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Representations of Nilpotent Groups: Extensions, Neutral Cohomology, and Pontryagin Spaces

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Abstract. Spectral criteria for the cohomological triviality of extensions of representations of connected nilpotent groups are obtained. They are applied to the study of symmetrized extensions of unitary representations by finite-dimensional representations and to the theory of J-unitary representations of groups on Pontryagin spaces.

Key words: extension of representation, Pontryagin space, Engel element, neutral cocycle.

1. Introduction. Irreducible unitary representations of nilpotent groups have attracted attention of many experts. It is known that such representations are either one-dimensional or infinite-dimensional; for nilpotent Lie groups, Kirillov [3] developed the orbit method, which has made it possible to obtain a specific comprehensible classification in many cases. Arbitrary unitary representations uniquely decompose into direct integrals of irreducible ones.

In the case of nonsymmetric representations, the situation is significantly less certain. The irreducible finite-dimensional representations are still one-dimensional (by the Lie–Kolchin theorem [4]), but arbitrary representations no longer decompose into sums of irreducible ones. They can be decomposed into sums of indecomposable representations, but the latter can rarely be classified.

There is a natural situation in which it is necessary to study mixtures of finite-dimensional and unitary representations of a group; this is the theory of group representations in the Pontryagin spaces Π_k. As Naimark showed in [5], a J-symmetric representation of a connected nilpotent group in Π_k has an invariant subspace of dimension k; this allows us to consider such representations as special constructions of "symmetrized extensions" of finite-dimensional representations by unitary ones. Although there remains the problem with the finite-dimensional part, it becomes simpler, because the dimensions of the arising finite-dimensional components are bounded by the degree of indefiniteness. In this paper we successively consider (1) decomposability conditions for representations of nilpotent groups, (2) the decomposability of extensions of finite-dimensional representations by unitary ones, and (3) the structure of Π_k-representations.

2. Cohomology and extensions. Let λ and U be weakly continuous representations of a group G on Hilbert spaces L and $\mathfrak{H}$, respectively. We use $B(\mathfrak{H}, L)$ to denote the space of bounded linear operators from $\mathfrak{H}$ to L treated as a G-bimodule. The cohomology of G with coefficients in $B(\mathfrak{H}, L)$ is defined in a standard way (note that in the classical case, which is considered in the literature most frequently, λ is equal to the trivial representation ι on a one-dimensional space). In particular, a 1-cocycle is a weakly continuous function $\xi\colon G \to B(\mathfrak{H}, L)$ satisfying the condition

$$\xi(gh) = \lambda(g)\xi(h) + \xi(g)U(h) \quad \text{for any } g, h \in G.$$

The triviality condition for 1-cohomology is particularly important in representation theory, because it ensures the triviality of extensions. Recall that an extension of a representation λ by U is any representation for which there exists a closed invariant subspace L such that the restriction to L is equivalent to λ and the representation induced on the quotient space is equivalent to U. An extension is considered trivial if L has an invariant complement.

The standard construction of extensions is as follows. Let $\mathfrak{Z} = L \oplus \mathfrak{H}$. With each (λ, U)-cocycle ξ we associate the representation π on $\mathfrak{Z}$ acting by the rule

$$\pi(g) = \begin{pmatrix} \lambda(g) & \xi(g) \\ 0 & U(g) \end{pmatrix} \quad \text{for } g \in G.$$

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This is an extension of λ by U; we denote it by $\mathfrak{e}(\lambda, U, \xi)$. The extension $\mathfrak{e}(\lambda, U, \xi)$ is trivial if and only if the cocycle ξ is a coboundary.

Given $h \in G$, we define a map $\mathrm{ad}_h : G \to G$ by

$$\mathrm{ad}_h(g) = ghg^{-1}h^{-1} \quad \text{for all } g \in G.$$

An element h is said to be Engel if, for any $g \in G$, there exists a number $n = n_g$ for which $\mathrm{ad}_h^n(g) = e$.

Theorem 1. *Let λ and U be representations of a group G. If*

$$\mathrm{Sp}(\lambda(h)) \cap \mathrm{Sp}(U(h)) = \varnothing \tag{1}$$

for some Engel element $h \in G$, then $\mathscr{H}^1(\lambda, U) = 0$.

Since all elements of a nilpotent group are Engel, we have the following corollary, which plays an important role throughout the paper.

Corollary 2. *Let λ and U be representations of a connected nilpotent group G. If $\mathrm{Sp}(\lambda(h)) \cap \mathrm{Sp}(U(h)) = \varnothing$ for some $h \in G$, then $\mathscr{H}^1(\lambda, U) = 0$.*

Corollary 3. *Let π be a representation of a connected nilpotent group on a Banach space X, and let L be a closed invariant subspace for π. If $\lambda = \pi|_L$, U is the representation induced on X/L, and condition* (1) *holds for some $h \in G$, then Y has an invariant complement in X.*

It is easy to see that this result does not carry over to solvable groups.

Let χ be a character of a group G (i.e., a multiplicative functional on G). We say that χ is adjoint to the representation λ if $\lambda(g)x = \chi(g)x$ for some vector $x \neq 0$ and all $g \in G$. We denote the set of all characters adjoint to λ by $\mathrm{sign}(\lambda)$. We say that a representation λ is *monothetic* if $\mathrm{sign}(\lambda)$ is one-point. The following result (it is undoubtedly well known, but the authors do not know where it was published) is an immediate consequence of Lie's theorem and Corollary 3.

Corollary 4. *Any finite-dimensional representation of a connected nilpotent group is a direct sum of monothetic representations.*

In what follows, we always assume that G is a connected nilpotent group.

Suppose that λ is a representation of G on a finite-dimensional space L, U is a unitary representation of G on a Hilbert space $\mathfrak{H}$, and ξ is a (λ, U)-cocycle.

Theorem 5. *Let $\pi = \mathfrak{e}(\lambda, U, \xi)$ be a representation of a group G on $\mathfrak{Z} = L \dotplus \mathfrak{H}$ and suppose that $\dim \mathfrak{H} = \infty$.*

Then there exist closed π-invariant subspaces $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ such that

$$\mathfrak{Z} = X_n \dotplus Y_n, \qquad X_{n+1} \subset X_n,\ Y_n \subset Y_{n+1} \ \text{for any } n,$$

all representations $\pi|_{Y_n}$ are similar to unitary representations, each subspace X_n contains L, the intersection of all subspaces X_n is finite-dimensional, and the closure of the union of all Y_n has finite codimension in $\mathfrak{Z}$.

Corollary 6. *If the representation $\mathfrak{e}(\lambda, U, \xi)$ does not decompose into the direct sum of two subrepresentations, then $\dim \mathfrak{H} < \infty$, $\mathrm{sign}(\lambda) = \{\chi\}$, and $U = \chi \mathbf{1}_{\mathfrak{H}}$.*

Studying representations of groups on spaces with an indefinite metric, Ismagilov [1] introduced a special class of cocycles; now we proceed to consider this class. We use $\lambda^\sharp$ to denote the representation conjugate to λ: $\lambda^\sharp(g) = \lambda(g^{-1})^*$. By $C(\mathfrak{H}, L)$ we denote the space of continuous maps from G to $B(\mathfrak{H}, L)$. For each $c \in C(\mathfrak{H}, L)$, we define a map $c^\sharp(g) \in C(L, \mathfrak{H})$ by setting $c^\sharp(g) = c(g^{-1})^*$.

Clearly, if ξ is a (λ, U)-cocycle, then $\xi^\sharp$ is a $(U, \lambda^\sharp)$-cocycle, and the map $\xi\xi^\sharp : G \times G \to B(L)$ defined by $\xi\xi^\sharp(g,h) = \xi(g)\xi^\sharp(h)$ is a $(\lambda, \lambda^\sharp)$-2-cocycle. We say that a cocycle ξ is *neutral* if $\xi\xi^\sharp$ is a coboundary, i.e., if there exists a map $\gamma \in C(L, L)$ (a *prechain* of the cocycle ξ) such that

$$\xi(g)\xi^\sharp(h) = (d^1_{\lambda,\lambda^\sharp}\gamma)(g,h) = \lambda(g)\gamma(h) - \gamma(gh) + \gamma(g)\lambda^\sharp(h).$$

We denote the set of all neutral cocycles by $\mathscr{Z}^1_\nu(\lambda, U)$. It can be shown that any cocycle cohomologous to a neutral one is neutral; therefore, we can define the neutral 1-cohomology set

$\mathscr{H}^1_\nu(\lambda, U)$ in a natural way. This set is not always a subgroup in $\mathscr{H}^1(\lambda, U)$ and may have very complex structure even in the case where $\lambda = \iota$ and $U = \iota_m$ is a representation by unit operators on $\mathbb{C}^m$. Thus, for the group $G = T_n$ of upper triangular $n \times n$ matrices with 1's on the diagonal, the exact description of neutral 1-(ι, ι_m)-cohomology (= cocycles) is as follows.

Let A be an $m \times (n-1)$ matrix such that $B = A^*A = (b_{ij})$ satisfies the condition

$$\operatorname{Im} b_{ij} = 0 \quad \text{if } |i-j| > 1;$$

then $\xi(g) = A(g_{12}, \dots, g_{n-1,n})^T$ is a neutral (ι, ι_m)-cocycle, and all neutral cocycles have this form.

The following theorem, which is important for further considerations, asserts that, for a large class of representations, the set $\mathscr{H}^1_\nu(\lambda, U)$ is massive.

Theorem 7. *Let G be a connected locally compact nilpotent group, and let U be its unitary representation weakly containing but not containing ι. Then the neutral (ι, U)-cocycles are dense in the space $\mathscr{Z}^1(\iota, U)$ of all 1-cocycles with respect to the topology of uniform convergence on compact sets.*

3. J-Unitary representations in Π_k. Recall that the Pontryagin space Π_k is a space H with indefinite inner product $[\,\cdot\,,\,\cdot\,]$ which can be decomposed into the direct sum of a negative subspace of dimension k and a subspace being a Hilbert space with respect to $[\,\cdot\,,\,\cdot\,]$. An operator acting on H is said to be J-unitary if it is invertible and preserves $[\,\cdot\,,\,\cdot\,]$. A J-unitary representation of a group G on H is a homomorphism of G to the group of all J-unitary operators.

According to Naimark's theorem mentioned in the introduction, any J-unitary representation of a connected nilpotent group on Π_k has a nonpositive invariant subspace of dimension k. If this subspace is negative, then the representation is similar to a unitary representation. Therefore, of interest are only those representations which have invariant neutral (i.e., consisting of vectors x with $[x, x] = 0$) subspaces.

In what follows, by L we denote the maximal invariant neutral subspace of a representation π. Its J-orthogonal complement $L^{[\perp]}$ is invariant as well, but the decomposition $H = L[+]L^{[\perp]}$ does not take place; moreover, $L \subset L^{[\perp]}$. Let us choose a closed subspace $\mathfrak{H} \subset L^{[\perp]}$ complementary to L and a neutral subspace $M \subset H$ complementary to $L^{[\perp]}$; then the representation can be written in the block-matrix form with respect to the decomposition $H = L \oplus \mathfrak{H} \oplus M$ as

$$\pi = \begin{pmatrix} \lambda & \xi & \gamma \\ 0 & U & \eta \\ 0 & 0 & \mu \end{pmatrix}.$$

Here the representation U on $\mathfrak{H}$ is similar to a unitary representation (and becomes unitary after a change of inner product in $\mathfrak{H}$), ξ is a (λ, U)-cocycle, $\mu(g) = \lambda(g^{-1})^*$, $\eta(g) = \xi(g^{-1})^*$, $\gamma(g)^* = \gamma(g^{-1})$, and

$$\gamma(gh) = \lambda(g)\gamma(h) + \xi(g)\eta(h) + \gamma(g)\mu(h).$$

We see that the cocycle ξ is neutral and $-\gamma$ is its prechain. Conversely, given a neutral cocycle, we can construct a J-unitary representation, choosing a prechain. This allows us to apply results on extensions to the theory of Π_k-representations. We mention some of such applications below. Recall that a group G is assumed to be connected and nilpotent.

Theorem 8. *Let L be a neutral invariant subspace of a J-unitary representation π. Then there exists a decreasing sequence of closed invariant subspaces H_m containing L and such that all $H_m^{[\perp]}$ are positive and $\bigcap_m H_m$ is finite-dimensional.*

This fact (approximative splittability) can be regarded as a theorem on the decomposability of any representation into the "direct sum" of a sequence of unitary representations (on $E_k = H_{k-1} \ominus H_k$) and a representation on a finite-dimensional (possibly degenerate) subspace (on $\bigcap_m H_m$). Note that, for the Lorentz group, a similar result was obtained in [2].

Of interest is also another version of decomposability (into a finite J-orthogonal direct sum). We say that a representation on a Π_k-space H is Π-decomposable if $H = H_1[+]H_2$, where H_1 and

H_2 are invariant and not positive. Otherwise, we say that the representation is Π-indecomposable. Clearly, any representation is a finite direct sum of Π-indecomposable ones. For representations on finite-dimensional spaces, such a decomposition is unique (up to isomorphism), but its uniqueness in the general case has been neither proved nor disproved. The problem of classifying finite-dimensional Π-indecomposable representations is far from being completely solved, too.

We say that a representation is primary if its restriction to the maximal invariant neutral subspace is monothetic. The following result answers one of the questions that have led to writing this note.

Theorem 9. (i) *If a group G is commutative, then all of its Π-indecomposable representations are primary.*

(ii) *There exists a Π-indecomposable representation of the Heisenberg group T_3 which is not primary.*

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