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Edward Kissin and Victor S. Shulman

May 17, 2015

Dear Professor Delorme,

We are submitting to the Journal of Functional Analysis the revised version of our paper Ms. Ref. No.: JFA-15-137

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We are very grateful to the referee for many helpful suggestions and we made some changes following his suggestions.

Yours sincerely,
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1) The theorem

Theorem 0.1 *Let π be a representation of a group G in the Banach space X and let Y be an invariant subspace. For an Engel element $h \in G$, let $T = \pi(h)$, T_Y the be restriction of T to Y and $\widehat{T}$ be the operator naturally generated by T on X/Y. Let $\mathrm{Sp}\ T_Y \cap \mathrm{Sp}\ \widehat{T} = \emptyset$. Then π splits; this means that there is a π-invariant subspace Z in X with $X = Y \oplus Z$.*

that was formulated and nicely proved by the referee coincides with our Corollary 2.17 (in the initial submission). Now we call it Theorem 2.17.

2) "*Lemma* 2.3 *is a commonplace for anyone familiar with homologies and can be omitted*"
We omitted the proof of Lemma 2.3, but left the statement (it is Lemma 3.5) to use it for the proofs of Proposition 3.6, Corollary 3.7 and Theorem 3.12.

3) "*The text on pages* 8, 9, 11, 12 *can be shortened..*"
We shortened the text on these pages.

4) "*My opinion is that Section* 3.2 *is not necessary.*"
We omitted Section 3.2 ("L-decomposition"), but kept Proposition 3.5 (it is now Proposition 3.6), since we use its results in the proofs of Corollary 3.8 and Theorem 3.18.

5) Following the referee's advice, we proved Theorem 3.18 in greater generality – for nilpotent groups in whose dual objects characters are not separated. We deduced from Theorem 3.18 the case of the Heisenberg group as a consequence.

Yours sincerely,
Edward Kissin and Victor Shulman

Non-unitary representations of nilpotent groups, I: cohomologies, extensions and neutral cocycles.

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May 17, 2015

Abstract

Let λ be a finite-dimensional representation of a connected nilpotent group G and U be a unitary representation of G. We investigate the structure of the extensions of λ by U and, correspondingly, the group $\mathcal{H}^1(\lambda, U)$ of 1-cohomologies. A spectral criterion of triviality of $\mathcal{H}^1(\lambda, U)$ is proved and systematically used in the study of various types of decomposition of the extensions. We consider a special type of (λ, U)-cocycles – neutral cocycles, which play a crucial role in the theory of J-unitary representations of groups on Pontryagin Π_k-spaces.

1 Introduction and Preliminaries

Irreducible unitary representations of connected nilpotent groups G were studied in works of Dixmier, Lenglends, Guichardet, Pukanski, Kirillov and other mathematicians. For Lie groups Kirillov [Kir] developed the famous method of orbits relating irreducible representations with symplectic geometry. The study of general unitary representations is simplified by the fact that they uniquely decompose in direct integrals of the unitary ones.

The situation becomes more complicated if one considers non-unitary representations. In this case all irreducible finite-dimensional representations are still one-dimensional and correspond to characters of the group, but the general finite-dimensional representations do not decompose in the sums of irreducible ones. Therefore it is natural to take non-decomposable representations as building blocks — by the Krull-Schmidt Theorem, the decomposition of an arbitrary representation in the sum of non-decomposable ones is unique up to isomorphism. Unfortunately the classification of non-decomposable representations is a "wild" problem even for such a simple commutative group as $G = \mathbb{R}^2$.

An intermediate, or mixed situation – the combination of finite-dimensional and unitary representations – naturally arises when one deals with a J-unitary representation on Pontryagin Π_k-spaces. In brief, such a space is a direct sum $H = H_+ + H_-$ of Hilbert spaces supplied with an "indefinite scalar product" $[x, y] = (x_+, y_+) - (x_-, y_-)$, where $\dim H_- = k < \infty$ (for details see [KS]). A representation π of G on H is J-unitary if all operators $\pi(g)$ preserve this form: $[\pi(g)x, \pi(g)y] = [x, y]$. The properties of J-unitary representations of nilpotent groups and, in particular, of the Heisenberg group are used in the study of various problems in the quantum theory (see, for example, [DT], [MPS], [Sc], [Sc1], [St], [SW]).

J-unitary representations of semisimple and solvable groups were studied by Araki [A], Ismagilov [Is1, Is2, Is3], Kissin and Shulman [KS], Naimark [N1, N2], Naimark and Ismagilov [NI], Sakai [Sa] and other mathematicians. By a theorem of Naimark [N], each J-unitary representation of G has a k-dimensional non-positive invariant subspace. This allows us to restrict the study of J-unitary representations to the study of representations that admit a triangular form

$$\pi(g) = \begin{pmatrix} \lambda(g) & \xi(g) & \gamma(g) \\ 0 & U(g) & \xi(g^{-1})^* \\ 0 & 0 & \lambda(g^{-1})^* \end{pmatrix} \tag{1.1}$$

doubleext

corresponding to some decomposition $H = L \oplus \mathfrak{H} \oplus M$, where L and M are k-dimensional neutral subspaces, $\mathfrak{H}$ is a positive subspace, λ is the restriction of π to L and U is the unitary representation defined by π in $\mathfrak{H}$.

The left upper corner of π – the representation $\mathfrak{e}(\lambda, U, \xi) = \begin{pmatrix} \lambda & \xi \\ 0 & U \end{pmatrix}$ of G is an extension of a finite-dimensional representation λ by a unitary representation U; and π is a "double extension" of U by λ. It is natural, therefore, that the first step in the investigation of J-unitary representations should be the study of the extensions $\mathfrak{e}(\lambda, U, \xi)$. This is the subject of the present work. In [KS1], we apply the results of this paper to the investigation of the structure of J-symmetric representations of nilpotent groups on Π_k-spaces.

Clearly, the most significant obstacle in the study of the extensions $\mathfrak{e}(\lambda, U, \xi)$ remains the complexity of the structure of the finite-dimensional representations λ. For nilpotent groups, this is partly alleviated (see Corollary 2.18) by the fact that λ decomposes into the direct sum of monothetic representations λ_χ (each λ_χ is a representation such that $\chi(g)$ is the unique eigenvalue of the matrix $\lambda_\chi(g)$). In the J-unitary representations on Π_k-spaces, the dimensions of λ_χ are regulated by the indefiniteness index k: $\dim \lambda = \sum_\chi \dim \lambda_\chi \le k$. In the most important for physical applications case of Π_1-space $\dim \lambda = 1$. Moreover, sometimes we have additional information about the structure of λ – e.g. its semisimplicity.

In this paper we mostly concentrate on the way the extensions are obtained, that is, on the connecting cocycles. Recall that if λ and U are representations of a group G on Banach spaces L, $\mathfrak{H}$, then a continuous function $\xi\colon G \to B(\mathfrak{H}, L)$ is a 1-cocycle if $\xi(gh) = \lambda(g)\xi(h) + \xi(g)U(h)$ for all $g, h \in G$. It defines a representation π, which we denoted above by $\mathfrak{e}(\lambda, U, \xi)$, that acts on the direct sum $L \dotplus \mathfrak{H}$ by the formula

$$\pi(g)(x \oplus y) = (\lambda(g)x + \xi(g)y) \oplus U(g)y.$$

The space L splits π (i.e., L has an invariant complement) if and only if ξ is a coboundary: $\xi(g) = \lambda(g)T - TU(g)$ for some $T \in B(\mathfrak{H}, L)$. The quotient of the space of all cocycles by the subspace of all coboundaries is denoted $\mathcal{H}^1(\lambda, U)$ and called the space of 1-cohomologies.

In Section 2 of the paper we study $\mathcal{H}^1(\lambda, U)$ without the assumption that L is finite-dimensional and U is unitary. The results of this section (and, to some extent, of subsequent sections) are based on a spectral criterion of triviality of cocycles (Corollary 2.9) that states that if G is nilpotent and there is $g \in G$ such that spectra of $\lambda(g)$ and $U(g)$ do not intersect, then 1-cohomologies are trivial.

Section 3 is devoted to the investigation of various types of decomposition of an extension $\mathfrak{e}(\lambda, U, \xi)$ of a finite-dimensional representation λ by a unitary one. It provides some sufficient conditions of decomposability of $\mathfrak{e}(\lambda, U, \xi)$ into the sum of two subrepresentations which are not extensions of λ. We find some sufficient conditions of decomposability of $\mathfrak{e}(\lambda, U, \xi)$ into the direct sum of a unitary and a finite-dimensional representations. It is also proved that $\mathfrak{e}(\lambda, U, \xi)$ always has an "approximate decomposition", that is, there exists a finite-dimensional subspace $\mathcal{L}$, containing L, and a sequence of pairs of invariant subspaces $(X_n, Y_n)_{n=1}^\infty$ such that $L \oplus \mathfrak{H} = X_n \dotplus Y_n$ for all n,

$(Y_n)_{n=1}^{\infty}$ is an increasing and $(X_n)_{n=1}^{\infty}$ is a decreasing sequences such that $\mathcal{L} = \cap_n X_n$. We say that $\mathcal{L}$ approximately splits π.

We also consider the problem of decomposition of $\mathfrak{e}(\lambda, U, \xi)$ into primary components: $\mathfrak{e}(\lambda, U, \xi) = \sum_{\chi} \mathfrak{e}(\lambda_\chi, U_\chi, \xi_\chi)$, where λ_χ are monothetic components of λ (we call the extensions $\mathfrak{e}(\lambda_\chi, U_\chi, \xi_\chi)$ primary). It is proved that for commutative groups such a decomposition always exists (Theorem 3.17). However, this result does not extend to all nilpotent groups. We show that this decomposition fails for nilpotent groups in whose dual objects characters are not separated, and construct an example of an extension for the Heisenberg group which does not admit primary decomposition.

In Section 4 we study (λ, U)-cocycles ξ with a special property: the product $\xi(g)\xi(h^{-1})^*$ is a $(\lambda, \lambda^\sharp)$-coboundary. We call these cocycles neutral. Our interest in neutral cocycles is due to the fact (discovered by Ismagilov [Is3]) that these particular cocycles appear in J-unitary representations on Π_k-spaces (see (1.1)). It is proved that the set $Z^1_\nu(\lambda, U)$ of all neutral 1-cocycles is dense in $Z^1(\lambda, U)$ if $\lambda = \iota$ and U has no fixed vectors. We give a precise description of $Z^1_\nu(\iota, U)$ in the opposite case – when U is the identity representation on a Hilbert space.

Applications of these results to J-unitary representations of nilpotent groups on Π_k-spaces are considered in [KS1] and in the next part of our work [KS2].

Acknowledgement. We are very grateful to the referee for many helpful suggestions and to Ekaterina Shulman for the example after Lemma 2.12.

2 Cohomologies of Engel groups

2.1 Cohomologies of groups with coefficients in bimodules.

We recall the definition of continuous cohomologies of a topological group G with coefficients in a topological G-bimodule $\mathfrak{X}$. Let $C^0 = \mathfrak{X}$, and, for $n \geq 1$, $C^n = C^n(G, \mathfrak{X})$ be the space of all continuous functions (n-cochains) from $G^n = G \times ... \times G$ to $\mathfrak{X}$. The coboundary operators $d^n : C^n \to C^{n+1}$ are defined by the rule

$$d^n c(g_1, ..., g_{n+1}) = g_1 c(g_2, ..., g_{n+1}) + \sum_{i=1}^{n} (-1)^i c(g_1, ..., g_i g_{i+1}, g_{i+2}, ..., g_{n+1}) + (-1)^{n+1} c(g_1, ..., g_n) g_{n+1}.$$

Let $\mathcal{Z}^n = \mathcal{Z}^n(G, \mathfrak{X}) = \ker d^n$ be the set of all n-cocycles and $\mathcal{B}^n = \mathcal{B}^n(G, \mathfrak{X}) = \mathrm{im}\ d^{n-1}$ – the set of all n-coboundaries. Then $\mathcal{B}^n \subseteq \mathcal{Z}^n$. The quotient space $\mathcal{H}^n = \mathcal{Z}^n / \mathcal{B}^n$ is called the n-th group of cohomologies of G with coefficients in $\mathfrak{X}$.

With few later exceptions we consider cohomologies with coefficients in operator bimodules defined by a pair of representations. Namely, let L and $\mathfrak{H}$ be Banach spaces, $B(\mathfrak{H}, L)$ be the space of all bounded operators from $\mathfrak{H}$ to L and $B(\mathfrak{H}) = B(\mathfrak{H}, \mathfrak{H})$. Let λ and U be weakly continuous representations of a group G on L and $\mathfrak{H}$ respectively. The space $B(\mathfrak{H}, L)$ supplied with the weak topology is a G-bimodule with respect to the operations $gT = \lambda(g)T$, $Tg = TU(g)$. The corresponding cohomologies are denoted by $\mathcal{H}^n(\lambda, U)$, and similarly for other constructions.

For the reader's convenience we write separately the formulas for low-dimensional operators d^n:

$$\begin{aligned} d^0(T)(g) &= \lambda(g)T - TU(g), \ \text{for } T \in C^0 = B(\mathfrak{H}, L), \\ d^1(c)(g_1, g_2) &= \lambda(g_1)c(g_2) - c(g_1 g_2) + c(g_1)U(g_2), \ \text{for } c \in C^1, \\ d^2(c)(g_1, g_2, g_3) &= \lambda(g_1)c(g_2, g_3) - c(g_1 g_2, g_3) + c(g_1, g_2 g_3) - c(g_1, g_2)U(g_3), \ \text{for } c \in C^2. \end{aligned} \quad (2.1)$$

Thus a function $\xi\colon\ G \to B(\mathfrak{H}, L)$ is a 1-cocycle (we also write (λ, U)-cocycle, if it is necessary to indicate which representations we consider) if

$$\xi(gh) = \lambda(g)\xi(h) + \xi(g)U(h) \text{ for all } g, h \in G. \tag{2.2}$$

g2.2

It is a 1-coboundary if there is $X_\xi \in B(\mathfrak{H}, L)$ such that

$$\xi(g) = \lambda(g)X_\xi - X_\xi U(g), \text{ for all } g \in G. \tag{2.3}$$

g2.2'

By the above definition, $\mathcal{H}^1(\lambda, U) = 0$ if each (λ, U)-cocycle is a (λ, U)-coboundary.

Consider the following example. Let $L = \mathbb{C}$ and λ and U be trivial representations of G on L and $\mathfrak{H}$, respectively. Then a (λ, U)-cocycle can be identified with a weakly continuous map α: $G \to \mathfrak{H}$ satisfying the condition

$$\alpha(gh) = \alpha(g) + \alpha(h) \text{ for } g, h \in G. \tag{2.4}$$

additive

If $G = \mathbb{R}^n$, it is easy to see that the map α is $\mathbb{Q}$-linear and, by continuity, $\mathbb{R}$-linear. In other words, there are fixed vectors $u_1, ..., u_n$ in $\mathfrak{H}$ such that

$$\alpha(x_1, ..., x_n) = x_1u_1 + ... + x_nu_n, \text{ for } (x_1, ..., x_n) \in \mathbb{R}^n, \text{ and } \dim \alpha(\mathbb{R}^n) \le n. \tag{2.5}$$

t

It is important that this can be extended to a wide class of groups.

For a locally compact group G, let $G^{[1]}$ be the commutator of G, that is, the closed subgroup generated by all commutators $ghg^{-1}h^{-1}$ in G. As the quotient homomorphism

$$\theta\colon\ G \longrightarrow \widetilde{G} = G/G^{[1]} \tag{2.6}$$

q

is continuous, $\widetilde{G}$ is a locally compact commutative group. If G is connected, $\widetilde{G}$ is connected. Hence, by [M, Theorem 26], $\widetilde{G}$ is the direct product of a compact subgroup C and a subgroup isomorphic to some $\mathbb{R}^n$. Setting $G_0 = \theta^{-1}(C)$, we obtain the following lemma-definition:

L5.1i **Lemma 2.1** *Let G be a connected locally compact group. Then there is a normal subgroup G_0 of G containing $G^{[1]}$ such that $G_0/G^{[1]}$ is compact and the group G/G_0 is isomorphic to $\mathbb{R}^n$ for some $n = n_G \in \mathbb{N}$.*

Fix an isomorphism of G/G_0 onto $\mathbb{R}^{n_G}$ and denote by ω_G (or just ω if no confusion is possible) its composition with the canonical homomorphism from G onto G/G_0. Thus

$$\omega : G \to \mathbb{R}^{n_G} \text{ with } \ker \omega = G_0. \tag{2.7}$$

1.1e

L5.1 **Corollary 2.2** *Let α be a weakly continuous map from G into a Banach space $\mathfrak{H}$ satisfying* (2.4). *Then $G_0 \subseteq \ker(\alpha)$ and there is a linear weakly continuous map β from $\mathbb{R}^{n_G}$ to $\mathfrak{H}$ such that $\alpha(g) = \beta(\omega(g))$. The set $\alpha(G) = \beta(\mathbb{R}^{n_G})$ is a real linear subspace of $\mathfrak{H}$ and* $\dim_{\mathbb{R}}(\alpha(G)) \le n_G$.

Proof. As α is continuous, $\mathrm{Ker}(\alpha)$ is a closed normal subgroup of G. Since $\alpha(ghg^{-1}h^{-1}) = 0$, for $g, h \in G$, we have $G^{[1]} \subseteq \mathrm{Ker}(\alpha)$. Hence (see (2.6)) $\widetilde{\alpha}\colon\ \theta(g) \to \alpha(g)$ is a continuous map from $\widetilde{G} = C \times \mathbb{R}^{n_G}$ into $\mathfrak{H}$. Since $\widetilde{\alpha}(\widetilde{g}^k) = k\widetilde{\alpha}(\widetilde{g})$, for all $k \in \mathbb{N}$ and $\widetilde{g} \in \widetilde{G}$, and since C is compact, we have $\widetilde{\alpha}(C) = \{0\}$. Hence $G_0 \subseteq \mathrm{Ker}(\alpha)$. Then $\beta\colon\ \omega(g) \to \alpha(g)$ is a weakly continuous map from $\mathbb{R}^{n_G}$ into $\mathfrak{H}$, satisfying (2.4). As in (2.5), we have $\dim_{\mathbb{R}}(\alpha(G)) \le n_G$. ∎

Let λ and U be representations of G on L and $\mathfrak{H}$ respectively.We need the following lemma.

L2.2 **Lemma 2.3** *Let, with respect to a decomposition* $L = L_1 \dotplus ... \dotplus L_n$, λ *have form*

$$\lambda(g) = \begin{pmatrix} \lambda_1(g) & \lambda_{12}(g) & ... & \lambda_{1n}(g) \\ 0 & \lambda_2(g) & ... & \lambda_{2n}(g) \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & ... & \lambda_n(g) \end{pmatrix} \tag{2.8}$$ h2.5

If $\mathcal{H}^1(\lambda_i, U) = 0$ *for all* i, *then* $\mathcal{H}^1(\lambda, U) = 0$. *Similarly, if all* $\mathcal{H}^1(U, \lambda_i) = 0$ *then* $\mathcal{H}^1(U, \lambda) = 0$.

Proof. Let $n = 2$. Then $\lambda = \begin{pmatrix} \lambda_1 & \lambda_{12} \\ 0 & \lambda_2 \end{pmatrix}$. Set $\sigma = \lambda_{12}$. Let $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ be a (λ, U)-cocycle, where ξ_i are maps from G into $B(\mathfrak{H}, L_i)$. By (2.2), $\xi_i(e) = 0$ and

$$\begin{aligned} \xi_1(gh) &= \lambda_1(g)\xi_1(h) + \sigma(g)\xi_2(h) + \xi_1(g)U(h), \\ \xi_2(gh) &= \lambda_2(g)\xi_2(h) + \xi_2(g)U(h). \end{aligned}$$

Thus ξ_2 is a (λ_2, U)-cocycle. As $\mathcal{H}^1(\lambda_2, U) = 0$, there is $X \in B(\mathfrak{H}, L_2)$ such that $\xi_2(g) = \lambda_2(g)X - XU(g)$ for all $g \in G$.

Set $\omega(g) = \xi_1(g) - \sigma(g)X$. Then $\omega(g) \in B(\mathfrak{H}, L_1)$ and $\omega(e) = 0$, as $\sigma(e) = 0$. Since $\sigma(gh) = \lambda_1(g)\sigma(h) + \sigma(g)\lambda_2(h)$, we have

$$\begin{aligned} \omega(gh) = \xi_1(gh) - \sigma(gh)X &= (\lambda_1(g)\xi_1(h) + \sigma(g)\xi_2(h) + \xi_1(g)U(h)) - (\lambda_1(g)\sigma(h) + \sigma(g)\lambda_2(h))X \\ &= \lambda_1(g)(\xi_1(h) - \sigma(h)X) + \sigma(g)(\lambda_2(h)X - XU(h)) + \xi_1(g)U(h) - \sigma(g)\lambda_2(h)X \\ &= \lambda_1(g)\omega(h) + \omega(g)U(h). \end{aligned}$$

Thus ω is a (λ_1, U)-cocycle. As $\mathcal{H}^1(\lambda_1, U) = 0$, there is $Y \in B(\mathfrak{H}, L_1)$ such that $\omega(g) = \lambda_1(g)Y - YU(g)$ for $g \in G$. Hence $\xi_1(g) = \lambda_1(g)Y - YU(g) + \sigma(g)X$ for all $g \in G$. Set $X_\xi = \begin{pmatrix} Y \\ X \end{pmatrix}$. Then $\xi(g) = \lambda(g)X_\xi - X_\xi U(g)$, so that the lemma holds for $n = 2$.

Assume, by induction, that the lemma holds for $n = k$. For $n = k + 1$, set

$$\widehat{\lambda}_1(g) = \begin{pmatrix} \lambda_1(g) & \lambda_{12}(g) & ... & \lambda_{1k}(g) \\ 0 & \lambda_2(g) & ... & \lambda_{2k}(g) \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & ... & \lambda_k(g) \end{pmatrix} \text{ and } \sigma(g) = \begin{pmatrix} \lambda_{1k+1}(g) \\ \lambda_{2k+1}(g) \\ \vdots \\ \lambda_{kk+1}(g) \end{pmatrix}.$$

Then $\lambda(g) = \begin{pmatrix} \widehat{\lambda}_1(g) & \sigma(g) \\ 0 & \lambda_{k+1}(g) \end{pmatrix}$ and $\mathcal{H}^1(\lambda_{k+1}, U) = 0$. By the assumption of induction, $\mathcal{H}^1(\widehat{\lambda}_1, U) = 0$. Using now the same argument as above, we obtain that $\mathcal{H}^1(\lambda, U) = 0$. ■

C3.2 **Corollary 2.4** *Let* λ *have form* (2.8). *Suppose that, for each* λ_i *on the diagonal,* U *can be written in the form* (2.8) *with some* $\{U_j^i\}$ *on the diagonal such that* $\mathcal{H}^1(\lambda_i, U_j^i) = 0$ *for all* j. *Then* $\mathcal{H}^1(\lambda, U) = 0$.

Proof. Fix i. As $\mathcal{H}^1(\lambda_i, U_j^i) = 0$, for all j, it follows from Lemma 2.3 that $\mathcal{H}^1(\lambda_i, U) = 0$. Since this holds for all i, we have from Lemma 2.3 that $\mathcal{H}^1(\lambda, U) = 0$. ■

Let λ, μ be similar representations on L and L' respectively: $\mu(g) = S^{-1}\lambda(g)S$, for some bounded operator S from L' to L with a bounded inverse. Then

$$\mathcal{H}^1(\lambda, U) = 0 \text{ implies } \mathcal{H}^1(\mu, U) = 0. \tag{2.9}$$ 2.j

Indeed, the bijective map $\xi \mapsto S\xi$ from $C^1(G, B(\mathfrak{H}, L))$ to $C^1(G, B(\mathfrak{H}, L'))$ sends $\mathcal{Z}^1(\mu, U)$ onto $\mathcal{Z}^1(\lambda, U)$ and $\mathcal{B}^1(\mu, U)$ onto $\mathcal{B}^1(\lambda, U)$. Therefore it induces an isomorphism of 1-cohomologies.

P3.7 **Proposition 2.5** *Let, with respect to a decomposition $L = L_1 \dotplus ... \dotplus L_n$, λ have form* (2.8). *Suppose that $\{\Omega_1, \Omega_2\}$ is a partition of $[1, ..., n]$ such that $\mathcal{H}^1(\lambda_i, \lambda_j) = 0$, if $i < j$ and i and j belong to different sets Ω_1, Ω_2. Then there are subspaces $\{M_k\}_{k=1}^n$ of L such that*

(i) $L_1 \dotplus ... \dotplus L_k = M_1 \dotplus ... \dotplus M_k$ *for all $k = 1, ..., n$;*

(ii) *λ has upper triangular form with respect to the decomposition $L = M_1 \dotplus ... \dotplus M_n$*

$$\lambda(g) = \begin{pmatrix} \mu_1(g) & \mu_{12}(g) & ... & \mu_{1n}(g) \\ 0 & \mu_2(g) & ... & \mu_{2n}(g) \\ 0 & 0 & ... & ... \\ 0 & 0 & ... & \mu_n(g) \end{pmatrix},$$

where each μ_i is similar to λ_i and $\mu_{ij} = 0$ if i and j belong to different sets Ω_1, Ω_2;

(iii) *the subspaces $\mathcal{M}_m = \sum_{i \in \Omega_m} \dotplus M_i$, $m = 1, 2$, are invariant for λ and $L = \mathcal{M}_1 \dotplus \mathcal{M}_2$.*

Proof. We argue by induction. Set $\mathcal{L}_k = L_1 \dotplus ... \dotplus L_k$, for $k = 1, ..., n$. All $\mathcal{L}_k$ are invariant for λ. Assume that, for some k, $\mathcal{L}_k = M_1 \dotplus ... \dotplus M_k$ and with respect to this decomposition

$$\lambda(g)|_{\mathcal{L}_k} = \begin{pmatrix} \mu_1(g) & ... & \mu_{1k}(g) \\ 0 & \ddots & \vdots \\ 0 & 0 \cdots 0 & \mu_k(g) \end{pmatrix}, \text{ where each } \mu_i \text{ is similar to } \lambda_i$$

and $\mu_{ij} = 0$ if i and j belong to different sets Ω_1, Ω_2. Hence the subspaces $\mathcal{M}_1^k = \sum_{i \in \Omega_1, i \leq k} \dotplus M_i$, $\mathcal{M}_2^k = \sum_{i \in \Omega_2, i \leq k} \dotplus M_i$ are invariant for λ and $\mathcal{L}_k = \mathcal{M}_1^k \dotplus \mathcal{M}_2^k$.

Then $\mathcal{L}_{k+1} = \mathcal{L}_k \dotplus L_{k+1} = \mathcal{M}_1^k \dotplus \mathcal{M}_2^k \dotplus L_{k+1}$ and with respect to this decomposition

$$\lambda(g)|_{\mathcal{L}_{k+1}} = \begin{pmatrix} \lambda(g)|_{\mathcal{M}_1^k} & 0 & \xi_1(g) \\ 0 & \lambda(g)|_{\mathcal{M}_2^k} & \xi_2(g) \\ 0 & 0 & \lambda_{k+1}(g) \end{pmatrix}.$$

Clearly, ξ_1 is a $(\lambda|_{\mathcal{M}_1^k}, \lambda_{k+1})$-cocycle and ξ_2 is a $(\lambda|_{\mathcal{M}_2^k}, \lambda_{k+1})$-cocycle.

Assume that $k + 1 \in \Omega_2$. Then $\mathcal{H}^1(\lambda_i, \lambda_{k+1}) = 0$ for all $i \in \Omega_1$, $i < k + 1$. As μ_i are similar to λ_i, $\mathcal{H}^1(\mu_i, \lambda_{k+1}) = 0$. Hence, by Lemma 2.3, $\mathcal{H}^1(\lambda|_{\mathcal{M}_1^k}, \lambda_{k+1}) = 0$, so that there is a bounded operator X from L_{k+1} into $\mathcal{M}_1^k$ such that $\xi_1(g) = \lambda(g)|_{\mathcal{M}_1^k} X - X\lambda_{k+1}(g)$ for all $g \in G$. Set $M_{k+1} = \{-Xy \dotplus y : y \in L_{k+1}\}$. The operator S: $y \to -Xy \dotplus y$ from L_{k+1} onto M_{k+1} is bounded and has a bounded inverse. Hence the representation $\mu_{k+1} = S\lambda_{k+1}S^{-1}$ on M_{k+1} is similar to λ_{k+1}. The subspace $\mathcal{M}_2^k \dotplus M_{k+1}$ is invariant for λ since, for each $y \in L_{k+1}$, $\ddots\ddots$.

$$\begin{aligned} \lambda(g)Sy &= \lambda(g)(-Xy \dotplus y) = (-\lambda(g)|_{\mathcal{M}_1^k} Xy + \xi_1(g)y) \dotplus \xi_2(g)y \dotplus \lambda_{k+1}(g)y \\ &= \xi_2(g)y \dotplus (-X\lambda_{k+1}(g)y \dotplus \lambda_{k+1}(g)y) = \xi_2(g)y \dotplus S\lambda_{k+1}(g)y = \xi_2(g)y \dotplus \mu_{k+1}(g)Sy \end{aligned}$$

belongs to $\mathcal{M}_2^k \dotplus M_{k+1}$. We have $\mathcal{L}_{k+1} = \mathcal{M}_1^k \dotplus \mathcal{M}_2^k \dotplus M_{k+1}$ and with respect to this decomposition

$$\lambda(g)|_{\mathcal{L}_{k+1}} = \begin{pmatrix} \lambda(g)|_{\mathcal{M}_1^k} & 0 & 0 \\ 0 & \lambda(g)|_{\mathcal{M}_2^k} & \eta(g) \\ 0 & 0 & \mu_{k+1}(g) \end{pmatrix}, \text{ where } \eta = \xi_2 S^{-1}.$$

Similarly, we can consider the case when $k+1 \in \Omega_1$. This completes the proof. ■

R2.2 **Remark 2.6** *The result similar to Proposition* 2.5 *holds if* $\{\Omega_1, ..., \Omega_k\}$ *is a partition of* $[1, ..., n]$ *and all* $H^1(\lambda_i, \lambda_j) = 0$ *when* $i < j$ *and* i *and* j *belong to different sets* $\Omega_1, ..., \Omega_k$.

2.2 Trivial 1-cohomologies of Engel groups.

Let λ and U be representations of a topological group G on L and $\mathfrak{H}$ respectively. We shall consider some sufficient condition for $\mathcal{H}^1(\lambda, U) = 0$. For $h \in G$, define a map $\mathrm{ad}_h\colon G \to G$ by the formula

$$\mathrm{ad}_h(g) = ghg^{-1}h^{-1} \text{ for all } g \in G. \tag{2.10}$$ f2.2

Definition 2.7 (i) *$h \in G$ is an Engel element if* $\mathrm{ad}_h^n(g) = e$ *for each* $g \in G$ *and some* $n = n(g)$.
(ii) *We say that G is an Engel group, if each element of G is an Engel element.*

Clearly, all elements in the center of G are Engel elements. For a subgroup H of a topological group G, let $K(G, H)$ be the minimal closed subgroup of G that contains all commutators $ghg^{-1}h^{-1}$, $g \in G$, $h \in H$. Set

$$G^{[1]} = K(G, G),\ G^{[2]} = K(G, G^{[1]}), ...,\ G^{[n]} = K(G, G^{[n-1]}). \tag{2.11}$$ 2.2n

A group G is *nilpotent* if $G^{[n]} = \{e\}$ for some n. Clearly, all nilpotent groups are Engel groups.
For $g \in G$, consider the operator S_g on $B(\mathfrak{H}, L)$ defined by

$$S_g(T) = \lambda(g)T - TU(g) \text{ for } T \in B(\mathfrak{H}, L). \tag{2.12}$$ 2.1g

The map $g \to S_g(T) = \lambda(g)T - TU(g)$ is a (λ, U)-coboundary for each $T \in B(\mathfrak{H}, L)$ (see (2.3)).

P2new **Proposition 2.8** *Let ξ be a (λ, U)-cocycle. If there exists an Engel element $h \in G$ satisfying*

$$\xi(h) \in S_h B(\mathfrak{H}, L) \text{ and } \ker S_h = \{0\}, \tag{2.13}$$ 2new

then ξ is a coboundary.

Proof. As ξ is a (λ, U)-cocycle, it satisfies (2.2). By (2.13), there is $X \in B(\mathfrak{H}, L)$ such that $\xi(h) = S_h(X)$. Define $\eta(g) = \xi(g) - S_g(X)$ for all $g \in G$. Then η is a (λ, U)-cocycle satisfying

$$\eta(e) = \xi(e) = 0 \text{ and } \eta(h) = \xi(h) - S_h(X) = 0. \tag{2.14}$$ 2.1new

If we prove that $\eta = 0$, then $\xi(g) = S_g(X)$ for all $g \in G$, so that ξ is a (λ, U)-coboundary.
For each $g \in G$, we have from (2.2) and (2.14) that

$$\eta(hg) = \lambda(h)\eta(g) + \eta(h)U(g) = \lambda(h)\eta(g),$$
$$\eta(gh) = \lambda(g)\eta(h) + \eta(g)U(h) = \eta(g)U(h).$$

Set $z(g) = \mathrm{ad}_h(g) = ghg^{-1}h^{-1}$. Then $gh = z(g)hg$. Hence, by the above formulae,

$$\begin{aligned}\eta(g)U(h) &= \eta(gh) = \eta(z(g)hg)\\ &= \lambda(z(g))\eta(hg) + \eta(z(g))U(hg) = \lambda(z(g))\lambda(h)\eta(g) + \eta(z(g))U(hg).\end{aligned}$$

Therefore

$$\eta(z(g))U(hg) = \eta(g)U(h) - \lambda(z(g)h)\eta(g) = W_g(\eta(g)), \qquad (2.15)$$ temp

where the operator W_g on $B(\mathfrak{H}, L)$, for each $g \in G$, is defined by

$$W_g(T) = TU(h) - \lambda(z(g)h)T = TU(h) - \lambda(ghg^{-1})T = -\lambda(g)S_h(\lambda(g^{-1})T).$$

It follows from (2.13) that $\ker W_g = \{0\}$ for each $g \in G$.

Fix $g \neq e$. Set $g_0 = z^0(g) = g$ and inductively, $g_n = z^n(g) = \mathrm{ad}_h(z^{n-1}(g))$ for $n \in \mathbb{N}$. Replacing in (2.15) g by g_{n-1}, we have

$$\eta(g_n)U(hg_{n-1}) = W_{g_{n-1}}(\eta(g_{n-1})) \text{ for all } n \geq 1. \qquad (2.16)$$ 3new

As h is an Engel element, there is n such that $g_n = z^n(g) = \mathrm{ad}_h^n(g) = e$ and $g_{n-1} = z^{n-1}(g) \neq e$. As $\eta(e) = 0$ and $\ker W_{g_{n-1}} = \{0\}$, we have from (2.16) that $\eta(g_{n-1}) = 0$. Using (2.16) again, we have $\eta(g_{n-2}) = 0$. Repeating this n times, we establish that $\eta(g) = \eta(z^0(g)) = 0$. Thus $\eta = 0$. ■

P2.4n **Corollary 2.9** *$\mathcal{H}^1(\lambda, U) = \mathcal{H}^1(U, \lambda) = 0$ if there is an Engel element $h \in G$ satisfying*

$$\mathrm{Sp}(\lambda(h)) \cap \mathrm{Sp}(U(h)) = \varnothing. \qquad (2.17)$$ f2.9

Proof. It follows from Rosenblum's Theorem (see [RR, Theorem 0.12]) that the operator S_h defined in (2.12) satisfies

$$\mathrm{Sp}(S_h) \subseteq \{\alpha - \beta\colon\ \alpha \in\ \mathrm{Sp}(\lambda(h)), \beta \in\ \mathrm{Sp}(U(h))\}.$$

From this and (2.17) we have that S_h is invertible, so that condition (2.13) holds for all (λ, U)-cocycles. By Proposition 2.8, they are coboundaries. ■

D2.n **Definition 2.10** *We say that representations λ and U of a group G are **spectrally disjoint** if*

$$\mathrm{Sp}(\lambda(h)) \cap \mathrm{Sp}(U(h)) = \varnothing \textit{ for some } h \in G. \qquad (2.18)$$ g2.9

Corollaries 2.4 and 2.9 yield

P2.4 **Corollary 2.11** *Let λ and U be representations of an Engel group G. Let λ have an upper block-triangular form (λ_{ij}). Suppose that, for each i, U has an upper block-triangular form such that λ_{ii} and every diagonal block of U are spectrally disjoint. Then $\mathcal{H}^1(\lambda, U) = \mathcal{H}^1(U, \lambda) = 0$.*

A complex-valued function χ on G is a character if $\chi(gh) = \chi(g)\chi(h)$ for all $g, h \in G$. Then

$$\chi^*(g) = \overline{\chi(g^{-1})} = \overline{\chi(g)^{-1}} \text{ for } g \in G, \qquad (2.19)$$ 4.11

is also a character. Denote by χ_e the identity character: $\chi_e(g) \equiv 1$. A character χ is unitary if

$$\chi = \chi^*, \text{ that is, } |\chi(g)| = 1 \text{ for all } g \in G. \qquad (2.20)$$ 4.12

Each character χ generates a one-dimensional representation of G.

We will need the following general result.

L4.2 **Lemma 2.12** *Let χ and $\{\chi_i\}_{i=1}^r$ be continuous characters on a connected topological group G. If $\chi(g) \in \{\chi_i(g)\}_{i=1}^r$, for each $g \in G$, then χ coincides with one of the characters $\chi_1, ..., \chi_r$.*

Proof. The sets $G_i = \{g \in G\colon\ \chi(g) = \chi_i(g)\}$ are closed subgroups of G and $\cup_{i=1}^r G_i = G$. Let n be the smallest number such that $\cup_{i=1}^n G_i = G$. If $G_n \neq G$ then $F = \cup_{i=1}^{n-1} G_i$ is a closed proper subset of G and G_n contains the open subset $G \backslash F$. For each $g \in G\backslash F$, the open set $V = g^{-1}(G\backslash F)$ belongs to G_n and $e \in V$. As G is connected, $G = \cup_k V^k \subseteq G_n$. Thus $G = G_n$, so that $\chi = \chi_n$. ■

K **Example 2.13** The condition in the above lemma that a group G is connected is essential. Indeed, consider the group $G = Z \times Z$ and the characters

$$\chi_1(m,n) = (-1)^m,\ \chi_2(m,n) = (-1)^{m+n},\ \chi_3(m,n) = (-1)^n \text{ and } \chi_4(m,n) = 1.$$

Then $\chi_4(m,n) = 1 \in \{\chi_1(m,n), \chi_2(m,n), \chi_3(m,n)\}$ for all $(m,n) \in G$.

A representation π of G on a Banach space is *operator irreducible* if the only bounded operators commuting with all $\pi(g)$ are scalar operators. Then there is a character χ_π on the centre Z of G such that $\pi(z) = \chi_\pi(z)\mathbf{1}$ for $z \in Z$. As elements in Z are Engel elements, we have

C2.14 **Corollary 2.14** *Let λ and U be operator irreducible representations of a group G and let χ_λ and χ_U be the corresponding characters on its centre. If $\chi_\lambda \neq \chi_U$ then $\mathcal{H}^1(\lambda, U) = \mathcal{H}^1(U, \lambda) = 0$.*

Let χ be a character of a group G. Denote by $\mathfrak{H}^\chi$ the eigen-space of U:

$$\mathfrak{H}^\chi = \{x \in \mathfrak{H} : U(g)x = \chi(g)x, \text{ for all } g \in G\}. \tag{2.21}$$ 2.eg

We say that χ and U are *eigen-disjoint* if $\mathfrak{H}^\chi = 0$; they are *spectrally disjoint* if

$$\chi(h) \notin \operatorname{Sp}(U(h)) \text{ for some } h \in G; \tag{2.22}$$ 2.f

(that is, U is spectrally disjoint with the representation $\chi\iota$ on $\mathbb{C}$ defined by χ). Furthermore, χ and U are *sectionally spectrally disjoint* if U has an upper block-triangular form with respect to some decompositions of $\mathfrak{H}$ such that χ is spectrally disjoint with each diagonal block U_i.

A set Ω of characters of G is *eigen-disjoint* (*spectrally disjoint*, *sectionally spectrally disjoint*) with U if this is true for each $\chi \in \Omega$.

D2 **Definition 2.15** (i) *We call a representation λ on a Banach space L **monothetic**, or a χ-representation, if there is a character χ such that $\operatorname{Sp}(\lambda(g)) = \{\chi(g)\}$ for all $g \in G$.*

(ii) *We call λ **elementary** if with respect to a decomposition $L = L_1 \dotplus ... \dotplus L_n$, λ has form*

$$\lambda(g) = \begin{pmatrix} \lambda_1(g) & ... & \lambda_{1n}(g) \\ 0 & \ddots & \vdots \\ 0 & 0 & \lambda_n(g) \end{pmatrix} \text{ for } g \in G, \tag{2.23}$$ h2.5'

where all λ_i are χ_i-representations for some characters χ_i (they may repeat). Set

$$\operatorname{sign}(\lambda) = \{\chi_i\}_{i=1}^r,\ r \leq n, \text{ where now } \chi_i \text{ do not repeat.}$$

The definition of sign does not depend on the choice of triangularization if G is connected. Indeed, if ω is a character which arises in another triangular form, then $\omega(g) \in \mathrm{Sp}(\lambda(g)) = \{\chi_i(g)\colon 1 \le i \le n\}$ for all $g \in G$. By Lemma 2.12, ω coincides with some χ_i.

In particular, if all $\lambda_i(g) = \chi_i(g)\mathbf{1}_{L_i}$ then λ is elementary. For example, if λ is finite-dimensional and G is solvable then, by Lie Theorem, λ has form (2.23) in some basis. Corollary 2.11 yields

C3.4 **Corollary 2.16** *Let λ be an elementary representation of an Engel group G. If* $\mathrm{sign}(\lambda)$ *is sectionally spectrally disjoint with a representation U of G then $\mathcal{H}^1(\lambda, U) = \mathcal{H}^1(U, \lambda) = 0$.*

2.3 Applications to decomposability

Let Y be a subspace of a Banach space X and $q\colon X \to \widehat{X} = X/Y$ the quotient map. If an operator $T \in B(X)$ preserves Y, let T_Y be its restriction to Y and $\widehat{T}$ the operator on $\widehat{X}$ induced by T.

Let $\Pi(T)$ be the approximate spectrum of T and $\partial(\mathrm{Sp}(T))$ the boundary of $\mathrm{Sp}(T)$. Then

$$\partial(\mathrm{Sp}(T)) \subseteq \Pi(T) \subseteq \mathrm{Sp}(T) \text{ and } \Pi(T_Y) \subseteq \Pi(T), \text{ so that } \partial(\mathrm{Sp}(T_Y)) \subseteq \Pi(T_Y) \subseteq \mathrm{Sp}(T) \qquad (2.24)$$ 3.02

(see [RR, Theorem 0.7 and 0.8]). Set $\widehat{x} = q(x)$ for $x \in X$. We will need the fact that

$$\mathrm{Sp}(T_Y) \cap \Pi(\widehat{T}) = \varnothing \text{ implies } \partial(\mathrm{Sp}(\widehat{T})) \subseteq \Pi(T). \qquad (2.25)$$ 3.03

First, let us show that $\Pi(\widehat{T}) \subseteq \Pi(T)$. For $t \in \Pi(\widehat{T})$, set $S = T - t\mathbf{1}$ and let $x_n \in X$ be such that

$$\|\widehat{x}_n\| = \inf_{y \in Y} \|x_n + y\| = 1 \text{ and } \left\|\widehat{S}\widehat{x}_n\right\| = \inf_{z \in Y} \|Sx_n + z\| = \inf_{u \in Y} \|S(x_n + u)\| \to 0, \text{ as } n \to \infty,$$

since $\mathrm{Sp}(T_Y) \cap \Pi(\widehat{T}) = \varnothing$. Hence there are $u_n \in Y$ such that $\|S(x_n + u_n)\| \to 0$, as $n \to \infty$. Set $v_n = (x_n + u_n)/\|x_n + u_n\|$. Then $\|v_n\| = 1$ and $\|x_n + u_n\| \ge \|\widehat{x}_n\| = 1$. Thus

$$\|(T - t\mathbf{1})v_n\| = \|S(x_n + u_n)\| / \|x_n + u_n\| \le \|S(x_n + u_n)\| \to 0, \text{ as } n \to \infty.$$

Hence $t \in \Pi(T)$. As $\partial(\mathrm{Sp}(\widehat{T})) \subseteq \Pi(\widehat{T})$, by (2.24), we obtain that $\partial(\mathrm{Sp}(\widehat{T})) \subseteq \Pi(T)$.

Let π be a representation of G on X and Y be a π-invariant subspace. Let $X = Y \dotplus \mathfrak{H}$ and $\pi^{\mathfrak{H}}$ be the representation induced on $\mathfrak{H}$. By Proposition 2.5, if $\mathcal{H}^1(\pi_Y, \pi^{\mathfrak{H}}) = 0$ then Y has an invariant complement. We consider now the case when apriori it is not given that Y has a direct complement.

qout **Theorem 2.17** *Let π be a representation of a group G on a Banach space X and let Y be a π-invariant subspace. If $\mathrm{Sp}(\pi(h)_Y) \cap \mathrm{Sp}(\widehat{\pi(h)}) = \varnothing$, for some Engel element $h \in G$, then there is a π-invariant subspace $\mathfrak{H}$ such that $X = Y \dotplus \mathfrak{H}$.*

Proof. Let $T = \pi(h)$. Set $\alpha_1 = \mathrm{Sp}(T_Y)$ and $\alpha_2 = \mathrm{Sp}(\widehat{T})$. Note that $\mathrm{Sp}(T) \subseteq \alpha_1 \cup \alpha_2$. Indeed, let $t \notin \alpha_1 \cup \alpha_2$ and $S = T - t\mathbf{1}$. If $x \in \ker S$ then $Tx = tx$ and $\widehat{T}\widehat{x} = \widehat{Tx} = t\widehat{x}$. Hence $\widehat{x} = 0$, as $t \notin \alpha_2$. Thus $x \in Y$ and $T_Y x = Tx = tx$. Hence $x = 0$, as $t \notin \alpha_1$. Thus $\ker S = \{0\}$.

Let $z \in X$. As $\widehat{S}$ is invertible, $\widehat{z} = \widehat{S}\widehat{x} = \widehat{Sx}$ for some $x \in X$. Thus $z - Sx \in Y$. As S_Y is invertible, $z - Sx = S_Y y = Sy$ for some $y \in Y$. Hence $z = S(x + y)$. Thus S is surjective. Hence S is invertible, so that $t \notin \mathrm{Sp}(T)$.

As $\alpha_1 \cap \alpha_2 = \varnothing$, there is a contour Γ containing α_1 such that $\Gamma \cap (\alpha_1 \cup \alpha_2) = \varnothing$ and α_2 lies outside Γ. Let $P = -(2\pi i)^{-1} \int_\Gamma (T - t\mathbf{1})^{-1} dt$ be the corresponding Riesz spectral projection. Then the subspace $R := PX$ is T-invariant, $PT = TP$, $\mathrm{Sp}(T_R)$ lies inside Γ and $\mathrm{Sp}(T_R) \subseteq \mathrm{Sp}(T)$. Thus $\mathrm{Sp}(T_R) \subseteq \alpha_1$. Let $y \in Y$ and $t \in \Gamma$. As $t \notin \alpha_1$, there is $z \in Y$ such that $y = (T_Y - t\mathbf{1}_Y)z = (T - t\mathbf{1})z$. Hence $(T_Y - t\mathbf{1}_Y)^{-1}y = (T - t\mathbf{1})^{-1}y \in Y$. As $\alpha_1 = \mathrm{Sp}(T_Y)$, we have $-(2\pi i)^{-1}\int_\Gamma (T_Y - t\mathbf{1}_Y)^{-1}dt = \mathbf{1}_Y$. Therefore

$$P|_Y = -(2\pi i)^{-1}\int_\Gamma (T - t\mathbf{1})^{-1}|_Y dt = -(2\pi i)^{-1}\int_\Gamma (T_Y - t\mathbf{1}_Y)^{-1}dt = \mathbf{1}_Y.$$

Hence $PY = Y \subseteq PX$.

If $Y \neq R$, the subspace $\widehat{R} = R/Y \neq \{0\}$ of $\widehat{X}$ is $\widehat{T}$-invariant, as R is T-invariant. Hence, by (2.24), the restriction $\widehat{T}_{\widehat{R}}$ of $\widehat{T}$ to $\widehat{R}$ satisfies

$$\partial(\mathrm{Sp}(\widehat{T}_{\widehat{R}})) \subseteq \Pi(\widehat{T}_{\widehat{R}}) \subseteq \ \mathrm{Sp}(\widehat{T}) = \alpha_2.$$

On the other hand, $\widehat{T}_{\widehat{R}}$ can be considered as the operator $\widehat{T_R}$ on $\widehat{R}$ induced by T_R. Then $\Pi(\widehat{T_R}) = \Pi(\widehat{T}_{\widehat{R}}) \overset{(2.24)}{\subseteq} \Pi(\widehat{T}) \subseteq \mathrm{Sp}(\widehat{T}) = \alpha_2$. As $(T_R)_Y = T_Y$, we have $\mathrm{Sp}((T_R)_Y) \cap \Pi(\widehat{T_R}) \subseteq \alpha_1 \cap \alpha_2 = \varnothing$. Hence $\partial(\mathrm{Sp}(\widehat{T_R})) \overset{(2.25)}{\subseteq} \Pi(T_R) \overset{(2.24)}{\subseteq} \mathrm{Sp}(T_R) = \alpha_1$, so that $\partial(\mathrm{Sp}(\widehat{T_R})) \subseteq \alpha_1 \cap \alpha_2 = \varnothing$, a contradiction. Thus $Y = R$, P is the projection on Y, $(\mathbf{1} - P)$ is the projection on a T-invariant complement $\mathfrak{H}$ of Y and $\mathrm{Sp}(T_{\mathfrak{H}})$ lies outside Γ. As $\mathrm{Sp}(T_Y) \cup \mathrm{Sp}(T_{\mathfrak{H}}) = \mathrm{Sp}(T) \subseteq \alpha_1 \cup \alpha_2$, we have $\mathrm{Sp}(T_{\mathfrak{H}}) \subseteq \alpha_2$.

With respect to the decomposition $X = Y \dot{+} \mathfrak{H}$, π has the block-matrix form $\pi(g) = \begin{pmatrix} \lambda(g) & \xi(g) \\ 0 & U(g) \end{pmatrix}$ and $\mathrm{Sp}(\lambda(h)) \cap \mathrm{Sp}(U(h)) = \mathrm{Sp}(T_Y) \cap \mathrm{Sp}(T_{\mathfrak{H}}) \subseteq \alpha_1 \cap \alpha_2 = \varnothing$. Applying Corollary 2.9, we have $\mathcal{H}^1(\lambda, U) = 0$. By Proposition 2.5, Y has a π-invariant complement. ■

The result below must be known, at least for the most important case of nilpotent groups and finite-dimensional representations, but we could not find a reference.

C3.4ii **Corollary 2.18** (i) *Each elementary representation λ of an Engel group G on a Banach space L uniquely decomposes in a direct sum of χ-representations λ_{L_χ}:*

$$L = \sum_{\chi \in \mathrm{sign}(\lambda)} \dot{+} L_\chi \quad \textit{and} \quad \lambda = \sum_{\chi \in \mathrm{sign}(\lambda)} \dot{+} \lambda_{L_\chi}.$$

(ii) *Let $n := \dim L < \infty$. Then*

$$L_\chi = \{x \in L : (\lambda(g) - \chi(g)\mathbf{1})^n x = 0 \textit{ for all } g \in G\}. \tag{2.26}$$ monospace

(iii) *Let G be connected. If K is a λ-invariant subspace of L then $K = \sum_{\chi \in \mathrm{sign}(\lambda)} \dot{+} (L_\chi \cap K)$.*

Proof. (i) By definition, λ has form (2.23) and $\mathrm{Sp}(\lambda_i(g)) = \{\chi_i(g)\}$, where $\chi_i \in \mathrm{sign}(\lambda)$. For $\chi \in \mathrm{sign}(\lambda)$, set $\Omega_\chi = \{i \in [1, ..., n] : \chi_i = \chi\}$. Then if i and j belong to different sets Ω_χ and Ω_ω, the representations λ_i and λ_j are spectrally disjoint. Hence $\mathcal{H}^1(\lambda_i, \lambda_j) = 0$, by Corollary 2.11. Applying Proposition 2.5 and Remark 2.6, we get the needed decomposition.

(ii) Denote by M_χ the right hand side of (2.26). As λ_{L_χ} is a χ-representation and $\dim L_\chi \leq n$, the matrix $\lambda(g)|_{L_\chi}$ has only one eigenvalue $\chi(g)$ for each $g \in G$. Hence $L_\chi \subseteq M_\chi$.

If $\omega \neq \chi$ then $\chi(h) \neq \omega(h)$ for some $h \in G$. Hence there does not exist $x \neq 0$ such that $(\lambda(h) - \omega(h)\mathbf{1})^n x = (\lambda(h) - \chi(h)\mathbf{1})^n x = 0$. Thus $M_\chi \cap M_\omega = \{0\}$.

Let $x = \sum_{\omega \in \operatorname{sign}(\lambda)} x_\omega \in M_\chi$ with $x_\omega \in L_\omega$. Then

$$0 = (\lambda(g) - \chi(g)\mathbf{1})^n x = \sum_{\omega \in \operatorname{sign}(\lambda)} (\lambda(g) - \chi(g)\mathbf{1})^n x_\omega.$$

As $(\lambda(g) - \chi(g)\mathbf{1})^n x_\omega \in L_\omega$ and the spaces L_ω are linear independent, all $(\lambda(g) - \chi(g)\mathbf{1})^n x_\omega = 0$, so that all $x_\omega \in M_\chi$. As $L_\omega \cap M_\chi \subseteq M_\omega \cap M_\chi = \{0\}$, if $\omega \neq \chi$, we have $x \in L_\chi$. Thus $M_\chi \subseteq L_\chi$.

(iii) Fix χ. For each $\omega \neq \chi$, $\chi(g_\omega) \neq \omega(g_\omega)$ for some $g_\omega \in G$. Hence the operator

$$S_\chi = \prod_{\omega \in \operatorname{sign}(\lambda), \omega \neq \chi} (\lambda(g_\omega) - \omega(g_\omega)\mathbf{1})^n$$

is invertible on L_χ and $S_\chi L_\omega = \{0\}$. Hence the projection P_χ on L_χ along all other L_ω is a polynomial of S_χ: $P_\chi = p(S_\chi)$. Set $K_\chi = P_\chi K$. As K is λ-invariant, $S_\chi K \subseteq K$. Hence $K_\chi = p(S_\chi)K \subseteq K$. As $P_\chi L = L_\chi$, we have $K_\chi \subseteq L_\chi \cap K$. Conversely, if $x \in L_\chi \cap K$ then $x = P_\chi x \in K_\chi$. Thus $L_\chi \cap K \subseteq K_\chi$, so that $K_\chi = L_\chi \cap K$. Let now $y \in K$. As $\mathbf{1} = \sum_{\chi \in \operatorname{sign}(\lambda)} \dot{+} P_\chi$, we have $y = \sum_{\chi \in \operatorname{sign}(\lambda)} \dot{+} P_\chi y$ and $P_\chi y \in K_\chi$. Thus $K = \sum_{\chi \in \operatorname{sign}(\lambda)} \dot{+} K_\chi$. ■

Corollary 2.18 does not extend to solvable groups. Indeed, if λ is the identity representation of

$$G = \left\{ g = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} : \ a, b, c \in \mathbb{C}, \ ab \neq 0 \right\} \text{ on } L = \mathbb{C}^2 \text{: } \lambda(g)x = gx \text{ for } x \in L.$$

on $\mathbb{C}^2$, then $\operatorname{sign}(\lambda) = \{\chi_1, \chi_2\}$, where $\chi_1(g) = a$ and $\chi_2(g) = b$ are characters on G. However, λ has only one invariant subspace. Thus Corollary 2.18 does not hold.

A representation on L is *non-decomposable* if L is not the direct sum of invariant subspaces. By Corollary 2.18, non-decomposable elementary representations of Engel groups are monothetic.

Let χ_e be the identity character on an Engel group G and Λ_G the set of all non-decomposable finite-dimensional χ_e-representations of G. By Corollary 2.18, each finite-dimensional representation π of G is a finite direct sum of representations from Λ_G multiplied by characters from $\operatorname{sign}(\pi)$.

3 Decomposition of extensions of finite-dimensional representations of nilpotent groups by unitary representationes

From now on G is a connected, locally compact **nilpotent** group, λ is an **elementary** (in many cases finite-dimensional) representation of G on a Banach space L and U is a **unitary** representation of G on a separable Hilbert space $\mathfrak{H}$. With each (λ, U)-cocycle ξ, one can associate the representation π of G:

$$\pi(g) = \mathfrak{e}(\lambda, U, \xi)(g) = \begin{pmatrix} \lambda(g) & \xi(g) \\ 0 & U(g) \end{pmatrix} \text{ for } g \in G, \text{ on } \mathfrak{Z} = L \dot{+} \mathfrak{H}, \qquad (3.1)$$ 3.22

called the *extension* of λ by U performed by ξ. It is *decomposable* if $\mathfrak{Z} = \mathfrak{Z}_1 \dot{+} \mathfrak{Z}_2$, where $\mathfrak{Z}_1, \mathfrak{Z}_2$ are π-invariant subspaces. In our approach we study decompositions of the extensions $\mathfrak{e}(\lambda, U, \xi)$ depending on the kind of disjointness of $\operatorname{sign}(\lambda)$ and U.

After some work on a special type of decomposition of unitary representations of nilpotent groups, we consider in subsection 3.2 the weakest type of disjointness of $\operatorname{sign}(\lambda)$ and U – their eigen-disjointness (the general case can be reduced to this case). We obtain a cohomological criterion of decomposability of π and show that in any decomposition $\mathfrak{Z} = \mathfrak{Z}_1 \dotplus \mathfrak{Z}_2$ in the sum of π-invariant components, either one of them contains L, or $L = (L \cap \mathfrak{Z}_1) \dotplus (L \cap \mathfrak{Z}_2)$ and both representations $\pi|_{\mathfrak{Z}i}$ are extensions of the representations $\lambda|_{L\cap\mathfrak{Z}i}$ by some representations similar to unitary ones.

We proceed to investigate the "spectral" decomposition of $\mathfrak{e}(\lambda, U, \xi)$ which arises when $\mathrm{sign}(\lambda)$ is sectionally spectrally disjoint with U, or with some subrepresentation of U. As a consequence, we show that each extension $\pi = \mathfrak{e}(\lambda, U, \xi)$ "approximately" decomposes in the following sense: $\mathfrak{Z} = X_n \dotplus Y_n$ for some pairs of invariant subspaces such that $Y_n \subseteq Y_{n+1}$, $\pi|_{Y_n}$ are similar to unitary representations, $X_{n+1} \subseteq X_n$ and the space $\cap_n X_n$ is finite-dimensional and contains L.

A glance at the finite-dimensional situation leads to a conjecture that if λ is not monothetic, $\mathfrak{e}(\lambda, U, \xi)$ always decomposes in a sum of extensions with monothetic λ_i. In subsection 3.4 we show that this is true for commutative G but fails in general (for example, for the Heisenberg group).

3.1 Decompositions of unitary representations with respect to sets of characters

L2.n **Lemma 3.1** *Let U be a unitary representation of a group G and χ be a character of G.*
(i) *If χ is non-unitary then χ and U are spectrally disjoint.*
(ii) *If χ and U are sectionally spectrally disjoint, then χ^* and U are sectionally spectrally disjoint.*

Proof. (i) By (2.20), $|\chi(g)| \neq 1$ for some $g \in G$. Hence $\chi(g) \notin \mathrm{Sp}(U(g))$.
(ii) Clear, if χ is unitary, as $\chi^* = \chi$. If χ is non-unitary, χ^* is non-unitary and (i) gives (ii). ■

For a unitary representation U of G on $\mathfrak{H}$, let $\mathfrak{H}^\chi$ be the χ-eigenspace of U.

P2.1e **Lemma 3.2** *Let χ be a unitary character and M be a U-invariant subspace of $\mathfrak{H}$. Then $M \subseteq \mathfrak{H}^\chi$ if and only if, for each invariant subspace $K \neq \{0\}$ of M, χ and U_K are not spectrally disjoint, i.e.,*

$$\chi(g) \in \mathrm{Sp}(U(g)|_K) \text{ for all } g \in G. \tag{3.2}$$ 7.2n

Proof. Replace U by $\chi^{-1}U$. Then it suffices to get a proof for χ_e. If $M \subseteq \mathfrak{H}^{\chi_e}$, (3.2) holds. Conversely, let (3.2) hold. Then, for each invariant subspace $K \neq \{0\}$ of M,

$$1 \in \mathrm{Sp}(U(g)|_K) \text{ for all } g \in G. \tag{3.3}$$ 2.1p

Let Z be the centre of G. We claim that $Z \subseteq \ker U$. Indeed, let $z \in Z$ and $P(\Delta)$ be the spectral measure of $U(z)$ on $\mathbb{D} = \{\lambda \in \mathbb{C}\colon |\lambda| = 1\}$. If $P(\Delta) \neq 0$, for some $\Delta \subseteq \mathbb{D}$, then $P(\Delta)$ belongs to the W*-algebra generated by $U(z)$ and $U(z)^* = U(z^{-1})$ and commutes with all $U(g)$, $g \in G$. Thus the subspace $M_\Delta = P(\Delta)M$ is invariant for U. By (3.3), $1 \in \mathrm{Sp}(U(z)|_{M_\Delta}) \subseteq \Delta$. Hence $P(\Delta) \neq 0$ implies $1 \in \Delta$ which means that $\mathrm{Sp}(U(z)) = \{1\}$, so that $U(z) = \mathbf{1}_M$. Therefore $Z \subseteq \ker U$.

If $\ker U \neq G$ then $\widetilde{G} = G/\ker U \neq \{e\}$ is a connected, locally compact nilpotent group. Hence its centre $\widetilde{Z} \neq \{e\}$. Let $\phi\colon G \to \widetilde{G}$ be the quotient map. Define the unitary representation $\widetilde{U}$ of $\widetilde{G}$ on M by $\widetilde{U}(\phi(g)) = U(g)$. Then $\ker \widetilde{U} = \{e\}$. On the other hand, by (3.3), $1 \in \mathrm{Sp}(\widetilde{U}(\widetilde{g})|_K)$ for each invariant subspace $\{0\} \neq K \subseteq M$ and all $\widetilde{g} \in \widetilde{G}$. Hence, as above, $\widetilde{Z} \subseteq \ker \widetilde{U}$. This contradiction shows that $\ker U = G$ and $U(g) = \mathbf{1}_M$ for all $g \in G$. Thus $M \subseteq \mathfrak{H}^\chi$. ■

Note that the assumption that G is nilpotent is essential for the validity of Lemma 3.2

P4.1e **Proposition 3.3** *Let $\Omega = \{\chi_i\}_{i=1}^n$ be a finite set of characters of G.*

(i) *Ω and U are eigen-disjoint if and only if*

$$\mathfrak{H} = \oplus_{n=1}^N \mathfrak{H}_n \text{ for } N \le \infty, \tag{3.4}$$ 7.7

where $\mathfrak{H}_n$ are U-invariant subspaces such that each $U_{\mathfrak{H}_n}$ and Ω are spectrally disjoint.

(ii) *Let Ω and U be eigen-disjoint. If $\dim \mathfrak{H} < \infty$ then U and Ω are spectrally disjoint.*

Proof. (i) Let Ω and U be eigen-disjoint. Then each invariant subspace $K \neq \{0\}$ of $\mathfrak{H}$ contains an invariant subspace M such that U_M and Ω are spectrally disjoint. Indeed, K is not contained in $\mathfrak{H}^{\chi_1}$. Hence it follows from Lemma 3.2 that there is an invariant subspace K_1 of K such that U_{K_1} and χ_1 are spectrally disjoint. Similarly, there exists an invariant subspace K_2 of K_1 such that U_{K_2} and χ_2 are spectrally disjoint. Then U_{K_2} and $\{\chi_1, \chi_2\}$ are spectrally disjoint. Continuing this process, we obtain a subspace $M = K_n$ such that U_M and Ω are spectrally disjoint.

Let $\mathcal{R}$ be the set of all families $R = \{M_\alpha\}$ of mutually orthogonal invariant subspaces of $\mathfrak{H}$ such that each U_{M_α} and Ω are spectrally disjoint. As $\mathfrak{H}$ is separable, each R is at most countable. Order $\mathcal{R}$ by inclusion. If $\{R_j\}$ is a linearly ordered subset in $\mathcal{R}$, the family $R = \cup R_j$ belongs to $\mathcal{R}$ and majorizes all R_j. Hence $\mathcal{R}$ has a maximal family $\{\mathfrak{H}_n\}_{n=1}^N$. Set $K = \mathfrak{H} \ominus \left(\oplus_{n=1}^N \mathfrak{H}_n\right)$. If $K \neq \{0\}$ then, by the above argument, K has an invariant subspace $M \neq \{0\}$ such that Ω and U_M are spectrally disjoint, so that $\{\mathfrak{H}_n\}_{n=1}^N$ is not maximal. Hence $K = \{0\}$ and $\mathfrak{H} = \oplus_{n=1}^N \mathfrak{H}_n$.

Conversely, let (3.4) hold. Assume that $\mathfrak{H}^\chi \neq \{0\}$ for some $\chi \in \Omega$. Let $x = \sum_{n=1}^N x_n \in \mathfrak{H}^\chi$, $x_n \in \mathfrak{H}_n$. Then $U(g)x_n = \chi(g)x_n$ for all $g \in G$ and n. Choose n such that $x_n \neq 0$. Then χ and $U_{\mathfrak{H}_n}$ are not spectrally disjoint – a contradiction.

(ii) If $\dim \mathfrak{H} < \infty$ then, as U is unitary and G is nilpotent, $\mathfrak{H} = \oplus_{\omega \in \Omega_1} \mathfrak{H}^\omega$ for some finite set Ω_1 of unitary characters on G. As Ω and U are eigen-disjoint, $\Omega_1 \cap \Omega = \varnothing$. Hence, by Lemma 2.12, each $\chi \in \Omega$ is spectrally disjoint with U. ■

C4.1 **Corollary 3.4** *Let Ω be a finite set of characters of G. Then*

(i) *$\mathfrak{H} = \mathfrak{H}^0 \oplus \mathfrak{H}_\Omega$, where $\mathfrak{H}^0$ and $\mathfrak{H}_\Omega$ are U-invariant spaces, $\mathfrak{H}_\Omega = \oplus_{\chi \in \Omega} \mathfrak{H}^\chi$ and the representation $U_{\mathfrak{H}^0}$ is eigen-disjoint with Ω.*

(ii) *If U and each $\chi \in \Omega$ are sectionally spectrally disjoint then U and Ω are eigen-disjoint.*

(iii) *Let $\mathfrak{H}^\chi = \{0\}$. Then χ and U are sectionally spectrally disjoint if and only if χ and $U_{\mathfrak{H}^0}$ are sectionally spectrally disjoint.*

Proof. (i) Clearly, $\mathfrak{H}^\chi$ and $\mathfrak{H}^{\chi'}$ are orthogonal if $\chi \neq \chi'$. Set $\mathfrak{H}_\Omega = \oplus_{\chi \in \Omega} \mathfrak{H}^\chi$ and $\mathfrak{H}^0 = \mathfrak{H} \ominus \mathfrak{H}_\Omega$.

(ii) is evident.

(iii) By (i), $\mathfrak{H} = \mathfrak{H}^0 \oplus \left(\oplus_{\omega \in \Omega, \omega \neq \chi} \mathfrak{H}^\omega\right)$, and χ and $U_{\mathfrak{H}^\omega}$ are spectrally disjoint. Thus if χ and $U_{\mathfrak{H}^0}$ are sectionally spectrally disjoint, χ and U are sectionally disjoint.

Conversely, if χ and U are sectionally spectrally disjoint, $\mathfrak{H} = \oplus_{i=1}^n \mathfrak{H}_i$, where each $\mathfrak{H}_i$ is U-invariant and spectrally disjoint with χ. Let P_i be the projections on $\mathfrak{H}_i$ and $\mathfrak{H}_i^\omega = P_i \mathfrak{H}^\omega$ for $\omega \in \Omega$. For $x \in \mathfrak{H}^\omega$, we have $U(g)P_i x = P_i U(g)x = \omega(g)P_i x$. Hence $\mathfrak{H}_i^\omega \subseteq \mathfrak{H}_i \cap \mathfrak{H}^\omega$ and $\mathfrak{H}^\omega = \oplus_{i=1}^n \mathfrak{H}_i^\omega$.

Set $\mathfrak{H}_i' = \mathfrak{H}_i \ominus \left(\oplus_{\chi \in \Omega} \mathfrak{H}_i^\omega\right)$. Then $\mathfrak{H}_i'$ are invariant subspaces and

$$\mathfrak{H} = \oplus_{i=1}^n \mathfrak{H}_i = \oplus_{i=1}^n \left(\left(\oplus_{\chi \in \Omega} \mathfrak{H}_i^\omega\right) \oplus \mathfrak{H}_i'\right) = \left(\oplus_{\chi \in \Omega} \mathfrak{H}^\omega\right) \oplus \left(\oplus_{i=1}^n \mathfrak{H}_i'\right) = \mathfrak{H}_\Omega \oplus \left(\oplus_{i=1}^n \mathfrak{H}_i'\right).$$

By (i), $\mathfrak{H}^0 = \oplus_{i=1}^n \mathfrak{H}_i'$. As χ is spectrally disjoint with each $U_{\mathfrak{H}_i}$, it is also spectrally disjoint with each $U_{\mathfrak{H}_i'}$. Thus χ is sectionally spectrally disjoint with $U_{\mathfrak{H}^0} = \oplus_{i=1}^n U_{\mathfrak{H}_i'}$. ■

3.2 Spectral and approximate decompositions of the extensions $\mathfrak{e}(\lambda, U, \xi)$

It is well known that ξ is a coboundary if and only if the extension $\mathfrak{e}(\lambda, U, \xi)$ (see (3.1)) has an invariant subspace complementing L (more general results were established in Proposition 2.5 and **frequently** used in the previous section), but it is convenient to formulate it precisely here.

L2.3 **Lemma 3.5** (i) *A (λ, U)-cocycle ξ is a coboundary if and only if $\mathfrak{Z} = L \dotplus H$, where H is $\mathfrak{e}(\lambda, U, \xi)$-invariant. The restriction of $\mathfrak{e}(\lambda, U, \xi)$ to H is similar to U.*

(ii) *If (λ, U)-cocycles ξ, η are cohomological then the representations $\mathfrak{e}(\lambda, U, \xi)$ and $\mathfrak{e}(\lambda, U, \eta)$ are similar.*

Recall that a representation $\pi = \mathfrak{e}(\lambda, U, \xi)$ on $\mathfrak{Z}$ is decomposable if $\mathfrak{Z} = \mathfrak{Z}_1 \dotplus \mathfrak{Z}_2$ where $\mathfrak{Z}_1, \mathfrak{Z}_2$ are π-invariant subspaces. We proceed now with the following cohomological criterion of decomposability of π that illustrates a dichotomy of decompositions of $\mathfrak{e}(\lambda, U, \xi)$.

P2.dec **Proposition 3.6** *Let $\pi = \mathfrak{e}(\lambda, U, \xi)$.*

(i) *If there are projections $p \in B(L)$ and $q \in B(\mathfrak{H})$ commuting with λ and U such that*

$$\xi - p\xi q - (\mathbf{1}_L - p)\xi(\mathbf{1}_{\mathfrak{H}} - q) \text{ is a coboundary,} \tag{3.5}$$ 2.e

then π is decomposable and $L = L_1 \dotplus L_2$, where $L_1 = pL = \mathfrak{Z}_1 \cap L$, $L_2 = (\mathbf{1}_L - p)L = \mathfrak{Z}_2 \cap L$.

(ii) *Let U and $\operatorname{sign}(\lambda)$ be eigen-disjoint and $\dim L < \infty$. If π is decomposable then there are projections $p \in B(L)$ and $q \in B(\mathfrak{H})$ commuting with λ and U and satisfing (3.5). Moreover, $\mathfrak{Z}_1 \cap L = pL$ and $\mathfrak{Z}_2 \cap L = (\mathbf{1}_L - p)L$, so that $L = (\mathfrak{Z}_1 \cap L) \dotplus (\mathfrak{Z}_2 \cap L)$.*

Proof. (i) It follows from the assumptions that $\eta = p\xi q + (\mathbf{1}_L - p)\xi(\mathbf{1}_{\mathfrak{H}} - q)$ is a (λ, U)-cocycle. Set $L_1 = pL$, $L_2 = (\mathbf{1}_L - p)L$, $\mathfrak{H}_1 = q\mathfrak{H}$ and $\mathfrak{H}_2 = (\mathbf{1}_{\mathfrak{H}} - q)\mathfrak{H}$. Then $\mathfrak{X}_i = L_i \dotplus \mathfrak{H}_i$, $i = 1, 2$, are π_η-invariant subspaces, $\mathfrak{Z} = \mathfrak{X}_1 \dotplus \mathfrak{X}_2$ and $L = L_1 \dotplus L_2$.

If $\xi - \eta$ is a coboundary, π is similar to $\pi_\eta = \mathfrak{e}(\lambda, U, \eta)$ by Lemma 3.5, i.e., $\pi = \widetilde{T}^{-1}\pi_\eta\widetilde{T}$, where $\widetilde{T} = \begin{pmatrix} \mathbf{1}_L & T \\ \mathbf{0} & \mathbf{1}_{\mathfrak{H}} \end{pmatrix}$ and $T \in B(\mathfrak{H}, L)$. Then, for some $T_1 \in B(\mathfrak{H}_1, L_2)$, $T_2 \in B(\mathfrak{H}_2, L_1)$,

$$\mathfrak{Z}_i = \widetilde{T}^{-1}\mathfrak{X}_i = L_i \dotplus (-T + \mathbf{1}_{\mathfrak{H}})\mathfrak{H}_i = L_i \dotplus (-T_i + \mathbf{1}_{\mathfrak{H}})\mathfrak{H}_i \tag{3.6}$$ 4.211

are π-invariant subspaces, $\mathfrak{Z} = \mathfrak{Z}_1 \dotplus \mathfrak{Z}_2$, $L_i := \mathfrak{Z}_i \cap L$ and $L = L_1 \dotplus L_2$.

(ii) Let $n := \dim L < \infty$. If U and $\operatorname{sign}(\lambda)$ are eigen-disjoint, U and λ have no non-zero intertwining operators. Indeed, let $W\lambda(g) = U(g)W$ for all $g \in G$. By Corollary 2.18, $L = \sum_{\chi \in \operatorname{sign}(\lambda)} \dotplus L_\chi$ and $(\lambda(g) - \chi(g)\mathbf{1}_L)^n x = 0$ for each $x \in L_\chi$ and all $g \in G$. Hence $(U(g) - \chi(g)\mathbf{1}_L)^n Wx = W(\lambda(g) - \chi(g)\mathbf{1}_L)^n x = 0$. Thus $Wx = 0$, because $U(g)$ has no χ-eigenvectors. Therefore $W = 0$.

Let $\mathfrak{Z} = \mathfrak{Z}_1 \dotplus \mathfrak{Z}_2$, where $\mathfrak{Z}_1, \mathfrak{Z}_2$ are π-invariant subspaces. The projection P on $\mathfrak{Z}_1$ along $\mathfrak{Z}_2$ commutes with π. Write P as a block-matrix with respect to the decomposition $\mathfrak{Z} = L \dotplus \mathfrak{H}$. Then P_{21} intertwines λ and U. Hence $P_{21} = 0$ by the above. Then $p := P_{11}$, $q := P_{22}$ are projections commuting with λ and U, respectively, and $p\xi(g) - \xi(g)q = \lambda(g)P_{12} - P_{12}U(g)$. Hence $p\xi(g)(\mathbf{1}_{\mathfrak{H}} - q)$ is a (λ, U)-coboundary, since

$$p\xi(g)(\mathbf{1}_{\mathfrak{H}} - q) = (p\xi(g) - \xi(g)q)(\mathbf{1}_{\mathfrak{H}} - q) = \lambda(g)P_{12}(\mathbf{1}_{\mathfrak{H}} - q) - P_{12}(\mathbf{1}_{\mathfrak{H}} - q)U(g).$$

Similarly, $(\mathbf{1}_L - p)\xi(g)q$ is a coboundary. Thus $\xi - (p\xi q + (\mathbf{1}_L - p)\xi(\mathbf{1}_{\mathfrak{H}} - q))$ is a coboundary.

As P is a projection, $P = P^2$, so that $P_{12} = pP_{12} + P_{12}q$. We also have

$$\mathfrak{Z}_1 = P(L \dotplus \mathfrak{H}) = pL + \{P_{12}x + qx : x \in \mathfrak{H}\},$$
$$\mathfrak{Z}_2 = (\mathbf{1}_{\mathfrak{Z}} - P)(L \dotplus \mathfrak{H}) = (\mathbf{1}_L - p)L + \{-P_{12}y + (\mathbf{1}_{\mathfrak{H}} - q)y\colon\ y \in \mathfrak{H}\}.$$

If $qx = 0$ then $P_{12}x = pP_{12}x + P_{12}qx = pP_{12}x$. Hence $pL = \mathfrak{Z}_1 \cap L$. Similarly, if $(\mathbf{1}_{\mathfrak{H}} - q)y = 0$ then $P_{12}y = pP_{12}y + P_{12}qy = pP_{12}y + P_{12}y$, so that $pP_{12}y = 0$. Thus $(\mathbf{1}_L - p)L = \mathfrak{Z}_2 \cap L$. ■

Proposition 3.6(ii) shows that, if λ is finite-dimensional and eigen-disjoint with U, then the decomposition of $\pi = \mathfrak{e}(\lambda, U, \xi)$ is determined by a pair of projections (p, q), commuting with λ and U and satisfying condition (3.5), and $L = L \cap \mathfrak{Z}_1 \dotplus L \cap \mathfrak{Z}_2$ with $L \cap \mathfrak{Z}_1 = pL$. Depending on triviality or non-triviality of p the decomposition belongs to one of two classes. In the first class one of the summands $\mathfrak{Z}_1$, $\mathfrak{Z}_2$ contains L. In the second class neither of them contains L and $\pi|_{\mathfrak{Z}_i}$ are extensions of $\lambda|_{L\cap\mathfrak{Z}_i}$ by representations similar to $U|_{q\mathfrak{H}}$ and $U|_{(1-q)\mathfrak{H}}$.

The simplest type of the decomposition of the extension $\pi = \mathfrak{e}(\lambda, U, \xi)$ arises when the representations λ and U are sectionally spectrally disjoint. Lemma 3.5 and Corollary 2.16 yield

C3.0 **Corollary 3.7** *If U and* $\mathrm{sign}(\lambda)$ *are sectionally spectrally disjoint then $\mathfrak{Z} = L \dotplus H$, where H is π-invariant and $\pi|_H$ is similar to U.*

As λ is an elementary representation of a nilpotent group on L, we have from Corollary 2.18

$$L = \sum_{\chi\in\mathrm{sign}(\lambda)} \dotplus L_\chi,\ \text{each } L_\chi \text{ is invariant for } \lambda \text{ and } \mathrm{sign}(\lambda_{L_\chi}) = \chi. \tag{3.7}$$ 3.11

Assume that $\mathrm{sign}(\lambda) = \Omega_1 \cup \Omega_2$, where $\Omega_1 \cap \Omega_2 = \varnothing$, and that $\mathfrak{H} = \mathfrak{H}_1 \dotplus \mathfrak{H}_2$, where $\mathfrak{H}_i$ are U-invariant. One of the sets Ω_j can be empty and one of the subspaces $\mathfrak{H}_j$ can be $\{0\}$. Let

$$\mathfrak{Z} = L \dotplus \mathfrak{H} \text{ and } L_j = \sum_{\chi\in\Omega_j} \dotplus L_\chi \text{ for } j = 1, 2.$$

C2.k **Corollary 3.8** *Let $\pi = \mathfrak{e}(\lambda, U, \xi)$, let Ω_1 and $U_{\mathfrak{H}_2}$ be sectionally spectrally disjoint, and let Ω_2 and $U_{\mathfrak{H}_1}$ be sectionally spectrally disjoint. Then $\mathfrak{Z} = \mathfrak{Z}_1 \dotplus \mathfrak{Z}_2$ is the direct sum of π-invariant subspaces and there are operators $T_1 \in B(\mathfrak{H}_1, L_2)$, $T_2 \in B(\mathfrak{H}_2, L_1)$ such that*

$$\mathfrak{Z}_j = L_j \dotplus (-T_j + \mathbf{1}_{\mathfrak{H}_j})\mathfrak{H}_j,\quad \pi_{\mathfrak{Z}_j} = \begin{pmatrix} \lambda_{L_j} & \eta_j \\ 0 & \sigma_j \end{pmatrix},\ j = 1, 2,$$

and the representation σ_j on $(-T_j + \mathbf{1}_{\mathfrak{H}_j})\mathfrak{H}_j$ is similar to $U_{\mathfrak{H}_j}$.

Proof. Let p be the projection on L_1 along L_2, and q be the projection on $\mathfrak{H}_1$ along $\mathfrak{H}_2$. Then p commutes with λ and q commutes with U. Therefore $\xi_{12} = p\xi(\mathbf{1}_{\mathfrak{H}} - q)$ is a $(p\lambda, (\mathbf{1}_{\mathfrak{H}} - q)U)$-cocycle and $\xi_{21} = (\mathbf{1}_L - p)\xi q$ is a $((\mathbf{1}_L - p)\lambda, qU)$-cocycle. As the pairs (Ω_1, U_2) and (Ω_2, U_1) are sectionally spectrally disjoint, $\mathcal{H}^1((\mathbf{1}_L - p)\lambda, qU) = \mathcal{H}^1(p\lambda, (\mathbf{1}_{\mathfrak{H}} - q)U) = 0$ by Corollary 2.16. Hence ξ_{21}, ξ_{12} are coboundaries and (3.5) holds. Applying Proposition 3.6(i) and (3.6) we conclude the proof. ■

R4.1 **Remark 3.9** Corollary 3.8 can be generalized as follows. Let $\pi = \mathfrak{e}(\lambda, U, \xi)$ and $\operatorname{sign}(\lambda) = \cup_{j=1}^n \Omega_j$ where all Ω_j are mutually disjoint. Let $\mathfrak{H} = \sum_{j=1}^n \dotplus \mathfrak{H}_j$ where all $\mathfrak{H}_j$ be U-invariant. Let

$$\mathfrak{Z} = L \dotplus \mathfrak{H} \text{ and } L_j = \sum_{\chi \in \Omega_j} \dotplus L_\chi,\ j = 1, ..., n.$$

If all pairs $(\Omega_i, U_{\mathfrak{H}_j})$, $i \neq j$, are sectionally spectrally disjoint, then $\mathfrak{Z} = \sum_{j=1}^n \dotplus \mathfrak{Z}_j$ is the direct sum of π-invariant subspaces and there are operators $T_j \in B(\mathfrak{H}_j, \sum_{i\neq j} \dotplus L_i)$ such that

$$\mathfrak{Z}_j = L_j \dotplus (-T_j + 1_{\mathfrak{H}_j})\mathfrak{H}_j, \quad \pi_{\mathfrak{Z}_j} = \begin{pmatrix} \lambda|_{L_j} & \eta_j \\ 0 & \sigma_j \end{pmatrix} \text{ and } \sigma_j \text{ is similar to } U|_{\mathfrak{H}_j}. \qquad \blacksquare$$

For $x \in \mathfrak{H}$, $u \in L$, a rank one operator $x \otimes u$ acts from $\mathfrak{H}$ to L by

$$(x \otimes u)z = (z, x)u \text{ for } z \in \mathfrak{H}. \tag{3.8}$$ 0.1

Then $\|x \otimes u\| = \|x\|\|u\|$ and $(x \otimes u)^* = u \otimes x$. For $R \in B(L), T \in B(\mathfrak{H})$ and $u \in L$, $v \in \mathfrak{H}$,

$$R(x \otimes u)T = (T^*x) \otimes Ru \text{ and } (x \otimes u)(y \otimes v) = (v, x)(y \otimes u) \in B(L). \tag{3.9}$$ 0.2

We need now the following result on the kernels of (λ, U)-cocycles for the simplest pairs of representations λ, U.

P3.1nn **Proposition 3.10** *Let χ be a character of a connected locally compact group G. Let λ be a χ-representation of G on L, $\dim L < \infty$, and U be the representation $\chi \mathbf{1}_{\mathfrak{H}}$ on $\mathfrak{H}$. If $\xi \in \mathcal{Z}^1(\lambda, U)$ then the codimension of the subspace $E^\xi = \cap_{g\in G} \ker \xi(g)$ in $\mathfrak{H}$ does not exceed $n_G \dim L$.*

Proof. Considering the representations $\chi^{-1}(g)\lambda(g)$ and $\mathbf{1}_{\mathfrak{H}}$ instead of λ and $\chi\mathbf{1}_{\mathfrak{H}}$, we may assume that $\chi = \chi_e$ is the identity character. Let $\{e_i\}_{i=1}^m$ be a basis in L in which all operators $\lambda(g)$, $g \in G$, have lower triangular form with $\lambda_{ii}(g) = 1$. As ξ is continuous, there are continuous maps $\{x_i(g)\}_{i=1}^m$ from G into $\mathfrak{H}$ such that $\xi(g) \overset{(3.8)}{=} \sum_{i=1}^m x_i(g) \otimes e_i$. We have from (2.2) that $\xi(e) = 0$. Hence all $x_i(e) = 0$.

As $\xi(gh) = \lambda(g)\xi(h) + \xi(g)$ and $\lambda(g)e_i = \sum_{j=i}^m \lambda_{ji}(g)e_j$, we have

$$\begin{aligned}\xi(gh) &= \lambda(g)\sum_{i=1}^m x_i(h) \otimes e_i + \sum_{i=1}^m x_i(g) \otimes e_i \overset{(3.9)}{=} \sum_{i=1}^m x_i(h) \otimes \lambda(g)e_i + \sum_{i=1}^m x_i(g) \otimes e_i \\ &= \sum_{i=1}^m x_i(h) \otimes \left(\sum_{j=i}^m \lambda_{ji}(g)e_j\right) + \sum_{i=1}^m x_i(g) \otimes e_i \overset{(3.9)}{=} \sum_{i=1}^m \left(\sum_{k=1}^i \overline{\lambda_{ik}(g)} x_k(h) + x_i(g)\right) \otimes e_i,\end{aligned}$$

for $g, h \in G$. Hence

$$x_i(gh) = \sum_{k=1}^i \overline{\lambda_{ik}(g)} x_k(h) + x_i(g) = x_i(h) + \sum_{k=1}^{i-1} \overline{\lambda_{ik}(g)} x_k(h) + x_i(g).$$

For $i = 1$, $x_1(gh) = x_1(g) + x_1(h)$. Let R_1 be the complex subspace of $\mathfrak{H}$ generated by the set $\{x_1(g)\colon\ g \in G\}$. Then, by Corollary 2.2, $\dim R_1 \leq n_G$. For $i = 2$, we have

$$x_2(gh) = x_2(g) + x_2(h) + \overline{\lambda_{21}(g)} x_1(h), \text{ for } g, h \in G. \tag{3.10}$$ 5.2’

Let $\widehat{x}_2(g)$ be the projection of $x_2(g)$ on $\mathfrak{H} \ominus R_1$. Then $\widehat{x}_2(gh) = \widehat{x}_2(g) + \widehat{x}_2(h)$ for $g, h \in G$, and $\widehat{x}_2(e) = 0$. Hence, by Corollary 2.2, the complex subspace R_2 of $\mathfrak{H} \ominus R_1$ generated by the set $\{\widehat{x}_2(g)\colon\ g \in G\}$ is finite-dimensional and $\dim R_2 \le n_G$. Then all $x_2(g) \in R_1 \oplus R_2$. Continuing this process, we obtain subspaces $\{R_j\}_{j=1}^m$ of $\mathfrak{H}$ such that $x_i(g) \in \sum_{j=1}^i \oplus R_j$ and $\dim R_j \le n_G$. Set $R = \sum_{j=1}^m \oplus R_j$. Then $x_i(g) \in R$, for all g and i, and $\dim R \le n_G \dim L$.

Let $M = \mathfrak{H} \ominus R$. Then the codimension of M equals $\dim R$, and $\xi(g)x = 0$ for all $g \in G$, $x \in M$. Thus $M \subset E^\xi$ whence the codimension of E^ξ does not exceed $n_G \dim L$. ■

We return to the decomposition of the extensions $\pi = \mathfrak{e}(\lambda, U, \xi)$. From now on $\dim L < \infty$, so that L has decomposition (3.7). Let $\Omega \subseteq \operatorname{sign}(\lambda)$ be such that $\mathfrak{H}^\chi \neq \{0\}$ if $\chi \in \Omega$, and $\mathfrak{H}^\chi = \{0\}$ if $\chi \in \operatorname{sign}(\lambda) \backslash \Omega$. By Corollary 3.4,

$$\mathfrak{H} = \mathfrak{H}^0 \oplus \sum_{\chi \in \Omega} \oplus \mathfrak{H}^\chi, \text{ where } \operatorname{sign}(\lambda) \text{ and } U_{\mathfrak{H}^0} \text{ are eigen-disjoint.} \tag{3.11}$$ 3.111

The subspace $L \dotplus \mathfrak{H}^0$ and all subspaces $L \dotplus \mathfrak{H}^\chi$, $\chi \in \Omega$, are π-invariant.

We start with a special case when $\operatorname{sign}(\lambda) = \{\chi\}$ and $U = \chi \mathbf{1}_{\mathfrak{H}}$. Then π is a χ-representation. Let $\mathfrak{Z}^\chi = \{x \in \mathfrak{Z}\colon\ \pi(g)x = \chi(g)x \text{ for } g \in G\}$ be the χ-eigenspace of π. Then the space $E^\chi = \cap_{g \in G} \ker \xi(g)$ coincides with $\mathfrak{Z}^\chi \cap \mathfrak{H}$. Let $K_\chi = \mathfrak{H} \ominus E^\chi$ and $\mathcal{L} = L \dotplus K_\chi$. Proposition 3.10 yields

P3.1n **Lemma 3.11** *Let $\pi = \mathfrak{e}(\lambda, \chi \mathbf{1}_{\mathfrak{H}}, \xi)$ and λ be a χ-representation. Then $\mathfrak{Z} = \mathcal{L} \dotplus E^\chi$ is the direct sum of π-invariant subspaces, $\dim K_\chi \le n_G \dim L$ and $\pi|_{\mathcal{L}}$ is a χ-representation.*

We consider now the case when $\operatorname{sign}(\lambda)$ and $U_{\mathfrak{H}^0}$ (the part of U eigen-disjoint with λ) are sectionally spectrally disjoint. As above, let $\mathfrak{Z}^\chi$ be the χ-eigenspace of π.

C3.6 **Theorem 3.12** *Suppose that $\operatorname{sign}(\lambda)$ and $U_{\mathfrak{H}^0}$ (see (3.11)) are sectionally spectrally disjoint. Then $\mathfrak{Z}$ is the direct sum of π-invariant subspaces*

$$\mathfrak{Z} = \mathcal{L} \dotplus E^\Omega \dotplus H_0 \text{ and } E^\Omega = \sum_{\chi \in \Omega} \dotplus E^\chi, \text{ where } \dim \mathcal{L} \le (n_G + 1) \dim L,$$

E^χ are some subspaces of $\mathfrak{Z}^\chi$ and $\pi|_{H_0}$ is similar to the unitary representation $U_{\mathfrak{H}^0}$.

Moreover, there are subspaces $K_\chi \subseteq L \dotplus \mathfrak{H}^\chi$, $\dim K_\chi \le n_G \dim L_\chi$ such that

$$\mathcal{L} = \left(\sum_{\chi \notin \Omega} \dotplus L_\chi \right) \dotplus \left(\sum_{\chi \in \Omega} \dotplus (L_\chi \dotplus K_\chi) \right) \text{ and } L_\chi \dotplus K_\chi \text{ are } \pi\text{-invariant subspaces.}$$

Proof. Set $\Omega' = \Omega \cup \{0\}$. The cocycle $\xi = (\xi_{\chi\omega})$, $\chi \in \operatorname{sign}(\lambda), \omega \in \Omega'$, where each $\xi_{\chi\omega}(g) \in B(\mathfrak{H}^\omega, L_\chi)$ is a $(\lambda_\chi, U_{\mathfrak{H}^\omega})$-cocycle, since λ and U are block-diagonal. By Corollary 2.16, $\mathcal{H}^1(\lambda_\chi, U_{\mathfrak{H}^\omega}) = 0$ if $\chi \neq \omega$. Hence, by (2.3), there are operators $T_{\chi\omega} \in B(\mathfrak{H}^\omega, L_\chi)$ such that $\xi_{\chi\omega}(g) = \lambda_\chi(g) T_{\chi\omega} - T_{\chi\omega} U_{\mathfrak{H}^\omega}(g)$. Set

$$T_\omega = \sum_{\chi \in \operatorname{sign}(\lambda), \chi \neq \omega} T_{\chi\omega} \text{ and } H_\omega = \{-T_\omega y \dotplus y\colon\ y \in \mathfrak{H}^\omega\} \subseteq L \dotplus \mathfrak{H}^\omega \text{ for each } \omega \in \Omega'.$$

Then, as in Lemma 3.5, the spaces H_0 and all $L_\omega \dotplus H_\omega$ are π-invariant, π_{H_0} is similar to $U_{\mathfrak{H}^0}$ and

$$\mathfrak{Z} = L \dotplus \sum_{\chi\in\Omega'} \oplus \mathfrak{H}^\chi = L \dotplus \left(\sum_{\chi\in\Omega'} H_\chi\right) = \left(\sum_{\chi\notin\Omega} \dotplus L_\chi\right) \dotplus \left(\sum_{\chi\in\Omega} \dotplus (L_\chi \dotplus H_\chi)\right) \dotplus H_0.$$

Since each $\mathfrak{H}^\chi$ is a χ-eigenspace of U, the restriction of π to $L_\chi \dotplus H_\chi$ has form $\pi|_{L_\chi \dotplus H_\chi} = \begin{pmatrix} \lambda_{L_\chi} & \eta_\chi \\ 0 & \chi \mathbf{1}_{H_\chi} \end{pmatrix}$, for some $(\lambda_\chi, \chi\mathbf{1}_{H^\chi})$-cocycle η_χ, and is a χ-representation. As in Lemma 3.11, set $E^\chi = \cap_{g\in G} \ker \eta_\chi(g)$ and $K_\chi = H_\chi \ominus E^\chi$. The proof is complete. ■

To investigate the structure of the extensions $\pi = \mathfrak{e}(\lambda, U, \xi)$ removing the restriction that $\operatorname{sign}(\lambda)$ and $U_{\mathfrak{H}^0}$ in (3.11) are sectionally spectrally disjoint, we need the following notion.

D3.36 **Definition 3.13** *A π-invariant subspace $\mathcal{L}$ of $\mathfrak{Z}$ approximately splits π, if there are pairs $(X_n, Y_n)_{n=1}^\infty$ of π-invariant subspaces such that, for each n, $\pi|_{Y_n}$ is similar to a unitary representation,*

$$\mathfrak{Z} = X_n \dotplus Y_n, \ X_{n+1} \subseteq X_n, \ Y_n \subseteq Y_{n+1} \text{ and } \mathcal{L} = \cap_n X_n.$$

T3.7 **Theorem 3.14** *For each extension $\pi = \mathfrak{e}(\lambda, U, \xi)$, there is an invariant finite-dimensional subspace $\mathcal{L}$ containing L, $\dim \mathcal{L} \le (n_G + 1)\dim L$, which approximately splits π.*

Proof. Let $Z = L \dotplus \sum_{\chi\in\Omega} \oplus \mathfrak{H}^\chi$. As $\pi|_Z$ satisfies conditions of Theorem 3.12, it follows that

$$Z = \mathcal{L} \dotplus E^\Omega \text{ and } E^\Omega = \sum_{\chi\in\Omega} \dotplus E^\chi, \text{ where } \mathcal{L} = \left(\sum_{\chi\notin\Omega} \dotplus L_\chi\right) \dotplus \left(\sum_{\chi\in\Omega} \dotplus (L_\chi \dotplus K_\chi)\right)$$

is a π-invariant subspace, $\dim \mathcal{L} \le (1 + n_G)\dim L$, E^χ are some subspaces of Z^χ,

$$\operatorname{sign}(\pi|_{\mathcal{L}}) = \operatorname{sign}(\lambda) \text{ and } \mathfrak{Z} = L \dotplus \mathfrak{H} = L \dotplus \mathfrak{H}^0 \dotplus \sum_{\chi\in\Omega} \oplus \mathfrak{H}^\chi = (\mathcal{L} \dotplus \mathfrak{H}^0) \dotplus E^\Omega. \tag{3.12}$$ 3.a

Let $\operatorname{sign}(\pi|_{\mathcal{L}})$ and $U_{\mathfrak{H}^0}$ be sectionally spectrally disjoint. By Corollary 3.7, $\mathcal{L} \dotplus \mathfrak{H}^0 = \mathcal{L} \dotplus H$, H is π-invariant and $\pi|_H$ is similar to a unitary representation. Set $X_n = \mathcal{L}$ and $Y_n = H \dotplus E^\Omega$ for all n.

Let $\operatorname{sign}(\pi|_{\mathcal{L}})$ and $U_{\mathfrak{H}^0}$ be not sectionally spectrally disjoint. By Proposition 3.3, $\mathfrak{H}^0 = \sum_{k=1}^\infty \oplus \mathfrak{H}_k$, each $\mathfrak{H}_k$ is U-invariant and $U_{\mathfrak{H}_k}$ is spectrally disjoint with $\operatorname{sign}(\pi|_{\mathcal{L}})$. For each k, the subspace $\mathcal{L} \dotplus \mathfrak{H}_k$ is π-invariant and, by Corollary 3.7, there is a π-invariant subspace H_k such that $\mathcal{L} \dotplus \mathfrak{H}_k = \mathcal{L} \dotplus H_k$ and $\pi|_{H_k}$ is similar to $U_{\mathfrak{H}_k}$. Set

$$X_n = \mathcal{L} \dotplus \sum_{k=n+1}^\infty \oplus \mathfrak{H}_k \text{ and } Y_n = E^\Omega \dotplus \sum_{k=1}^n \dotplus H_k.$$

The subspaces X_n, Y_n are π-invariant and π_{Y_n} is similar to a unitary representation. For each n,

$$\mathfrak{Z} = L \dotplus \mathfrak{H} \overset{(3.12)}{=} \mathcal{L} \dotplus \mathfrak{H}^0 \dotplus E^\Omega = \left(\mathcal{L} \dotplus \sum_{k=1}^n \oplus \mathfrak{H}_k\right) \dotplus \sum_{k=n+1}^\infty \oplus \mathfrak{H}_k \dotplus E^\Omega$$
$$= \left(\mathcal{L} \dotplus \sum_{k=1}^n \dotplus H_k\right) \dotplus \sum_{k=n+1}^\infty \oplus \mathfrak{H}_k \dotplus E^\Omega = X_n \dotplus Y_n.$$

Moreover, $X_{n+1} \subseteq X_n$, $Y_n \subseteq Y_{n+1}$ and $\cap_n X_n = \mathcal{L}$. Thus $\mathcal{L}$ approximately splits π. ■

Remark 3.15 In Definition 3.13, the condition $\dim(\mathcal{L}) < \infty$ does not always imply that $\overline{\cup_n Y_n}$ has finite codimension. In Theorem 3.14, however, $\overline{\cup_n Y_n}$ has finite codimension.

Theorem 3.14 and Lemma 3.11 yield

Corollary 3.16 *If $\mathfrak{e}(\lambda, U, \xi)$ is non-decomposable then $\dim \mathfrak{H} < \infty$, $\mathrm{sign}(\lambda) = \{\chi\}$ and $U = \chi \mathbf{1}_{\mathfrak{H}}$.*

3.3 Decomposition of $\mathfrak{e}(\lambda, U, \xi)$ into primary components

As we know (see Corollary 2.18) each finite-dimensional representation λ of a nilpotent group decomposes in the direct sum of monothetic representations λ_k, i.e. all $\mathrm{sign}(\lambda_k)$ are singletons. We call extensions $\mathfrak{e}(\lambda, U, \xi)$ with monothetic λ *primary*. In this subsection we discuss the possibility to decompose an arbitrary extension $\mathfrak{e}(\lambda, U, \xi)$ in the direct sum of primary extensions.

We regard a monothetic representations λ as a primary extension $\mathfrak{e}(\lambda, 0, 0)$.

T3.7n **Theorem 3.17** *Let G be a commutative, connected locally compact group. Then any extension $\pi = \mathfrak{e}(\lambda, U, \xi)$ decomposes in a finite direct sum of primary extensions.*

Proof. If $\mathrm{sign}(\lambda)$ contains a non-unitary character χ, one can split it off by Corollary 3.8: $\mathfrak{e}(\lambda, U, \xi) = \mathfrak{e}(\lambda_\chi, 0, 0) + \mathfrak{e}(\lambda', U', \xi')$ and $\chi \notin \mathrm{sign}(\lambda')$. Thus we may assume that $\mathrm{sign}(\lambda)$ consists of unitary characters. We will prove that $\mathfrak{Z}$ is the direct sum of π-invariant subspaces $\mathfrak{Z}_\chi$:

$$\mathfrak{Z} = \sum_{\chi \in \mathrm{sign}(\lambda)} \dotplus \mathfrak{Z}_\chi \text{ such that } L_\chi \subseteq \mathfrak{Z}_\chi \text{ and } \pi_{\mathfrak{Z}_\chi} = \mathfrak{e}(\lambda_{L_\chi}, U_k, \xi_k) \text{ for each } \chi. \tag{3.13}$$ 3.13

Let $\chi \in \mathrm{sign}(\lambda)$ and G^* be the dual group of G. Then

$$\mathfrak{H} = \int_{G^*}^{\oplus} \mathfrak{H}_\omega dP(\omega) \text{ and } U(g) = \int_{G^*}^{\oplus} \omega(g) dP(\omega), \text{ for } g \in G,$$

where P is a spectral measure on G^*. Set $\Omega_1 = \{\chi\}$ and $\Omega_2 = \mathrm{sign}(\lambda) \setminus \{\chi\}$. By Lemma 2.12, there is $h \in G$ such that $\chi(h) \notin \{\phi(h)\}_{\phi \in \Omega_2}$. Set $\varepsilon = \frac{1}{3} \min\{|\chi(h) - \phi(h)| \colon \phi \in \Omega_2\}$ and consider the sets

$$V = \{\omega \in G^* \colon |\chi(h) - \omega(h)| < \varepsilon\} \text{ and } G^* \setminus V = \{\omega \in G^* \colon |\chi(h) - \omega(h)| \geq \varepsilon\} \tag{3.14}$$ 3.6e

in G^*. Then $\Omega_2 \subset G^* \setminus V$. The subspaces

$$\mathfrak{H}_1 = \int_V^{\oplus} \mathfrak{H}_\omega dP(\omega) \text{ and } \mathfrak{H}_2 = \int_{G^* \setminus V}^{\oplus} \mathfrak{H}_\omega dP(\omega)$$

are invariant for U, $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and

$$\chi(h) \overset{(3.14)}{\notin} \mathrm{Sp}(U(h)|_{\mathfrak{H}_2}) = \overline{\{\omega(h)\}_{\omega \in G^* \setminus V}} \text{ and } \phi(h) \overset{(3.14)}{\notin} \mathrm{Sp}(U(h)|_{\mathfrak{H}_1}) = \overline{\{\omega(h)\}_{\omega \in V}},$$

for each $\phi \in \Omega_2$. Thus Ω_1, $U_{\mathfrak{H}_2}$ are spectrally disjoint, and Ω_2, $U_{\mathfrak{H}_1}$ are spectrally disjoint. By Corollary 3.8, π is decomposable: $\mathfrak{Z} = \mathfrak{Z}_1 \dotplus \mathfrak{Z}_2$ is the direct sum of π-invariant subspaces:

$$\mathfrak{Z}_m = L_m \dotplus \mathfrak{X}_m, \text{ where } L_m = \sum_{\chi \in \Omega_m} \dotplus L_\chi \text{ and } \mathfrak{X}_m = \{-T_m x \dotplus x : x \in \mathfrak{H}_m\},$$

for some operators $T_1 \in B(\mathfrak{H}_1, L_2)$, $T_2 \in B(\mathfrak{H}_2, L_1)$. The representations $\pi|_{\mathfrak{Z}_m}$ have form

$$\pi|_{\mathfrak{Z}_m} = \begin{pmatrix} \lambda_{L_m} & \eta_m \\ 0 & \sigma_m \end{pmatrix}$$

and the representations σ_m of G on $\mathfrak{X}_m$ are similar to $U_{\mathfrak{H}_m}$. Hence we can assume that they are unitary. Setting $\mathfrak{Z}_\chi = \mathfrak{Z}_1$ and continuing this process, we conclude the proof. ■

For each unitary representation π of G on $\mathfrak{H}$, a matrix element of π is a function $g \mapsto (\pi(g)x, x)$, where $x \in \mathfrak{H}$. The set of all matrix elements of π will be denoted by $E(\pi)$. Unitarily equivalent representations have the same sets of matrix elements. The dual object of G is the set $\widehat{G}$ of all unitary equivalence classes $\widetilde{\pi}$ of irreducible unitary representations π of G, supplied with the topology of unitary convergence of matrix elements. More precisely, $\widetilde{\pi}$ belongs to the closure of $M \subset \widehat{G}$ if each element of $E(\widetilde{\pi})$ can be uniformly on compacts approximated by matrix elements of representations in M. This topology can be non-Hausdorff.

The space $\widehat{G}$ contains all unitary characters χ of G, as we can identify χ and the equivalence class of one-dimensional representations $\widetilde{\chi\iota}$. So we may speak about separating of characters in $\widehat{G}$. The local topology for characters can be described in a simpler way than for arbitrary representations. Namely, choosing a compact $K \subset G$ and an $\varepsilon > 0$, define a neighborhood $W_{K,\varepsilon}(\chi)$ of χ by

$$W_{K,\varepsilon}(\chi) = \{\widetilde{\pi} \in \widehat{G}\colon\ |\varphi(g) - \chi(g)| < \varepsilon \text{ for some } \varphi \in E(\pi) \text{ and all } g \in K\}. \tag{3.15}$$ neighb

This family of open sets forms a base of neighborhoods for χ. We say that characters χ and ω are *separated* in $\widehat{G}$ if they have non-intersecting neighborhoods in $\widehat{G}$.

non-Haus **Theorem 3.18** *Suppose that G is a connected locally compact separable nilpotent group. If there are characters χ_1, χ_2 which are not separated in $\widehat{G}$, then there is a finite-dimensional representation λ, a unitary representation U and a (λ, U)-cocycle ξ such that the extension $\mathfrak{e}(\lambda, U, \xi)$ cannot be decomposed in a sum of primary extensions. Moreover, for each decomposition of the representation space $L \oplus \mathfrak{H}$ in a sum of invariant subspaces, one of summands contains L.*

Proof. Note that χ_1, χ_2 are not separated in $\widehat{G}$ if and only if the trivial character χ_e and the unitary character $\chi = \overline{\chi_1}\chi_2$ are not separated in $\widehat{G}$. Let $L = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ be a two-dimensional Hilbert space and λ be the orthogonal sum of the representations ι and $\chi\iota$:

$$\lambda(g) = e_1 \otimes e_1 + \chi(g)(e_2 \otimes e_2) \text{ for } g \in G.$$

Since connected locally compact groups are σ-compact, we choose compacts $\{e\} \in K_1 \subset K_2 \subset ...$ such that $G = \cup_{n=1}^\infty K_n$. By our assumption, $W_{K_n, 2^{-n}}(\chi_e) \cap W_{K_n, 2^{-n}}(\chi) \neq \varnothing$ (see (3.15)). This means that there are irreducible unitary representations π_n of G on $\mathfrak{H}_n$ and $u_n, v_n \in \mathfrak{H}_n$ such that

$$|(\pi_n(g)u_n, u_n) - 1| < 2^{-n} \text{ and } |(\pi_n(g)v_n, v_n) - \chi(g)| < 2^{-n} \text{ for all } g \in K_n. \tag{3.16}$$ powtwo

Since $e \in K_n$, we have $|\|u_n\|^2 - 1| < 1/2^n$. So changing u_n,v_n if necessary, we may assume that $\|u_n\| = \|v_n\| = 1$. It is known (see [Kir]) that connected nilpotent locally compact groups are of type 1. So, being separable, they are GCR-groups (see [Dixm, 13.9.4]). It follows [Dixm, 4.4.1] that $\widehat{G}$ is a T_0-space, that is, the intersection of all neighborhoods of each point contains only this point. Therefore the representations π_n can be chosen pairwise non-equivalent.

Set $\mathfrak{H} = \oplus\mathfrak{H}_n$, $U = \oplus_{n=1}^{\infty}\pi_n$,

$$u_n(g) = u_n - \pi_n(g)^* u_n \text{ and } v_n(g) = \overline{\chi(g)} v_n - \pi_n(g)^* v_n. \tag{3.17} \qquad \boxed{\texttt{3.g}}$$

Then

$$\|u_n(g)\|^2 = 2\,\mathrm{Re}(1 - (u_n, \pi_n(g) u_n)) \overset{(3.16)}{\leq} 2^{-(n-1)} \text{ for } g \in K_n. \tag{3.18} \qquad \boxed{\texttt{3.n}}$$

Set $u(g) = \oplus_{n=1}^{\infty} u_n(g)$ and $v(g) = \oplus_{n=1}^{\infty} v_n(g)$ for $g \in G$. Then $u(g) \in \mathfrak{H}$, since

$$\|u(g)\|^2 \overset{(3.18)}{\leq} \sum_{k=1}^{n-1} \|u_k(g)\|^2 + \sum_{k=n}^{\infty} 2^{-(k-1)} < \infty \text{ for } g \in K_n.$$

As $u_n(gh) = u_n(h) + \pi_n(h)^* u_n(g)$, we have

$$u(gh) = u(h) + U(h)^* u(g) \text{ for } g, h \in G. \tag{3.19} \qquad \boxed{\texttt{3.j}}$$

Similarly, $v(g) \in \mathfrak{H}$ and $v(gh) = \overline{\chi(g)} v(h) + U(h)^* v(g)$.

Let us define a map $\xi : G \to B(\mathfrak{H}, L)$ by

$$\xi(g) = u(g) \otimes e_1 + v(g) \otimes e_2. \tag{3.20} \qquad \boxed{\texttt{3.k}}$$

Using (3.19), we get that ξ is a (λ, U)-cocycle. Let $\pi = \mathfrak{e}(\lambda, U, \xi)$ and $\mathfrak{Z} = L \dotplus \mathfrak{H}$. Let us show that if $\mathfrak{Z} = \mathfrak{Z}_1 \dotplus \mathfrak{Z}_2$ is the sum of π-invariant subspaces then one of them contains L.

Assume to the contrary that neither of them contains L. Each π_n is eigen-disjoint with χ_e and χ, as it is irreducible. Hence U is eigen-disjoint with χ_e, χ and we have from Proposition 3.6(ii) that there are projections $p \neq \mathbf{0}, \mathbf{1}$ and q commuting with λ and U, respectively, such that $\eta = \xi - (p\xi q + (\mathbf{1} - p)\xi(\mathbf{1} - q))$ is a (λ, U)-coboundary. Since p commutes with λ, either $p = e_1 \otimes e_1$, or $p = e_2 \otimes e_2$. Assume that $p = e_1 \otimes e_1$. Hence, by (3.9),

$$p\eta(g)(\mathbf{1} - q) = p\xi(g)(\mathbf{1} - q) = (\mathbf{1} - q)u(g) \otimes e_1 \text{ and } (\mathbf{1} - p)\eta(g)q = (\mathbf{1} - p)\xi(g)q = qv(g) \otimes e_2$$

are also (λ, U)-coboundaries. Then, for some $x, y \in \mathfrak{H}$, the operator $T = x \otimes e_1 + y \otimes e_2$ satisfies

$$(\mathbf{1} - q)u(g) \otimes e_1 = \lambda(g)T - TU(g) \overset{(3.9)}{=} (1 - U(g)^*)x \otimes e_1 + (\overline{\chi_2(g)} - U(g)^*)y \otimes e_2.$$

Hence $(\mathbf{1} - q)u(g) = (1 - U(g)^*)x$. As q commutes with U and all π_n are pairwise non-equivalent, q is the projection on a subspace $\oplus_{n \in E}\mathfrak{H}_n$ for some $E \subseteq \mathbb{N}$. Let $x = \oplus_{n=1}^{\infty} x_n$, $x_n \in \mathfrak{H}_n$. Then

$$(1 - \pi_n(g)^*)u_n \overset{(3.17)}{=} u_n(g) = (1 - \pi_n(g)^*)x_n, \text{ for } n \notin E \text{ and all } g \in G.$$

As χ_e is eigen-disjoint with all π_n, we have $u_n = x_n$ for $n \notin E$. Taking into account that $\|u_n\| = 1$ and $\|x\|^2 = \sum \|x_n\|^2 < \infty$, we conclude that the set $\mathbb{N} \setminus E$ is finite.

Similarly, as $qv(g) \otimes e_2$ is a (λ, U)-coboundary, $qv(g) = (\overline{\chi_2(g)} - U(g)^*)z$ for some $z = \oplus_{n=1}^{\infty} z_n \in \mathfrak{H}$, $z_n \in \mathfrak{H}_n$. Repeating the above argument, we get that $v_n = z_n$ for $n \in E$. As $\|v_n\| = 1$ and $\|z\|^2 = \sum \|z_n\|^2 < \infty$, we conclude that the set E is finite, a contradiction. This contradiction shows that in any decomposition $\mathfrak{Z} = \mathfrak{Z}_1 \dotplus \mathfrak{Z}_2$ into invariant subspaces, either $\mathfrak{Z}_1$ or $\mathfrak{Z}_2$ contains L. Thus $\mathfrak{e}(\lambda, U, \xi)$ does not decompose in a sum of primary extensions. ■

To see an example of a connected nilpotent group whose characters cannot be separated in the dual object, let us consider the real Heisenberg group

$$G = H_3(\mathbb{R}) = \left\{ g(x, y, z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : \, x, y, z \in \mathbb{R} \right\}. \tag{3.21}$$ 6.7

It is known (see, for example, [ShZ]) that the unitary characters χ of G and the corresponding one-dimensional unitary representations ι_χ on $\mathbb{C}u$ have form

$$\chi_{\alpha,\beta}(g(x, y, z)) = e^{i(\alpha x+\beta y)}, \text{ for } \alpha, \beta \in \mathbb{R}, \text{ and } \iota_{\chi_{\alpha,\beta}}(g)u = \chi_{\alpha,\beta}(g)u.$$

In particular $\chi_{0,0} = 1$ – the trivial character and $\iota_{\chi_{0,0}} = \iota$ – the trivial representation.

Infinite-dimensional unitary irreducible representations of G act on $L^2(\mathbb{R})$ by the formula

$$U_\sigma(g(x, y, z))f(t) = e^{i\sigma(z+ty)} f(t + x), \text{ for } f \in L^2(\mathbb{R}), \text{ where } 0 \neq \sigma \in \mathbb{R}. \tag{3.22}$$ 3.20

Let us show that the class $\widetilde{\iota}$ belongs to the closure of the set $\{\widetilde{U_{\sigma_n}}\colon \sigma_n = n^{-6}, n \in \mathbb{N}\}$. Define f_n in $L^2(\mathbb{R})$ by $f_n(t) = n^{-2}$ for $t \in [0, n^4]$, and $f_n(t) = 0$ for $t \notin [0, n^4]$. Then $\|f_n\| = 1$. For $g = g(x, y, z)$,

$$\begin{aligned} |(U_{\sigma_n}(g)f_n, f_n) - 1| &\le \|U_{\sigma_n}(g)f_n - f_n\| \le \left\| \left(e^{i(z+ty)/n^6} - 1\right) f_n(t + x)\right\| + \|f_n(t + x) - f_n(t)\| \\ &= n^{-2} \left(\int_{-x}^{n^4-x} \left|e^{i(z+ty)/n^6} - 1\right|^2 dt \right)^{1/2} + n^{-2} \left| \int_{-x}^{0} dt + \int_{n^4-x}^{n^4} dt \right|^{1/2} \\ &\le \max_{-x \le t \le n^4-x} \left|e^{i(z+ty)/n^6} - 1\right| + n^{-2}(2\,|x|)^{1/2} \le (|y| + (2\,|x|)^{1/2})n^{-2}. \end{aligned}$$

Consider the increasing sequence of compacts $K_m = \{g = g(x, y, z)\colon\ |x| + |y| + |z| \le m\}$. Then $G = \cup_m K_m$ and on each K_m the matrix elements $(U_{\sigma_n}(g)f_n, f_n)$ uniformly tend to 1. This means that any neighborhood $W_{K_m,\varepsilon}(\chi_{0,0})$ of $\chi_{0,0}$ contains a representation U_{σ_n} for some n.

On the other hand, it should be noted that if $U_\sigma \in W_{K,\varepsilon}(\chi_{0,0})$ then $U_\sigma \in W_{K,\varepsilon}(\chi)$ for each character $\chi = \chi_{\alpha,\beta}$. To see this, note that the unitary operator V on $L^2(\mathbb{R})$ that acts by

$$(V f)(t) = e^{i\alpha(t-\frac{\beta}{\sigma})} f\left(t - \frac{\beta}{\sigma}\right), \text{ for } f \in L^2(\mathbb{R}),$$

satisfies $V\chi(g)U_\sigma(g) = U_\sigma(g)V$ for all $g \in G$. Hence $|(U_\sigma V x, V x) - \chi(g)| = |(U_\sigma x, x) - 1|$. Thus if $U_{\sigma_n} \in W_{K_m,\varepsilon}(\chi_{0,0})$ then $U_{\sigma_n} \in W_{K_m,\varepsilon}(\chi)$, so that $\chi_{0,0}$ and χ cannot be separated.

4 Neutral cocycles

4.1 Definitions and general results

In this section we consider a connected, locally compact group G and its representations λ and U on separable Hilbert spaces L and $\mathfrak{H}$. Let $C^1(G, B(\mathfrak{H}, L))$ be the space of all weakly continuous functions from G to $B(\mathfrak{H}, L)$. We introduce an involution map from $C^1(G, B(\mathfrak{H}, L))$ into $C^1(G, B(L, \mathfrak{H}))$ by

$$c^\sharp(g) = c(g^{-1})^* \text{ for } c \in C^1(\mathfrak{H}, L). \tag{4.1}$$ 3.3e

If $c=\lambda\in C^1(G,B(L))$ then $\lambda^\sharp$ is a representation of G on L; if $c=U\in C^1(G,B(\mathfrak{H}))$ then $U^\sharp=U$. If ξ is a (λ,U)-cocycle, then $\xi^\sharp$ is a $(U,\lambda^\sharp)$-cocycle:

$$\xi^\sharp(gh)\overset{(4.1)}{=}\xi(h^{-1}g^{-1})^*=U(g)\xi^\sharp(h)+\xi^\sharp(g)\lambda^\sharp(h). \tag{4.2}$$ 3.4e

We may consider $B(L)$ as a G-bimodule with respect to the representations λ and $\lambda^\sharp$. Then the (low-dimensional) cohomologies of G with coefficients in $B(L)$ are defined in a standard way (see (2.1)): $C^1(G,B(L))$, $C^2(G^2,B(L))$ and $C^3(G^3,B(L))$ are the spaces of all weakly continuous functions (n-cochains, $n=1,2,3$) from G, $G\times G$ and $G\times G\times G$ to $B(L)$, respectively, while the coboundary operators $d^1=d^1_{\lambda,\lambda^\sharp}\colon C^1\to C^2$ and $d^2=d^2_{\lambda,\lambda^\sharp}\colon C^2\to C^3$ act by the rules

$$\begin{aligned} d^1(c)(g,h)&=\lambda(g)c(h)-c(gh)+c(g)\lambda^\sharp(h) \text{ for } c\in C^1,\\ d^2(c)(g,h,k)&=\lambda(g)c(h,k)-c(gh,k)+c(g,hk)-c(g,h)\lambda^\sharp(k) \text{ for } c\in C^2, \end{aligned} \tag{4.3}$$ 3.0

$g,h,k\in G$. Denote by $\mathcal{Z}^n(\lambda,\lambda^\sharp)=\ker d^n$ the set of all n-cocycles. For a (λ,U)-cocycle ξ, direct calculations show that the map $\xi\xi^\sharp\colon (g,h)\to\xi(g)\xi^\sharp(h)$ from $G\times G$ to $B(L)$ belongs to $\mathcal{Z}^2(\lambda,\lambda^\sharp)$.

Dn **Definition 4.1** *A (λ,U)-cocycle ξ is **neutral** if $-\xi\xi^\sharp\in$ im $d^1_{\lambda,\lambda^\sharp}$, i.e., there is a cochain $\gamma\in C^1(G,B(L))$ called a **prechain** of ξ such that*

$$-\xi(g)\xi^\sharp(h)=(d^1_{\lambda,\lambda^\sharp}\gamma)(g,h)=\lambda(g)\gamma(h)-\gamma(gh)+\gamma(g)\lambda^\sharp(h) \tag{4.4}$$ 3.1e

(*we put minus in the left hand side for convenience*). *Clearly, γ is defined up to a summand which is a $(\lambda,\lambda^\sharp)$-cocycle. Denote by $\mathcal{Z}^1_\nu(\lambda,U)$ the subset in $\mathcal{Z}^1(\lambda,U)$ consisting of all neutral cocycles.*

The study of neutral cocycles is motivated by applications to the theory of representations in Pontryagin spaces (see [Is3]).

It can be shown that $\gamma\in C^1(G,B(L))$ is a prechain of a (λ,U)-cocycle, for some unitary representations U, if and only if the 2-coboundary $\beta=-d^1_{\lambda,\lambda^\sharp}\gamma$ is completely positive:

$$\sum_{i,j}\beta(g_i,g_j^{-1})t_i\overline{t_j}\ge 0,\ \text{for all } g_1,..,g_n\in G \text{ and } t_1,..,t_n\in\mathbb{C}.$$

The "only if" part is straightforward while the "if" part can be proved similarly to its analogue for *-algebras established in [KS, Theorem 21.22]. It follows that the set of all prechains of neutral (λ,U)-cocycles, for various unitary representations U, form a subcone in $C^1(G,B(L))$.

L3.2.1 **Lemma 4.2** (i) *If γ is a prechain of ξ then $\gamma^\sharp$ is also a prechain of ξ.*

(ii) *Each coboundary $\eta\in\mathcal{B}^1(\lambda,U)$ is neutral and $\xi+\eta\in\mathcal{Z}^1_\nu(\lambda,U)$ if $\xi\in\mathcal{Z}^1_\nu(\lambda,U)$.*

Proof. (i) Set $x=h^{-1}$, $y=g^{-1}$. Then $\xi(g)\xi^\sharp(h)=(\xi(h^{-1})\xi(g)^*)^*=(\xi(x)\xi^\sharp(y))^*$. Hence

$$\begin{aligned} -\xi(g)\xi^\sharp(h)&=-(\xi(x)\xi^\sharp(y))^*\overset{(4.4)}{=}(\lambda(x)\gamma(y)-\gamma(xy)+\gamma(x)\lambda^\sharp(y))^*\\ &=\lambda(y^{-1})\gamma(x)^*-\gamma(xy)^*+\gamma(y)^*\lambda(x)^*=\lambda(g)\gamma^\sharp(h)-\gamma^\sharp(gh)+\gamma^\sharp(g)\lambda^\sharp(h). \end{aligned}$$

(ii) As $\eta(g)=\lambda(g)T-TU(g)$ for some $T\in B(\mathfrak{H},L)$, we have $\eta^\sharp(g)=T^*\lambda^\sharp(g)-U(g)T^*$. Set $\gamma_\eta(g)=TU(g)T^*-\frac{1}{2}\lambda(g)TT^*-\frac{1}{2}TT^*\lambda^\sharp(g)$. It is easy to check that

$$-\eta(g)\eta^\sharp(h)=\lambda(g)\gamma_\eta(h)-\gamma_\eta(gh)+\gamma_\eta(g)\lambda^\sharp(h),\ \text{so that } \eta\in\mathcal{Z}^1_\nu(\lambda,U).$$

Set $\gamma_1(g) = \xi(g)T^* + T\xi^\sharp(g)$. As ξ is a (λ, U)-cocycle, we have from (2.2) and (4.2) that

$$(d^1_{\lambda,\lambda^\sharp}\gamma_1)(g,h) = \lambda(g)\gamma_1(h) - \gamma_1(gh) + \gamma_1(g)\lambda^\sharp(h) = \xi(g)\eta^\sharp(h) + \eta(g)\xi^\sharp(h).$$

By definition, $-\xi(g)\xi^\sharp(h) = (d^1_{\lambda,\lambda^\sharp}\gamma_0)(g,h)$, for some 1-cochain $\gamma_0 \in C^1(\lambda, \lambda^\sharp)$. Then

$$\begin{aligned} -(\xi(g)+\eta(g))(\xi(h)+\eta(h))^\sharp &= -\xi(g)\xi^\sharp(h) - (\xi(g)\eta^\sharp(h) + \eta(g)\xi^\sharp(h)) - \eta(g)\eta^\sharp(h) \\ &= (d^1_{\lambda,\lambda^\sharp}\gamma_0)(g,h) - (d^1_{\lambda,\lambda^\sharp}\gamma_1)(g,h) + (d^1_{\lambda,\lambda^\sharp}\gamma_\eta)(g,h). \end{aligned}$$

Setting $\gamma = \gamma_0 - \gamma_1 + \gamma_\eta$, we obtain that $\xi + \eta \in \mathcal{Z}^1_\nu(\lambda, U)$. ■

It follows from Lemma 4.2(i) that one can always choose a prechain γ satisfying

$$\gamma = \gamma^\sharp, \text{ that is, } \gamma(g^{-1}) = \gamma(g)^* \text{ for all } g \in G. \tag{4.5}$$
$\boxed{\texttt{3.5e}}$

Denote by $\mathcal{H}^1_\nu(\lambda, U)$ the image of $\mathcal{Z}^1_\nu(\lambda, U)$ in $\mathcal{H}^1(\lambda, U)$. It completely determines $\mathcal{Z}^1_\nu(\lambda, U)$; we call it the *set of neutral cohomologies* of G. Our aim is to find conditions for $\mathcal{H}^1_\nu(\lambda, U) \neq 0$ and, more generally, to describe properties of $\mathcal{H}^1_\nu(\lambda, U)$.

In the next two subsections we consider neutral cohomologies in the case when $\lambda = \iota$ – the trivial representation on a one-dimensional space $L = \mathbb{C}u$. This case is "classical" (see [G]), but in our situation the general case cannot be reduced to this one because λ can be non-unitary.

It follows from (2.2) and (3.8) that ξ is a (ι, U)-cocycle if and only if

$$\xi(g) = r(g) \otimes u, \text{ where } r(g) \in \mathfrak{H},\ r(e) = 0 \text{ and } r(gh) = r(h) + U(h)^* r(g), \tag{4.6}$$
$\boxed{\texttt{3.24'}}$

for $g, h \in G$. If ξ is a coboundary then there is $T \overset{(3.8)}{=} w \otimes u$, for some $w \in \mathfrak{H}$, such that

$$\xi(g) = T - TU(g) = r(g) \otimes u, \text{ where } r(g) = w - U^*(g)w, \text{ for } g \in G. \tag{4.7}$$
$\boxed{\texttt{3.35}}$

By (4.2), $\xi^\sharp(h) = \xi(h^{-1})^* = (r(h^{-1}) \otimes u)^* \overset{(3.9)}{=} u \otimes r(h^{-1})$, so that

$$\xi(g)\xi^\sharp(h) = (r(g) \otimes u)(u \otimes r(h^{-1})) \overset{(3.9)}{=} (r(h^{-1}), r(g))(u \otimes u) = (r(h^{-1}), r(g))\mathbf{1}_L.$$

Let $\gamma \in C^1(G, B(L))$. Then $\gamma(g) \in B(L, L)$, for $g \in G$, so that $\gamma(g) = \phi(g)\mathbf{1}_L$, where ϕ is a complex-valued continuous function on G. As $\lambda = \lambda^\sharp = \iota$, we have from (4.4)

$$(d^1_{\lambda,\lambda^\sharp}\gamma)(g,h) = \gamma(h) - \gamma(gh) + \gamma(g) = (\phi(h) - \phi(gh) + \phi(g))\mathbf{1}_L.$$

Thus a cocycle $\xi = r \otimes u$ is neutral if and only if there is a complex-valued function ϕ on G such that

$$(r(h^{-1}), r(g)) = -\phi(h) + \phi(gh) - \phi(g), \quad \phi(e) = 0,\ \phi(g^{-1}) \overset{(4.5)}{=} \overline{\phi(g)} \tag{4.8}$$
$\boxed{\texttt{3.36}}$

for all $g, h \in G$. Then $\gamma = \phi\mathbf{1}_L$ is a prechain.

Our study is divided in two complementary parts: 1) when $U = I$ is the trivial representation of G on $\mathfrak{H}$, so that the χ_e-eigenspace $\mathfrak{H}^{\chi_e} = \mathfrak{H}$, and 2) when U has no trivial subrepresentations, i.e., $\mathfrak{H}^{\chi_e} = 0$. It might seem that the first part is trivial. However, this is not true; the description of neutral 1-cocycles and the corresponding 1-prechains in this case is quite complicated. In particular, from these examples we will see that $\mathcal{H}^1_\nu(\lambda, U)$ does not need to constitute a subgroup of $\mathcal{H}^1(\lambda, U)$.

4.2 Neutral (ι,I)-cocycles

Let G be a connected, locally compact group and $U = I$ be the trivial representation of G on a Hilbert space $\mathfrak{H}$. Set $\dim \mathfrak{H} = m \le \infty$. We fix an orthonormal basis in $\mathfrak{H}$ and realize operators from $\mathbb{R}^k$, $k < \infty$, to $\mathfrak{H}$ as complex $m \times k$ (infinite, if $m = \infty$) matrices from $M_{m\times k}(\mathbb{C})$.

In this setting all (ι, I)-coboundaries are zero and each (ι, I)-cocycle has form

$$\xi(g) \overset{(4.6)}{=} r(g) \otimes u, \text{ where } r(g) \in \mathfrak{H}, \text{and } r(gh) = r(g) + r(h).$$

In particular $r(e) = 0$ and $r(g^{-1}) = -r(g)$. It follows from Corollary 2.2 that there is a linear map β: $\mathbb{R}^{n_G} \to \mathfrak{H}$ such that $r(g) = \beta(\omega(g))$, where $\omega = \omega_G : G \to G/G_0 \cong \mathbb{R}^{n_G}$. Thus

$$r(g) = A\omega(g) \text{ for } g \in G, \tag{4.9}$$ 6.2N

where $A \in M_{m\times n_G}(\mathbb{C})$. The set $r(G) = \{r(g)\}_{g\in G} = A\mathbb{R}^{n_G}$ is a real linear subspace of $\mathfrak{H}$.

Let ξ be neutral and $\gamma = \phi(g)\mathbf{1}_L$ be its prechain. Setting $h = g^{-1}$ in (4.8), we have $0 = \phi(e) = \phi(g) + (r(g), r(g)) + \overline{\phi(g)}$. Hence there exists a real-valued continuous function ε on G such that

$$\phi(g) = -\|r(g)\|^2/2 + i\varepsilon(g). \tag{4.10}$$ 6.5eN

Substituting (4.10) in (4.8) and taking into account (4.9) and the fact that

$$\operatorname{Im}(A^*A\omega(h), \omega(g)) = -\operatorname{Im}(A^*A\omega(g), \omega(h)) \text{ for } g, h \in G, \tag{4.11}$$ 3.40

we have

$$\varepsilon(gh) = \varepsilon(g) + \varepsilon(h) + \operatorname{Im}(A^*A\omega(g), \omega(h)) \text{ for } g, h \in G. \tag{4.12}$$ 6.5N

If a function ε_0 is some solution of (4.12), then all solutions have the form $\varepsilon = \varepsilon_0 + \widetilde{\varepsilon}$, where $\widetilde{\varepsilon}$ satisfies $\widetilde{\varepsilon}(gh) = \widetilde{\varepsilon}(g) + \widetilde{\varepsilon}(h)$. By Corollary 2.2, $\widetilde{\varepsilon}(g) = (\zeta, \omega(g))$ for some $\zeta \in \mathbb{R}^{n_G}$. Thus ϕ has form

$$\phi(g) = -\|A\omega(g)\|^2/2 + i(\zeta, \omega(g)) + i\varepsilon_0(g). \tag{4.13}$$ form

It follows from (4.11) and (4.12) that

$$\varepsilon(gh) - \varepsilon(hg) = 2\operatorname{Im}(A\omega(g), A\omega(h)). \tag{4.14}$$ 3.41

As $\omega(g^{-1}) = -\omega(g)$, we have

$$0 \overset{(4.12)}{=} \varepsilon(e) = \varepsilon(gg^{-1}) \overset{(4.12)}{=} \varepsilon(g) + \varepsilon(g^{-1}) - \operatorname{Im}\|A\omega(g)\|^2 = \varepsilon(g) + \varepsilon(g^{-1}).$$

Hence

$$\varepsilon(g^{-1}) = -\varepsilon(g) \text{ for } g \in G. \tag{4.15}$$ 3.30

Recall that $G^{[1]} = K(G, G)$ is the closed subgroup of G generated by all commutators $[g, h] = ghg^{-1}h^{-1}$. By Lemma 2.1, $G^{[1]} \subseteq G_0$, so $\omega(z) = 0$ for $z \in G^{[1]}$. Hence (4.12) implies

$$\varepsilon(gz) = \varepsilon(zg) = \varepsilon(g) + \varepsilon(z), \text{ for } g \in G. \tag{4.16}$$ 1N

Also $\omega(gz) = \omega(g) + \omega(z) = \omega(g) = \omega(zg)$. Hence $\omega((zg)^{-1}) = -\omega(g)$, so that

$$\begin{aligned} \varepsilon(gz(zg)^{-1}) &\overset{(4.12)}{=} \varepsilon(gz) + \varepsilon((zg)^{-1}) + \operatorname{Im}(A^*A\omega(gz), \omega((zg)^{-1})) \\ &\overset{(4.15)}{=} \varepsilon(gz) - \varepsilon(zg) - \operatorname{Im}\|A\omega(g)\|^2 \overset{(4.16)}{=} 0. \end{aligned}$$

As $G^{[2]} = K(G, G^{[1]})$ is generated by commutators $gzg^{-1}z^{-1}$ and (see (4.16)) ε is additive on $G^{[2]}$,

$$\varepsilon|_{G^{[2]}} = 0. \tag{4.17}$$ 3.33

Let $S = A^*A = (s_{ij})_{i,j=1}^{n_G}$. As $\omega(g)$,$\omega(h) \in \mathbb{R}^{n_G}$, we have

$$\mathrm{Im}(A^*A\omega(g), \omega(h)) = (\mathrm{I}(A^*A)\omega(g), \omega(h)) \text{ where } \mathrm{I}(S) = (\mathrm{Im}\, s_{ij})_{i,j=1}^{n_G}. \tag{4.18}$$ 4.16

Hence (4.12) has form

$$\varepsilon(gh) = \varepsilon(g) + \varepsilon(h) + (\mathrm{I}(A^*A)\omega(g), \omega(h)) \text{ for } g, h \in G, \text{ and } \varepsilon(e) = 0. \tag{4.19}$$ 4.17

Summing up previous observations we have:

P3.11 **Proposition 4.3** (i) *Each (ι, I)-cocycle has form $\xi(g) = A\omega(g) \otimes e$, where $A \in M_{m\times n_G}(\mathbb{C})$ and $\omega = \omega_G$ (see* (2.7)*).*

(ii) *A (ι, I)-cocycle $\xi = A\omega \otimes e$ is neutral if and only if there is a continuous function $\varepsilon_0 : G \to \mathbb{R}$ satisfying* (4.12)*. The prechains γ of ξ have form $\gamma(g) = \phi(g)\mathbf{1}_L$, where*

$$\phi(g) = -\|A\omega(g)\|^2/2 + i(\zeta, \omega(g)) + i\varepsilon_0(g) \text{ and } \zeta \text{ is some vector in } \mathbb{R}^{n_G}. \tag{4.20}$$ 6.5fN

(iii) *If the matrix A^*A has real entries, then the cocycle $\xi = A\omega \otimes e$ is neutral and all prechains have the form* (4.20) *with $\varepsilon_0 = 0$.*

(iv) *If $G^{[2]} = G^{[1]}$ (for example, if G is commutative), then a cocycle $\xi = A\omega \otimes e$ is neutral if and only if the matrix A^*A has real entries.*

Proof. Part (i) and "only if" part of (ii) were proved above. Part "if" of (ii) can be proved by substituting (4.20) into (4.13).

(iii) If A^*A has real entries, i.e., $\mathrm{I}(A^*A) = 0$, then $\varepsilon_0 = 0$ satisfies (4.19) and (iii) follows from (ii).

(iv) By (iii), it suffices to prove the part "only if". Let $G^{[2]} = G^{[1]}$. Then $g^{-1}h^{-1}gh \in G^{[1]} = G^{[2]}$ for all $g, h \in G$. Therefore if $\xi = A\omega \otimes e$ is a neutral cocycle and $\varepsilon(g) = \mathrm{Im}\,\phi(g)$ (see (4.10), then

$$\varepsilon(gh) = \varepsilon(hg(g^{-1}h^{-1}gh)) \overset{(4.16)}{=} \varepsilon(hg) + \varepsilon(g^{-1}h^{-1}gh) \overset{(4.17)}{=} \varepsilon(hg).$$

By (4.14) and (4.18), $(\mathrm{I}(A^*A)\omega(g), \omega(h)) = 0$ for all $g, h \in G$. As $\omega(g)$ span $\mathbb{R}^{n_G}$, $\mathrm{I}(A^*A) = 0$. ■

To continue our study of neutral cocycles we need some general facts on locally compact, connected nilpotent groups of index 2. Let E be such a group, $Z = E^{[1]}$ and $H = E/Z$. Then H, Z are commutative and Z lies in the center of E. Let q: $E \to H$ be the quotient map.

By [M, Theorem 26], H is isomorphic to the direct product $\mathbb{R}^{n_H} \times F$ and Z to the direct product $\mathbb{R}^{n_Z} \times K$, where F and K are compact subgroups. Set

$$n = n_{_H},\ k = n_{_Z},\ \text{and let } \omega_{_H}\colon H \to \mathbb{R}^n,\ \omega_{_Z}\colon Z \to \mathbb{R}^k \tag{4.21}$$ 4.1

be the corresponding continuous epimorphisms with $F = \ker \omega_{_H}$, $K = \ker \omega_{_Z}$. Clearly

$$\omega_{_E}(a) = \omega_{_H}(q(a)) \text{ for each } a \in E. \tag{4.22}$$ omegaH

The epimorphism $q : E \to H$ has a locally bounded Borel right inverse $\rho : H \to E$ (see [Keh]). We are going to show that ρ can be chosen with some additional properties.

For an arbitrary right inverse ρ of q, set

$$x \diamond_\rho y = \rho(xy)^{-1}\rho(x)\rho(y) \text{ for } x, y \in H. \tag{4.23}$$ 3.47

Then $x \diamond_\rho y$ belongs to Z, since

$$q(x \diamond_\rho y) = q(\rho(xy))^{-1}q(\rho(x))q(\rho(y)) = (xy)^{-1}xy = e_H.$$

selection **Lemma 4.4** *There is a Borel locally bounded right inverse ρ of q such that the map $\varphi(x,y) = \omega_Z(x \diamond_\rho y)$ from $H \times H$ to $\mathbb{R}^k$ is "biadditive":*

$$\varphi(xz, y) = \varphi(x, y) + \varphi(z, y) \textit{ and } \varphi(x, yz) = \varphi(x, y) + \varphi(x, z) \textit{ for all } x, y, z \in H. \tag{4.24}$$ bilinear

Proof. Let us fix an arbitrary Borel locally bounded right inverse ρ_1 of q and set, for brevity, $x \diamond y := x \diamond_{\rho_1} y$. Since $x \diamond y$ belongs to the center of E,

$$\begin{aligned}((xy) \diamond z)^{-1}(x \diamond (yz)) &= \rho_1(z)^{-1}\rho_1(xy)^{-1}\rho_1(xyz)\rho_1(xyz)^{-1}\rho_1(x)\rho_1(yz)\\ &= \rho_1(z)^{-1}\rho_1(xy)^{-1}\rho_1(x)\rho_1(y)\rho_1(y)^{-1}\rho_1(yz) = \rho_1(z)^{-1}(x \diamond y)\rho_1(y)^{-1}\rho_1(yz)\\ &= \rho_1(z)^{-1}\rho_1(y)^{-1}\rho_1(yz)(x \diamond y) = (y \diamond z)^{-1}(x \diamond y).\end{aligned}$$

Therefore

$$(y \diamond z)((xy) \diamond z)^{-1}(x \diamond (yz))(x \diamond y)^{-1} = e. \tag{4.25}$$ 3.21

It follows that the map $\psi(x,y) = \omega_Z(x \diamond y)$ from $H \times H$ to $\mathbb{R}^k$ satisfies the condition

$$\psi(y, z) - \psi(xy, z) + \psi(x, yz) - \psi(x, y) = 0 \text{ for } x, y, z \in H. \tag{4.26}$$ 3.23

In other words, ψ is a Borel 2-cocycle of H with coefficients in $\mathbb{R}^k$, where the left and right actions of H on $\mathbb{R}^k$ are trivial (cf. (2.1)).

Denote by $\mathcal{H}^j_{bor}(H, \mathbb{R}^k)$ the Borel cohomologies and by $\mathcal{H}^j(H, \mathbb{R}^k)$ the continuous cohomologies. It is known [Wig, Theorem 2] (see also [MooreIII], page 32) that the natural homomorphism of $\mathcal{H}^j(H, \mathbb{R}^k)$ to $\mathcal{H}^j_{bor}(H, \mathbb{R}^k)$ is an isomorphism.

As $\mathcal{H}^j(F, \mathbb{R}^k) = 0$ for all $j \in \mathbb{N}$ (see [G, Corollary 3.2.1]), it follows from Proposition 1.8.1 [G] that $\mathcal{H}^2(\mathbb{R}^n \times F, \mathbb{R}^k) = \mathcal{H}^2(\mathbb{R}^n, \mathbb{R}^k)$ and the isomorphism is realized by the restriction of a chain to $\mathbb{R}^n$-components. This means that $\psi(x, y) = \widehat{\psi}(\omega_H(x), \omega_H(y)) + d^1v(x, y)$ for all $x, y \in H$, where $v \in C^1(H, \mathbb{R}^k)$ and $\widehat{\psi} \in \mathcal{Z}^2(\mathbb{R}^n, \mathbb{R}^k)$ is the restriction of ψ to $\mathbb{R}^n \times \mathbb{R}^n$ ($\mathbb{R}^n$ is identified with $\mathbb{R}^n \times e_F \subset H$).

It is known that $\mathcal{H}^2(\mathbb{R}^n, \mathbb{R}^k)$ is naturally isomorphic (see e.g. [G]) to the group L_a of all bilinear antisymmetric maps from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^{n_G}$: each class in $\mathcal{H}^2(\mathbb{R}^n, \mathbb{R}^k)$ contains a unique cocycle from L_a. Thus each cocycle in $\mathcal{Z}^2_{bor}(\mathbb{R}^n, \mathbb{R}^k)$ is equivalent to a unique bilinear antisymmetric map in L_a. Hence there is a Borel function u: $\mathbb{R}^n \to \mathbb{R}^k$ and a bilinear map B on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$B(r, t) = \widehat{\psi}(r, t) + d^1u(r, t) \text{ for } r, t \in \mathbb{R}^n. \tag{4.27}$$ 3.48

Therefore, setting $\eta(x) = u(\omega_H(x)) - v(x)$, we have that η is a Borel function on H and

$$\begin{aligned}\psi(x, y) &= \widehat{\psi}(\omega_H(x), \omega_H(y)) + d^1v(x, y) = B(\omega_H(x), \omega_H(y)) - d^1\eta(x, y)\\ &= B(\omega_H(x), \omega_H(y)) - \eta(x) + \eta(xy) - \eta(y).\end{aligned} \tag{4.28}$$ 4.35

As Z is isomorphic to the direct product $\mathbb{R}^k \times S$, where S is a compact group, the continuous homomorphism s: $r \to r \times e_S$ from $\mathbb{R}^k$ to Z is a right inverse of ω_Z: $\omega_Z(s(r)) = r$.

Now we set

$$\rho(x) = \rho_1(x)s(\eta(x)) \tag{4.29}$$

and as above $x \diamond_\rho y = \rho(xy)^{-1}\rho(x)\rho(y)$ for $x, y \in H$.

As $s(\eta(x)) \in Z$, it commutes with all $g \in E$, so that ρ is also a right inverse of q, because

$$q(\rho(x)) = q(\rho_1(x)s(\eta(x))) = q(\rho_1(x))q(s(\eta(x))) = q(\rho_1(x)) = x.$$

Furthermore

$$x \diamond_\rho y = (x \diamond y)s(\eta(xy))^{-1}s(\eta(x))s(\eta(y)) \text{ for } x, y \in H.$$

As $\omega_Z(s(\eta(x))) = \eta(x)$, we have, for $x, y \in H$,

$$\varphi(x, y) = \omega_Z(x \diamond_\rho y) = \psi(x, y) - \eta(xy) + \eta(x) + \eta(y) \overset{(4.28)}{=} B(\omega_H(x), \omega_H(y)). \tag{4.30}$$

As B is a bilinear map and $\omega_H(xy) = \omega_H(x) + \omega_H(y)$, the equality (4.24) is proved. ■

Corollary 4.5 *There are real-valued $n_H \times n_H$-matrices $T_1, ..., T_{n_Z}$ such that, for $x, y \in H$,*

$$\omega_Z(x \diamond_\rho y) = ((T_1\omega_H(x), \omega_H(y)), (T_2\omega_H(x), \omega_H(y)), .., (T_{n_Z}\omega_H(x), \omega_H(y))) \in \mathbb{R}^{n_Z}. \tag{4.31}$$

Proof. Follows from (4.30) and the fact that each bilinear functional on a real Euclidian space E can be written in the form $(x, y) \mapsto (Tx, y)$ where T is a linear operator on E. ■

For $a \in E$, set

$$\mathfrak{z}(a) = \rho(q(a))^{-1}a. \tag{4.32}$$

Since $q(\mathfrak{z}(a)) = q(a)^{-1}q(a) = e_H$, we have $\mathfrak{z}(a) \in Z$, so $\mathfrak{z}$ can be considered as a map from E to Z.

Let us illustrate our constructions with the following example.

Example 4.6 Let $k \in \mathbb{N}$. For $x, y \in \mathbb{R}^k$, set

$$x \boxtimes y = (x_1y_2, x_2y_3, ..., x_{k-1}y_{k-2}) \in \mathbb{R}^{k-1}. \tag{4.33}$$

The set $E = E_k := \mathbb{R}^k \oplus \mathbb{R}^{k-1}$ supplied with the operation

$$(x \oplus v)(y \oplus w) = (x + y) \oplus (v + w + x \boxtimes y) \tag{4.34}$$

is a nilpotent group of index 2. It is easy to check that

$$(x \oplus v)^{-1} = (-x) \oplus (x \boxtimes x - v).$$

Then $Z = 0 \oplus \mathbb{R}^{k-1}$, H is identified with $\mathbb{R}^k$ via the map $(x, v) + Z \mapsto x$ and $q : E \to H$ is given by $q(x, v) = x$. The special inverse ρ of q is the map $x \mapsto x \oplus 0$. Then $\rho(x)^{-1} = -x \oplus x \boxtimes x$ and

$$x \diamond_\rho y = \rho(x + y)^{-1}\rho(x)\rho(y) \overset{(4.34)}{=} ((-(x + y)) \oplus (x + y) \boxtimes (x + y))((x + y) \oplus x \boxtimes y) \overset{(4.34)}{=} 0 \oplus x \boxtimes y.$$

Clearly, $\omega_E(x \oplus v) = \omega_H(q(x \oplus v)) = x$ and $\omega_Z(0 \oplus v) = v$. Therefore

$$\omega_Z(x \diamond_\rho y) = x \boxtimes y. \tag{4.35}$$

Hence the matrices T_j, introduced in Corollary 4.5, are equal to $e_{j+1,j}$, $1 \le j \le k-1$, where by $(e_{pq} : 1 \le p, q \le k)$ we denote the usual matrix units in $M_k(\mathbb{R})$.

Note that $\rho(q(x \oplus v)) = x \oplus 0$ and $(\rho(q(x \oplus v)))^{-1} = (-x) \oplus (x \boxtimes x)$. Hence

$$\mathfrak{z}(x \oplus v) \overset{(4.32)}{=} ((-x) \oplus (x \boxtimes x))(x \oplus v) \overset{(4.34)}{=} (0, v) \quad \text{and} \quad \omega_Z(\mathfrak{z}(x \oplus v)) = v. \tag{4.36}$$

zeta

Now we come to the general case of a connected locally compact group G. If $G^{[2]} \neq G^{[1]}$, then the group $E := G/G^{[2]}$ is nilpotent of index 2. By p we denote the canonical epimorphism of G onto E. We set as above $Z = E^{[1]}$, $H = E/Z$ and preserve all notations introduced in Lemma 4.4 and Corollary 4.5. In particular, by ρ we denote the special inverse to the quotient map $q : E \to H$.

Note that

$$n_{_H} = n_{_E} = n_{_G},\ \omega(g) = \omega_{_G}(g) = \omega_E(p(g)) \text{ and } n_{_Z} = n_{_{G^{[1]}}},\ \text{since } Z = p(G^{[1]}). \tag{4.37}$$

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We call the matrices $T_1, ..., T_{n_Z}$ constructed for E in Corollary 4.5 *the standard suit of matrices of* G. Returning to our study of neutral cocycles we obtain now their description in the general case.

den-nil **Theorem 4.7** *Let G be a connected, locally compact group and $G^{[2]} \neq G^{[1]}$. Let*

$$E = G/G^{[2]},\ Z = E^{[1]} \cong G^{[1]}/G^{[2]},\ H = E/Z \ \textit{and}\ n := n_{_H},\ k := n_{_Z}.$$

Let ι and I be the identity representations of G on $L = \mathbb{C}u$ and $\mathfrak{H} = \mathbb{C}^m$, respectively. Then

(i) *A (ι, I)-cocycle $\xi(g) = A\omega(g) \otimes u$ is neutral if and only if the $n \times n$ matrix* $\mathrm{I}(A^*A)$ *(see* (4.18)*) is a linear combination of $T_j - T_j^*$, where $\{T_j\}_{j=1}^k$ is the standard suit of matrices of G :*

$$\mathrm{I}(A^*A) = \frac{1}{2}\sum_{j=1}^{k} \sigma_j (T_j - T_j^*) \ \textit{for some}\ \sigma_1, ..., \sigma_k \in \mathbb{R}; \tag{4.38}$$

combin

(ii) *If condition* (4.38) *is satisfied then all prechains of ξ have form $\gamma(g) = \phi(g)\mathbf{1}_L$, where*

$$\phi(g) = -\|A\omega(g)\|^2/2 + i(\zeta, \omega(g)) + i(\sigma, \omega_{_Z}(\mathfrak{z}(p(g)))) - \frac{1}{2}(\sigma, \omega_{_Z}(q(p(g)) \diamond q(p(g)))), \tag{4.39}$$

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$\sigma = (\sigma_1, ..., \sigma_k)$, σ_j are coefficients in (4.38)*, $\zeta \in \mathbb{R}^n$ and the map $\mathfrak{z} : E \to Z$ is defined in* (4.32)*.*

Proof. If ε is a solution of (4.12), then $\varepsilon(gv) \overset{(4.16)}{=} \varepsilon(g) + \varepsilon(v) \overset{(4.17)}{=} \varepsilon(g)$, for $g \in G$ and $v \in G^{[2]}$. Hence the number $\varepsilon(g)$ depends on $p(g)$ only, and we may define a function $\delta : E \to \mathbb{R}$ by setting

$$\delta(p(g)) = \varepsilon(g). \tag{4.40}$$

factor

It follows from (4.37) that δ satisfies the condition similar to (4.12)

$$\delta(ab) = \delta(a) + \delta(b) + \ \mathrm{Im}(A^*A\omega_{_E}(a), \omega_{_E}(b)) \text{ for } a, b \in E. \tag{4.41}$$

6.51

As $Z \subseteq G_0$, we have $\omega_E(z) = \{0\}$ for $z \in Z$. Therefore, by (4.41),

$$\delta(uz) = \delta(u) + \delta(z) \text{ for } u, z \in Z. \tag{4.42}$$

6.53

By Corollary 2.2, there is a linear map $\beta\colon \mathbb{R}^k \to \mathbb{R}$ such that $\delta(z) = \beta(\omega_Z(z))$; equivalently

$$\delta(z) = (\sigma, \omega_Z(z)) \text{ for some } \sigma = (\sigma_1, ..., \sigma_k) \in \mathbb{R}^k \text{ and all } z \in Z. \tag{4.43} \qquad \boxed{\texttt{6.52}}$$

Let $a = \rho(x)$, $b = \rho(y)$ in (4.41), for $x, y \in H$. As $x \diamond y \in Z$, we have $\omega_E(x \diamond y) = 0$. Then

$$\delta(\rho(x)\rho(y)) \overset{(4.41)}{=} \delta(\rho(x)) + \delta(\rho(y)) - \mathrm{Im}(A\omega_E(\rho(x)), A\omega_E(\rho(y))),$$

$$\delta(\rho(x)\rho(y)) \overset{(4.23)}{=} \delta(\rho(xy)(x \diamond y)) = \delta(\rho(xy)) + \delta(x \diamond y).$$

Hence

$$\delta(\rho(x)) + \delta(\rho(y)) - \mathrm{Im}(A\omega_E(\rho(x)), A\omega_E(\rho(y))) = \delta(\rho(xy)) + \delta(x \diamond y). \tag{4.44} \qquad \boxed{\texttt{3.50}}$$

Invert x and y in (4.44) and subtract the two equalities. It follows from (4.11) and (4.42) that

$$2\,\mathrm{Im}(A\omega_E(\rho(x)), A\omega_E(\rho(y))) = \delta(y \diamond x) - \delta(x \diamond y) \overset{(4.43)}{=} (\sigma, \omega_Z(y \diamond x) - \omega_Z(x \diamond y)). \tag{4.45} \qquad \boxed{\texttt{rel}}$$

From Corollary 4.5 we obtain that

$$\begin{aligned} \omega_Z(y \diamond x) - \omega_Z(x \diamond y) &= ((T_1\omega_H(x), \omega_H(y)), ..., (T_k\omega_H(x), \omega_H(y))) \\ &\quad - ((T_1\omega_H(y), \omega_H(x)), ..., (T_k\omega_H(y), \omega_H(x))) \\ &= (((T_1 - T_1^*)\omega_H(x), \omega_H(y)), ..., ((T_k - T_k^*)\omega_H(x), \omega_H(y))). \end{aligned} \tag{4.46} \qquad \boxed{\texttt{1}}$$

By (4.22), $\omega_E(\rho(x)) = \omega_H(q(\rho(x))) = \omega_H(x)$. Hence (4.45) can be rewritten in the form

$$2(\mathrm{I}(A^*A)\omega_H(x), \omega_H(y)) = \sum_{j=1}^{k} \sigma_j((T_j - T_j^*)\omega_H(x), \omega_H(y)).$$

Taking into account that ω_H maps H onto $\mathbb{R}^n$, we get (4.38).

Conversely, let A satisfy condition (4.38). We have to prove that there is a solution ε of the equation (4.41). For $\sigma = (\sigma_1, ..., \sigma_k) \in \mathbb{R}^k$, we set

$$\delta_0(a) = (\sigma, \omega_Z(\mathfrak{z}(a))) - \frac{1}{2}(\sigma, \omega_Z(q(a) \diamond q(a))) \text{ for } a \in E. \tag{4.47} \qquad \boxed{\texttt{3.26}}$$

As Z belongs to the center of E,

$$\mathfrak{z}(ab) = \rho(q(a)q(b))^{-1}ab \overset{(4.23)}{=} (q(b) \diamond q(a))\mathfrak{z}(a)\mathfrak{z}(b) \tag{4.48} \qquad \boxed{\texttt{3.24}}$$

for $a, b \in E$. Hence $\omega_Z(\mathfrak{z}(ab)) \overset{(4.48)}{=} \omega_Z(q(b) \diamond q(a)) + \omega_Z(\mathfrak{z}(a)) + \omega_Z(\mathfrak{z}(b))$. By Lemma 4.4,

$$\omega_Z(q(ab) \diamond q(ab)) = \omega_Z(q(a) \diamond q(a)) + \omega_Z(q(a) \diamond q(b)) + \omega_Z(q(b) \diamond q(a)) + \omega_Z(q(b) \diamond q(b))$$

It follows that

$$\begin{aligned} \omega_Z(\mathfrak{z}(ab))) - \frac{1}{2}\omega_Z(q(ab) \diamond q(ab)) &= \omega_Z(\mathfrak{z}(a)) + \omega_Z(\mathfrak{z}(b)) + \frac{1}{2}\omega_Z(q(b) \diamond q(a)) \\ &\quad - \frac{1}{2}\omega_Z(q(a) \diamond q(a)) - \frac{1}{2}\omega_Z(q(a) \diamond q(b)) - \frac{1}{2}\omega_Z(q(b) \diamond q(b)). \end{aligned}$$

Therefore, as $\omega_{E}(a) \overset{(4.22)}{=} \omega_{H}(q(a))$,

$$\begin{aligned}
\delta_0(ab) &= (\sigma, \omega_{Z}(\mathfrak{z}(ab))) - \frac{1}{2}(\sigma, \omega_{Z}(q(ab) \diamond q(ab))) \\
&\overset{(4.47)}{=} \delta_0(a) + \delta_0(b) + \frac{1}{2}(\sigma, \omega_{Z}(q(b) \diamond q(a))) - \frac{1}{2}(\sigma, \omega_{Z}(q(a) \diamond q(b))) \\
&\overset{(4.46)}{=} \delta_0(a) + \delta_0(b) + \frac{1}{2}\sum_j \sigma_j((T_j - T_j^*)\omega_{H}(q(a)), \omega_{H}(q(b)) \\
&\overset{(4.38)}{=} \delta_0(a) + \delta_0(b) + (\mathrm{I}(A^*A)\omega_{E}(a), \omega_{E}(b)) \overset{(4.18)}{=} \delta_0(a) + \delta_0(b) + \mathrm{Im}(A\omega_{E}(a), A\omega_{E}(b)).
\end{aligned}$$

Thus δ_0 satisfies (4.41). To complete the proof of part (i), it remains to set $\varepsilon_0(g) = \delta_0(p(g))$ and to take into account that $\omega_{E}(p(g)) \overset{(4.37)}{=} \omega_{G}(g)$.

Part (ii) follows from (4.13) and our construction of ε_0. ■

dobav **Remark 4.8** If $G^{[2]} = G^{[1]}$ then $Z = \{e_Z\}$ and $E = H$, so that $n_Z = 0$ and $\omega_Z(x \diamond y) = 0$ for all $x, y \in H$. Thus the standard suit of operators is zero and Proposition 4.3(iv) can be considered as a partial case of Theorem 4.7(i).

Now we apply the above result to an important example: the group $G = \mathcal{T}_n$ of all $n \times n$ real upper triangular matrices $g = (g_{ij})$ with identity on the main diagonal. We denote by $\widehat{g}_k$ the diagonals of a matrix $g \in G$:

$$\widehat{g}_k = (g_{1,1+k}, ..., g_{n-k,n}) \in \mathbb{R}^{n-k} \text{ for } k = 1, ..., n-1.$$

3.7 **Corollary 4.9** *Let $G = \mathcal{T}_n$. Let ι and I be the identity representations of G on $L = \mathbb{C}u$ and $\mathfrak{H} = \mathbb{C}^m$. Then each (ι, I)-cocycle has form $\xi(g) = A\widehat{g}_1 \otimes u$ where $A \in M_{m\times(n-1)}(\mathbb{C})$. The cocycle is neutral if and only if the $(n-1)\times(n-1)$ matrix $S = A^*A = (s_{ij})$ satisfies the condition*

$$\mathrm{Im}\, s_{ij} = 0, \ \text{when } |i-j| > 1. \tag{4.49}$$ rN

If (4.49) holds, the corresponding prechains have the form $\gamma(g) = \phi(g)\mathbf{1}_L$, where

$$\phi(g) = \phi_{S,\sigma,\zeta}(g) = -\frac{1}{2}(S\widehat{g}_1, \widehat{g}_1) + i(\zeta, \widehat{g}_1) + i(\sigma, \widehat{g}_2 - \frac{1}{2}\widehat{g}_1 \boxtimes \widehat{g}_1), \tag{4.50}$$ 8N

$\sigma = (\sigma_1, ..., \sigma_{n-2}) \in \mathbb{R}^{n-2}$, with $\sigma_i = 2s_{i,i+1}$, and ζ is arbitrary vector in $\mathbb{R}^{n-1}$.

Proof. It is easy to see that

$$G^{[1]} = K(G,G) = \{g \in G\colon \widehat{g}_1 = 0\}, \ G^{[2]} = \{g \in G\colon \widehat{g}_1 = \widehat{g}_2 = 0\}, \ G/G^{[1]} \cong \mathbb{R}^{n-1} \text{ and } n_{G} = n-1.$$

Furthermore, the group $E = G/G^{[2]}$ is naturally identified as a set with $\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-2}$ via the map $p\colon g \mapsto \widehat{g}_1 \oplus \widehat{g}_2$. Direct calculations show that $\widehat{(gh)}_1 = \widehat{g}_1 + \widehat{h}_1$ and $\widehat{(gh)}_2 = \widehat{g}_2 + \widehat{h}_2 + \widehat{g}_1 \boxtimes \widehat{g}_1$. It follows that the map p is an isomorphism of E on the group E_{n-1} considered in Example 4.6. We may identify p with the standard epimorphism from G to E, that is, $p(g) = \widehat{g}_1 \oplus \widehat{g}_2$.

By (4.37), $\omega_G(g) = \omega_E(p(g)) = \widehat{g}_1$, so $\xi(g) = A\widehat{g}_1$. By Example 4.6, the standard suit of operators for E is $\{e_{j+1,j}\colon 1 \le j \le n-2\} \subset M_{n-1}(\mathbb{R})$. Hence, by Theorem 4.7, ξ is neutral if and only if $\mathrm{Im}(S)$ is a real linear combination of matrices $e_{j,j+1} - e_{j+1,j}$. Since S is selfadjoint, this is equivalent to the condition (4.49). Note that the coefficients of this linear combinations (the coefficients σ_j in (4.38)) are expressed via the entries of S by the formula $\sigma_j = 2\,\mathrm{Im}\, s_{j,j+1}$.

To deduce the last statement of the corollary from (4.47) we have to calculate $\omega_Z(q(p(g))\diamond q(p(g))$ and $\omega_Z(\mathfrak{z}(p(g)))$. By (4.35), $\omega_Z(q(a)\diamond q(a)) = x \boxtimes x$ for $a = x \oplus v \in E$. It follows that

$$\omega_Z(q(p(g)) \diamond q(p(g))) = \widehat{g}_1 \boxtimes \widehat{g}_1.$$

Similarly by (4.36), $\omega_Z(\mathfrak{z}(p(g))) = \widehat{g}_2$. It remains to substitute this in (4.47). ∎

Remark 4.10 The functions $\phi_{S,\sigma,\zeta}$ in (4.50) form a subcone in $C^1(G) \cong C_{\mathbb{C}}(G)$. Indeed, let (S^1,σ^1,ζ^1) and (S^2,σ^2,ζ^2) satisfy (4.49) and $\phi_{S^1,\sigma^1,\zeta^1}$, $\phi_{S^2,\sigma^2,\zeta^2}$ be the corresponding prechains. For $\lambda_1, \lambda_2 > 0$, set $S = \lambda_1 S^1 + \lambda_2 S^2$, $\sigma = \lambda_1\sigma^1 + \lambda_2\sigma^2$ and $\zeta = \lambda_1\zeta^1 + \lambda_2\zeta^2$. Then (S,σ,ζ) satisfies (4.49) and, by (4.50), $\phi_{S,\sigma,\zeta} = \lambda_1\phi_{S^1,\sigma^1,\zeta^1} + \lambda_2\phi_{S^2,\sigma^2,\zeta^2}$ is a prechain of a neutral (ι, I)-cocycle.

We shall now consider two particular cases: $n = 3$ and $n = 4$.

Let $n = 3$. Then $G = H_3(\mathbb{R})$ is the real Heisenberg group. For each $m \times 2$ matrix A, $m = 1, 2$, S is 2×2 matrices and condition (4.49) holds. Thus (ι, I)-cocycles $\xi(g) = A\widehat{g}_1 \otimes u$ are neutral for all matrices A and $\mathcal{H}^1_\nu(\iota, I) = \mathcal{H}^1(\iota, I) \neq 0$.

Let $n = 4$. Then A are $m \times 3$ matrices. For $m = 1$, $A = (a_{11}, a_{12}, a_{13})$ and (4.49) reduces to the condition $s_{13} = \overline{a_{11}}a_{13} \in \mathbb{R}$. The matrices $A_1 = (1, 0, 1)$, $A_2 = (i, 0, 2i)$ satisfy (4.49) while $A_1 + A_2$ does not. This implies that the sum of two neutral cocycles need not be neutral. Thus

Corollary 4.11 *$\mathcal{H}^1_\nu(\iota, I)$ does not, in general, form a subspace in $\mathcal{H}^1(\iota, I)$.*

4.3 Density of neutral cocycles

In this subsection we study the neutral (λ, U)-cocycles for representations λ and U which are spectrally related in a more flexible way: U *weakly contains* λ, that is, all matrix functions $g \mapsto (\lambda(g)x, x)$, $x \in L$, of λ belong to the closure in $C(G)$ with respect to the topology of uniform convergence on compacts of the subspace generated by the matrix functions of U.

Let $\dim L = 1$ and ι be the trivial representation of G on L. Let U be a unitary representation of G on $\mathfrak{H}$ without fixed vectors, i.e., $\mathfrak{H}^{\chi_e} = \{0\}$, where χ_e is the trivial character. Then (see [G, Corollary III.2.3]) U weakly contains the representation ι if and only if there are unit vectors $\{e_n\}_{n=1}^\infty$ in $\mathfrak{H}$ such that

$$\|U(g)e_n - e_n\| \to 0 \text{ uniformly on each compact subset of } G. \qquad (4.51)$$ 2.11

Clearly, χ_e and U are not spectrally disjoint.

Recall that a topological group G is σ-compact, if it has a sequence of compact subsets

$$K_1 \subset K_2 \subset ... \subset K_n \subset ... \text{ such that } G = \cup_{n=1}^\infty K_n. \qquad (4.52)$$ 2.1n

For example, countable discrete groups and connected locally compact groups are σ-compact.

P.2n **Proposition 4.12** *Let $\{U_n\}_{n=1}^\infty$ be unitary representations of a σ-compact group G on $\{\mathfrak{H}_n\}_{n=1}^\infty$ and let $\mathfrak{H}_n^{\chi_e} = \{0\}$. Suppose that, for each N, the representation $\oplus_{n=N}^\infty U_n$ weakly contains ι. Let $U = \oplus_{n=1}^\infty U_n$ and $\mathfrak{H} = \oplus_{n=1}^\infty \mathfrak{H}_n$. Then G has a neutral (ι, U)-cocycle which is not a coboundary.*

Proof. Let $\{K_n\}_{n=1}^{\infty}$ satisfy (4.52). The projections P_N on $\oplus_{n=1}^{N}\mathfrak{H}_n$ commute with U. As U weakly contains ι, there is a unit vector $z_1 \in \mathfrak{H}$ such that $\sup_{g\in K_1}\|U(g)z_1 - z_1\| < 2^{-1}$. Let N_1 be such that $\|z_1 - P_{N_1}z_1\| < 2^{-1}$. Then $\|P_{N_1}z_1\| > 2^{-1}$. Set $y_1 = P_{N_1}z_1/\|P_{N_1}z_1\|$. Then $\|y_1\| = 1$,

$$\sup_{g\in K_1}\|U(g)y_1 - y_1\| = \sup_{g\in K_1}\|P_{N_1}(U(g)z_1 - z_1)\|/\|P_{N_1}z_1\| < 2^{-1}/\|P_{N_1}z_1\| < 1.$$

As the representation $U' = \oplus_{n=N_1+1}^{\infty}U_n$ also weakly contains ι, there is $z_2 \in \oplus_{n=N_1+1}^{\infty}\mathfrak{H}_n$, $\|z_2\| = 1$, such that $\sup_{g\in K_2}\|U'(g)z_2 - z_2\| < 2^{-2}$. Let $N_2 > N_1$ be such that $\|z_2 - P_{N_2}z_2\| < 2^{-1}$. As above, $y_2 = P_{N_2}z_2/\|P_{N_2}z_2\|$ is a unit vector, $\sup_{g\in K_2}\|U(g)y_2 - y_2\| < 2^{-1}$ and $y_2 \in \oplus_{n=N_1+1}^{N_2}\mathfrak{H}_n$.

Arguing in this way, we get unit vectors $\{y_i\}_{i=1}^{\infty}$ such that

$$\sup_{g\in K_i}\|U(g)y_i - y_i\| < 2^{1-i} \text{ and } y_i = x_{N_{i-1}+1}\oplus \dots \oplus x_{N_i}, \text{ for some } x_n \in \mathfrak{H}_n. \tag{4.53} \quad \boxed{\texttt{2.2f}}$$

Then

$$r(g) = \oplus_{i=1}^{\infty}(y_i - U^*(g)y_i) = \oplus_{n=1}^{\infty}(x_n - U_n^*(g)x_n) \tag{4.54} \quad \boxed{\texttt{3.32}}$$

belongs to $\mathfrak{H}$ for each $g \in G$, since, for $g \in K_m$,

$$\|r(g)\|^2 = \sum_{i=1}^{\infty}\|y_i - U^*(g)y_i\|^2 \overset{(4.53)}{\leq} \sum_{i=1}^{m-1}\|y_i - U^*(g)y_i\|^2 + \sum_{i=m}^{\infty}2^{2-2i} < \infty.$$

Similarly, one can show that r is continuous on G. Since

$$\begin{aligned} r(h) + U^*(h)r(g) &= \oplus_{n=1}^{\infty}(x_n - U_n^*(h)x_n) + U^*(h)\oplus_{n=1}^{\infty}(x_n - U_n^*(g)x_n) \\ &= \oplus_{n=1}^{\infty}(x_n - U_n^*(gh)x_n) = r(gh), \end{aligned}$$

we have (see (4.6)) that $\xi = r \otimes u$ is a (ι, U)-cocycle. To show that ξ is not a coboundary we have to establish (see (4.7)) that there does not exist $w \in \mathfrak{H}$ such that $r(g) = w - U^*(g)w$ for all $g \in G$. Assume, to the contrary, that such $w = \oplus_{n=1}^{\infty}w_n$, where $w_n \in \mathfrak{H}_n$, exists. Then, by (4.54), $x_n - U_n^*(g)x_n = w_n - U_n^*(g)w_n$, so that $U_n^*(g)(x_n - w_n) = x_n - w_n$ for all n and g. As $\mathfrak{H}_n^{\chi_e} = \{0\}$, we have $x_n = w_n$ for all n, which is impossible, since $\infty \neq \|w\|^2 = \sum_{n=1}^{\infty}\|x_n\|^2 = \sum_{i=1}^{\infty}\|y_i\|^2 = \infty$.

To see that ξ is neutral, consider $\gamma(g) = \phi(g)(u \otimes u)$, where

$$\phi(g) = \sum_{n=1}^{\infty}(U_n(g)x_n - x_n, x_n) = \sum_{i=1}^{\infty}(U(g)y_i - y_i, y_i) \text{ for } g \in G.$$

By (4.53), ϕ is a well defined continuous map from G to $\mathfrak{H}$, $\phi(e) = 0$, $\phi(g^{-1}) = \overline{\phi(g)}$ and

$$-\phi(h) + \phi(gh) - \phi(g) = \sum_{n=1}^{\infty}((\mathbf{1} - U_n(h) + U_n(gh) - U_n(g))x_n, x_n).$$

As

$$\begin{aligned} (r(h^{-1}), r(g)) &= (\oplus_{n=1}^{\infty}(x_n - U_n^*(h^{-1})x_n), \oplus_{n=1}^{\infty}(x_n - U_n^*(g)x_n)) \\ &= \sum_{n=1}^{\infty}((\mathbf{1} - U_n(h) + U_n(gh) - U_n(g))x_n, x_n), \end{aligned}$$

it follows from (4.8) that $d^1_{\lambda,\lambda^\sharp}\gamma = -\xi(g)\xi^\sharp(h)$, so that ξ is neutral. ■

E2.2 **Example 4.13** Let $G=\mathbb{R}$, let $\mathfrak{H}_n=\mathbb{C}y_n$ and $\mathfrak{H}=\oplus_{n=1}^{\infty}\mathfrak{H}_n$. For $t\in\mathbb{R}$, let $U_n(t)y_n=e^{it/n^2}y_n$ and $U=\oplus_{n=1}^{\infty}U_n$. For each N, the representation $\oplus_{n=N}^{\infty}U_n$ weakly contains the trivial representation ι, as $\|U(t)y_n-y_n\| \underset{n\to\infty}{\to} 0$ uniformly on each compact set in $\mathbb{R}$. As in (4.54), set

$$r(t)=\oplus_{n=1}^{\infty}(y_n-e^{-it/n^2}y_n) \text{ and } \phi(t)=\sum_{n=1}^{\infty}(e^{it/n^2}-1)$$

Then r and ϕ satisfy (4.8), so that $\xi(t)=r(t)\otimes u$ is a neutral (ι,U)-cocycle and not a coboundary, and $\gamma(t)=\phi(t)(u\otimes u)$ is a prechain. Thus $\mathcal{H}_\nu^1(\iota,U)\neq 0$. ■

T3.11 **Theorem 4.14** *Let G be a connected, locally compact nilpotent group, let U be a unitary representation of G on $\mathfrak{H}$ and $\mathfrak{H}^{\chi_e}=\{0\}$. Then $\mathcal{H}_\nu^1(\iota,U)\neq 0$ if and only if $\mathcal{H}^1(\iota,U)\neq 0$. Moreover, $\mathcal{Z}_\nu^1(\iota,U)$ is dense in $\mathcal{Z}^1(\iota,U)$: for each non-trivial (ι,U)-cocycle ξ, there is a net of non-trivial neutral (ι,U)-cocycles which converges to ξ uniformly on compacts.*

Proof. Let $\overline{\mathcal{B}^1(\iota,U)}$ be the closure of the set $\mathcal{B}^1(\iota,U)$ of all boundaries in the topology of the uniform convergence on compacts. Note first that, since $\mathfrak{H}^{\chi_e}=\{0\}$, the group $\mathcal{H}_{\rm red}^1(\iota,U):=\mathcal{Z}^1(\iota,U)/\overline{\mathcal{B}^1(\iota,U)}$ of the *reduced* 1-*cohomologies* is trivial, so that $\mathcal{Z}^1(\iota,U)=\overline{\mathcal{B}^1(\iota,U)}$. Indeed, decomposing U into the irreducible representations

$$U=\int_\Omega^{\oplus}U_\omega d\mu,$$

we see that $U_\omega\neq\iota$ for almost all ω. Hence, by Corollary 2.14 (see also [G2, Corollary 5]), $\mathcal{H}^1(\iota,U_\omega)=\{0\}$ for almost all ω. Now it follows from [G, Proposition 3.2.6] that $\mathcal{H}_{\rm red}^1(\iota,U)=\{0\}$.

If U does not weakly contain ι then (see [G2, D]) $\mathcal{B}^1(\iota,U)=\overline{\mathcal{B}^1(\iota,U)}$. Therefore $\mathcal{H}^1(\iota,U)=\mathcal{H}_{\rm red}^1(\iota,U)=0$ and there is nothing to prove.

Suppose now that U weakly contains ι. Then there are unit vectors $\{e_n\}_{n=1}^{\infty}$ in $\mathfrak{H}$ such that $\|U(g)e_n-e_n\|\to 0$ uniformly on compact subsets of G. As $\mathfrak{H}^{\chi_e}=\{0\}$, it follows from Proposition 3.3 that $\mathfrak{H}=\oplus_{k=1}^{N}\mathfrak{H}_k$, $N\leq\infty$, where $\mathfrak{H}_k$ are invariant subspaces and each $U_k=U_{\mathfrak{H}_k}$ is spectrally disjoint with χ_e. Let P_k be the projections on $\mathfrak{H}_k$. If $\lim_{n\to\infty}\|P_ke_n\|\neq 0$, for some k, there is $\varepsilon>0$ and a subsequence $\{n_m\}_{m=1}^{\infty}$ such that $\|P_ke_{n_m}\|>\varepsilon$. Set $y_m=P_ke_{n_m}/\|P_ke_{n_m}\|$. Then

$$\|U_k(g)y_m-y_m\|\leq\|P_k(U(g)e_{n_m}-e_{n_m})\|/\varepsilon\leq\|U(g)e_{n_m}-e_{n_m}\|/\varepsilon\to 0, \text{ as } m\to\infty,$$

for all $g\in G$. Thus U_k and χ_e are not spectrally disjoint - a contradiction. Therefore

$$\lim_{n\to\infty}\|P_ke_n\|=0 \text{ for all } k. \tag{4.55}$$ 2.13

If $N<\infty$ then $1=\|e_n\|\leq\sum_{k=1}^{N}\|P_ke_n\|$ for all n. Thus there is $k\leq N$ such that $\lim_{n\to\infty}\|P_ke_n\|\neq 0$ which contradicts (4.55). Hence $N=\infty$. Set $Q_m=\sum_{k=m+1}^{\infty}P_k$. Then for each m

$$\|U(g)Q_me_n-Q_me_n\|\leq\|Q_m(U(g)e_n-e_n)\|\leq\|U(g)e_n-e_n\|\to 0, \text{ as } n\to\infty.$$

Let $\lim_{n\to\infty}\|Q_me_n\|=0$ for some m. Then, as $1=\|e_n\|\leq\sum_{k=1}^{m}\|P_ke_n\|+\|Q_me_n\|$, we have that $\lim_{n\to\infty}\|P_ke_n\|\neq 0$, for some k, which contradicts (4.55). Thus $\lim_{n\to\infty}\|Q_me_n\|\neq 0$ for all m. Hence all representations $\oplus_{k=m+1}^{\infty}U_k$ weakly contain ι. As G is connected and locally compact, it is σ-compact. Applying Proposition 4.12, we conclude that G has a neutral (ι,U)-cocycle (not a coboundary) ξ_ν. By Lemma 4.2, the set $\xi_\nu+\mathcal{B}^1(\iota,U)$ consists of neutral cocycles. Since (see above) $\mathcal{Z}^1(\iota,U)=\overline{\mathcal{B}^1(\iota,U)}$, the set $\xi_\nu+\mathcal{B}^1(\iota,U)$ is dense in $\mathcal{Z}^1(\iota,U)$. ■

R4.16 **Remark 4.15** In Theorem 3.18 it was shown that, for the groups whose dual objects contain characters that cannot be separated (in particular, for the Heisenberg group), one can construct extensions that do not decompose in a direct sum of primary extensions. For applications to representations on Pontryagin spaces, it is important that the cocycles ξ (see (3.20)) that define such representations, are neutral. To show this we have to present the corresponding prechains.

Using notations of Theorem 3.18, we have $L = \mathbb{C}e_1 \oplus \mathbb{C}e_2$, $\lambda(g) = e_1 \otimes e_1 + \chi(g)(e_2 \otimes e_2)$,

$$u_n(g) = u_n - \pi_n(g)^* u_n,\ v_n(g) = \overline{\chi(g)} v_n - \pi_n(g)^* v_n \text{ and } \xi(g) = \sum_{n=1}^{\infty} (u_n(g) \otimes e_1 + v_n(g) \otimes e_2).$$

Then $\xi^\sharp(g) = \xi(g^{-1})^* = \sum_{n=1}^{\infty} ((e_1 \otimes u_n(g^{-1}), e_2 \otimes v_n(g^{-1}))$. Set $\gamma = -\sum_{n=1}^{\infty} \gamma_n$, where all

$$\gamma_n(g) = (u_n, u_n(g))(e_1 \otimes e_1) + (u_n(g^{-1}), v_n)(e_1 \otimes e_2) + (v_n, u_n(g))(e_2 \otimes e_1) + (v_n, v_n(g))(e_2 \otimes e_2)$$

belong to $B(L)$. The series in $\gamma(g)$ converge uniformly on compacts because of condition (3.18). As $\lambda = \lambda^\sharp$, we obtain by direct calculations, using (3.9) and (4.3), that ξ is neutral (see (4.4)), since

$$(d_{\lambda,\lambda}\gamma)(g,h) = \lambda(g)\gamma(h) - \gamma(gh) + \gamma(g)\lambda(h) = \sum_{n=1}^{\infty} (\lambda(g)\gamma_n(h) - \gamma_n(gh) + \gamma_n(g)\lambda(h))$$

$$= \sum_{n=1}^{\infty} (a_n^{11}(e_1 \otimes e_1) + a_n^{12}(e_1 \otimes e_2) + a_n^{21}(e_2 \otimes e_1) + a_n^{22}(e_2 \otimes e_2)) = -\xi(g)\xi^\sharp(h),$$

where

$$a_n^{11} = (u_n, u_n(gh) - u_n(h) - u_n(g)),\ \ a_n^{12} = (u_n(h^{-1}g^{-1}) - \chi(g) u_n(h^{-1}) - u_n(g^{-1}), v_n),$$
$$a_n^{21} = (v_n, u_n(gh) - u_n(h) - \overline{\chi(h)} u_n(g)),\ \ a_n^{22} = (v_n, v_n(gh) - \overline{\chi(g)} v_n(h) - \overline{\chi(h)} v_n(g)).$$

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