

$$\max\{1, 2\varepsilon(T)\}\left(\|F\|_1(T,\infty), \|G\|_\infty(T,\infty)\right) U^{m-1} V < 1.$$

Then it follows from (8.23) that $A^m$ on S is contractive. By a contraction mapping principle (see Theorem 5.2.1 in Smart [35]), (8.20) has a solution $x_k(t)$ in S and (8.20) itself implies $x_k(t) = e_k + o(1)$ as $t \to \infty$ #

**LEMMA** 8.4 Assume that the following conditions hold:

(i) $\alpha$: $[t_0, \infty) \to (0, 1)$ is decreasing,

(ii) $f \in C([t_0, \infty), [0, \infty))$ and there is a $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that, for each $t_n \ge t_0 + \rho$ and all $t \in [t_0 + \rho, t_n]$,

$$f(t) \le \alpha(t)\max\{f(s):\ t - \rho \le s \le t_n\}.$$

Then, for $t \ge t_0 + \rho$, $f(t)$ satisfies

$$f(t) \le \max\{f(s):\ t_0 \le s \le t_0 + \rho\} \exp\left( \int_{t_0+\rho}^{t} \frac{1}{\rho} \ln \alpha(s)\, ds \right). \tag{8.24}$$

**Proof**. For any $t_n \ge t_0 + \rho$ and any $t \in [t_0 + \rho, t_n]$, (i) and (ii) imply that

$$f(t) \le \max\{f(s):\ t \le s \le t_n\} \le \alpha(t) \max\{f(s):\ t - \rho \le s \le t_n\}.$$

Suppose $\max\{f(s):\ t - \rho \le s \le t_n\} > 0$. Then, since $\alpha(t) < 1$, we must have

$$\max\{f(s):\ t \le s \le t_n\} < \max\{f(s):\ t - \rho \le s \le t_n\},$$

from which it follows that

$$\max\{f(s):\ t - \rho \le s \le t_n\} = \max\{f(s):\ t - \rho \le s \le t\}. \tag{8.25}$$

Equality (8.25) is obvious if $\max\{f(s):\ t - \rho \le s \le t_n\} = 0$. Hence, for $t \in [t_0 + \rho, t_n]$ we obtain

$$f(t) \le \alpha(t) \max\{f(s):\ t - \rho \le s \le t\}. \tag{8.26}$$

Since $t_n \to \infty$ as $n \to \infty$, (8.26) actually holds for $t \ge t_0 + \rho$. Again, applying the same reasoning as above to (8.26), we obtain

$$\max\{f(s):\ t \le s \le t + \rho\} \le \alpha(t) \max\{f(s):\ t - \rho \le s \le t\} \tag{8.27}$$

for $t \ge t_0 + \rho$. As each $t \ge t_0 + \rho$ has a unique representation $t = t_1 + k\rho$, where $t_1 \in [t_0, t_0 + \rho)$ and $k \ge 1$ is an integer, on applying (8.27) to (8.26) repeatedly, we have

$$\begin{aligned} f(t) &\le \alpha(t_1 + k\rho) \max\{f(t_1 + k\rho + s):\ -\rho \le s \le 0\} \\ &\le \prod_{i=1}^{k} \alpha(t_1 + i\rho) \max\{f(t_1 + \rho + s):\ -\rho \le s \le 0\} \\ &\le \max\{f(s):\ t_0 \le s \le t_0 + 2\rho\} \exp\left\{ \sum_{i=1}^{k} \ln \alpha(t_1 + i\rho) \right\} \\ &\le \max\{f(s):\ t_0 \le s \le t_0 + \rho\} \exp\left( \int_{t_1}^{t_1+k\rho} \frac{1}{\rho} \ln \alpha(s)\, ds \right) \end{aligned}$$

$$\leq \max\{f(s):\ t_0 \leq s \leq t_0 + \rho\} \exp\left(\int_{t_0+\rho}^{t} \frac{1}{\rho} \ln \alpha(s)\, ds\right)$$

Thus (8.24) holds for $t \geq t_0 + \rho$. #

With the help of Lemmas 8.3 and 8.4 and an adaptation of the method used in Driver [10], we now complete the proof of Theorem 8.1.

**Proof of Theorem 8.1**. Since each solution of (8.20) is also a solution of (8.4), by Lemma 8.3 (8.4) has $N$ solutions $x_n(t)$, $n = 1, 2, \cdots, N$, of the form (8.7) on $[T - \rho, \infty)$ and

$$X(t) = (x_1(t), x_2(t), \cdots, x_N(t))$$

is a matrix solution of (8.4). The matrix $X^{-1}(t)$ exists for large $t \geq T$ and $X^{-1}(t) \to I$ as $t \to \infty$ since $X(t) \to I$ as $t \to \infty$. Without loss of generality, we assume that $X^{-1}(t)$ is bounded on some interval $[T_1 - 2\rho, \infty) \subset [T - \rho, \infty)$ and $x(t) = X(t)y(t)$ for $t \geq T_1 - 2\rho$. Then, by substitution, (8.4) is changed to

$$\left(\frac{d}{dt}X(t)\right) y(t) + X(t)\frac{d}{dt}y(t) = F(t)X(t)y(t) + G(t)\,(X(t - r(t))y(t - r(t)) - X(t)y(t)).$$

Since $X(t)$ satisfies

$$\frac{d}{dt}X(t) = F(t)\, X(t) + G(t)\,(X(t - r(t)) - X(t)),$$

by substitution again we obtain

$$\frac{d}{dt}y(t) = X^{-1}(t)G(t)X(t - r(t))\,(y(t - r(t)) - y(t)) \tag{8.28}$$

for $t \geq T_1 - \rho$. Integration of (8.28) from $t$ to $t - r(t)$ leads to

$$z(t) = \int_{t}^{t-r(t)} X^{-1}(s)G(s)X(s - r(s))z(s)\, ds \tag{8.29}$$

for $t \geq T_1$, where $z(t) = y(t - r(t)) - y(t)$. Put

$$\alpha(t) = \|X^{-1}G\|_{\infty}\|X\|_{\infty}\varepsilon(t),$$

where $\varepsilon(t)$ is given by (8.5). Since $\alpha(t) \downarrow 0$ as $t \to \infty$ and $T_1 \geq T$ can be as large as we choose, we also assume $\alpha(t) < 1$ for $t \geq T_1$. We recall that the existence of $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, a sequence of ZAPs of the discussed equation, is assumed. Thus, for any $t_n \geq T_1$ and $t \in [T_1, t_n]$, (8.29) implies

$$|z(t)| \leq \alpha(t) \max\{|z(s)|: \ t - \rho \leq s \leq t_n\}. \tag{8.30}$$

Then the application of Lemma 8.4 yields

$$|z(t)| \leq c_0 \exp\left(\int_{T_1+\rho}^{t} \frac{1}{\rho} \ln \alpha(s)\, ds\right) \tag{8.31}$$

for $t \geq T_1 + \rho$, where $c_0 = \max\{|z(s)|: T_1 \leq s \leq T_1 + \rho\}$. Hence, for any $\delta > 0$ and $t \geq T_1 + \rho$, (8.28) and (8.31) imply

$$\left|y(t+\delta) - y(t)\right| \leq c_0 \|X^{-1}G\|_\infty \|X\|_\infty \int_t^{t+\delta} ds \left(\exp\left(\int_{T_1+\rho}^{s} \rho^{-1} \ln \alpha(\vartheta)\, d\vartheta\right)\right).$$

As $\alpha(t)$ is decreasing and $\alpha(t) < 1$,

$$\int_t^{t+\delta} ds \left(\exp\left(\int_{T_1+\rho}^{s} \rho^{-1} \ln \alpha(\vartheta)\, d\vartheta\right)\right)$$

$$\leq \left(\exp\left(\int_{T_1+\rho}^{t} \rho^{-1} \ln \alpha(\vartheta)\, d\vartheta\right)\right) \int_t^{t+\delta} \exp\left[(s-t)\rho^{-1} \ln \alpha(T_1+\rho)\right] ds$$

$$\leq \rho(-\ln \alpha(T_1+\rho))^{-1} \exp\left(\int_{T_1+\rho}^{t} \rho^{-1} \ln \alpha(\vartheta)\, d\vartheta\right).$$

Thus

$$\left|y(t+\delta) - y(t)\right| \leq c_1 \exp\left(\int_{T_1+\rho}^{t} \rho^{-1} \ln \alpha(\vartheta)\, d\vartheta\right) \tag{8.32}$$

for $t \geq T_1 + \rho$ and $\delta > 0$, where

$$c_1 = c_0 \rho(-\ln \alpha(T_1+\rho))^{-1} \|X^{-1}G\|_\infty \|X\|_\infty.$$

Since $\alpha(t) \downarrow 0$ as $t \to \infty$, for any $\beta \geq 0$, we have

$$\exp\left(\int_{T_1+\rho}^{t} \rho^{-1} \ln \alpha(\vartheta)\, d\vartheta\right) = o\left(e^{-\beta t}\right) \tag{8.33}$$

as $t \to \infty$. Then, by (8.32), (8.33) and the Cauchy convergence principle, there is a $c \in \mathbb{C}^N$ such that $y(t) \to c$ as $t \to \infty$. Letting $\delta \to \infty$ in (8.32) and taking (8.33) into account, we obtain

$$|y(t) - c| = o\left(e^{-\beta t}\right)$$

as $t \to \infty$, where $\beta \geq 0$ can be arbitrary. Therefore,

$$x(t) = X(t)(y(t) - c) + X(t)c = X(t)c + o\left(e^{-\beta t}\right),$$

i.e., every solution of (8.4) has the form (8.8). #

## § 8.4 EXTENSION

As a consequence of Theorem 8.1 and the proof of Theorem 8.2, some results for the scalar equation (8.2) can be easily obtained. Let

$$b_0(t) = b(t), \; b_k(t) = b(t) \exp\left(\int_t^{t+\omega} (b_{k-1}(s) + a(s) - b(s))\, ds\right), \; k = 1, 2, \cdots \tag{8.34}$$

for $t \geq \tau$ and let $a(t)$, $b(t)$ and all $b_k(t)$ be zero for $t < \tau$.

**COROLLARY** 8.5 Assume that $a(t), b(t) \in L^\infty(\tau, \infty)$, $r(t) \to 0$ as $t \to \infty$, and $\|r_t\| \in L^p(\tau, \infty)$ for some $p \in [1, \infty)$. Then, for sufficiently large $T \geq \tau$, (8.2) has a solution on $[T - \rho, \infty)$ with the form

$$X(t) = \left\{\exp\left(\int_\tau^t (b_M(s) + a(s) - b(s))\,ds\right)\right\}(1 + o(1)) \tag{8.35}$$

as $t \to \infty$, where $M$ satisfies $M < p \le M + 1$ and $b_M(t)$ is determined by (8.34). Moreover, every solution of (8.2) has the form

$$x(t) = X(t)c + o\left(e^{-\beta t}\right), \tag{8.36}$$

where $X(t)$ is given by (8.35), $c \in \mathbb{C}$ depends on $x$, and $\beta \ge 0$ is arbitrary.

The proof of Corollary 8.5 is parallel to that of Theorem 8.2.

When the condition $a(t) \in L^\infty(\tau, \infty)$ does not hold, the conclusion of Corollary 8.5 may not be true. However, it is still possible to determine the asymptotic behaviour of the solutions of (8.2) under some other conditions.

**COROLLARY** 8.6 Assume that $b(t)$ and $\int_t^{t+\omega} a(s)\,ds$ are in $L^\infty(\tau, \infty)$, $r(t) \to 0$ as $t \to \infty$, and $\|r_t\| \in L^p(\tau, \infty)$ for some $p \in [1, \infty)$. Then, for sufficiently large $T \ge \tau$, (8.2) has a solution on $[T - r, \infty)$ with the form

$$X(t) = \left\{\exp\left(\int_\tau^t (b_{M+1}(s) + a(s) - b(s))\,ds\right)\right\}(1 + o(1)) \tag{8.37}$$

as $t \to \infty$, where $M$ satisfies $M < p \le M + 1$ and $b_{M+1}(t)$ is given by (8.35). Further, every solution of (8.2) has the asymptotic form

$$x(t) = X(t)c + o\left(\exp\left(\int_\tau^t a(s)\,ds - \beta t\right)\right), \tag{8.38}$$

where $c \in \mathbb{C}$ depends on $x$ and $\beta \ge 0$ is arbitrary.

**Proof.** Let

$$x(t) = \left\{\exp\left(\int_\tau^t a(s)\,ds\right)\right\} y(t).$$

Then (8.2) is changed to

$$\frac{d}{dt}y(t) = \bar{a}(t)y(t) + \bar{b}(t)\big(y(t - r(t)) - y(t)\big), \tag{8.39}$$

where $\bar{a} = b_1 - b_0$ and $\bar{b} = b_1$. Imitating (8.34), we define

$$\bar{b}_0(t) = \bar{b}(t), \ \bar{b}_k(t) = \bar{b}(t)\exp\left(\int_t^{t-r(t)} \big(\bar{b}_{k-1}(s) + \bar{a}(s) - \bar{b}(s)\big)\,ds\right), \ k = 1, 2, \cdots. \tag{8.40}$$

Since $\bar{a}(s)$ and $\bar{b}(s)$ are bounded, by Corollary 8.5 (8.39) has a solution of the form

$$Y(t) = \left\{\exp\left(\int_\tau^t \big(\bar{b}_M(s) + \bar{a}(s) - \bar{b}(s)\big)\,ds\right)\right\}(1 + o(1)) \tag{8.41}$$

as $t \to \infty$ and every solution of (8.39) has the asymptotic form

$$y(t) = Y(t)c + o\big(e^{-\beta t}\big).$$

Then, by the transformation from (8.2) to (8.39), we only need show that

$$\bar{b}_k(t) + \bar{a}(t) - \bar{b}(t) = b_{k+1}(t) - b(t) \tag{8.42}$$

for $k = M$. Obviously, (8.42) holds for $k = 0$. If (8.42) holds for $k$, then, from (8.39), (8.40) and (8.34),

$$\bar{b}_{k+1}(t) + \bar{a}(t) - \bar{b}(t) = \bar{b}(t)\exp\left(\int_t^{t-r(t)} \big(\bar{b}_k(s) + \bar{a}(s) - \bar{b}(s)\big)\,ds\right) - b(t)$$

$$= b(t)\left\{\exp\left(\int_t^{t-r(t)} a(s)\,ds\right)\right\}\exp\left(\int_t^{t-r(t)} \big(b_{k+1}(s) - b(s)\big)\,ds\right) - b(t)$$

$$= b_{k+2}(t) - b(t).$$

By induction, (8.42) holds for all $k \geq 0$ and thus for $k = M$. #

**COROLLARY** 8.7 Assume that $b(t) \in L^\infty(\tau, \infty)$, $\|a_t\| \in L^q(\tau, \infty)$ for some $q \in (1, \infty)$ and $\|r_t\| \in L^p(\tau, \infty)$ for some $p \in [1, \infty)$. Then Corollary 8.5 holds if we replace $b_M(t)$ by $b_m(t)$ in (8.35), where $m$ satisfies $m < p(1 - q^{-1}) \leq m + 1$ and $b_m(t)$ is given by (8.34).

The proof of Corollary 8.7 is similar to that of Theorem 8.2.

When $a(t) \in L(\tau, \infty)$ and $b(t) \in L^\infty(\tau, \infty)$, the equation (8.2) can be treated by Theorem 8.1. Thus, from Theorem 8.1 and Corollaries 8.5-8.7, the asymptotic behaviour of solutions of (8.2) is known clearly if $b(t) \in L^\infty(\tau, \infty)$, $\|r_t\| \in L^p(\tau, \infty)$ for some $p \in [1, \infty)$, and either $\|a_t\| \in L^q(\tau, \infty)$ for some $q \in [1, \infty]$ or $\int_t^{t+\infty} a(s)\, ds \in L^\infty(\tau, \infty)$.

**Remark.** In some circumstances, Corollaries 8.5-8.7 for the scalar equation (8.2) can be extended to vector equations. For instance, the equation

$$\frac{d}{dt}x(t) = a(t)A\, x(t) + b(t)B\,(x(t - r(t)) - x(t)), \tag{8.43}$$

where $A$ and $B$ are commutable $N \times N$ constant matrices and $a(t)$ and $b(t)$ are scalar functions, can be treated in the same way as (8.2) and the results obtained in this section hold for (8.43) with some obvious amendments.

# CHAPTER 9
# ASYMPTOTIC SOLUTIONS OF EQUATIONS WITH A DOMINATING DIAGONAL MATRIX

## § 9.1 INTRODUCTION

The equation to be discussed in this chapter has the form

$$\frac{d}{dt}\{x(t) - D(t, x_t)\} = \Lambda(t)x(t) + L(t,x_t) + L_2(t, x^t) \tag{9.1}$$

for $t \geq \tau$, where $\Lambda(t) = dg\{\lambda_1(t), \lambda_2(t), \cdots, \lambda_N(t)\}$ is a diagonal matrix of locally integrable functions and, for each $t \geq \tau$, $D(t, .)$, $L(t, .)$: $C([-r, 0], \mathbb{C}^N) \to \mathbb{C}^N$ and $L_2(t, .)$: $C([0, \sigma], \mathbb{C}^N) \to \mathbb{C}^N$ are bounded linear operators as described in Part II (p.76). Without loss of generality, we assume $r = \sigma$.

As we mentioned at the beginning of this part, the delay differential equation

$$\frac{d}{dt}x(t) = \Lambda(t)x(t) + L(t, x_t) \tag{9.2}$$

was studied in [4]. With a dichotomy condition on $\Lambda(t)$ and appropriate requirements for smallness of $\|L(t, .)\|$, $N$ special solutions of (9.2) with the forms

$$x_n(t) = (e_n + o(1))\exp\left(\int_\tau^t (\lambda_n(s) + \eta(s))\, ds\right), \quad n = 1, 2, \cdots, N, \tag{9.3}$$

were obtained and it was shown that each solution of (9.2) could be represented as

$$x(t) = \sum_{i=1}^{N} \gamma_i x_i(t) + o\left(\exp\left(\int_\tau^t \lambda_n(s)\, ds - \beta t\right)\right) \tag{9.4}$$

for every $\lambda_n$ and arbitrary $\beta \geq 0$.

In this chapter, we shall investigate the equation (9.1) and obtain some results parallel to those in [4]. In proving the existence of special solutions of (9.1) with the forms (9.3), the method used in [4] is adapted. However, since the terms $D(t, x_t)$

and $L_2(t, x^t)$ are involved in (9.1), a method different from that in [4] needs to be developed in order to obtain representations similar to (9.4). The norms

$$|v| = \max\{|v_1|, |v_2|, \cdots, |v_N|\}, \quad |M| = \max\left\{\sum_{j=1}^{N} |m_{ij}|: \ 1 \le i \le N\right\}$$

for $v = (v_1, v_2, \cdots, v_N)^T \in \mathbb{C}^N$ and $M = (m_{ij}) \in \mathbb{C}^{N \times N}$ will be used. Although other, equivalent, norms could be used, there may be different multipliers corresponding to different norms in some norm-involved inequalities.

## § 9.2 SPECIAL SOLUTIONS

Let $(d_{ij})$ and $(\mathfrak{L}_{kij})$, $k = 1, 2$, be the matrix representations of $D$ and $L_k$ where

$$L_1(t, \varphi) = A(t)D(t, \varphi) + L(t, \varphi). \tag{9.5}$$

For any function $f$ on $\mathbf{R} \equiv (-\infty, \infty)$, we define $\bar{f}_t$ and $\bar{f}^t$ by

$$\bar{f}_t(\theta) = \exp\left(\int_t^{t+\theta} f(s)\, ds\right), \quad \bar{f}^t(\alpha) = \exp\left(\int_t^{t+\alpha} f(s)\, ds\right) \tag{9.6}$$

for $\theta \in [-r, 0]$ and $\alpha \in [0, r]$. For convenience, we extend the domain of definition of $\Lambda(t)$, $L_1(t, .)$ and $L_2(t, .)$ to $\mathbf{R}$ by letting them vanish for $t < \tau$. Then, for $n \in \{1, 2, \cdots, N\}$, the functions

$$H_n(t) =_{def} \|D(t, \widetilde{(\lambda_n)}_t\, .)\| = \max\left\{\sum_{j=1}^{N} \|d_{ij}(t, \widetilde{(\lambda_n)}_t\, .)\|: \ 1 \le i \le N\right\} \tag{9.7}$$

for $t \ge \tau$, where $\|d_{ij}(t, \widetilde{(\lambda_n)}_t\, .)\|$ is the ordinary uniform norm on $[-r, 0]$, and

$$\beta_n(t) =_{def} \|L_1(t, \widetilde{(\lambda_n)}_t .)\| + \|L_2(t, \widetilde{(\lambda_n)}^t .)\| \tag{9.8}$$

for $t \in \mathbf{R}$ are well defined. For any $T \ge \tau$, let

$$x(t) = \left(e_n + y(t)\right)\exp\left(\int_T^t \left(\lambda_n(s) + \eta(s)\right) ds\right) \tag{9.9}$$

for $t \geq T - r$, where $e_n$ is the $n$th coordinate vector and $\eta(s) = 0$ for $s \leq T$. Then (9.1) becomes

$$\frac{d}{dt}\left\{e_n + y(t) - D(t, \overline{(\lambda_n + \eta)}_t(e_n + y_t))\right\}$$

$$= (\Lambda - (\lambda_n + \eta)I)\{\cdots\} + L_1(t, \overline{(\lambda_n + \eta)}_t(e_n + y_t)) + L_2(t, \overline{(\lambda_n + \eta)}^t(e_n + y^t)) \tag{9.10}$$

for $t \geq T$, where $\{\cdots\}$ is equal to the term under $d/dt$. Put

$$u_k(t, n, \eta, y) = d_{kn}(t, \overline{(\lambda_n + \eta)}_t) + \sum_{j=1}^{N} d_{kj}(t, \overline{(\lambda_n + \eta)}_t(y_j)_t), \tag{9.11}$$

$$v_k(t, n, \eta, y) = \mathfrak{L}_{1kn}(t, \overline{(\lambda_n + \eta)}_t) + \mathfrak{L}_{2kn}(t, \overline{(\lambda_n + \eta)}^t)$$

$$+ \sum_{j=1}^{N} \left(\mathfrak{L}_{1kj}(t, \overline{(\lambda_n + \eta)}_t(y_j)_t) + \mathfrak{L}_{2kj}(t, \overline{(\lambda_n + \eta)}^t(y_j)^t)\right). \tag{9.12}$$

Then, in terms of $u_k$ and $v_k$, (9.10) can be written as

$$\frac{d}{dt}\{1 + y_n(t) - u_n(t, n, \eta, y)\} = -\eta(t)(\cdots) + v_n(t, n, \eta, y), \tag{9.13}$$

$$\frac{d}{dt}\{y_k(t) - u_k(t, n, \eta, y)\} = \left(\lambda_k(t) - \lambda_n(t) - \eta(t)\right)(\cdots) + v_k(t, n, \eta, y), \quad k \neq n. \tag{9.14}$$

Instead of (9.13) and (9.14), we consider the system

$$\eta(t) = v_n(t, n, \eta, y), \tag{9.15}$$

$$y_n(t) = u_n(t, n, \eta, y), \tag{9.16}$$

$$y_k(t) = u_k(t, n, \eta, y) + \int_{T_k}^{t} ds \left(\exp\left(\int_s^t \left(\lambda_k(\vartheta) - \lambda_n(\vartheta)\right) d\vartheta\right)\right) \times$$

$$\{v_k(s, n, \eta, y) - v_n(s, n, \eta, y)(y_k(s) - u_k(s, n, \eta, y))\}, \quad k \neq n, \tag{9.17}$$

where $T_k$ is to be defined below. Each solution of (9.15)-(9.17) also satisfies (9.13) and (9.14).

At this stage we suppose that, for some $T \geq \tau$, the following conditions hold.

(i). The indices $1, \cdots, n-1, n+1, \cdots, N$ can be partitioned into two classes $J_1$ and $J_2$ such that the function

$$g_n(t) =_{\mathrm{def}} \max\left\{\left|\int_{T_k}^{t} ds\left(\beta_n(s)\exp\left(\int_{s}^{t} \operatorname{Re}\left(\lambda_k(\vartheta) - \lambda_n(\vartheta)\right) d\vartheta\right)\right)\right| : k \neq n\right\} \tag{9.18}$$

exists on $[T, \infty)$ and $g_n(t) \to 0$ as $t \to \infty$, where $T_k = T$ or $\infty$ according as $k \in J_1$ or $k \in J_2$, and $\beta_n(t)$ is given by (9.8).

(ii). $H_n(t)$ given by (9.7) satisfies $H_n(t) \to 0$ as $t \to \infty$.

(iii). $H_n(t)$, $\beta_n(t)$ and $g_n(t)$ satisfy

$$\sup\left\{\max\left\{3\int_{t-r}^{t} \beta_n(s)\,ds,\ 3H_n(t) + 13g_n(t)\right\} : t \geq T\right\} < e^{-1}. \tag{9.19}$$

**THEOREM** 9.1 Suppose that the conditions (i)-(iii) are met. Then (9.1) has a solution on $[T - r, \infty)$ with the asymptotic form

$$x_n(t) = (e_n + o(1))\exp\left(\int_T^t \left(\lambda_n(s) + \eta(s)\right) ds\right) \tag{9.20}$$

as $t \to \infty$, where $|\eta(s)| \leq 2e\beta_n(s)$ for $s \geq T$.

Although the proof of this theorem is an adaptation of [4], we present an outline of it here because some of the estimates obtained will be referred to later.

**Proof**. With $y(t) = (y_1(t), y_2(t), \cdots, y_N(t))^T$ bounded and continuous on $[T - r, \infty)$ and constant on $[T - r, T]$, and with $\eta(t)$ locally integrable on $[T, \infty)$, vanishing for $t < T$, and $\int_{t-r}^{t} |\eta(s)|\,ds$ bounded, functions of the form $\bar{y}(t) = (y^T(t), \eta(t))^T$ with

$$\|\bar{y}\| =_{def} \sup\left\{\max\left\{|y(t)|, \int_{t-r}^{t} |\eta(s)|\, ds\right\}: t \ge T\right\}$$

form a Banach space. We show that the system of the equations (9.15)-(9.17) has a solution in the closed unit ball **B** of this space.

For any $\bar{y} \in \mathbf{B}$, we define $A\bar{y}(t)$ by the right hand sides of (9.15)-(9.17) for $t \ge T$, $A\bar{y}_k(t) = A\bar{y}_k(T)$ for $t$ in $[T - r, T]$ and $k = 1, 2, \cdots, N$, and $A\bar{y}_{N+1}(t) = 0$ for $t < T$. Then, for $t \ge T$, the equalities (9.15), (9.16), (9.11), (9.12), (9.7), and (9.8) produce

$$|A\bar{y}_{N+1}(t)| = |v_n(t, n, \eta, y)| \le 2e\beta_n(t) \tag{9.21}$$

and

$$|A\bar{y}_n(t)| = |u_n(t, n, \eta, y)| \le 2eH_n(t). \tag{9.22}$$

Since (9.19) implies $3eH_n(t) < 1$, from (9.17) and (9.18)

$$|A\bar{y}_k(t)| \le 2eH_n(t) + \left|\int_{T_k}^{t} ds \left(2e\beta_n(s)\,(2 + 2eH_n(s)) \exp\left(\int_s^t \mathrm{Re}\big(\lambda_k(\vartheta) - \lambda_n(\vartheta)\big)\, d\vartheta\right)\right)\right|$$

$$\le 2eH_n(t) + 6eg_n(t) \tag{9.23}$$

for $k \ne n$ and $t \ge T$. By (9.19) again, $A$ maps **B** into itself.

For any $\bar{y} = (y^T, \eta)^T \in \mathbf{B}$ and $\bar{z} = (z^T, \mu)^T \in \mathbf{B}$, by letting

$$\|\bar{y}\|(t) = \max\left\{\int_{t-r}^{t} |\eta(s)|\, ds, \int_{t}^{t+r} |\eta(s)|\, ds, \|y_t\|, \|y^t\|\right\} \tag{9.24}$$

for $t \ge T$, we have

$$\|\tilde{\eta}_t - \tilde{\mu}_t\| \le e\int_{t-r}^{t} |\eta(s) - \mu(s)|\, ds \le e\|\bar{y} - \bar{z}\|(t),$$

$$\|\tilde{\eta}_t(y_j)_t - \tilde{\mu}_t(z_j)_t\| \le \|\tilde{\eta}_t - \tilde{\mu}_t\| + e\|(y_j)_t - (z_j)_t\| \le 2e\|\bar{y} - \bar{z}\|(t).$$

Similarly,

$$\|\tilde{\eta}^t - \tilde{\mu}^t\| \le e\|\bar{y} - \bar{z}\|(t),\quad \|\tilde{\eta}^t(y_j)^t - \tilde{\mu}^t(z_j)^t\| \le 2e\|\bar{y} - \bar{z}\|(t).$$

Then, from (9.15)-(9.17), (9.11), (9.12), (9.7), and (9.8),

$$|A\bar{y}_{N+1}(t) - A\bar{z}_{N+1}(t)| \le 3e\beta_n(t)\,\|\bar{y} - \bar{z}\|(t), \tag{9.25}$$

$$|A\bar{y}_n(t) - A\bar{z}_n(t)| \le 3eH_n(t)\,\|\bar{y} - \bar{z}\|(t), \tag{9.26}$$

and for $k \neq n$,

$$|A\bar{y}_k(t) - A\bar{z}_k(t)| \le 3eH_n(t)\,\|\bar{y} - \bar{z}\|(t) +$$

$$\left|\int_{T_k}^{t} \{9e\beta_n(s)\|\bar{y} - \bar{z}\|(s) + |v_n(s, n, \eta, y)|(1 + 3eH_n(s)\|\bar{y} - \bar{z}\|(s))\}\exp\left(\int_s^t \mathrm{Re}(\lambda_k - \lambda_n)d\vartheta\right)ds\right|$$

$$\le 3eH_n(t)\|\bar{y} - \bar{z}\|(t) + 13e\left|\int_{T_k}^{t} \beta_n(s)\,\|\bar{y} - \bar{z}\|(s)\exp\left(\int_s^t \mathrm{Re}(\lambda_k - \lambda_n)\,d\vartheta\right)ds\right|. \tag{9.27}$$

Hence, since $\|\bar{y}\|(t) \le \|\bar{y}\|$, (9.25)-(9.27) and (9.19) show that $A$ is contractive. Thus, within **B**, the system of (9.15)-(9.17) possesses a unique solution $\bar{y}(t) = (y^T(t), \eta(t))^T$. Further, (9.21)-(9.23) and (i) and (ii) ensure that $y(t) = o(1)$ as $t \to \infty$ and $|\eta(t)| \le 2e\beta_n(t)$. Then (9.9) leads to a solution of (9.1) with the form (9.20). #

Suppose, instead of (i) and (iii), we impose the conditions (iv) and (v).

(iv). There is a $K \ge 0$ and a division of $\{1, 2, \cdots, N\}$ such that $Q(k, n, s, t) \ge -K$ for $t \ge s \ge \tau$ and $k \in J_2$, and $Q(k, n, s, t) \le K$ for $t \ge s \ge \tau$ and $Q(k, n, s, t) \to -\infty$ as $t \to \infty$ when $k \in J_1$, where

$$Q(k, n, s, t) = \int_s^t \mathrm{Re}\big(\lambda_k(\vartheta) - \lambda_n(\vartheta)\big)\,d\vartheta.$$

(v). $\beta_n(t) \in L(\tau, \infty)$.

Then, from (ii), (iv) and (v), we have

$$13e^K\int_T^\infty \beta_n(t)\,dt + 3\sup\{H_n(t) : t \ge T\} < e^{-1} \tag{9.28}$$

for sufficiently large $T \geq \tau$, which implies (iii). Also, by (9.18), $g_n(t) \to 0$ as $t \to \infty$.

**THEOREM** 9.2 Assume that (ii), (iv) and (v) hold for some $n \in \{1, 2, \cdots, N\}$. Then, for $T \geq \tau$ satisfying (9.28), (9.1) has a solution on $[T - r, \infty)$ with the form

$$x_n(t) = (e_n + o(1)) \exp\left(\int_\tau^t \lambda_n(s)\, ds\right) \tag{9.29}$$

as $t \to \infty$.

**Proof**. Since $\eta \in L(T, \infty)$, (9.29) is obtained by multiplying (9.20) by a scalar constant. #

When the conditions of Theorem 9.1 are met, $\eta(t)$ and the $o(1)$ term in (9.20) can be obtained by successive approximations. Let

$$\bar{y}^0 = 0, \ \bar{y}^{k+1}(t) = A\bar{y}^k(t), \ k = 0, 1, 2, \cdots \tag{9.30}$$

for $t \geq T - r$ and let $\bar{y} \in \mathbf{B}$ be the fixed point of $A$. Since $A$ is contractive and $\bar{y}^0 \in \mathbf{B}$, we must have $\|\bar{y}^k - \bar{y}\| \to 0$ as $k \to \infty$. Thus, in some circumstances, we can obtain more precise representations than (9.20) if we can estimate $\|\bar{y}^k - \bar{y}\|$ effectively.

**THEOREM** 9.3 Assume that the following conditions hold:

(vi) there is a number $\delta > 0$ such that, for each $k \neq n$, either $\mathrm{Re}(\lambda_k(t) - \lambda_n(t)) \geq \delta$ for all $t \geq \tau$ or $\mathrm{Re}(\lambda_k(t) - \lambda_n(t)) \leq -\delta$ for all $t \geq \tau$,

(vii) $\beta_n(t) \in L^p(\tau, \infty)$ for some $p \in (1, \infty)$,

(viii) $\|(H_n)_t\| \in L^p(\tau, \infty)$ and $H_n(t) \to 0$ as $t \to \infty$.

Then, for sufficiently large $T \geq \tau$, (9.1) has a solution on $[T - r, \infty)$ with the asymptotic form

$$x_n(t) = (e_n + o(1)) \exp\left( \int_{\tau}^{t} (\lambda_n(s) + \eta^m(s))\, ds \right) \tag{9.31}$$

as $t \to \infty$, where $m$ is determined by $m < p \le m + 1$ and $\eta^m(t) = \bar{y}^m_{N+1}(t)$ by (9.30).

**Proof.** Since (vi)-(viii) imply (i)-(iii) for sufficiently large $T \ge \tau$, the $o(1)$ term and $\eta(t)$ in (9.20) (from the fixed point $\bar{y} = (y^T, \eta)^T$ of $A$ ) can be obtained by successive approximations (9.30). If we show that $|\eta(t) - \bar{y}^m_{N+1}(t)| \in L(T, \infty)$, then, multiplying (9.20) by a scalar constant and replacing $\eta(t)$ by $\eta^m(t)$, we obtain a solution of (9.1) with the form (9.31).

From (vi), (vii) and (9.18), we have $\|(g_n)_t\| \in L^p(T, \infty)$. Hence, in view of (9.21)-(9.24) and (viii), $\|\bar{y}\|(t) \in L^p(T, \infty)$. Moreover, it follows from (9.25)- (9.27) that

$$\left| \eta(t) - \bar{y}^{j+1}_{N+1}(t) \right| \le 3e\beta_n(t) \|\bar{y} - \bar{y}^j\|(t), \tag{9.32}$$

$$\left| y_n(t) - \bar{y}^{j+1}_n(t) \right| \le 3eH_n(t) \|\bar{y} - \bar{y}^j\|(t), \tag{9.33}$$

and for $k \ne n$,

$$\left| y_k(t) - \bar{y}^{j+1}_k(t) \right| \le 3eH_n(t) \|\bar{y} - \bar{y}^j\|(t) + 13e \left| \int_{T_k}^{t} e^{-\delta|t-s|} \beta_n(s) \|\bar{y} - \bar{y}^j\|(s)\, ds \right| \tag{9.34}$$

for $t \ge T$ and $j = 0, 1, 2, \cdots$. Since $\|\bar{y} - \bar{y}^0\|(t) = \|\bar{y}\|(t) \in L^p(T, \infty)$, inductively we obtain $\|\bar{y} - \bar{y}^j\|(t) \in L^{p_j}(T, \infty)$ for $p_j = \max(1, p(j+1)^{-1})$ and $j = 0, 1, 2, \cdots$. Then, by (9.32),

$$|\eta(t) - \bar{y}^m_{N+1}(t)| \le 3e\beta_n(t) \|\bar{y} - \bar{y}^{m-1}\|(t) \in L(T, \infty).$$

The proof is complete. #

**Remarks.** (a). When $D(t, .) = L_2(t, .) = 0$, the results of this, as well as next, section agree with those in [4] except that our condition (iii) slightly differs from that of [4] due to the difference of norms.

(b). Theorems 9.2 and 9.3 are analogues for (9.1) of the Levinson and Hartman-Wintner theorems for ordinary differential equations (see Ch.1 in [12]). Another main theorem for ordinary differential equations, Eastham's theorem (Theorem 1.6.1 in [12]), cannot be extended to (9.1) by the technique used here.

(c). In some cases, we can impose conditions directly on $\|D(t,.)\|$, $\|L_1(t,.)\|$ and $\|L_2(t,.)\|$ instead of on $H_n(t)$ and $\beta_n(t)$. For instance, if $\exp\int_t^{t+\alpha} \lambda_n(s)\,ds$ is bounded for all $\alpha \in [-r, r]$ and $t \geq \tau$, then we have

$$H_n(t) \leq c_0\|D(t,.)\| \text{ and } \beta_n(t) \leq c_0(\|L_1(t,.)\| + \|L_2(t,.)\|)$$

for some $c_0 > 0$. If $\int_t^{t+\theta} \lambda_n(s)\,ds$ is bounded above for all $\theta \in [-r, 0]$ and $t \geq \tau$, we also have

$$H_n(t) \leq c_0\|D(t,.)\| \text{ and } \|L_1(t, \widetilde{(\lambda_n)_t}\,.)\| \leq c_0\|L_1(t,.)\|.$$

(d). The remarks (ii), (iv) and (v) ( p479 in [4]) also apply to this section.

## § 9.3 DECOMPOSITION OF THE SOLUTION SPACE

If the conditions of Theorem 9.1 hold for all $n = 1, 2, \cdots, N$, then (9.1) has $N$ solutions of the form (9.20) on $[T - r, \infty)$, which compose a matrix solution

$$X(t) = (x_1(t), x_2(t), \cdots, x_N(t)) =_{def} Y(t)\exp\left(\int_T^t \Lambda_1(s)\,ds\right) \tag{9.35}$$

of (9.1). Since $Y(t) \to I$ as $t \to \infty$, there is a $T_1 \geq T$ and an $a_0 \geq 1$ such that

$$|Y(t)|.|Y^{-1}(s)| \leq a_0 \quad \text{for } t, s \geq T_1 - r. \tag{9.36}$$

From the proof of Theorem 9.1, we know that

$$\left|\int_{t+\theta}^{t}\left(\Lambda_1(s)-\Lambda(s)\right)ds\right| \le 1,\ \ \theta\in[-r,\ 0], \tag{9.37}$$

and

$$\left|\int_{T}^{t}\left(\Lambda_1(s)-\Lambda(s)\right)ds\right| \le 2e\int_{T}^{t}\max\left\{\beta_n(s):\ 1\le n\le N\right\}ds$$

for $t\ge T$. Thus, by the boundedness of each $\int_{t-r}^{t}\beta_n(s)\,ds$ in (9.19), we also have

$$\left|\int_{T}^{t}\left(\Lambda_1(s)-\Lambda(s)\right)ds\right| \le 2e\int_{T-r}^{t}\beta(s)\,ds \tag{9.38}$$

for $t\ge T$, where

$$\beta(t) = \sup\left\{\frac{1}{r}\int_{u-r}^{u}\max\left\{\beta_n(s):\ 1\le n\le N\right\}ds:\ u\ge t\right\}. \tag{9.39}$$

Obviously, $\beta(t)$ is decreasing.

We impose two more conditions below in order to analyse the structure of the solution space.

(ix). There are $q, M\in\{1, 2, \cdots, N\}$ such that

$$\left|\exp\left(\int_{u}^{t}\left(\lambda_q(s)I-\Lambda(s)\right)ds\right)\right| \le a_1,\ \left|\exp\left(\int_{u}^{t}\left(\Lambda(s)-\lambda_M(s)I\right)ds\right)\right| \le a_1 \tag{9.40}$$

for $t\ge u\ge \tau$.

(x). $\int_{t-r}^{t}\left(\beta_q(s)+\beta_M(s)\right)ds\to 0$ as $t\to\infty$ ($q, M$ as in (ix)).

**Remark**. The condition (ix) automatically holds under the Levinson dichotomy condition (i.e., (iv) holds for all $n\in\{1, 2, \cdots, N\}$).

If (ii) for $n=q$ and (x) hold, then $\alpha(t)$ defined by

$$\alpha(t) = \sup\left\{(1+(1+a_0e)a_1)\,\|(H_q)_s\| + 2a_0e(a_1)^2\int_{s-r}^{\infty}\left(\beta_q(\vartheta)+\beta_M(\vartheta)\right)d\vartheta:\ s\ge t\right\} \tag{9.41}$$

decreases to zero as $t \to \infty$. We therefore also assume that $\alpha(t) < 1$ for $t \geq T_1$.

**THEOREM 9.4** Assume that (i)-(iii), together with (ix) and (x), hold for all $n = 1, 2, \cdots, N$. Then every solution of (9.1) on $[T_1 - r, \infty)$ has the forms

$$x(t) = X(t)\left\{c + O\left(\exp\left(\int_{T_1+r}^{t+r} \left(2e\beta(s) + \frac{1}{r}\ln \alpha(s)\right) ds\right)\right)\right\} \tag{9.42}$$

and

$$x(t) = X(t)c + O\left(\alpha(t) \exp\left(\int_{T_1+r}^{t} \left(\lambda_q(s) + \frac{1}{r}\ln \alpha(s)\right) ds\right)\right) \tag{9.43}$$

as $t \to \infty$, where $c \in \mathbb{C}^N$ depends on $x$, $\alpha(t)$ is given by (9.41) and $\beta(t)$ by (9.39).

**Remark**. Since both $\alpha(t)$ and $\beta(t)$ are decreasing and $\alpha(t) \to 0$ as $t \to \infty$, the order terms in (9.42) and (9.43) can be written as $o(e^{-\beta t})$ and

$$o\left(\exp\left(\int_{\tau}^{t} \lambda_q(s)\, ds - \beta t\right)\right),$$

respectively, for arbitrary $\beta \geq 0$. Thus the solution space $S$ of (9.1) can be decomposed by $X(t)$ as $S = S_1 \oplus S_2$, where

$$S_1 = \left\{X(t)c\colon\ c \in \mathbb{C}^N\right\},$$

$$S_2 = \left\{x(t)\colon\ x(t) = o\left(\exp\left(\int_{\tau}^{t} \lambda_q(s)\, ds - \beta t\right)\right) \text{ for any } \beta \geq 0\right\}.$$

Homomorphically, after the transformation $x(t) = X(t)y(t)$, the solution space has the decomposition $S = \mathbb{C}^N \oplus S_0$, where $S_0$ is composed of all small solutions (tending to zero faster than any exponential function) of the transformed equation. From this decomposition it is clear enough that the asymptotic behaviour of solutions is dominated by $X(t)$.

**LEMMA** 9.5 Assume that $y(t) \in C([t_0, \infty), \mathbb{C}^N)$ satisfies $\|y(t) - y_t\| \le \gamma(t)$ for $t \ge t_0 + r$, where $\gamma(t)$ is continuous, strictly decreasing, and $\gamma(t) \to 0$ as $t \to \infty$. Also, for any $\delta > 0$, $\gamma(t+\delta)(\gamma(t))^{-1}$ is decreasing. Then there is a $c \in \mathbb{C}^N$ such that $y(t) \to c$ as $t \to \infty$ and

$$|y(t) - c| \le \alpha_0 \gamma(t + r) \tag{9.44}$$

for $t \ge t_0$, where $\alpha_0 = \gamma(t_0 + r)(\gamma(t_0 + r) - \gamma(t_0 + 2r))^{-1}$.

**Proof**. For any $s > 0$, there is an integer $k \ge 0$ satisfying $kr < s \le (k+1)r$. Then, for $t \ge t_0$ with $t + s \ge t_0 + r$,

$$\begin{aligned} |y(t+s) - y(t)| &\le |y(t+s) - y(t+(k+1)r)| + |y(t+(k+1)r) - y(t+kr)| \\ &\quad + \cdots + |y(t+r) - y(t)| \\ &\le \gamma(t+(k+1)r) + \gamma(t+(k+1)r) + \cdots + \gamma(t+r). \end{aligned}$$

Since $\gamma(t)$ and $\gamma(t+r)(\gamma(t))^{-1}$ are decreasing, we have

$$\begin{aligned} |y(t+s) - y(t)| &\le \gamma(t+s) + \gamma(t+r)\{1 + \gamma(t+2r)(\gamma(t+r))^{-1} \\ &\quad + \cdots + (\gamma(t+2r)(\gamma(t+r))^{-1})^k\} \\ &\le \gamma(t+s) + \gamma(t+r)\left(1 - \gamma(t+2r)(\gamma(t+r))^{-1}\right)^{-1} \\ &\le \gamma(t+s) + \alpha_0\gamma(t+r) \to 0 \end{aligned}$$

as $t \to \infty$. Hence $y(t)$ has a limit $c \in \mathbb{C}^N$ as $t \to \infty$ and (9.44) is obtained by letting $s \to \infty$. #

**Proof of Theorem 9.4**. Applying the variation of parameters formula to (9.1), we obtain

$$x(t) = D(t, x_t) + \left(\exp\left(\int_{t+\theta}^{t} A(s)\, ds\right)\right)\left(x(t+\theta) - D(t+\theta, x_{t+\theta})\right) +$$

$$+\int_{t+\theta}^{t} ds \left(\exp\left(\int_{s}^{t} \Lambda(\vartheta)\, d\vartheta\right)\right)\left(L_1(s, x_s) + L_2(s, x^s)\right) \tag{9.45}$$

for $t \geq T_1$ and $\theta \in [-r, 0]$. Let

$$x(t) = X(t)y(t). \tag{9.46}$$

Then the substitution of (9.46) into (9.45) yields

$$X(t)y(t) = \left(\exp\left(\int_{t+\theta}^{t} \Lambda(\vartheta)\, d\vartheta\right)\right)\left(X(t+\theta)y(t+\theta) - D(t+\theta, X_{t+\theta}y_{t+\theta})\right)$$

$$+ D(t, X_t y_t) + \int_{t+\theta}^{t} ds \left(\exp\left(\int_{s}^{t} \Lambda(\vartheta)\, d\vartheta\right)\right)\left(L_1(s, X_s y_s) + L_2(s, X^s y^s)\right). \tag{9.47}$$

Since $X(t)$ satisfies (9.45), replacing $x(t)$ by $X(t)$ in (9.45) and substituting it into the left hand side of (9.47), we change (9.47) to

$$\left(\exp\left(\int_{t+\theta}^{t} \Lambda(\vartheta)\, d\vartheta\right)\right) X(t+\theta)\left(y(t) - y(t+\theta)\right)$$

$$= -D(t, X_t(y(t) - y_t)) + \left(\exp\left(\int_{t+\theta}^{t} \Lambda(\vartheta)\, d\vartheta\right)\right) D\left(t+\theta, X_{t+\theta}(y(t) - y_{t+\theta})\right)$$

$$- \int_{t+\theta}^{t} ds \left(\exp\left(\int_{s}^{t} \Lambda(\vartheta)\, d\vartheta\right)\right)\left(L_1(s, X_s(y(t) - y_s)) + L_2(s, X^s(y(t) - y^s))\right). \tag{9.48}$$

Let $\omega(t) = \sup\{|W(t, \theta)|: \theta \in [-r, 0]\}$, where

$$W(t, \theta) =_{def} X(t+\theta)\left(y(t) - y(t+\theta)\right) \exp\left(-\int_{0}^{t+\theta} \lambda_q(s)\, ds\right) \tag{9.49}$$

for $t \geq T_1$ and $\theta \in [-r, 0]$. Then, multiplying (9.48) by

$$\left(\exp\left(-\int_{0}^{t+\theta} \lambda_q(s)\, ds\right)\right) \exp\left(-\int_{t+\theta}^{t} \Lambda(s)\, ds\right),$$

we obtain an equation

$$W(t,\theta) = D\left(t+\theta, \widetilde{(\lambda_q)}_{t+\theta} W(t+\theta, .)\right) + D\left(t+\theta, X_{t+\theta}X^{-1}(t+\theta)W(t,\theta)\right)$$

$$- \left(\exp\left(\int_{t+\theta}^{t} (\lambda_q I - \Lambda)\, d\vartheta\right)\right) D\left(t, \widetilde{(\lambda_q)}_t W(t, .)\right)$$

$$- \int_{t+\theta}^{t} ds \left(\exp\left(\int_{t+\theta}^{s} (\lambda_q I - \Lambda)\, d\vartheta\right)\right) \left\{ L_1\left(s, \widetilde{(\lambda_q)}_s W(s, .)\right) + [\cdots] \right\} \tag{9.50}$$

for $W(t, \theta)$ for $t \geq T_1 + r$ and $\theta \in [-r, 0]$, where

$$[\cdots] = L_1\left(s, X_s X^{-1}(s) W(t, s-t)\right) + L_2\left(s, X^s X^{-1}(s)\left(W(t, s-t) - W(s + ., - .)\right)\right).$$

From (9.50) an estimate of $\omega(t)$ will be derived. From (9.36), (9.37) and (9.40) it follows that

$$\|\widetilde{(-\lambda_q)}_t X_t X^{-1}(t)\| = \sup\left\{ \left| Y(t+\theta) \left(\exp\left(\int_{t+\theta}^{t} \left(\lambda_q(s) I - \Lambda_1(s)\right) ds\right)\right) Y^{-1}(t) \right| : \theta \in [-r, 0] \right\}$$

$$\leq a_0 a_1 e, \tag{9.51}$$

$$\|\widetilde{(-\lambda_M)}^t X^t X^{-1}(t)\| \leq a_0 a_1 e \tag{9.52}$$

for $t \geq T_1$, so

$$|W(t,\theta)| \leq a_1 H_q(t)\omega(t) + H_q(t+\theta)\omega(t+\theta) + a_0 a_1 e H_q(t+\theta)\omega(t)$$

$$+ a_1 \int_{t+\theta}^{t} \left\{ \|L_1(s, \widetilde{(\lambda_q)}_s .)\| \left(\omega(s) + a_0 a_1 e\, \omega(t)\right) + \|L_2(s, \widetilde{(\lambda_M)}^s .)\| \left(\omega(t) + \|\omega^s\|\right) a_0 a_1 e \right\} ds$$

for $t \geq T_1 + r$ and $\theta \in [-r, 0]$. Then, in view of (9.41),

$$\omega(t) \leq \alpha(t) \max\{\omega(s) : t - r \leq s \leq t + r\} \tag{9.53}$$

for $t \geq T_1 + r$. Recall that $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$, is a sequence of ZAPs of (9.1). Thus, for any $t_k > T_1 + r$ and all $t \in [T_1 + r, t_k]$, we also have

$$\omega(t) \leq \alpha(t) \max\{\omega(s) : t - r \leq s \leq t_k\}. \tag{9.54}$$

Since $\alpha(t) < 1$, $\alpha(t) \downarrow 0$ as $t \to \infty$, and $\omega(t)$ is continuous, by Lemma 8.4 $\omega(t)$ satisfies

$$\omega(t) \le \|\omega^{T_1}\| \exp\left(\int_{T_1+r}^{t} \tfrac{1}{r}\ln \alpha(s)\, ds\right) \tag{9.55}$$

for $t \ge T_1 + r$. Then, from (9.49), (9.35), (9.36), (9.38), and (9.40),

$$\|y(t) - y_t\| \le \|(Y^{-1})_t\|\, \omega(t) \sup\left\{\left\|\exp\left(\int_T^{t+\theta} (\lambda_q I - \Lambda_1)\, ds + \int_0^T \lambda_q(s) I\, ds\right)\right\| : \theta \in [-r, 0]\right\}$$

$$\le c_1 \exp\left(\int_{T_1+r}^{t} \left(2e\beta(s) + \tfrac{1}{r}\ln \alpha(s)\right) ds\right) =_{def} \gamma(t) \tag{9.56}$$

for $t \ge T_1 + r$, where

$$c_1 = a_0 a_1 \|\omega^{T_1}\| \exp\left(\int_{T-r}^{T_1+r} 2e\beta(s)\, ds + \int_0^T \mathrm{Re}\, \lambda_q(s)\, ds\right).$$

As both $\alpha(t)$ and $\beta(t)$ are decreasing and $\alpha(t) \downarrow 0$ as $t \to \infty$, by Lemma 9.5 $y(t)$ has a limit $c \in \mathbb{C}^N$ as $t \to \infty$ and $|y(t) - c| = O(\gamma(t + r))$, and this gives the form (9.42) for $x$. Again, from (9.35) and (9.49),

$$X(t)(y(t) - c) = -X(t) \sum_{j=0}^{\infty} \left(y(t + (j+1)r) - y(t + jr)\right)$$

$$= -X(t) \sum_{j=0}^{\infty} X^{-1}(t + jr) W(t + (j+1)r, -r) \exp\left(\int_0^{t+jr} \lambda_q(s)\, ds\right)$$

$$= -\sum_{j=0}^{\infty} Y(t)\left(\exp\left(\int_t^{t+jr} \left(\lambda_q I - \Lambda_1\right) ds\right)\right) Y^{-1}(t + jr) W(t + (j+1)r, -r) \exp\left(\int_0^t \lambda_q(s)\, ds\right)$$

Thus, by (9.36), (9.38), (9.40), and (9.55), we obtain

$$|X(t)(y(t) - c)| \le a_0 a_1 \|\omega^{T_1}\| \sum_{j=0}^{\infty} \left(\exp\left(\int_t^{t+jr} \left(2e\beta(s) + \tfrac{1}{r}\ln \alpha(s)\right) ds\right)\right) \times$$

$$\exp\left(\int_0^t \mathrm{Re}\, \lambda_q(s)\, ds + \int_{T_1+r}^{t+jr} \tfrac{1}{r} \ln \alpha(s)\, ds\right). \tag{9.57}$$

Since $(2c\beta(s) + r^{-1}\ln \alpha(s)) \downarrow -\infty$ as $s \to \infty$, the series in (9.57) is bounded for $t \geq T_1$. Also,

$$\exp\left(\int_t^{t+r} \frac{1}{r} \ln \alpha(s)\, ds\right) \leq \exp(\ln \alpha(t)) = \alpha(t).$$

Then (9.57) implies the form (9.43) for $x$. #

**COROLLARY** 9.6 Assume that the conditions of Theorem 9.2 are met for all $n = 1, 2, \cdots, N$. Then every solution of (9.1) has the forms

$$x(t) = X(t)\left\{c + O\left(\alpha(t) \exp\left(\int_{T_1+r}^{t} \frac{1}{r} \ln \alpha(s)\, ds\right)\right)\right\} \tag{9.58}$$

and

$$x(t) = X(t)c + O\left(\alpha(t) \exp\left(\int_{T_1+r}^{t} \left(\lambda_q(s) + \frac{1}{r} \ln \alpha(s)\right) ds\right)\right) \tag{9.59}$$

as $t \to \infty$, where $\alpha(t)$ is given by (9.41).

**Proof**. Since the conditions of Theorem 9.2 for all $n$ ensure the existence of $X(t)$ satisfying (9.35) with $\Lambda_1 = \Lambda$ and imply (ix) and (x), from the proof of Theorem 9.4, we can replace $\beta(t)$ by 0. Then (9.58) and (9.59) follow from (9.42) and (9.43). #

**COROLLARY** 9.7 Assume that the conditions of Theorem 9.3 hold for all $n = 1, 2, \cdots, N$. Then every solution of (9.1) has the forms (9.42) and (9.43).

**Proof**. As the conditions of Theorem 9.3 for all $n$ imply (ix), (x) and the conditions of Theorem 9.1 for all $n$, the conclusion follows from Theorem 9.4. #

## § 9.4 TWO EXAMPLES

As an application of the above theorems, we consider two simple cases of (9.1).

**COROLLARY** 9.8 Assume that (9.1) satisfies the following:

(a) the condition (iv) (on p.138) holds for all $n = 1, 2, \cdots, N$,

(b) $\mathrm{Re}\int_t^{t+\alpha} \lambda_n(s)\,ds$ is bounded for all $t \geq \tau$, $\alpha \in [-r, r]$ and $n = 1, 2, \cdots, N$,

(c) $\|\Lambda(t)D(t, .)\|$, $\|L(t, .)\|$ and $\|L_2(t, .)\|$ are in $L^p(\tau, \infty)$ for some $p \in (1, 2]$,

(d) $\|D(t, .)\| \to 0$ as $t \to \infty$ and $\sup\{\|D(t + \theta, .)\|: \theta \in [-r, 0]\} \in L^p$.

Then, for sufficiently large $T \geq \tau$, (9.1) has $N$ solutions on $[T - r, \infty)$ with the forms

$$x_n(t) = (e_n + o(1))\exp\left(\int_T^t \left(\lambda_n(s) + \eta_n(s)\right) ds\right), \quad n = 1, 2, \cdots, N, \tag{9.60}$$

as $t \to \infty$, where

$$\eta_n(t) = \lambda_n(t)\, d_{nn}(t, \widetilde{(\lambda_n)}_t) + \mathfrak{L}_{nn}(t, \widetilde{(\lambda_n)}_t) + \mathfrak{L}_{2nn}(t, \widetilde{(\lambda_n)}^t). \tag{9.61}$$

Moreover, each solution of (9.1) has the form

$$x(t) = \sum_{n=1}^{N} c_n x_n(t) + o(e^{-\beta t}) \tag{9.62}$$

as $t \to \infty$, where $(c_1, c_2, \cdots, c_N)^T \in \mathbb{C}^N$ depends on $x$ and $\beta \geq 0$ is arbitrary.

**Proof.** Since (a)-(d) imply (vi)-(viii) for all $n$, the existence of the $N$ solutions in (9.60) with (9.61) follows from Theorem 9.3. According to (b), there are constants $\alpha > 0$ and $q > 0$ such that

$$\left|\exp\left(\int_T^t \lambda_n(s)\, ds\right)\right| \leq \alpha\, e^{qt} \tag{9.63}$$

for all $t \geq T$ and $n = 1, 2, \cdots, N$. Then, as (ix) and (x) are implied by (a) and (c), (9.62) follows from Theorem 9.4 and (9.63). #

As another example, we consider the scalar equation

$$\frac{d}{dt}\{x(t) - e^{-t^2}x(t-1)\} = \lambda(t)x(t) + b(t)x(t + \sin t) \tag{9.64}$$

for $t \geq 0$, where $\lambda(t) = -t^2 \sin t$ and $b(t) \in L(0, \infty)$. We claim that, for sufficiently large $T \geq 0$, (9.64) has a solution on $[T-1, \infty)$ with the form

$$X(t) = (1 + o(1))\exp\left(-\int_T^t s^2 \sin s \, ds\right) \tag{9.65}$$

as $t \to \infty$, and every solution of (9.64) has the form

$$x(t) = X(t)c + o\left(\exp\left(-\int_T^t s^2 \sin s \, ds - \beta t\right)\right) \tag{9.66}$$

for any $\beta \geq 0$. In fact, $\{t_n = n\pi:\ n = 0, 1, 2, \cdots\}$ is a sequence of ZAPs and, if we view the terms $b(t)x(t + \sin t)$ as $L(t, x_t) + L_2(t, x^t)$ and $e^{-t^2}x(t-1)$ as $D(t, x_t)$, we have

$$\|L(t, \tilde{\lambda}_t\,.)\| + \|L_2(t, \tilde{\lambda}\,.)\| \leq |b(t)| \in L(0, \infty),$$

$$\|D(t, \tilde{\lambda}_t\,.)\| \leq \exp\left(-t^2 + \int_{t-1}^t s^2 \, ds\right) \leq e^{1-t} \to 0 \text{ as } t \to \infty,$$

and

$$\|\lambda(t)D(t, \tilde{\lambda}_t\,.)\| \leq t^2 e^{1-t} \in L(0, \infty).$$

Then (9.65) and (9.66) are obtained from Corollary 9.6.

# CHAPTER 10
# MIXED TYPE DIFFERENTIAL EQUATIONS WITH A DOMINATING BOUNDED MATRIX

## § 10.1 INTRODUCTION

In this chapter, we study the asymptotic behaviour of solutions of the equation

$$\frac{d}{dt}\{x(t) - D(t, x_t)\} = A(t)\,x(t) + L_1(t, x_t) + L_2(t, x^t) \tag{10.1}$$

for $t \geq \tau$, where $A(t)$ is a bounded $N \times N$ matrix of locally integrable functions and, for each $t \geq \tau$, the linear operators $D(t, .)$ and $L_1(t, .)$ from $C([-r, 0], \mathbb{C}^N)$ to $\mathbb{C}^N$ and $L_2(t, .)$ from $C([0, \sigma], \mathbb{C}^N)$ to $\mathbb{C}^N$ are as in Part II (p.76). We assume $\sigma = r$ for convenience.

Driver [10] studied the delay differential equation

$$\frac{d}{dt}x(t) = L(t, x_t) \tag{10.2}$$

for $t \in \mathbf{R} \equiv (-\infty, \infty)$. Under the condition $rek < 1$, where $\|L(t, .)\| \leq k$, he proved that (10.2) has a special matrix solution $X(t, t_0)$ for $t \in \mathbf{R}$ and $t_0 \in \mathbf{R}$, which behaves like the fundamental matrix solution of an ordinary differential equation. This special matrix was first introduced by Rjabov [30] and Uvarov [36]. Moreover, each solution of (10.2) on $[-r, \infty)$ with $x_0 = \varphi$ has the asymptotic representation

$$x(t, \varphi) = X(t, 0)[\ell(\varphi) + o(1)] \tag{10.3}$$

as $t \to \infty$, where $\ell$ is an operator from $C$ to $\mathbb{C}^N$.

In the last chapter, we studied (10.1) when $A(t) = dg[\lambda_1(t), \lambda_2(t), \cdots, \lambda_N(t)]$ is diagonal and obtained $N$ special solutions with the forms

$$x_n(t) = \{e_n + o(1)\}\left(\exp\left(\int_\tau^t \widetilde{\lambda}_n(s)\, ds\right)\right), \quad n = 1, 2, \cdots, N, \tag{10.4}$$

where $e_n$ is the nth coordinate vector and each $\tilde{\lambda}_n - \lambda_n$ is small at infinity in some sense. These $N$ special solutions form a matrix solution $X(t)$ and, as $t \to \infty$, each solution of (10.1) has the asymptotic representation

$$x(t) = X(t)c + o\left(\exp\left(\int_T^t \lambda_n(s)\,ds - \beta t\right)\right)$$

for every $\lambda_n(t)$ and arbitrary $\beta \geq 0$.

In this chapter, we do not expect to obtain solutions of any special forms, such as (10.4), since we do not require that the matrix $A(t)$ be of any special form. However, under the assumption that $A(t)$ is bounded, it is still possible to obtain a matrix solution $X(t)$ which enables us to analyse the asymptotic behaviour of the solutions of (10.1).

Adapting the idea of [10], we shall show that, when $\|D(t, .)\|$, $\|L_1(t, .)\|$ and $\|L_2(t, .)\|$ are small in a suitable sense, (10.1) has a special matrix solution $X(t, t_0)$ for $t_0 \geq \tau$ and $t \geq \tau - r$, and that each solution on $[t_0 - r, \infty)$ of (10.1) with $x_{t_0} = \varphi$ has the form

$$x(t, t_0, \varphi) = X(t, t_0)[c(t_0, \varphi) + o(1)] \tag{10.5}$$

as $t \to \infty$. With some further requirements, there is an exponential gap between the solutions $X(t, t_0)c(t_0, \varphi)$ and $X(t, t_0)o(1)$ in (10.5). With additional smallness of $\|D(t, .)\|$ and $\|L_i(t, .)\|$, $i = 1, 2$, at infinity, each solution has the asymptotic representation

$$x(t, t_0, \varphi) = X(t, t_0)c(t_0, \varphi) + o(e^{-\beta t}) \tag{10.6}$$

as $t \to \infty$, where $\beta \geq 0$ is arbitrary.

## § 10.2 SPECIAL SOLUTIONS

Let $Y(t, s)$ be the fundamental matrix solution of the equation

$$\frac{d}{dt}y(t) = A(t)\,y(t) \tag{10.7}$$

for $t, s \in [\tau, \infty)$. Then $Y(t, s)$ satisfies

$$e^{-\|A\|\,|t-s|} \le |Y(t, s)| \le e^{\|A\|\,|t-s|} \tag{10.8}$$

for $t, s \in [\tau, \infty)$, where

$$\|A\| = \sup\{|A(t)|:\ t \ge \tau\}. \tag{10.9}$$

Throughout this chapter, we assume that the functions

$$U_k(t, s) =_{def} \left| \int_s^t \|L_k(p, .)\|\, e^{-r^{-1}|t-p|}\, dp \right| \ (k = 1, 2) \tag{10.10}$$

and $\|D(t, .)\|$ are bounded for $t, s \in [\tau, \infty)$. We put

$$\|D\| = \sup\{\|D(t, .)\|:\ t \ge \tau\}, \tag{10.11}$$

$$M_k = \sup\{U_k(t, s):\ t \ge \tau,\ s \ge \tau\}\ (k = 1, 2), \tag{10.12}$$

$$\Delta = \{\|D\| + \|D\| \max(1, r\|A\|) + M_1 + M_2\}e^{1 + r\|A\|}, \tag{10.13}$$

and also define functions

$$u(t, t_0) = e^{(r^{-1} + \|A\|)|t - t_0|}, \tag{10.14}$$

$$v(a, K, t, t_0) = Ke^{(a + \|A\|)|t - t_0|} \tag{10.15}$$

for $t_0 \ge \tau$, $t \ge \tau$, $a > 0$, and $K > 0$.

**THEOREM** 10.1 Assume that $\Delta < 1$. Then, for every $t_0 \ge \tau$ and $y^0 \in \mathbb{C}^N$, (10.1) with $x(t_0) = y^0$ has a unique solution on $[\tau - r, \infty)$ such that $x_\tau(\theta) = x(\tau)$ for $\theta \in [-r, 0]$ and $x(t)/u(t, t_0)$ is bounded for $t \ge \tau$.

**Proof.** By a variation of parameters formula, every solution on $[\tau - r, \infty)$ of (10.1) with $x(t_0) = y^0$ must satisfy

$$x(t) = D(t, x_t) + Y(t, t_0)(y^0 - D(t_0, x_{t_0})) + \int_{t_0}^{t} Y(t, s)\{L_3(s, x_s) + L_2(s, x^s)\}\, ds \tag{10.16}$$

for $t \geq \tau$, where $L_3(t, \varphi) = A(t)D(t, \varphi) + L_1(t, \varphi)$. Extend the domain of $u(t, t_0)$ to $[\tau - r, \infty)$ by $u_\tau(\theta, t_0) = u(\tau, t_0)$ for $\theta \in [-r, 0]$ and let

$$x(t) = u(t, t_0)y(t), \quad t \geq \tau - r. \tag{10.17}$$

Then $y(t)$ satisfies

$$y(t) = (Y(t, t_0)/u(t, t_0))(y^0 - D(t_0, u_{t_0} y_{t_0})) + D(t, (u_t/u(t, t_0))y_t)$$

$$+ \int_{t_0}^{t} (Y(t, s)/u(t, s))\{L_3(s, (u_s/u(s, t_0))y_s) + L_2(s, (u^s/u(s, t_0))y^s)\}\, ds \tag{10.18}$$

for $t \geq \tau$. Here and in the following pages, $u_t$ and $u^t$ stand for $u(t + ., t_0)$. Let $S$ be a subset of $C([\tau - r, \infty), \mathbb{C}^N) \cap L^\infty(\tau, \infty)$ such that each $y \in S$ satisfies $y(t_0) = y^0$ and $y_\tau(\theta) = y(\tau)$ for $\theta \in [-r, 0]$. For any $y \in S$, define $Ay(t)$ by the right hand side of (10.18) for $t \geq \tau$ and $Ay(t) = Ay(\tau)$ for $t \in [\tau - r, \tau]$. Then, with the usual norm in $L^\infty(\tau, \infty)$ as distance, $S$ is a complete metric space. We show that $A$ is a contraction mapping on $S$ so that the equation (10.18) has a unique solution in $S$.

From (10.8) and the definition of $u(t, t_0)$, we have

$$|Y(t, t_0)/u(t, t_0)| \leq e^{-r^{-1}|t - t_0|},$$

$$\|u_t/u(t, t_0)\| \leq e^{1 + r\|A\|} \quad \text{and} \quad \|u^t/u(t, t_0)\| \leq e^{1 + r\|A\|}$$

for $t \geq \tau$. Then, for any $y \in S$ and $t \geq \tau$,

$$|Ay(t)| \leq e^{-r^{-1}|t - t_0|}(|y^0| + \|D\|\, e^{1 + r\|A\|}\|y\|)$$

$$+ \left\{\|D\| + \left|\int_{t_0}^{t} e^{-r^{-1}|t - s|}\{\|A\|.\|D\| + \|L_1(s, .)\| + \|L_2(s, .)\|\}\, ds\right|\right\} e^{1 + r\|A\|}\|y\|.$$

It follows from (10.10), (10.12) and (10.13) that

$$|\mathcal{A}y(t)| \le |y^0| e^{-r^{-1}|t - t_0|}$$

$$+ \Big\{ \|D\|(1 + r\|A\|) + M_1 + M_2 + \|D\|(1 - r\|A\|)e^{-r^{-1}|t - t_0|} \Big\} e^{1 + r\|A\|} \|y\|$$

$$\le |y^0| e^{-r^{-1}|t - t_0|} + \Delta \|y\|$$

for $t \ge \tau$. Thus $\mathcal{A}$ maps $S$ into itself. Similarly, for any $y_1, y_2 \in S$ and $t \ge \tau$, we have

$$|\mathcal{A}y_1(t) - \mathcal{A}y_2(t)| \le \Delta \|y_1 - y_2\|.$$

The mapping $\mathcal{A}$ is contractive as $\Delta < 1$. #

Following the teminology of Rjabov, Uvarov and Driver, we call the solution of (10.1) given by Theorem 10.1 a special solution and denote it by $x(t, t_0, y^0)$. Let

$$F(a) = \Big\{ \|D\| + (ar)^{-1}[\|D\| \max(1, r\|A\|) + M_1 + M_2] \Big\} e^{r(a + \|A\|)}. \qquad (10.19)$$

Then, under the condition $\Delta < 1$, the equation

$$F(a) = 1 \qquad (10.20)$$

possesses a unique solution in $(0, r^{-1})$ and thus determines a unique value of

$$K = \{1 - (2\|D\| + M_1 + M_2) e^{r(a + \|A\|)}\}^{-1}. \qquad (10.21)$$

**THEOREM** 10.2 Assume that $\Delta < 1$. Then, for every $t_0 \ge \tau$ and $y^0 \in \mathbb{C}^N$, the special solution $x(t, t_0, y^0)$ of (10.1) on $[\tau - r, \infty)$ satisfies

$$|x(t, t_0, y^0)| \le |y^0| v(a, K, t, t_0). \qquad (10.22)$$

where $a$, $K$ and $v$ are given by (10.20), (10.21) and (10.15).

**Proof.** We derive (10.22) by the method of successive approximations. Let

$$x_0(t) = Y(t, t_0)y^0,$$

$$x_{k+1}(t) = Y(t, t_0)(y^0 - D(t_0, (x_k)_{t_0})) + D(t, (x_k)_t) +$$

$$+\int_{t_0}^{t} Y(t, s)\{L_3(s, (x_k)_s) + L_2(s, (x_k)^s)\}ds\,, \quad k = 0, 1, 2, \cdots \qquad (10.23)$$

for $t \geq \tau$ and $x_k(t) = x_k(\tau)$ for $t \in [\tau - r, \tau]$ and $k = 0, 1, 2, \cdots$. By induction we show that

$$|x_k(t)| \leq |y^0|\, v(a, K, t, t_0) \qquad (10.24)$$

for $t \geq \tau - r$ and $k = 0, 1, 2, \cdots$. From (10.8) and (10.15),

$$|x_0(t)| \leq |y^0|\, e^{\|A\|.|t - t_0|} \leq |y^0|\, v(a, K, t, t_0)$$

for $t \geq \tau - r$. Suppose that (10.24) holds for $k$ and $t \geq \tau - r$. Then, for $t \geq \tau$, since

$$\|v_t\| = \sup\{v_t(a, K, \theta, t_0) \colon \theta \in [-r, 0]\} \leq v(a, K, t, t_0)\, e^{r(a + \|A\|)}$$

and

$$\|v^t\| = \sup\{v^t(a, K, \alpha, t_0) \colon \alpha \in [0, r]\} \leq v(a, K, t, t_0)\, e^{r(a + \|A\|)},$$

we have

$$\begin{aligned}
|x_{k+1}(t)| &\leq e^{\|A\|.|t - t_0|}(|y^0| + \|D\|.|y^0|.\|v_{t_0}\|) + \|D\|.|y^0|.\|v_t\| \\
&\qquad + \left|\int_{t_0}^{t} e^{\|A\|.|t - s|}\{\|L_3(s, .)\|.\|v_s\| + \|L_2(s, .)\|.\|v^s\|\}\, ds\right| . |y^0| \\
&\leq e^{\|A\|.|t - t_0|}(1 + K\|D\|\, e^{r(a + \|A\|)})\, |y^0| + e^{r(a + \|A\|)}\|D\|\, v(a, K, t, t_0)\, |y^0| \\
&\qquad + e^{r(a + \|A\|)}\left|\int_{t_0}^{t} e^{\|A\|.|t - s|}\{\|L_3(s, .)\| + \|L_2(s, .)\|\}\, v(a, K, s, t_0)\, ds\right| . |y^0| \\
&\leq |y^0| v(a, K, t, t_0)\left\{K^{-1}e^{-a|t - t_0|} + e^{r(a + \|A\|)}\|D\|(1 + e^{-a|t - t_0|})\right\} \\
&\qquad + e^{r(a + \|A\|)}\left|\int_{t_0}^{t} e^{-a|t - s|}\{\|A\|.\|D\| + \|L_1(s, .)\| + \|L_2(s, .)\|\}\, ds\right| v(a, K, t, t_0)\, |y^0| \\
&=_{\text{def}} |y^0| v(a, K, t, t_0)\, \gamma(t).
\end{aligned}$$

From (7.18) and its proof in § 7.3 and (10.12),

$$\left| \int_{t_0}^{t} \|L_i(s,.)\| e^{-a|t-s|} \, ds \right| \le \left\{ (ar)^{-1} - \left( (ar)^{-1} - 1 \right) e^{-a|t-t_0|} \right\} M_i \tag{10.25}$$

for $t \ge \tau$ and $i = 1, 2$. As

$$\left| \int_{t_0}^{t} \|A\|.\|D\| e^{-a|t-s|} \, ds \right| = a^{-1} \|A\|.\|D\| \left(1 - e^{-a|t-t_0|}\right)$$

$$\le (ar)^{-1} \|D\| \max(1, r\|A\|) \left(1 - e^{-a|t-t_0|}\right), \tag{10.26}$$

substituting (10.25) and (10.26) into $\gamma(t)$ and taking (10.19)-(10.21) into account, we obtain

$$\gamma(t) \le e^{r(a+\|A\|)} \left[ \|D\| + (ar)^{-1} (\|D\| \max(1, r\|A\|) + M_1 + M_2) \right] + \Big\{ K^{-1}$$

$$+ e^{r(a+\|A\|)} \left[ \|D\| + M_1 + M_2 - (ar)^{-1} (\|D\| \max(1, r\|A\|) + M_1 + M_2) \right] \Big\} e^{-a|t-t_0|}$$

$$= F(a) + \{ K^{-1} + (1 - K^{-1}) - F(a) \} e^{-a|t-t_0|} = 1.$$

Thus (10.24) holds for $k + 1$ and, by induction, for all $k = 0, 1, 2, \cdots$.

Let $y_k(t) = x_k(t)/u(t, t_0)$. From the proof of Theorem 10.1, we know that

$$y_k \in S, \; y_{k+1} = Ay_k, \; k = 0, 1, 2, \cdots.$$

Since $A$ is contractive, we must have $\|y_k - y\| \to 0$ as $k \to \infty$, where $y$ is the unique solution of (10.18) in $S$. Then, for $t \ge \tau$,

$$|x_k(t) - x(t, t_0, y^0)| \le u(t, t_0) \|y_k - y\| \to 0$$

as $k \to \infty$ and (10.22) follows from (10.24). #

From Theorem 10.1 there is a matrix $X(t, s)$ for $t \ge \tau - r$ and $s \ge \tau$ such that $X(s, s) = I$, $X(t, s) = X(\tau, s)$ for $t \in [\tau - r, \tau]$, and each column of $X(t, s)$ is a special solution of (10.1) with respect to $t$. We call $X(t, s)$ the special matrix solution of (10.1) on $[\tau - r, \infty)$.

**THEOREM** 10.3 Assume that $\Delta < 1$. Then the special matrix solution of (10.1) satisfies

(i) for every $t_0 \geq \tau$, $y^0 \in \mathbb{C}^N$, and $t \geq \tau - r$,

$$x(t, t_0, y^0) = X(t, t_0)y^0,$$

(ii) for $t_0$, $t_1$ and $t$ in $[\tau, \infty)$,

$$X(t, t_0) = X(t, t_1)X(t_1, t_0), \; X^{-1}(t, t_0) = X(t_0, t),$$

(iii) for $t_0 \geq \tau$ and $t \geq \tau - r$,

$$\{v(a, K, t, t_0)\}^{-1} \leq |X(t, t_0)| \leq v(a, K, t, t_0). \tag{10.27}$$

**Proof**. From the definition of $x(t, t_0, y^0)$ and $X(t, t_0)$, (i) is obvious. For $t_0$, $t_1$ and $t$ in $[\tau, \infty)$,

$$\begin{aligned} X(t, t_0) &= (x(t, t_0, e_1), x(t, t_0, e_2), \cdots, x(t, t_0, e_N)) \\ &= (x(t, t_1, x(t_1, t_0, e_1)), x(t, t_1, x(t_1, t_0, e_2)), \cdots, x(t, t_1, x(t_1, t_0, e_N))) \\ &= (X(t, t_1)x(t_1, t_0, e_1), X(t, t_1)x(t_1, t_0, e_2), \cdots, X(t, t_1)x(t_1, t_0, e_N)) \\ &= X(t, t_1)(x(t_1, t_0, e_1), x(t_1, t_0, e_2), \cdots, x(t_1, t_0, e_N)) \\ &= X(t, t_1)X(t_1, t_0), \end{aligned}$$

where $e_n$ is the nth coordinate vector. Letting $t = t_0$ in the above, we have shown (ii) as $X(t_0, t_0) = I$. By (i) and Theorem 10.2,

$$|X(t, t_0)y^0| \leq |y^0| \, v(a, K, t, t_0)$$

for every $y^0 \in \mathbb{C}^N$. Since

$$|X(t, t_0)| = \sup\{|X(t, t_0)y^0| : y^0 \in \mathbb{C}^N, \; |y^0| = 1\},$$

we obtain

$$|X(t, t_0)| \leq v(a, K, t, t_0)$$

and, by (ii), (10.27) for $t_0 \geq \tau$ and $t \geq \tau$. Then (iii) follows from the fact that $X(t, t_0)$ is constant for $t \in [\tau - r, \tau]$. #

**Remarks**. (i). The boundedness of $U_k(t, s)$, $k = 1, 2$, is ensured if $\int_t^{t+r} \|L_k(s, .)\| \, ds$ is bounded for $k = 1, 2$, and this covers the case that the $\|L_k(t, .)\|$, $k = 1, 2$, are bounded. In the latter case, we may simply replace $M_k$ by $r\|L_k\|$ in the definitions of $\Delta$, a and K given by (10.13) and (10.19)-(10.21). However, for the purpose of weakening the condition and improving the result, we would rather define $\Delta'$, a' and K' by

$$\Delta' = \{\|D\| + \max[\|D\|, r(\|A\|.\|D\| + \|L_1\| + \|L_2\|)]\} e^{1 + r\|A\|}, \qquad (10.28)$$

$$F_0(a') = 1, \qquad (10.29)$$

where

$$F_0(a) =_{def} \{\|D\| + (ar)^{-1}\max[\|D\|, r(\|A\|.\|D\| + \|L_1\| + \|L_2\|)]\} e^{r(a + \|A\|)}, \qquad (10.30)$$

and

$$K' = (1 - 2\|D\| e^{r(a' + \|A\|)})^{-1} \qquad (10.31)$$

since the numbers $\Delta'$, a' and K' are smaller than $\Delta$, a and K if $r\|A\| < 1$, $\|D\| \neq 0$, and $\|L_1\| + \|L_2\| \neq 0$. In fact, using these parameters in the proofs of Theorems 10.1 and 10.2, we obtain

$$|Ay(t)| \leq |y^0| e^{-r^{-1}(t - t_0)} + \Delta'\|y\|$$

and

$$|x_{k+1}(t)| \leq |y^0| v(a', K', t, t_0)\{F_0(a') + e^{-a'(t - t_0)}(1 - F_0(a'))\}.$$

Thus we can replace $\Delta$, a and K by $\Delta'$, a' and K' given by (10.28)-(10.31). This remark also applies to later sections. When $D(t, .) = A(t) = L_2(t, .) = 0$, there is no difference between $\Delta$, a, K and $\Delta'$, a', K' and the condition $\Delta < 1$ coincides with that in [10] mentioned in § 10.1.

(ii). If the function

$$\{u(t, t_0)\}^{-1}\int_{t_0}^{t} e^{\|A\|(t-s)}|f(s)|\, ds$$

is bounded for $t,\ t_0 \in [\tau,\ \infty)$, then Theorem 10.1 also holds for the nonhomogeneous equation

$$\frac{d}{dt}\{x(t) - D(t, x_t)\} = A(t)\, x(t) + L_1(t, x_t) + L_2(t, x^t) + f(t) \qquad (10.32)$$

as the proof of Theorem 10.1 is still valid for (10.32). Moreover, Theorem 10.3 implies that each special solution of (10.32) satisfies

$$x(t, t_0, y^0) = X(t, t_0)y^0 + x(t, t_0, 0), \qquad (10.33)$$

where $X(t, t_0)$ is the special matrix solution of (10.1) and $x(t, t_0, 0)$ is the special solution of (10.32) with $x(t_0) = 0$.

(iii). We have assumed the existence and uniqueness on $[t_0 - r,\ \infty)$ of the solution $x(t, t_0, \varphi)$ of (10.1) with $x_{t_0} = \varphi$, for every $t_0 \geq \tau$ and $\varphi \in C$. Indeed, from the condition $\Delta < 1$, the existence of a sequence $\{t_k\}$ of ZAPs with $t_k \to \infty$ as $k \to \infty$, Theorem 7.2, and the proof of Theorem 10.1, we can prove such existence and uniqueness if necessary.

(iv). The concept of a special solution is related to the interval $[\tau - r,\ \infty)$ as we require the solution be constant on $[\tau - r,\ \tau]$. If we extend (10.1) to $\mathbf{R}$ by letting $D(t, .) = A(t) = L_1(t, .) = L_2(t, .) = 0$ for $t < \tau$, then this concept agrees with that in [10]. Since we have assumed existence and uniqueness, from Theorem 10.3 every special solution of (10.1) on $[\tau - r,\ \infty)$ is uniquely determined by the $N$ solutions $x(t, \tau, \varphi_n)$, $n = 1, 2, \cdots, N$, where $\varphi_n(\theta) = e_n$ for $\theta \in [-r,\ 0]$. Similarly, for any $T \geq \tau$, if we define special solutions on $[T - r,\ \infty)$ instead of $[\tau - r,\ \infty)$, then every special solution on $[T - r,\ \infty)$ is identical to a solution $x(t, T, \varphi)$ on $[T - r,\ \infty)$ with $\varphi$ constant on $[-r,\ 0]$.

(v). Theorem 10.3 indicates that the special matrix solution $X(t, t_0)$ of (10.1) has similar properties to the fundamental matrix solution $Y(t, t_0)$ of (10.7). Since

$\Delta < 1$ can be viewed as a smallness condition on $\|D(t,.)\|$ and $\|L_k(t,.)\|$ $(k = 1, 2)$, if we regard (10.1) as a perturbation of (10.7) then the similarity of the properties of $X(t, t_0)$ to those of $Y(t, t_0)$ seems natural. Indeed, from (10.8) and (10.27), there is a close relationship between their exponential bounds. However, no matter how small $\|D(t,.)\|$ and $\|L_k(t,.)\|$ $(k = 1, 2)$ are, we can not ensure that, after representing $X(t, t_0)$ in terms of $Y(t, t_0)$ by

$$X(t, t_0) = Y(t, t_0)\delta(t, t_0) \tag{10.34}$$

for some matrix function $\delta(t, t_0)$, $|\delta(t, t_0)|$ has a smaller exponential bound than $|Y(t, t_0)|$. For instance, the equation

$$\frac{d}{dt}x(t) = A\,x(t) + B\,x(t-1), \tag{10.35}$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ \varepsilon_0 & 0 \end{bmatrix}$$

with $\varepsilon_0 > 0$ sufficiently small, has a special matrix solution $X(t, 0) = (x_{ij}(t))_{2\times 2}$ on $[-1, \infty)$ given by $X(t, 0) = X(0, 0) = I$ for $t \in [-1, 0]$,

$$x_{11}(t) = e^{t},\ x_{12}(t) = 0,\ x_{22}(t) = e^{-t}$$

for $t \geq 0$, and

$$x_{21}(t) = \begin{cases} \frac{1}{2}(e^{t} - e^{-t}) + \varepsilon_0(1 - e^{-t}) & \text{for } t \in [0, 1), \\ \frac{1}{2}(1 + e + 2\varepsilon_0)(e - 1)e^{-t} + \frac{1}{2}(1 + \varepsilon_0 e^{-1})(e^{t} - e^{2-t}) & \text{for } t \geq 1. \end{cases}$$

On the other hand, the fundamental matrix solution of $dy(t)/dt = Ay(t)$ is

$$Y(t, 0) = e^{At} = \begin{bmatrix} e^{t} & 0 \\ \frac{1}{2}(e^{t} - e^{-t}) & e^{-t} \end{bmatrix}.$$

Since $Y^{-1}(t, 0) = Y(0, t) = e^{-At}$, for $t \geq 1$ (10.34) leads to

$$\delta(t, 0) = Y^{-1}(t, 0)\,X(t, 0) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2}\varepsilon_0(e - 2 + e^{2t-1}) & 1 \end{bmatrix}.$$

Clearly we have

$$\lim_{t \to \infty} e^{-t}|Y(t, 0)| = c_1 \text{ and } \lim_{t \to \infty} e^{-2t}|\delta(t, 0)| = c_2 \tag{10.36}$$

for some $c_1 > 0$ and $c_2 > 0$, so that the exponential bound of $|\delta(t, 0)|$ is twice that of $|Y(t, 0)|$, no matter how small $\varepsilon_0 > 0$ is.

## § 10.3 ASYMPTOTIC REPRESENTATION OF SOLUTIONS

Let

$$M_0 = (1 + K)(1 - K^{-1}e^{-ar}), \tag{10.37}$$

where $a$ and $K$ are given by (10.20) and (10.21).

**THEOREM** 10.4 Assume that $\Delta < 1$ and $M_0 < 1$. Then, for any $t_0 \geq \tau$, every solution of (10.1) on $[t_0 - r, \infty)$ has the asymptotic representation

$$x(t) = X(t, t_0)(c + O(e^{((1/r)\ln M_0)(t - t_0)})) \tag{10.38}$$

as $t \to \infty$, where $X(t, t_0)$ is the special matrix solution of (10.1) and $c \in \mathbb{C}^N$ depends on $x$.

To prove the theorem, we adapt the method of the proof of Theorem 9.4.

**Proof.** Suppose that $x(t)$ is a solution of (10.1) on $[t_0 - r, \infty)$. Then, by a variation of parameters formula, $x(t)$ satisfies

$$x(t) = D(t, x_t) + Y(t, t + \theta)(x(t + \theta) - D(t + \theta, x_{t+\theta}))$$

$$+ \int_{t+\theta}^{t} Y(t, s)\{L_3(s, x_s) + L_2(s, x^s)\}\, ds \tag{10.39}$$

for $t \geq t_0 + r$ and $\theta \in [-r, 0]$. Let

$$x(t) = X(t, t_0)z(t) \tag{10.40}$$

for $t \geq t_0 - r$. Substitution of (10.40) changes (10.39) to

$$X(t, t_0)z(t) = D(t, X_t z_t) + Y(t, t+\theta)(X(t+\theta, t_0)z(t+\theta) - D(t+\theta, X_{t+\theta}z_{t+\theta}))$$

$$+ \int_{t+\theta}^{t} Y(t, s)\{L_3(s, X_s z_s) + L_2(s, X^s z^s)\}\, ds \tag{10.41}$$

for $t \geq t_0 + r$ and $\theta \in [-r, 0]$. Since $X(t, t_0)$ satisfies (10.1) as well as (10.39), replacing $x(t)$ by $X(t, t_0)$ in (10.39) and substituting it into the left hand side of (10.41), we obtain

$$Y(t, t+\theta)X(t+\theta, t_0)(z(t) - z(t+\theta))$$

$$= Y(t, t+\theta)D(t+\theta, X_{t+\theta}(z(t) - z_{t+\theta})) - D(t, X_t(z(t) - z_t))$$

$$- \int_{t+\theta}^{t} Y(t, s)\{L_3(s, X_s(z(t) - z_s)) + L_2(s, X^s(z(t) - z^s))\}\, ds \tag{10.42}$$

for $t \geq t_0 + r$ and $\theta \in [-r, 0]$. Put

$$w(t) = \sup\{|W(t, \theta)|: \theta \in [-r, 0]\} \tag{10.43}$$

for $t \geq t_0$, where

$$W(t, \theta) = e^{(a + \|A\|)(t+\theta-t_0)} X(t+\theta, t_0)\,(z(t) - z(t+\theta)). \tag{10.44}$$

Then (10.42) is changed to

$$W(t, \theta) = D(t+\theta, X_{t+\theta}X^{-1}(t+\theta, t_0)W(t, \theta)) + D(t+\theta, e^{-(a + \|A\|)\cdot}W(t+\theta, .))$$

$$- Y(t+\theta, t)\, e^{(a + \|A\|)\theta} D(t, e^{-(a + \|A\|)\cdot}W(t, .))$$

$$- \int_{t+\theta}^{t} Y(t+\theta, s)\, e^{(a + \|A\|)(t+\theta-s)} \Big\{L_3\big(s, e^{-(a + \|A\|)\cdot}W(s, .)\big) + [\cdots]\Big\}\, ds \tag{10.45}$$

for $t \geq t_0 + r$ and $\theta \in [-r, 0]$, where

$$[\cdots] = L_3(s, X_s X^{-1}(s, t_0)W(t, s-t)) + e^{(a+\|A\|)(s-t_0)} L_2(s, X^s(z(t) - z^s)).$$

By (10.27) we have

$$\|X_t X^{-1}(t, t_0)\| = \sup\{|X(t+s, t)|:\ s \in [-r, 0]\} \le Ke^{r(a+\|A\|)}. \tag{10.46}$$

For $s \in [t+\theta, t]$ and $\alpha \in [0, r]$ with $s + \alpha \le t$,

$$Q(s, t) =_{def} |e^{(a+\|A\|)(s-t_0)} X(s+\alpha, t_0)(z(t) - z(s+\alpha))|$$

$$= e^{-\alpha(a+\|A\|)} |W(t, s+\alpha-t)| \le w(t).$$

If $s \in [t+\theta, t]$, $\alpha \in [0, r]$ and $s+\alpha > t$, then

$$Q(s, t) = e^{(a+\|A\|)(s-t)} |X(s+\alpha, t)W(s+\alpha, t-s-\alpha)|$$

$$\le Ke^{(a+\|A\|)(2(s-t)+\alpha)} w(s+\alpha) \le Ke^{r(a+\|A\|)} \|w^s\|.$$

Thus, for $s \in [t+\theta, t]$,

$$\|e^{(a+\|A\|)(s-t_0)} X^s(z(t) - z^s)\| \le Ke^{r(a+\|A\|)} \|w^s\|. \tag{10.47}$$

From (10.45)-(10.47) and (10.25) we obtain

$$|W(t, \theta)| \le e^{r(a+\|A\|)} \max\{w(p):\ t+\theta \le p \le t+r\} \times$$

$$\left\{(K + 1 + e^{a\theta})\|D\| + \int_{t+\theta}^{t} e^{a(t+\theta-s)}\{(1+K)\|L_3(s, .)\| + K\|L_2(s, .)\|\}\, ds\right\}$$

$$\le e^{r(a+\|A\|)} \max\{w(p):\ t+\theta \le p \le t+r\} \times$$

$$\{(K + 1 + e^{a\theta})\|D\| + (1+K)(ar)^{-1}\|D\|(1 - e^{a\theta}) \max(1, r\|A\|)$$

$$+ [(ar)^{-1} - ((ar)^{-1} - 1)e^{a\theta}]((1+K)M_1 + KM_2)\}$$

$$=_{def} (\varepsilon_1 + \varepsilon_2 e^{a\theta}) \max\{w(p):\ t+\theta \le p \le t+r\}.$$

In view of (10.19)-(10.21) and (10.37),

$$\varepsilon_1 = (1+K)F(a) - (ar)^{-1}M_2 e^{r(a+\|A\|)} = (1+K) - (ar)^{-1}M_2 e^{r(a+\|A\|)},$$

$$\begin{aligned}\varepsilon_2 &= (2+K)\|D\|\, e^{r(a+\|A\|)} - (1+K)F(a) + (ar)^{-1}M_2\, e^{r(a+\|A\|)} \\ &\quad + ((1+K)M_1 + KM_2)\, e^{r(a+\|A\|)} \\ &\le [(1+K)(2\|D\| + M_1 + M_2) + (ar)^{-1}M_2]\, e^{r(a+\|A\|)} - (1+K) \\ &= (1+K)(1-K^{-1}) - (1+K) + (ar)^{-1}M_2\, e^{r(a+\|A\|)} \\ &= -K^{-1}(1+K) + (ar)^{-1}M_2\, e^{r(a+\|A\|)},\end{aligned}$$

and

$$\begin{aligned}\varepsilon_1 + \varepsilon_2 e^{a\theta} &\le 1 + K - (ar)^{-1}M_2\, e^{r(a+\|A\|)}(1-e^{a\theta}) - K^{-1}(1+K)e^{a\theta} \\ &\le (1+K)(1-K^{-1}e^{-ar}) = M_0.\end{aligned}$$

Hence, for $t \ge t_0 + r$,

$$w(t) \le M_0 \max\{w(p):\ t-r \le p \le t+r\}. \tag{10.48}$$

Since (10.1) has a sequence $\{t_n\}$ of ZAPs with $t_n \to \infty$ as $n \to \infty$ and the involvement of the values of $w(p)$ on $(t, t+r]$ in (10.48) is caused by the term $L_2(s, X^s z^s)$, for any $t_n > t_0 + r$ and all $t \in [t_0 + r, t_n]$, we actually obtain

$$w(t) \le M_0 \max\{w(p):\ t-r \le p \le t_n\}. \tag{10.49}$$

Then the application of Lemma 8.4 to (10.49) yields

$$w(t) \le \|w^{t_0}\|\, e^{(t-t_0-r)r^{-1}\ln M_0} \tag{10.50}$$

for $t \ge t_0 + r$. Thus, by (10.44) and (10.50),

$$\begin{aligned}\|z(t) - z_t\| &\le \sup\{|e^{(a+\|A\|)(t_0-t-\theta)}X(t_0, t+\theta)|:\ \theta \in [-r, 0]\}w(t) \\ &\le K\,\|w^{t_0}\|\, e^{(t-t_0-r)r^{-1}\ln M_0}\end{aligned} \tag{10.51}$$

for $t \ge t_0 + r$. From (10.51) and Lemma 9.5, $z(t)$ has a finite limit $c \in \mathbb{C}^N$ as $t \to \infty$ and

$$|z(t) - c| \le K(1-M_0)^{-1}\|w^{t_0}\|\, e^{(r^{-1}\ln M_0)(t-t_0)} \tag{10.52}$$

for $t \ge t_0$. Therefore, $x(t)$ has the representation (10.38). #

Note that, since $M_0 < 1$, the order term in (10.38) can be replaced by $o(1)$ if we do not know the actual value of $M_0$.

By (10.13), the condition $\Delta < 1$ will hold for sufficiently small $\|D\|$ and $M_k$ $(k = 1, 2)$. Also, for $\|D\|$ and the $M_k$ $(k = 1, 2)$ small enough, a, determined by (10.20), will be sufficiently small and K, given by (10.21), sufficiently close to 1 that, by (10.37), $M_0 < 1$. Thus the conditions of Theorem 10.4 are met if $\|D\|$ and the $M_k$ $(k = 1, 2)$ are small enough.

Let

$$M^* = \{5(M_1 + M_2) + [7 + 2\max(1, r\,\|A\|)]\,\|D\|\}e^{1 + r\|A\|}. \tag{10.53}$$

**COROLLARY** 10.5 If $M^* \leq 1$, then the special matrix solution $X(t, s)$ of (10.1) on $[\tau - r, \infty)$ exists and, for every $t_0 \geq \tau$, every solution of (10.1) on $[t_0 - r, \infty)$ has the asymptotic representation

$$x(t) = X(t, t_0)(c + o(1)) \tag{10.54}$$

as $t \to \infty$.

**Proof**. Comparing (10.13) with (10.53), we have $\Delta < M^* \leq 1$. Then, by Theorem 10.4, we only need show that $M_0 < 1$. With the numbers $b_1$ and $b_2$ given by

$$b_1 =_{def} (2\|D\| + M_1 + M_2)\, e^{r(a + \|A\|)}$$

and

$$b_2 =_{def} (\|D\| + \|D\| \max(1, r\,\|A\|) + M_1 + M_2)\, e^{r(a + \|A\|)},$$

it follows from (10.53) that $b_i < M^* \leq 1$, $i = 1, 2$, and

$$3b_1 + 2b_2 - \|D\|e^{r(a + \|A\|)} \leq M^* \leq 1.$$

Then

$$1 - \|D\|e^{r(a + \|A\|)} \leq 2 - 3b_1 - 2b_2 \leq (1 - b_1)(1 - b_2)(2 - b_1),$$

and so

$$(1 - b_1)^{-1} \le (1 - \|D\|e^{r(a + \|A\|)})^{-1}(1 - b_2)(1 + 1 - b_1). \tag{10.55}$$

Since, from (10.19)-(10.21), $K = (1 - b_1)^{-1}$ and

$$ar = (1 - \|D\|e^{r(a + \|A\|)})^{-1}(b_2 - \|D\|e^{r(a + \|A\|)}),$$

(10.55) can be written as $K \le (1 - ar)(1 + K^{-1})$. Hence

$$(1 + K)(1 - (1 - ar)K^{-1}) \le 1,$$

which, together with $e^{-ar} > (1 - ar)$, implies $M_0 < 1$. #

## § 10.4 THE EXPONENTIAL GAP

**THEOREM** 10.6 Assume that $\Delta < 1$ and $M_0 < e^{-2r(a + \|A\|)}$. Then, for each $t_0 \ge \tau$, every solution of (10.1) on $[t_0 - r, \infty)$ has the asymptotic representation

$$x(t) = X(t, t_0)c + O(e^{-(a + \|A\| + \varepsilon)(t - t_0)}) \tag{10.56}$$

as $t \to \infty$, where

$$\varepsilon = -(2a + 2\|A\| + r^{-1}\ln M_0) > 0. \tag{10.57}$$

**Proof.** We know from Theorem 10.4 that every solution of (10.1) on $[t_0 - r, \infty)$ has the representation (10.38). Thus, by Theorem 10.3,

$$x(t) - X(t, t_0)c = X(t, t_0)\, O(e^{(r^{-1}\ln M_0)(t - t_0)}) = O(e^{(a + \|A\| + (1/r)\ln M_0)(t - t_0)})$$

as $t \to \infty$. From (10.57) we have

$$a + \|A\| + r^{-1}\ln M_0 = -(a + \|A\| + \varepsilon),$$

from which (10.56) follows. That $\varepsilon$ is positive is immediate from $M_0 < e^{-2r(a + \|A\|)}$. #

**Remark.** For any $c \in \mathbb{C}^N$, since

$$|c| = |X(t_0, t)X(t, t_0)c| \le Ke^{(a + \|A\|)(t - t_0)}|X(t, t_0)c|,$$

the solution $X(t, t_0)c$ satisfies

$$K^{-1}|c|e^{-(a + \|A\|)(t - t_0)} \le |X(t, t_0)c| \le K|c|e^{(a + \|A\|)(t - t_0)}$$

for $t \ge \tau - r$. This, together with Theorem 10.6, indicates that every solution of (10.1) on $[t_0 - r, \infty)$ can be separated by $X(t, t_0)$ into two solutions between which there is an exponential gap. In practice, it is almost impossible to make such a separation without further knowledge of $X(t, t_0)$. However, as far as the exponential estimates are concerned, Theorem 10.7, below, gives a more convenient separation. This will provide a means of decomposing the solution space as the direct sum of the special solutions and solutions smaller than any exponential as shown in § 10.5.

**THEOREM** 10.7 Assume that $\Delta < 1$ and $M_0e^{2r(a + \|A\|)} < 1$. Then, for each $t_0 \ge \tau$ and $\varphi \in C$, there is a unique $y^0 \in \mathbb{C}^N$ such that the solution on $[t_0 - r, \infty)$ of (10.1) with $x_{t_0} = \varphi$ has the representation

$$x(t, t_0, \varphi) = x(t, t_0, \varphi_0) + O(e^{-(a + \|A\| + \varepsilon)(t - t_0)}) \tag{10.58}$$

as $t \to \infty$, where $\varphi_0(\theta) = y^0$ for $\theta \in [-r, 0]$ and $\varepsilon$ is given by (10.57). For $t \ge t_0$, $x(t, t_0, \varphi_0)$ satisfies

$$K^{-1}|y^0|e^{-(a + \|A\|)(t - t_0)} \le |x(t, t_0, \varphi_0)| \le K|y^0|e^{(a + \|A\|)(t - t_0)}. \tag{10.59}$$

**Proof**. For any $t_0 \ge \tau$, we consider the equation (10.1) on $[t_0 - r, \infty)$ rather than on $[\tau - r, \infty)$. Then, for any $y^0 \in \mathbb{C}^N$, Remark (iv) in § 10.2 implies that the solution $x(t, t_0, \varphi_0)$ of (10.1) with $\varphi_0(\theta) = y^0$ for $\theta \in [-r, 0]$ is identical to the special solution on $[t_0 - r, \infty)$ of (10.1) with $x(t_0) = y^0$. The conclusion follows from Theorem 10.6. #

**Remark**. Following the idea of Corollary 10.5, sufficient conditions for $\Delta < 1$ and $M_0e^{2r(a + \|A\|)} < 1$ can be obtained in terms of $\|D\|$, $M_1$ and $M_2$.

## § 10.5 SMALLNESS AT INFINITY

In this section, we denote the special matrix solution $X(t, t_0)$ on $[\tau - r, \infty)$ by $X_\tau(t, t_0)$ and show the dependence on $[\tau, \infty)$ of the parameters $\|A\|$, $\|D\|$, $M_1$, $M_2$, $\Delta$, $a$, $K$, $M_0$, and $\varepsilon$ determined by (10.9)-(10.13), (10.19)-(10.21), (10.37), and (10.57) by $\|A\|(\tau)$, $\|D\|(\tau)$, $M_1(\tau)$, and so forth. As functions of $\tau$, all of $\|A\|$, $\|D\|$, $M_k$ $(k = 0, 1, 2)$, $\Delta$, $a$, $K$ and $-\varepsilon$ are decreasing as $\tau$ increases. If $\|D\|(T)$, $M_1(T)$ and $M_2(T)$ are $o(1)$ as $T \to \infty$, then also $\Delta(T) \to 0$, $a(T) \to 0$, $K(T) \to 1$, $M_0(T) \to 0$, and $\varepsilon(T) \to \infty$.

**THEOREM** 10.8 Assume that the following conditions are met:

(i) $\Delta(\tau) < 1$ and $M_0(\tau) \exp\{2r(a(\tau) + \|A\|(\tau))\} < 1$,

(ii) $\|D(t, .)\| \to 0$ as $t \to \infty$,

(iii) for $k = 1, 2$, $\int_t^{t+r} \|L_k(s, .)\| \, ds \to 0$ as $t \to \infty$.

Then, for every $t_0 \geq \tau$ and $\varphi \in C$, there is a unique $y^0 \in \mathbb{C}^N$ such that the solution on $[t_0 - r, \infty)$ of (10.1) with $x_{t_0} = \varphi$ has the asymptotic representation

$$x(t, t_0, \varphi) = X_{t_0}(t, t_0)y^0 + o(e^{-\beta t}) \tag{10.60}$$

as $t \to \infty$, where $\beta \geq 0$ is arbitrary. Moreover, for any $\delta > 0$, there are $T \geq t_0$, $c_1 > 0$ and $c_2 > 0$ such that, for $t \geq T$, $X_{t_0}(t, t_0)$ satisfies

$$c_1 e^{-(\delta + \|A\|(T))(t - t_0)} \leq |X_{t_0}(t, t_0)| \leq c_2 e^{(\delta + \|A\|(T))(t - t_0)}. \tag{10.61}$$

**Proof.** From Theorem 10.7 and Remark (iv) in § 10.2, there is a unique $y^0 \in \mathbb{C}^N$ such that

$$x(t, t_0, \varphi) = X_{t_0}(t, t_0)y^0 + x(t, t_0, \varphi_1)$$

and

$$x(t, t_0, \varphi_1) = O(\exp\{-(a(t_0) + \|A\|(t_0) + \varepsilon(t_0))(t - t_0)\}) \tag{10.62}$$

as $t \to \infty$, where $\varphi_1 = \varphi - y^0$.

We first show that, as $t \to \infty$,

$$x(t, t_0, \varphi_1) = o(e^{-\beta t}) \tag{10.63}$$

for arbitrary $\beta \geq 0$. For any $T \geq t_0$, by Theorem 10.7 again, $x(t, t_0, \varphi_1)$ on $[T - r, \infty)$ has the representation

$$x(t, t_0, \varphi_1) = X_T(t, T)c + O(\exp\{-(a(T) + \|A\|(T) + \varepsilon(T))(t - T)\}) \tag{10.64}$$

for some $c \in \mathbb{C}^N$. We claim that $c = 0$. In fact, Theorem 10.3 implies

$$|X_T(t, T)c| \geq (K(T))^{-1}|c|\exp\{-(a(T) + \|A\|(T))(t - T)\}$$

for $t \geq T$. If $c \neq 0$, then, for sufficiently large $t \geq T$,

$$|x(t, t_0, \varphi_1)| \geq (1/2)(K(T))^{-1}|c|\exp\{-(a(T) + \|A\|(T))(t - T)\}$$

$$\geq (1/2)(K(t_0))^{-1}|c|\exp\{-(a(t_0) + \|A\|(t_0))(t - t_0)\},$$

which contradicts (10.62). Thus $c = 0$. Since (ii) and (iii) ensure that $\|D\|(T)$, $M_1(T)$ and $M_2(T)$ are $o(1)$ as $T \to \infty$, we must have $(a(T) + \|A\|(T) + \varepsilon(T)) \to \infty$. Then, for any $\beta \geq 0$, the inequality

$$-(a(T) + \|A\|(T) + \varepsilon(T)) \leq -2\beta$$

holds for sufficiently large $T \geq t_0$ and (10.63) follows from (10.64).

As $X_{t_0}(t, t_0)$ is a matrix solution of (10.1) on $[t_0 - r, \infty)$, for any $T \geq t_0$ Theorem 10.7 and the above conclusion show that $X_{t_0}(t, t_0)$ on $[T - r, \infty)$ has the representation

$$X_{t_0}(t, t_0) = X_T(t, T)B + o(e^{-\beta t}) \tag{10.65}$$

as $t \to \infty$, where $\beta \geq 0$ is arbitrary and $B$ is an $N \times N$ constant matrix. Then, since the inequality

$$|X_{t_0}(t, t_0)| \geq (K(t_0))^{-1}\exp\{-(a(t_0) + \|A\|(t_0))(t - t_0)\}$$

for $t \geq t_0$ implies $B \neq 0$, from (10.65) we obtain

$$(1/2)(K(T))^{-1}|B|e^{-(a(T) + \|A\|(T))(t - T)} \leq |X_{t_0}(t, t_0)| \leq 2K(T)|B|e^{(a(T) + \|A\|(T))(t - T)}$$

for sufficiently large $t \geq T$. Inequality (10.61) follows since $a(T) \to 0$ as $T \to \infty$. #

**Remark**. Theorem 10.8 shows that, if $\|D\|(T)$ and $M_k(T)$ $(k = 1, 2)$ are $o(1)$ as $T \to \infty$, then, for sufficiently large $T$, the solution space $S_T$ can be decomposed as

$$S_T = \{X_T(t, T)c:\ c \in \mathbb{C}^N\} \oplus S_0,$$

where $S_0$ consists of solutions smaller than any exponential. Moreover, the exponential bound of $X_T(t, T)$ is asymptotic to that of $Y(t, T)$, the fundamental matrix solution of (10.7), as $t \to \infty$.

# CHAPTER 11 CONCLUSION

The aim of this thesis was to investigate some basic problems for a class of mixed type equations having the form

$$\frac{d}{dt}\{x(t) - D(t, x_t)\} = L_1(t, x_t) + L_2(t, x^t) + f(t) \qquad (11.1)$$

for $t \geq \tau$, where for each $t \geq \tau$, $D(t, .)$ and $L_1(t, .)$ are operators from $C([-r, 0], \mathbb{C}^N)$ to $\mathbb{C}^N$ and $L_2(t, .)$ is an operator from $C([0, \sigma], \mathbb{C}^N)$ to $\mathbb{C}^N$. (For detailed requirements on $D$, $L_k$ $(k = 1, 2)$ and $f$, refer to Part II (p.76)). We are interested in solutions on a half line $[t_0 - r, \infty)$.

Equations with piecewise constant arguments having the form

$$\frac{d}{dt}\{x(t) - D(x_t)\} = L_1(x_t) + L_2(x_{[t+\beta]}) + f(t), \ t \geq 0, \qquad (11.2)$$

were studied in Part I. Here $\beta \geq 0$ is a constant, $D$ and $L_k$ $(k = 1, 2)$ are from $C([-r, 0], \mathbb{C}^N)$ to $\mathbb{C}^N$ and $[.]$ is the integer part function. By applying the available theory for neutral differential equations to (11.2), we managed to represent its solutions in terms of $f$, $L_2$, the initial condition, and the matrix solution of the equation

$$\frac{d}{dt}\{X(t) - D(X_t)\} = L_1(X_t) + I, \ t \geq 0, \qquad (11.3)$$

with $X_0 = 0$. From this representation we obtained an exponential estimate and also analysed the relationship between the spectrum of the solution map $T_\beta$ and the asymptotic behaviour of the solutions of (11.2).

In Part II, some ways of posing the initial value problem were discussed in Chapter 6 and some simple properties of solutions obtained in Chapter 7, though we have not found an ideal way of imposing initial conditions for (11.1). As an alternative, we considered imposing conditions on $L_2(t, .)$ which, along with other conditions, ensure existence and uniqueness of the solution to the usual initial value problem. Note that the equation (11.2), viewed as a special case of (11.1), has the following property: $L_2(t, \psi)$ does not always involve the values of $\psi$ in a neighbourhood of $\sigma$. With this in mind, we assumed the existence of a sequence of Zero Advanced Points (ZAPs, see Definition 7.1) and discussed its relationship with uniqueness of the solution in Part II (Chapter 7).

Under the assumption of existence and uniqueness guaranteed by a sequence of ZAPs, in Part III we concentrated on asymptotic solutions and asymptotic representation of solutions of the homogeneous equation

$$\frac{d}{dt}\{x(t) - D(t, x_t)\} = L_1(t, x_t) + L_2(t, x^t). \tag{11.4}$$

As we have mentioned in each part, our investigation has been inspired by previous studies and some of our results can be viewed as generalizations of results in these.

For equations with piecewise constant arguments, previous studies have been mainly for differential difference equations involving only a few values of $x$ at single points $[at + b]$, while our equation (11.2) is functional differential with a term $L_2(t, x_{[t+\beta]})$ involving values of $x$ on intervals. As regards the asymptotic behaviour of solutions of (11.4), we have generalized some basic theorems for ordinary differential equations and some results for delay differential equations to this mixed type equation. However, the dimensional difference of $C$ and $\mathbb{C}^N$ leads to the existence of solutions smaller than any exponential (as $t \to \infty$) and hence to asymptotic representations different from those for ordinary and delay differential equations. Further, because of the appearance of the term $L_2(t, x^t)$ in (11.4), a completely different method was needed to obtain such results.

There are a number of outstanding problems for (11.1) and new efforts should be made on them to improve the existing theory. We list a few of them below. Some of them may be treated by the methods of this thesis, but some of them may require new techniques.

I. Nonautonomous equations with piecewise constant arguments. When the operator $L_2(.)$ in (11.2) is replaced by a one parameter operator $L_2(t, .)$, results analogous to those in Part I might be obtained by the same method. But for the equation

$$\frac{d}{dt}\{x(t) - D(t, x_t)\} = L_1(t, x_t) + L_2(t, x_{[t+\beta]}) + f(t), \tag{11.5}$$

a new method is needed as there is no variation of parameters formula available for nonautonomous neutral differential equations.

II. Initial value problem. Without the existence of a sequence $\{t_n\}$ of ZAPs, it is not clear how the initial condition for (11.1) should be imposed so that natural conditions for existence and uniqueness of the solution on $[t_0 - r, \infty)$ can be found. A simple but very important case is the autonomous equation

$$\frac{d}{dt}\{x(t) - D(x_t)\} = L_1(x_t) + L_2(x^t) + f(t). \tag{11.6}$$

Our theory does not apply to (11.6). However, the idea of IVP3 used in § 6.4 might be adapted to give initial data on $[-r, \sigma]$ for (11.6).

III. Weaker conditions for asymptotic solutions and asymptotic representation of solutions.

For a bounded linear operator $F(t, .)$ from $C$ to $\mathbb{C}^N$ with a parameter $t \in [\tau, \infty)$, there are two different meanings of the statement that $F(t, \varphi)$ is small in some sense as $t \to \infty$. We might either mean that the norm $\|F(t, .)\|$ is small at infinity, or that, for each $\varphi \in C$, $|F(t, \varphi)|$ is small as $t \to \infty$. Clearly, the smallness of $\|F(t, .)\|$ implies that of $|F(t, \varphi)|$ for each $\varphi \in C$, but not vice versa. In a particular problem, therefore, the condition that $|F(t, \varphi)|$ is small at infinity for each $\varphi \in C$ is weaker than that $\|F(t, .)\|$ is small at infinity.

In Chapter 8, the equation we discussed had a term

$$F(t, \varphi) = A(\varphi(-r(t)) - \varphi(0))$$

satisfying $|F(t, \varphi)| \to 0$ as $t \to \infty$ for each $\varphi \in C([-r, \sigma], \mathbb{C}^N)$. Under appropriate smallness conditions on $r(t)$ (i.e. on $|F(t, \varphi)|$) at infinity, we obtained the special solutions and the asymptotic representation of each solution. In Chapters 9 and 10, however, we obtained the results under the assumption that $\|D(t, .)\|$ and $\|L_2(t, .)\|$ (i.e. $\|F(t, .)\|$) are small at infinity. It is reasonable to ask whether similar results can be obtained for equations which contain terms of both the above types, so that the same equation involves terms $F_i(t, \varphi)$, $i = 1, 2$, with the natural assumptions that $\|F_1(t, .)\|$ and $|F_2(t, \varphi)|$ for each $\varphi \in C$ are small at infinity.

Indeed, by exploiting a different norm in a subspace of $C$, Cooke [7] achieved this aim for a class of delay differential equations having the form

$$\frac{d}{dt}x(t) = L(x_t) + F(t, x_t), \tag{11.7}$$

where $F(t, \varphi)$ is a combination of terms of $F_1$ and $F_2$ small as $t \to \infty$ in the above senses.

As a special case, for each $t \in [\tau, \infty)$, let $F(t, \varphi)$ from $C([-r, \sigma], \mathbb{C}^N)$ to $\mathbb{C}^N$ be defined by

$$F(t, \varphi) = \sum_{j \in J} f_j(t)(\varphi(-\gamma_j(t)) - \varphi(-r_j)), \tag{11.8}$$

where $J$ is a finite or infinite set of nature numbers and for each $j \in J$, $r_j \in [0, r]$, $\gamma_j(t) \in C([\tau, \infty), [-\sigma, r])$ and $f_j \in L^\infty(\tau, \infty)$. We assume that $\sum_{j \in J} \|f_j\|_\infty$ exists and that, as $t \to \infty$, $|\gamma_j(t) - r_j|$ is small in some sense uniformly for $j \in J$. Then $|F(t, \varphi)|$ is small at infinity for each $\varphi \in C$.

Now consider the equation

$$\frac{d}{dt}\{x(t) - D_0(t, x_t) - F_1(t, x_t)\} = A(t)x(t) + L_1(t, x_t) + L_2(t, x^t) + F_2(t, x_t) \tag{11.9}$$

for $t \geq \tau$, where $A(t)$ is an $N \times N$ matrix and $D_0(t, \varphi)$, $L_1(t, \varphi)$ and $L_2(t, \psi)$ are as in (11.1). We assume that $F_1(t, \varphi)$ is continuous in $(t, \varphi)$ and that, for each $t \geq \tau$, $F_1(t, \varphi)$ from $C([-r, 0], \mathbb{C}^N)$ to $\mathbb{C}^N$ and $F_2(t, \varphi*)$ from $C([-r, \sigma], \mathbb{C}^N)$ to $\mathbb{C}^N$ have the form (11.8). Note that $x_t$ in $F_2(t, x_t)$ is defined by

$$x_t(\theta) = x(t + \theta), \quad \theta \in [-r, \sigma]$$

and so is different from $x_t$ in $D_0(t, x_t)$, $F_1(t, x_t)$ and $L_1(t, x_t)$.

Recall that the representation of special solutions in terms of A and r(t) in Chapter 8 is obtained by transformations of the type

$$x(t) = \left\{\exp\left(\int_T^t \mathbf{F}(A, r)(s)\, ds\right)\right\} y(t), \qquad (11.10)$$

where $\mathbf{F}(A, r)(s)$ is a matrix function of s represented in terms of A and r(s). This provides a way for us to deal with (11.9) when $N = 1$. For a scalar equation of the form (11.9), with the requirements that $\|D_0(t, .)\|$, $\|A(t)D_0(t, .)\|$, $\|L_1(t, .)\|$, $\|L_2(t, .)\|$, and $|F_1(t, \varphi)|$, $|A(t)F_1(t, \varphi)|$ and $|F_2(t, \varphi^*)|$ for each $\varphi$ and $\varphi^*$ be small at infinity in a suitable sense, we expect there to be a special solution of the form

$$X(t) = (1 + o(1)) \exp\left(\int_T^t (A(s) + A_m(s))\, ds\right) \qquad (11.11)$$

as $t \to \infty$, where $A_m(s)$ can be determined by a repeated transformation method. Further, we expect to be able to represent each solution asymptotically in the form

$$x(t) = X(t)c + o\left(\exp\left(\int_T^t A(s)\, ds - \beta t\right)\right) \qquad (11.12)$$

as $t \to \infty$, where $\beta \geq 0$ is arbitrary.

When (11.9) stands for a vector equation, the same results as in Chapters 9 and 10 are to be anticipated if A(t) is either a diagonal or a bounded matrix. However, due to the involvement of $F_1(t, \varphi)$ and $F_2(t, \varphi^*)$ in (11.9), some tedious estimates can not be avoided and more elaborate proofs are needed.

We are not sure whether similar results can be obtained if $|F_2(t, \varphi^*)|$ (or $|F_1(t, \varphi)|$ or both) is small at infinity for each $\varphi^* \in C$ but $F_2(t, \varphi^*)$ does not have the form (11.8). The asymptotic behaviour of the solutions of (11.9) is nonetheless worth investigating under these weaker conditions.

IV. Asymptotic behaviour when A(t) in (11.9) has other forms. By borrowing a method used by Eastham (refer to § 1.10, [12]) for ordinary differential equations and applying the results in Chapter 9, it is possible to treat (11.9) when A(t) has the form

$$A(t) = dg[A_1(t), A_2(t), \cdots, A_k(t)],$$

where for $i = 1, 2, \cdots, k$, each $A_i(t)$ is either an $m_i \times m_i$ diagonal matrix or a matrix of Jordan form

$$A_i(t) = \begin{bmatrix} \lambda_i(t) & \mu_{i,1}(t) & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \mu_{i,m_i-1}(t) \\ 0 & & & \lambda_i(t) \end{bmatrix}.$$

Under suitable conditions it should also be possible to do so when $A(t)$ is a lower triangular matrix

$$A(t) = \begin{bmatrix} a_{11}(t) & & 0 \\ \vdots & \ddots & \\ a_{N1}(t) & \cdots & a_{NN}(t) \end{bmatrix}$$

or an upper triangular one.

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