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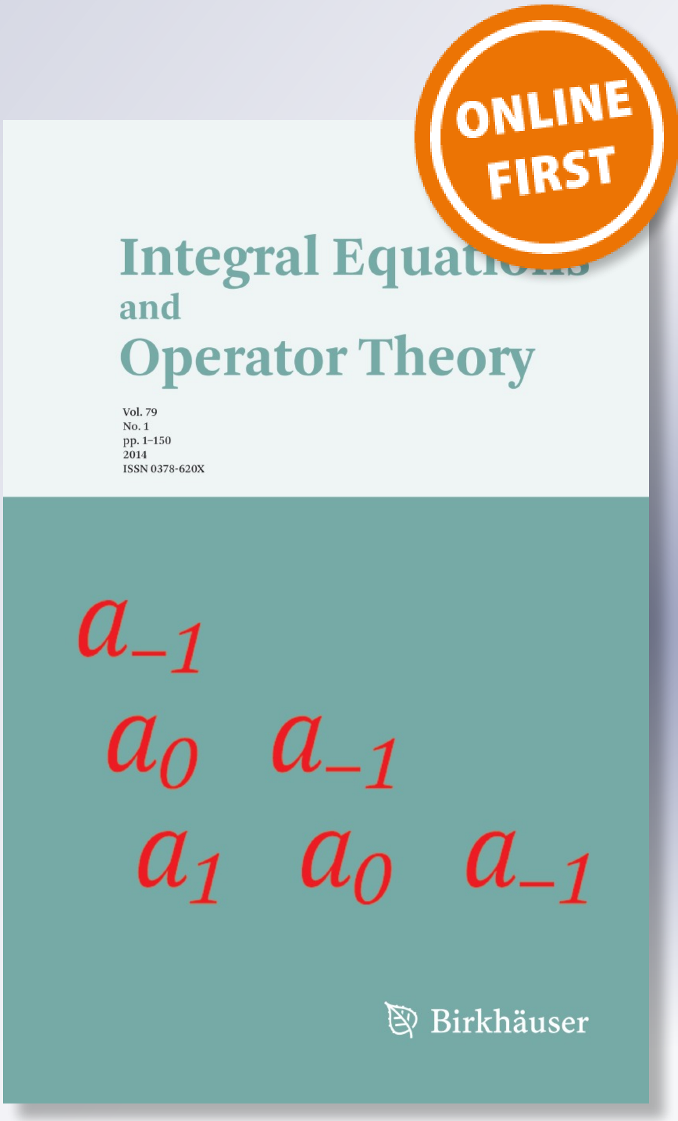
Integral Equations and Operator Theory

ISSN 0378-620X

Integr. Equ. Oper. Theory
DOI 10.1007/s00020-014-2145-x



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Clarkson–McCarthy Inequalities for l_p-Spaces of Operators in Schatten Ideals

Teresa Formisano and Edward Kissin

Abstract. In this paper we obtain generalized Clarkson–McCarthy inequalities for spaces $l_q(S^p)$ of operators from Schatten ideals S^p. We show that all Clarkson–McCarthy type inequalities are, in fact, some estimates on the norms of operators acting on the spaces $l_q(S^p)$ or from one such space into another. We also extend some inequalities for partitioned operators and for Cartesian decomposition of operators.

1. Introduction and Preliminaries

The original Clarkson inequalities for L_p spaces (summarized by Kato and Takahashi in [14]) were proved in Clarkson [8] in the context of uniform convexity of L_p spaces. Their non-commutative analogues for the Schatten ideals $S^p = S^p(H)$, where H is a separable Hilbert space, were obtained by McCarthy in [18]: For $A, B \in S^p, 2 \le p < \infty$,

$$(\|A+B\|_p^p + \|A-B\|_p^p)^{\frac{1}{p}} \le 2^{\frac{1}{p'}}(\|A\|_p^p + \|B\|_p^p)^{\frac{1}{p}}; \tag{1.1}$$

$$(\|A+B\|_p^p + \|A-B\|_p^p)^{\frac{1}{p}} \le 2^{\frac{1}{p}}(\|A\|_p^{p'} + \|B\|_p^{p'})^{\frac{1}{p'}}, \tag{1.2}$$

where $1/p + 1/p' = 1$. For $1 < p < 2$, these inequalities are reversed. Inequality (1.2) is stronger than (1.1), since (see, for example, [14, Lemma 2.3]) $\left(\frac{a^q+b^q}{2}\right)^{\frac{1}{q}} \le \left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}$, for $q \le p$ and nonnegative a, b. These Clarkson–McCarthy inequalities play an important role in analysis and operator theory and were used to prove that all Schatten ideals $S^p, 2 \le p < \infty$, are uniformly convex Banach spaces (see [18,19]). Bhatia and Holbrook [4] and Hirzallah and Kittaneh [11] generalized these inequalities for general symmetric norms. Bhatia and Kittaneh [7] obtained Clarkson–McCarthy inequalities for certain N-tuples of operators from S^p and the second author [15] extended them to all N-tuples of operators from S^p.

We start this paper with reinterpreting these inequalities as estimates on the norms of some operators acting on Banach spaces $l_q^N(S^p)$ for $N < \infty$. For the classical Clarkson inequalities in L_p spaces, this was done earlier by

Birkhäuser
Published online: 18 April 2014

Kato in [12] who considered the action of Littlewood matrices

$$A_{2^1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, A_{2^{n+1}} = \begin{pmatrix} A_{2^n} & A_{2^n} \\ A_{2^n} & -A_{2^n} \end{pmatrix}, \quad n = 1, 2, \ldots \tag{1.3}$$

from the space $l_r^{2^n}(L_p)$ into $l_s^{2^n}(L_p)$ and evaluated the norms of operators $\left\|A_{2^n}: l_r^{2^n}(L_p) \to l_s^{2^n}(L_p)\right\|$ in order to obtain various generalizations of Clarkson inequalities. This approach was later used and extended further in a number of papers by Kato and Takahashi in [14,21], Takahashi, Hashimoto and Kato in [20], Maligranda and Persson in [16,17], Tonge [22]. For arbitrary Banach spaces X, the action of Rademacher matrices R_n from the space $l_p^n(X)$ into $l_s^{2^n}(X)$ was investigated by Kato et al. in [13] to establish the relation between Clarkson inequalities in X and the type and cotype of X (see also [14]).

In this paper we study Clarkson–McCarthy inequalities for infinite sets of operators from S^p, that is, we consider Banach spaces $l_q(S^p)$ (they are symmetrically normed ideals of C*-algebra $l_\infty(B(H))$) and obtain for them analogues of Clarkson–McCarthy inequalities and other related inequalities. As in the finite case, these inequalities are estimates on the norms of some operators acting from the space $l_q(S^p)$ into $l_r(S^p)$. As a consequence, we prove that the spaces $l_p(S^p)$ are p-uniformly convex for $p \geq 2$, and p-uniformly smooth for $1 < p \leq 2$. We investigate the relation between the spaces $l_q(S^p)$ and the space $S^p(H, H^\infty)$ of compact operators A from H into the orthogonal sum H^∞ of an infinite number of copies of H satisfying $\|A\|_p < \infty$, and examine the embeddings of these spaces on to each other. We also consider infinite partition and Cartesian decomposition of operators from the Schatten ideals S^p.

Let H, K be separable Hilbert spaces, $B(H, K)$ be the space of all bounded operators from H to K and $C(H, K)$ the subspace of compact operators in $B(H, K)$. If $K = H$, set $B(H) = B(H, H)$ and $C(H) = C(H, H)$. Then $C(H)$ is the unique closed two-sided ideal of $B(H)$. For $A \in C(H, K)$, the operator $|A| = (A^*A)^{1/2}$ belongs to $C(H)$ and its eigenvalues $\{s_i\}$ converge to 0. For $p \in [1, \infty)$,

$$S^p(H, K) = \left\{ A \in C(H, K) \colon \|A\|_p := \||A|\|_p := \left(\sum_i s_i^p\right)^{1/p} < \infty \right\} \tag{1.4}$$

is a Banach space in norm $\|\cdot\|_p$ and $S^p := S^p(H) = S^p(H, H)$ is a two-sided *Schatten* ideal of $B(H)$ and

$$S^q \subset S^p \text{ and } \|A\|_p \leq \|A\|_q \text{ for } A \in S^q, \text{ if } 1 \leq q < p \leq \infty. \tag{1.5}$$

For a Banach space $(X, \|\cdot\|)$, the space $l_q^N(X)$ of sequences $x = (x_i)_{i=1}^N$, $x_i \in X$, satisfying

$$\|x\|_{l_q^N(X)} = \left(\sum_{i=1}^N \|x_i\|^q\right)^{1/q} < \infty, \text{ for } q < \infty,$$
$$\|x\|_{l_\infty^N(X)} = \sup \|x_i\| < \infty, \text{ for } q = \infty,$$

is a Banach space for each $N \in \mathbb{N} \cup \infty$. If $N < \infty$ then the norms $\|\cdot\|_{l_q^N(X)}$ are equivalent for all q. For $N = \infty$, set $l_q(X) = l_q^\infty(X)$. The norms $\|\cdot\|_{l_q(X)}$ are not equivalent:

$$l_p(X) \subsetneq l_q(X) \text{ and } \|\cdot\|_{l_q(X)} \le \|\cdot\|_{l_p(X)}, \text{ if } p < q. \tag{1.6}$$

To interpret (1.1) and (1.2) as inequalities in $l_q^2(S^p)$, consider the unitary matrix $\frac{1}{\sqrt{2}} A_{2^1}$ (see (1.3)) and the corresponding unitary operator

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1}_H & \mathbf{1}_H \\ \mathbf{1}_H & -\mathbf{1}_H \end{pmatrix} \text{ in } B(H \oplus H).$$

It acts on $l_q^2(S^p)$ (treat $A = (A_1, A_2), A_i \in S^p$, as a column) by $RA = \frac{1}{\sqrt{2}}(A_1 + A_2, A_1 - A_2)$. Then (1.1) and (1.2) can be written in the following form:

$$\|RA\|_{l_p^2(S^p)} \le 2^{\left|\frac{1}{2}-\frac{1}{p}\right|} \|A\|_{l_p^2(S^p)}, \text{ for } p \in [1, \infty); \tag{1.7}$$

$$\|RA\|_{l_p^2(S^p)} \le 2^{-\left|\frac{1}{2}-\frac{1}{p'}\right|} \|A\|_{l_{p'}^2(S^p)}, \text{ if } 2 \le p,$$

$$\|RA\|_{l_{p'}^2(S^p)} \le 2^{-\left|\frac{1}{2}-\frac{1}{p'}\right|} \|A\|_{l_p^2(S^p)} \text{ if } p \le 2. \tag{1.8}$$

Similarly, the inequality of Ball et al. [2] can be written for $p \in [2, \infty)$ as

$$\|RA\|_{l_p^2(S^p)} \le 2^{\frac{1}{p}} \|A\|_{l_2^2(S^p)}, \text{ where } R = \begin{pmatrix} \mathbf{1}_H & (p-1)^{-\frac{1}{2}}\mathbf{1}_H \\ \mathbf{1}_H & -(p-1)^{-\frac{1}{2}}\mathbf{1}_H \end{pmatrix}.$$

For $p \in [1, 2)$ it is reversed.

Let H^N be the orthogonal sum of $N < \infty$ copies of H. Each $R \in B(H^N)$ has matrix form $R = (R_{jk})_{j,k=1}^N, R_{jk} \in B(H)$, and acts on $l_q^N(S^p)$ (treat each $A = (A_1, \ldots, A_N), A_i \in S^p$, as a column). Some analogues of Clarkson–McCarthy inequalities (1.1) and (1.2) were obtained in [15]. Interpreting them as inequalities in $l_q^N(S^p)$ and setting $\lambda = \max \|R_{jk}\|$, we have, for $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\|RA\|_{l_t^N(S^p)} \le N^{\left|\frac{1}{p}-\frac{1}{2}\right|+\frac{1}{t}-\frac{1}{s}} \|R\| \, \|A\|_{l_s^N(S^p)} \text{ for } 1 \le p < \infty; \tag{1.9}$$

$$\|RA\|_{l_{p'}^N(S^p)} \le \lambda^{\frac{2}{p}-1} \|R\|^{\frac{2}{p'}} \|A\|_{l_p^N(S^p)} \text{ for } 1 \le p \le 2,$$

where $t, s \in [\min(p, 2), \max(p, 2)]$. Thus the norm of the operator R from $l_s^N(S^p)$ to $l_t^N(S^p)$ satisfies

$$\|R\|_{l_s^N(S^p) \to l_t^N(S^p)} \le N^{\left|\frac{1}{p}-\frac{1}{2}\right|+\frac{1}{t}-\frac{1}{s}} \|R\|,$$

$$\|R\|_{l_p^N(S^p) \to l_{p'}^N(S^p)} \le \lambda^{\frac{2}{p}-1} \|R\|^{\frac{2}{p'}}. \tag{1.10}$$

For unitary operators $R = \frac{1}{\sqrt{N}}(a_{jk}\mathbf{1}_H)_{j,k=1}^N$, where $a_{jk} = \exp\left(i\frac{2\pi(j-1)(k-1)}{N}\right)$, these inequalities were obtained in [7]. In this paper we obtain some analogues of inequalities (1.10) for $N = \infty$.

Each $A \in B(H, H^\infty)$ has form $A = (A_n)_{n=1}^\infty$ with $A_n \in B(H)$. Hence $B(H, H^\infty)$ can be considered as a subspace of $l_\infty(B(H))$ and $S^p(H, H^\infty)$ as a subspace of $l_\infty(S^p)$. By (1.5) and (1.6), $l_q(S^p) \subset l_r(S^p) \subset l_r(S^t)$ and $\|\cdot\|_{l_r(S^t)} \le \|\cdot\|_{l_r(S^p)} \le \|\cdot\|_{l_q(S^p)}$, if $1 \le q < r, 1 \le p < t$. Thus, for each

p, $\{l_q(S^p)\}_{1\le q<\infty}$ is an increasing range of spaces and $\{\|\cdot\|_{l_q(S^p)}\}_{1\le q<\infty}$ is a decreasing range of norms. In Sect. 2 we find the positions that $S^p(H,H^\infty)$ and $\|\cdot\|_p$ occupy in these ranges:

$$l_p(S^p)\subset S^p(H,H^\infty)\subset l_2(S^p) \text{ and } \|A\|_{l_2(S^p)}\le\|A\|_p\le\|A\|_{l_p(S^p)},$$

if $1\le p\le 2$. All inclusions and inequalities are reversed for $p\ge 2$.

In Sect. 3 we show that, for $p\in[1,2]$, operators $R\in B(H^\infty)$ map the spaces $l_p(S^p)$ into $l_2(S^p)$; for $p\ge 2$, they map $l_2(S^p)$ into $l_p(S^p)$, and the following analogue of (1.10) holds:

$$\|R\|_{l_p(S^p)\to l_2(S^p)}\le\|R\|_{B(H^\infty)} \text{ for } p\in[1,2],$$
$$\|R\|_{l_2(S^p)\to l_p(S^p)}\le\|R\|_{B(H^\infty)} \text{ for } 2\le p.$$

These results give, in turn, some analogues of inequalities (1.1). In particular, if $A,B\in l_q(S^p)$ then, for $p\in[1,\infty)$ and $q\in[\min(p,2),\max(p,2)]$, we have

$$(\|A+B\|^q_{l_q(S^p)}+\|A-B\|^q_{l_q(S^p)})^{\frac{1}{q}}\le 2^{\left|\frac{1}{p}-\frac{1}{2}\right|+\frac{1}{2}}(\|A\|^q_{l_q(S^p)}+\|B\|^q_{l_q(S^p)})^{\frac{1}{q}}.$$

Using this, we prove that the spaces $l_p(S^p)$ are p-uniformly convex for $p\ge 2$, and p-uniformly smooth for $p\in[1,2]$.

A set $\{\mathcal{P}_n\}_{n=1}^N$ of mutually orthogonal projections is a partition of $\mathbf{1}_H$ if $\sum_{n=1}^N\mathcal{P}_n=\mathbf{1}_H$. It is well known (see [10]) that $\sum_n\|\mathcal{P}_nA\mathcal{P}_n\|_p\le\|A\|_p$ for $A\in S^p$. For partitions $\{\mathcal{P}_n\}_{n=1}^N$ and $\{\mathcal{Q}_m\}_{m=1}^M$, it was established in [15] that, for $M,N<\infty$ and $2\le q\le p<\infty$,

$$(NM)^{\frac{q}{p}-1}\sum_{n,m}\|\mathcal{P}_nA\mathcal{Q}_m\|^q_p\le\|A\|^q_p\le(NM)^{\frac{q}{2}-1}\sum_{n,m}\|\mathcal{P}_nA\mathcal{Q}_m\|^q_p \quad (1.11)$$

and reversed for $1\le p\le q\le 2$. For $N=M$, $\mathcal{P}_n=\mathcal{Q}_n$ and $q=2,p$, this was proved in [5] and used to show that symmetrically normed ideals of $B(H)$ with Q^*-norms have the Radon–Riesz property.

In Sect. 4 we study infinite partitions $\mathcal{A}=\{\mathcal{P}_nA\mathcal{Q}_m\}_{n=1,m=1}^\infty$ of operators $A\in S^p$. Using results of Sects. 2 and 3, we prove that, for $2\le p<\infty$, the partition $\mathcal{A}$ belongs to $l_p(S^p)$ and

$$\|\mathcal{A}\|_{l_p(S^p)}\le\|A\|_p\le\|\mathcal{A}\|_{l_2(S^p)}.$$

For $1\le p\le 2$, the partition $\mathcal{A}$ belongs to $l_2(S^p)$ and the inequalities are reversed.

For a set $A=(A_n)_{n=1}^N$ of operators from S^p, consider the involution $A\to A^\sharp=(A_n^*)_{n=1}^N$. Then $X=\frac{1}{2}(A+A^\sharp)$ and $Y=\frac{1}{2i}(A-A^\sharp)$ are N-tuples of selfadjoint operators $X_k=\frac{1}{2}(A_k+A_k^*)$, $Y_k=\frac{1}{2i}(A_k-A_k^*)$ and $A=X+iY$ is the "Cartesian" decomposition of A. It was shown in [15]

$$(2N)^{-s}\left(\|X\|^q_{l_q^N(S^p)}+\|Y\|^q_{l_q^N(S^p)}\right)^{\frac{1}{q}}\le 2^{\frac{1}{q}-\frac{1}{2}}\|A\|_{l_q^N(S^p)}$$
$$\le(2N)^s\left(\|X\|^q_{l_q^N(S^p)}+\|Y\|^q_{l_q^N(S^p)}\right)^{\frac{1}{q}},$$

where $s = \left|\frac{1}{p} - \frac{1}{2}\right|, p \in [1, \infty)$ and $q \in [\min(p, 2), \max(p, 2)]$. For $N = 1$, this was proved in [6]. For other results of this kind and a discussion of their importance in the analysis of operators see [1,3,19,23].

In Sect. 5 we show that, for $N = \infty$, the sequences $A = (A_n)_{n=1}^{\infty} \in l_q(S^p)$, $X = \frac{1}{2}(A + A^\sharp)$ and $Y = \frac{1}{2i}(A - A^\sharp)$ satisfy the following inequalities

$$2^{\frac{1}{q}-\frac{1}{2}-\left|\frac{1}{p}-\frac{1}{2}\right|} \|A\|_{l_q(S^p)} \le \left(\|X\|_{l_q(S^p)}^q + \|Y\|_{l_q(S^p)}^q\right)^{1/q} \le 2^{\frac{1}{q}-\frac{1}{2}+\left|\frac{1}{p}-\frac{1}{2}\right|} \|A\|_{l_q(S^p)}.$$

The involution $\sharp\colon A \to A^\sharp$ preserves spaces $l_q(S^p)$, but not $S^p(H, H^\infty)$, if $p \ne 2$. Denoting by $D_p(\sharp)$ the domain of $\sharp$ in $S^p(H, H^\infty)$, we obtain for $A \in D_p(\sharp)$ and $p \in [1, 2]$ that

$$\|A\|_{l_2(S^p)} \le \left\|\left(\sum_{n=1}^{\infty}(X_n^2 + Y_n^2)\right)^{1/2}\right\|_p = 2^{-\frac{1}{2}} \left\|\left(|A|^2 + \left|A^\sharp\right|^2\right)^{1/2}\right\|_p \le 2^{\frac{1}{p}-\frac{1}{2}} \|A\|_{l_p(S^p)}.$$

For $p \in [2, \infty)$, the inequalities are reversed.

2. $S^p(H, H^\infty)$ lies Between the Spaces $l_p(S^p)$ and $l_2(S^p)$

Let $S^p(H, K), p \in [0, \infty)$, be the set of compact operators A in $B(H, K)$ with $\|A\|_p < \infty$ (see (1.4)). It is a linear space. Set $S^p = S^p(H) = S^p(H, H)$. For $A \in S^p(H, K)$, the operator $|A| = (A^*A)^{1/2} \in S^p$ is positive and $s_n(|A|^2) = s_n(|A|)^2$ are eigenvalues of $|A|^2$. Hence

$$\|A^*A\|_{p/2} = \left\||A|^2\right\|_{p/2} = \left(\sum_n s_n^{p/2}(|A|^2)\right)^{2/p} = \left(\sum_n s_n^p(|A|)\right)^{2/p} = \||A|\|_p^2 \overset{(1.4)}{=} \|A\|_p^2,$$

$$\text{and } A \in S^p(H, K) \text{ if and only if } A^*A \in S^{p/2}(H). \tag{2.1}$$

For $A, B \in S^p$ and $C, D \in B(H)$, we have (see [9])

$$\|A + B\|_p^p \le 2(\|A\|_p^p + \|B\|_p^p), \text{ if } p < 1, \tag{2.2}$$
$$\|A + B\|_p \le \|A\|_p + \|B\|_p \text{ if } 1 \le p,$$
$$\|CAD\|_p \le \|C\|\|A\|_p\|D\| \text{ and } \|A^*\|_p = \|A\|_p, \tag{2.3}$$
$$\|AB\|_{p/2} \le 2^{2/p} \|A\|_p \|B\|_p, \text{ if } p < 2, \tag{2.4}$$
$$\|AB\|_{p/2} \le \|A\|_p \|B\|_p, \text{ if } p \ge 2.$$

Operators $\{A_n\}$ in $B(H, K)$ converge to A in the *weak operator topology* (w.o.t) if $(A_n x, y) \to (Ax, y)$, and in the *strong operator topology* (s.o.t) if $\|Ax - A_n x\|_K \to 0$ for all $x \in H$, $y \in K$.

All Banach spaces $S^p(H, K)$ share the following important properties:

Lemma 2.1. [10, Theorems III.5.1 and III.6.3]

(i) *For $p \in [1,\infty)$, let operators $\{A_n\}$ in $S^p(H,K)$ converge to $A \in B(H,K)$ in* w.o.t. *If $\sup_n \|A_n\|_p = M < \infty$, then $A \in S^p(H,K)$ and $\|A\|_p \le M$.*

(ii) *Let $\{P_n\}_{n=1}^{\infty}$ be projections in $B(K)$ and $P_n \xrightarrow{\text{s.o.t.}} \mathbf{1}_K$. Then, for $p \in [1,\infty]$ and $A \in S^p(H,K)$,*

$$\|A - P_nA\|_p \to 0, \ \text{as } n \to \infty. \tag{2.5}$$

Let $H^\infty = H \oplus \ldots \oplus H \oplus \ldots$ Each operator $A \in B(H,H^\infty)$ has form $A = (A_n)_{n=1}^{\infty}$ where $A_n \in B(H)$ and $\|A_n\| \le \|A\|$. Thus we can identify $B(H,H^\infty)$ with a subspace of $l_\infty(B(H))$.

For $m \in \mathbb{N}$, the projection P_m on the first m components of H^∞ belongs to $B(H^\infty)$ and, for each $A = (A_n)_{n=1}^{\infty} \in l_\infty(B(H))$,

$$P_mA = (A_1,\ldots,A_n,\mathbf{0},\ldots) \in B(H,H^\infty).$$

Let $A \in B(H,H^\infty)$. As $P_m \overset{\text{s.o.t.}}{\to} \mathbf{1}_{H^\infty}$, we have

$$P_mA \overset{\text{s.o.t.}}{\to} A \ \text{ and } \ (P_mA)^*(P_mA) = A^*P_mA = \sum_{n=1}^{m} A_n^*A_n \overset{\text{w.o.t.}}{\to} A^*A, \tag{2.6}$$

as $m \to \infty$. If $P_mA \in S^p(H,H^\infty)$, for some $0 < p < \infty$, then

$$\|P_mA\|_p^2 \overset{(2.1)}{=} \|(P_mA)^*P_mA\|_{p/2} \overset{(2.6)}{=} \left\|\sum_{n=1}^{m} A_n^*A_n\right\|_{p/2}. \tag{2.7}$$

The next lemma gives some conditions for $A \in l_\infty(B(H))$ to belong to $B(H,H^\infty)$ and $S^p(H,H^\infty)$.

Lemma 2.2. *Let $A = (A_n)_{n=1}^{\infty} \in l_\infty(B(H))$. Then*

(i) *$A \in B(H,H^\infty)$ if and only if $\{P_mAx\}$ converges weakly in H^∞ for each $x \in H$, as $m \to \infty$.*

(ii) *$A \in S^p(H,H^\infty)$, for some $p \in [1,\infty)$, if and only if $A_n \in S^p(H)$, for all n, and there is $M > 0$ such that $\|P_mA\|_p \le M$ for all m. Moreover, $\|A\|_p \le M$.*

(iii) *$l_q(B(H)) \subset l_2(B(H)) \subset B(H,H^\infty)$ for $q \in [1,2)$, and*

$$\|A\|_{B(H,H^\infty)}^2 \le \sum_{n=1}^{\infty} \|A_n\|^2 = \|A\|_{l_2(B(H))}^2 \ \text{for } A \in l_2(B(H)). \tag{2.8}$$

(iv) *$l_q(S^p) \not\subseteq B(H,H^\infty)$, for $q > 2$ and all p.*

Proof. (i) Let $\{P_mAx\}$ weakly converge in H^∞ for each $x \in H$. By the uniform convergence theorem, there is $T = (T_n)_{n=1}^{\infty} \in B(H,H^\infty)$ such that $P_mAx \to Tx$. Choosing x in the n-th component of H^∞, we get $A_n = T_n$. Thus $A = T$. The part "only if" follows from (2.6).

(ii) Let $A \in S^p(H,H^\infty)$. As $S^p(H,H^\infty)$ is a left Banach $B(H^\infty)$-module, $(P_n - P_{n-1})A \in S^p(H,H^\infty)$ and $\|P_mA\|_p \le \|P_m\|\,\|A\|_p \le \|A\|_p$. Hence also $A_n \in S^p(H)$ for all n.

Conversely, we have $\|P_mA\| \le \|P_mA\|_p \le M$ and $P_mA \in S^p(H,H^\infty)$, as $A_n \in S^p(H)$ for all n. Then, for $x \in H$, $\|P_mAx\|$ is an increasing bounded sequence. Hence it converges and, for each k,

$$\begin{aligned}\|(P_{m+k}-P_m)Ax\|^2 &= (\|P_{m+k}Ax\| + \|P_mAx\|)(\|P_{m+k}Ax\| - \|P_mAx\|)\\ &\le 2M\,\|x\|\,(\|P_{m+k}Ax\| - \|P_mAx\|).\end{aligned}$$

Thus $\{P_mAx\}$ strongly converges in H^∞ for each $x \in H$. Hence, by (i), $A \in B(H,H^\infty)$. Then $P_mA \overset{\text{s.o.t.}}{\to} A$ and it follows from Lemma 2.1(i) that $A \in S^p(H,H^\infty)$ and $\|A\|_p \le M$.

(iii) follows from (1.6) and from the fact that $\|Ax\|^2_{H^\infty} = \sum_{n=1}^\infty \|A_nx\|^2 \le \|x\|^2 \sum_{n=1}^\infty \|A_n\|^2$ for each $x \in H$.

(iv) Let $q > 2, p \in [1,\infty)$ and $\alpha = \frac{2}{2+q}$. For some $0 \ne T \in S^p$, let $A_n = n^{-\alpha}T$. Then $A = (A_n)_{n=1}^\infty \in l_q(S^p)$, since

$$\|A\|^q_{l_q(S^p)} = \sum_{n=1}^\infty \|A_n\|_p^q = \|T\|_p^q \sum_{n=1}^\infty n^{-\alpha q} < \infty, \text{ as } \alpha q > 1.$$

On the other hand, as $2\alpha = \frac{4}{2+q} \le 1$, we have for each $x \notin \ker T$,

$$\|Ax\|^2 = \left\|(Tx,\ldots,n^{-\alpha}Tx,\ldots)\right\|^2 = \|Tx\|^2 \sum_n n^{-\frac{4}{2+q}} - \text{ diverges.}$$

Hence $A \notin B(H,H^\infty)$. Thus all spaces $l_q(S^p)$, $q > 2$, are not contained in $B(H,H^\infty)$. □

For positive operators $\{T_n\}_{n=1}^m$ in S^p, it was proved in [18] (also [5], [15, Theorem 1.22]) that

$$\sum_{n=1}^m \|T_n\|_p^p \le \left\|\sum_{n=1}^m T_n\right\|_p^p \text{ if } 1 \le p < \infty. \tag{2.9}$$

For $0 < p < 1$, it was shown Lemma 1 and formula (7) of [6] that

$$\left(\sum_{n=1}^m \|T_n\|_p\right)^p \le \left\|\sum_{n=1}^m T_n\right\|_p^p \le \sum_{n=1}^m \|T_n\|_p^p. \tag{2.10}$$

Proposition 2.3. *Let $A = (A_n)_{n=1}^\infty \in l_\infty(S^p)$. Then, for all $m \in \mathbb{N}$,*

$$\left(\sum_{n=1}^m \|A_n\|_p^2\right)^{p/2} \le \|P_mA\|_p^p = \left\|\sum_{n=1}^m A_n^*A_n\right\|_{p/2}^{p/2} \le \sum_{n=1}^m \|A_n\|_p^p, \tag{2.11}$$

if $p \in [1,2)$;

$$\sum_{n=1}^m \|A_n\|_p^p \le \|P_mA\|_p^p = \left\|\sum_{n=1}^m A_n^*A_n\right\|_{p/2}^{p/2} \le \left(\sum_{n=1}^m \|A_n\|_p^2\right)^{p/2}, \tag{2.12}$$

if $2 \le p$. If $p \in [1,\infty)$ and $A \in S^p(H,H^\infty)$ then

$$\lim_{m\to\infty} \left\|A^*A - \sum_{n=1}^m A_n^*A_n\right\|_{p/2} = 0. \tag{2.13}$$

Proof. If $1 \le p < 2$ then $\frac{p}{2} < 1$. Replacing T_n by $A_n^* A_n$ and p by $\frac{p}{2}$ in (2.10), we have

$$\left(\sum_{n=1}^m \|A_n^* A_n\|_{p/2}\right)^{p/2} \le \left\|\sum_{n=1}^m A_n^* A_n\right\|_{p/2}^{p/2} \le \sum_{n=1}^m \|A_n^* A_n\|_{p/2}^{p/2}.$$

Combining this with (2.7) and with $\|A_n\|_p^2 \overset{(2.1)}{=} \|A_n^* A_n\|_{p/2}$, we complete the proof of (2.11).

If $2 \le p$, then $1 \le \frac{p}{2}$ and $S^{p/2}$ is a Banach space. By the triangle inequality for norms,

$$\|P_m A\|_p^2 \overset{(2.7)}{=} \left\|\sum_{n=1}^m A_n^* A_n\right\|_{p/2} \le \sum_{n=1}^m \|A_n^* A_n\|_{p/2} \overset{(2.1)}{=} \sum_{n=1}^m \|A_n\|_p^2.$$

Replacing in (2.9), T_n by $A_n^* A_n$ and p by $\frac{p}{2}$, we obtain that

$$\sum_{n=1}^m \|A_n\|_p^p \overset{(2.1)}{=} \sum_{n=1}^m \|A_n^* A_n\|_{p/2}^{p/2} \le \left\|\sum_{n=1}^m A_n^* A_n\right\|_{p/2}^{p/2}.$$

Combining this with the above inequalities, we complete the proof of (2.12).

Let $A \in S^p(H, H^\infty)$. Then (2.5) implies $\|A - P_m A\|_p \to 0$. If $1 \le p < 2$ then $\frac{p}{2} < 1$ and it follows from (2.6) and (2.4) that

$$\begin{aligned}\left\|A^* A - \sum_{n=1}^m A_n^* A_n\right\|_{p/2} &= \|A^* A - A^* P_m A\|_{p/2} \\ &\le 2^{2/p} \|A^*\|_p \|A - P_m A\|_p \to 0,\end{aligned}$$

as $m \to \infty$. If $2 \le p$ then $1 \le \frac{p}{2}$. As $S^{p/2}$ is a Banach space,

$$\left\|A^* A - \sum_{n=1}^m A_n^* A_n\right\|_{p/2} = \|A^* A - A^* P_m A\|_{p/2} \le \|A^*\|_p \|A - P_m A\|_p \to 0,$$

as $m \to \infty$. Combining these inequalities and (2.6), we complete the proof of (2.13). $\square$

Let $C(H, H^\infty)$ be the subspace of all compact operators in $B(H, H^\infty)$. Recall that, for $1 \le q < \infty$,

$$l_q(S^p) = \left\{ \begin{array}{c} A = (A_n)_{n=1}^\infty \colon A_n \in S^p \text{ and} \\ \|A\|_{l_q(S^p)} = \left(\sum_{n=1}^\infty \|A_n\|_p^q\right)^{1/q} < \infty \end{array} \right\}, \tag{2.14}$$

are Banach spaces. If $A \in l_\infty(S^p)$ and $A \notin l_q(S^p)$, we set $\|A\|_{l_q(S^p)} = \infty$.

For $x \in H$ and $u \in K$, the rank one operator $x \otimes u$ in $B(H, K)$ acts by

$$(x \otimes u)z = (z, x)u \text{ for each } z \in H. \tag{2.15}$$

Then (see [10]) $x \otimes u \in S^p(H, K)$ and $\|x \otimes u\|_p = \|x\| \|u\|$, for all $p \in [1, \infty)$.

Theorem 2.4. (i) *Let* $1 \le p < 2$. *Then*

$$\begin{aligned} & l_p(S^p) \subset S^p(H, H^\infty) \subset l_2(S^p) \subset l_2(C(H)) \subset C(H, H^\infty), \\ & \|A\|_{l_2(S^p)} \le \|A\|_p \text{ for } A \in S^p(H, H^\infty), \\ & \text{and } \|A\|_p \le \|A\|_{l_p(S^p)} \text{ for } A \in l_p(S^p). \end{aligned} \tag{2.16}$$

For each $q \in (p, 2)$, *the space* $l_q(S^p)$ *neither contains, nor is contained in* $S^p(H, H^\infty)$.

(ii) *Let* $p \in (2, \infty)$. *Then* $l_2(S^p) \subset S^p(H, H^\infty) \subset l_p(S^p) \nsubseteq B(H, H^\infty)$,

$$\begin{aligned} & \|A\|_{l_p(S^p)} \le \|A\|_p \text{ for } A \in S^p(H, H^\infty), \\ & \text{and } \|A\|_p \le \|A\|_{l_2(S^p)} \text{ for } A \in l_2(S^p). \end{aligned} \tag{2.17}$$

For each $q \in (2, p)$, *the space* $l_q(S^p)$ *neither contains, nor is contained in* $S^p(H, H^\infty)$.

(iii) $l_2(S^2) = S^2(H, H^\infty)$ *and* $\|A\|_{l_2(S^2)} = \|A\|_2$ *for each* $A \in S^2(H, H^\infty)$.

(iv) *For* $q > 2$ *and any* $p \in [1, \infty)$, *the space* $l_q(S^p)$ *is not contained in* $B(H, H^\infty)$.

Proof. Let $A = (A_n)_{n=1}^\infty \in l_2(C(H))$. Then all $P_m A \in C(H, H^\infty)$. By Lemma 2.2(iii), $A \in B(H, H^\infty)$ and

$$\|A - P_m A\|_{B(H,H^\infty)} \overset{(2.8)}{\le} \|A - P_m A\|_{l_2(B(H))} = \sum_{n=m+1}^{\infty} \|A_n\|^2 \to 0,$$

as $m \to \infty$. Since $C(H, H^\infty)$ is complete, A belongs to $C(H, H^\infty)$. Therefore $l_2(S^p) \subset l_2(C(H)) \subseteq C(H, H^\infty)$.

(i) Let $p \in [1, 2)$ and $A \in l_p(S^p)$. By (2.11),

$$\|P_m A\|_p \le \left(\sum_{n=1}^{m} \|A_n\|_p^p \right)^{1/p} \le \|A\|_{l_p(S^p)},$$

for all m. Hence, by Lemma 2.2(ii), $A \in S^p(H, H^\infty)$ and $\|A\|_p \le \|A\|_{l_p(S^p)}$. Thus $l_p(S^p) \subseteq S^p(H, H^\infty)$.

If $A \in S^p(H, H^\infty)$ then

$$\|A\|_{l_2(S^p)}^p \overset{(2.14)}{=} \lim_{m \to \infty} \left(\sum_{n=1}^{m} \|A_n\|_p^2 \right)^{p/2} \overset{(2.11)}{\le} \lim_{m \to \infty} \|P_m A\|_p^p \overset{(2.5)}{=} \|A\|_p^p.$$

Thus $A \in l_2(S^p)$, so that $S^p(H, H^\infty) \subseteq l_2(S^p)$, and (2.16) holds.

Let us prove that $S^p(H, H^\infty) \neq l_2(S^p)$ and $l_q(S^p) \nsubseteq S^p(H, H^\infty)$, for $p < q \le 2$. Let $\{e_n\}_{n=1}^\infty$ be a basis in H and P_{e_n} projections on $\mathbb{C}e_n$. Set $A_n = n^{-\alpha} P_{e_n}$, for some $\alpha > 0$, and consider $A = (A_n)_{n=1}^\infty$. Then $A \in l_\infty(S^p)$, as $\|P_{e_n}\|_p = 1$ for all p.

Let $x = \sum_{n=1}^\infty \alpha_n e_n \in H$. Then $P_m A x = \sum_{n=1}^m \oplus n^{-\alpha} \alpha_n e_n$, where each $n^{-\alpha}\alpha_n e_n$ belongs to the n-th component of H^∞, converge to $\sum_{n=1}^\infty \oplus n^{-\alpha} \alpha_n e_n$ in H^∞, as $m \to \infty$. By Lemma 2.2(i), $A \in B(H, H^\infty)$ and

$$|A|^2 = A^* A = \sum_{n=1}^{\infty} A_n^* A_n = \sum_{n=1}^{\infty} n^{-2\alpha} P_{e_n}, \text{ so that } s_n(|A|) = n^{-\alpha}.$$

Therefore

$$\|A\|^q_{l_q(S^p)} = \sum_{n=1}^{\infty} n^{-q\alpha} \text{ and } \|A\|^p_p \overset{(2.1)}{=} |||A|||^p_p = \sum_{n=1}^{\infty} (n^{-\alpha})^p. \tag{2.18}$$

Setting $\alpha = \frac{1}{p}$ in (2.18), we obtain that $A \in l_q(S^p)$ and $A \notin S^p(H, H^\infty)$.

To prove that $l_2(C(H)) \neq C(H, H^\infty)$, set $\alpha = \frac{1}{2}$ in (2.18). Then A belongs to $C(H, H^\infty)$ and $A \notin l_2(C(H))$.

For $p \leq q < 2$, let us prove that $l_p(S^p) \neq S^p(H, H^\infty)$ and that $S^p(H, H^\infty) \nsubseteq l_q(S^p)$. Set $B_n = n^{-\frac{1}{q}} P_{e_1}$ and consider $B = (B_n)_{n=1}^{\infty}$. Then $Bx = \alpha_1 \sum_{n=1}^{\infty} \oplus n^{-\frac{1}{q}} e_1$, for $x = \sum_{n=1}^{\infty} \alpha_n e_n \in H$, where each $n^{-\frac{1}{q}} e_1$ belongs to the n-th component of H^∞. Hence B is bounded, since $\sum_{n=1}^{\infty} n^{-\frac{2}{q}} < \infty$.

Moreover, $B = e_1 \otimes u$ is a rank one operator in $B(H, H^\infty)$, where $u = \sum_{n=1}^{\infty} \oplus n^{-\frac{1}{q}} e_1 \in H^\infty$. Thus $B \in S^p(H, H^\infty)$, for all $p \in [1, \infty)$, and $B \notin l_q(S^p)$, since

$$\|B\|^q_{l_q(S^p)} = \sum_{n=1}^{\infty} \left\| n^{-\frac{1}{q}} P_{e_1} \right\|^q_p = \sum_{n=1}^{\infty} n^{-1} = \infty.$$

(ii) Let $2 \leq p$ and $A \in l_2(S^p)$. It follows from (2.14) that

$$\|A\|^p_{l_p(S^p)} = \lim_{m\to\infty} \sum_{n=1}^{m} \|A_n\|^p_p \overset{(2.12)}{\leq} \lim_{m\to\infty} \|P_m A\|^p_p \overset{(2.12)}{\leq} \lim_{m\to\infty} \left(\sum_{n=1}^{m} \|A_n\|^2_p \right)^{p/2}$$
$$= (\|A\|^2_{l_2(S^p)})^{p/2} = \|A\|^p_{l_2(S^p)}.$$

As all $\|P_m A\|_p \leq \|A\|_{l_2(S^p)}$, we have from Lemma 2.2(ii) that $A \in S^p(H, H^\infty)$ and $\|A\|_p \leq \|A\|_{l_2(S^p)}$. Hence, by (2.5), $\lim_{m\to\infty} \|P_m A\|_p = \|A\|_p$. Therefore $l_2(S^p) \subseteq S^p(H, H^\infty)$.

Let $A \in S^p(H, H^\infty)$. Then it follows from (2.12) that

$$\sum_{n=1}^{m} \|A_n\|^p_p \leq \|P_m A\|^p_p \overset{(2.5)}{\to} \|A\|^p_p,$$

as $m \to \infty$. Hence $A \in l_p(S^p)$, so that $S^p(H, H^\infty) \subseteq l_p(S^p)$ and (2.17) holds.

For $2 \leq q < p$, let us prove that $l_2(S^p) \neq S^p(H, H^\infty) \nsubseteq l_q(S^p)$. Set $A_n = n^{-\frac{1}{q}} P_{e_n}$ and consider $A = (A_n)_{n=1}^{\infty}$. Then $A \notin l_q(S^p)$ and $A \in S^p(H, H^\infty)$, since (see (2.18))

$$\|A\|^p_p = \sum_{n=1}^{\infty} n^{-\frac{p}{q}} < \infty \text{ and } \|A\|^q_{l_q(S^p)} = \sum_{n=1}^{\infty} (n^{-\frac{1}{q}})^q = \sum_{n=1}^{\infty} n^{-1} = \infty.$$

For $2 < q$, let us prove that $l_p(S^p) \neq S^p(H, H^\infty)$ and $l_q(S^p) \nsubseteq B(H, H^\infty)$ for all p. Set $B_n = n^{-\frac{1}{2}} P_{e_1}$ and consider $B = (B_n)_{n=1}^{\infty}$. Then $B \in l_q(S^p)$ and $B \notin B(H, H^\infty)$, since

$$\|B\|_{l_q(S^p)}^q = \sum_{n=1}^{\infty} \left\| n^{-\frac{1}{2}} P_{e_1} \right\|_p^q = \sum_{n=1}^{\infty} n^{-\frac{q}{2}} < \infty,$$

$$\|Be_1\|^2 = \left\| \sum_{n=1}^{\infty} \oplus n^{-\frac{1}{2}} e_1 \right\|^2 = \sum_{n=1}^{\infty} n^{-1} = \infty,$$

where each $n^{-\frac{1}{q}} e_1$ belongs to the n-th component of H^∞. This proves (iv) and completes the proof of (ii). To prove (iii), repeat the proof of (ii) for $p = 2$. □

3. Action of Operators from $B(H^\infty)$ on $l_q(S^p)$ Spaces

By Theorem 2.4(iii), $l_2(S^2) = S^2(H, H^\infty)$. Hence it is a left $B(H^\infty)$-module. In this section we show that, apart from $l_2(S^2)$, the Banach spaces $l_q(S^p)$ are not left $B(H^\infty)$-modules. We also establish the following analogue of inequality (1.10): for $R \in B(H^\infty)$,

$$\|R\|_{l_p(S^p) \to l_2(S^p)} \le \|R\|_{B(H^\infty)}, \text{ for } 1 \le p \le 2;$$
$$\|R\|_{l_2(S^p) \to l_p(S^p)} \le \|R\|_{B(H^\infty)}, \text{ for } 2 \le p.$$

Each operator R in $B(H^\infty)$ has matrix form $R = (R_{ij})_{i,j=1}^\infty$ with $R_{ij} \in B(H)$. It acts on each $A = (A_n)_{n=1}^\infty$ (consider it as a column) in its domain $D(R)$ in $l_\infty(B(H))$ by

$$RA = (R_{ij})(A_n)_{n=1}^\infty = \left(\sum R_{1j} A_j, \ldots, \sum R_{nj} A_j, \ldots \right). \tag{3.1}$$

The domain $D(R)$ of R consists of $A = (A_n)_{n=1}^\infty$ in $l_\infty(B(H))$ such that $\sum_{j=1}^m R_{nj} A_j \overset{\text{w.o.t.}}{\to} B_n \in B(H)$, for each n, and $(B_n)_{n=1}^\infty \in l_\infty(B(H))$.

Proposition 3.1. (i) $\cap\{D(R) : R \in B(H^\infty)\} = B(H, H^\infty)$.
(ii) *If* $(p, q) \neq (2, 2)$, *then the space* $l_q(S^p)$ *is not a left* $B(H^\infty)$*-module.*

Proof. (i) Set $\mathcal{D} = \cap\{D(R) \colon R \in B(H^\infty)\}$. Then $B(H, H^\infty) \subseteq \mathcal{D}$.

Let $\{Q_n\}_{n=1}^\infty$ be mutually orthogonal projections in $B(H)$ with infinite dimensional ranges satisfying $\sum_{n=1}^\infty Q_n = \mathbf{1}_H$. Let $\{U_n\}_{n=1}^\infty$ be isometries from H onto $Q_n H$. Then

$$U_k^* U_n = \delta_{kn} \mathbf{1}_H, U_n U_n^* = Q_n \text{ and } U_n^* Q_n = U_n^*, \tag{3.2}$$

where $\delta_{kk} = 1, \delta_{kn} = 0$ if $k \neq n$.

The operator $R = (R_{ij})$ on H^∞ such that all $R_{1n} = U_n$ and all $R_{in} = 0$ for $i \ge 2$, belongs to $B(H^\infty)$. The operator $L = (U_n^*)_{n=1}^\infty$ from H to $l_\infty(H)$ belongs to $B(H, H^\infty)$. Indeed, $Lx = (U_n^* x)_{n=1}^\infty \in l_\infty(H)$, for $x \in H$, and, by (3.2),

$$\sum_n \|U_n^* x\|^2 = \sum_n \|U_n^* Q_n x\|^2 \le \sum_n \|Q_n x\|^2 = \|x\|^2.$$

Let $A = (A_n)_{n=1}^\infty \in \mathcal{D}$. Then $RA \in l_\infty(B(H))$, so that, by (3.1), $\sum_{n=1}^m U_n A_n \overset{\text{w.o.t.}}{\longrightarrow} B \in B(H)$. Hence $L\left(\sum_{n=1}^m U_n A_n\right) \overset{\text{w.o.t.}}{\longrightarrow} LB \in B(H, H^\infty)$.

It follows from (3.2) that $P_mA = L\left(\sum_{n=1}^{m} U_nA_n\right)$. Therefore $P_mA \overset{\text{w.o.t.}}{\longrightarrow} LB$. By Lemma 2.2(i), $A = LB \in B(H, H^\infty)$. Thus $\mathcal{D} = B(H, H^\infty)$.

(ii) By (i), all left $B(H^\infty)$-modules lie in $B(H, H^\infty)$. Hence, by Lemma 2.2(iv), we only have to prove that $l_q(S^p)$ is not a left $B(H^\infty)$-module for $q \le 2$.

For $q < 2$, let us prove that $l_q(S^p)$ is not a left $B(H^\infty)$-module. Let $R_{nk} = 0$, for $k > 1$, and $R_{n1} = \alpha_n \mathbf{1}_H$, where $\alpha_n > 0$, $\sum_{n=1}^\infty \alpha_n^2 = 1$ and $\sum_{n=1}^\infty \alpha_n^q = \infty$. The operator $R = (R_{nk}) \in B(H^\infty)$, since

$$\|Rx\|^2 = \|x_1\|^2 \sum_{n=1}^\infty \alpha_n^2 = \|x_1\|^2 \le \|x\|^2 \text{ for } x = (x_n)_{n=1}^\infty \in H^\infty.$$

However, if $A = (A_n)_{n=1}^\infty \in l_q(S^p)$ and $A_1 \ne 0$ then $RA \notin l_q(S^p)$, since

$$\|RA\|_{l_q(S^p)}^q = \sum_{n=1}^\infty \|R_{n1}A_1\|_p^q = \|A_1\|_p^q \sum_{n=1}^\infty \alpha_n^q = \infty.$$

Thus $l_q(S^p)$ is not a left $B(H^\infty)$-module, for $q < 2$ and all $p \in [1, \infty)$.

Let $q = 2$. Let $\{e_n\}_{n=1}^\infty$ be a basis in H, let P_{e_n} be the projections on $\mathbb{C}e_n$ and $\{V_n\}_{n=1}^\infty$ be the partial isometries from $\mathbb{C}e_n$ onto $\mathbb{C}e_1$, i.e.,

$$V_ne_n = e_1, V_ne_j = 0 \text{ for } j \ne n, \text{ and } P_{e_n} = V_n^*V_n.$$

Let $R = (R_{nk}), R_{nk} = 0$, for $k > 1$, and $R_{n1} = V_n$. Then $\|Rx\|^2 = \sum_{n=1}^\infty \|V_nx_1\|^2$ for $x = (x_n)_{n=1}^\infty \in H^\infty$. If $x_1 = \sum_{k=1}^\infty \alpha_k e_k$, then $V_nx_1 = \alpha_n e_1$ and

$$\|Rx\|^2 = \sum_{n=1}^\infty |\alpha_n|^2 = \|x_1\|^2 \le \|x\|^2.$$

Thus $R \in B(H^\infty)$.

For $p \in (2, \infty)$, let us show that $l_2(S^p)$ is not a left $B(H^\infty)$-module. We have $A_1 = \sum_{n=1}^\infty n^{-\frac{1}{2}} P_{e_n} \in S^p(H)$, since $\|A_1\|_p^p = \sum_{n=1}^\infty n^{-\frac{p}{2}} < \infty$. Hence $A = (A_1, 0, 0, \dots) \in l_2(S^p)$. However, $l_2(S^p)$ is not a left $B(H^\infty)$-module, as $RA \notin l_2(S^p)$, because

$$\begin{aligned}\|RA\|_{l_2(S^p)}^2 &= \sum_{n=1}^\infty \|V_nA_1\|_p^2 \overset{(2.1)}{=} \sum_{n=1}^\infty \|A_1^*V_n^*V_nA_1\|_{p/2} \\ &= \sum_{n=1}^\infty \|A_1^*P_{e_n}A_1\|_{p/2} = \sum_{n=1}^\infty \left\|n^{-1}P_{e_n}\right\|_{p/2} = \sum_{n=1}^\infty n^{-1} = \infty.\end{aligned}$$

For $p \in [1, 2)$, let us show that $l_2(S^p)$ is not a left $B(H^\infty)$-module. Set $A = (n^{-\frac{1}{p}}V_n)_{n=1}^\infty$. As $R^* \in B(H^\infty)$ and $\|V_n^*V_n\|_{p/2} = \|P_{e_n}\|_{p/2} = 1$, we have

$$\|A\|^2_{l_2(S^p)} = \sum_{n=1}^{\infty} \|A_n\|_p^2 = \sum_{n=1}^{\infty} n^{-\frac{2}{p}} \|V_n^* V_n\|_{p/2} = \sum_{n=1}^{\infty} n^{-\frac{2}{p}} < \infty \text{ and}$$

$$\|R^* A\|_{l_2(S^p)} = \left\| \sum_{n=1}^{\infty} V_n^* A_n \right\|_p = \left\| \sum_{n=1}^{\infty} n^{-\frac{1}{p}} P_{e_n} \right\|_p = \left(\sum_{n=1}^{\infty} (n^{-\frac{1}{p}})^p \right)^{1/p}$$
$$= \left(\sum_{n=1}^{\infty} n^{-1} \right)^{1/p} = \infty.$$

Hence $A \in l_2(S^p)$ and $R^* A \notin l_2(S^p)$. Thus $l_2(S^p), p \in [1, 2)$, is not a left $B(H^\infty)$ -module. □

By Theorem 2.4, $l_p(S^p) \subset S^p(H, H^\infty) \subset l_2(S^p)$ for $p \in [1, 2]$. Thus the multiplication of the space $l_p(S^p)$ by operators from $B(H^\infty)$ leaves it in $S^p(H, H^\infty)$ and in $l_2(S^p)$, but not in any $l_q(S^p), p \le q < 2$.

Similarly, $l_2(S^p) \subset S^p(H, H^\infty) \subset l_p(S^p)$ for $2 \le p$. Thus the multiplication of the space $l_2(S^p)$ by operators from $B(H^\infty)$ leaves it in $S^p(H, H^\infty)$ and, hence, in $l_p(S^p)$, but not in any $l_q(S^p)$ for $2 < q$.

Theorem 3.2. *Let $R \in B(H^\infty)$.*

(i) *Let $p \in [1, 2]$ and $A \in l_p(S^p)$. Then $RA \in l_2(S^p)$ and*

$$\|RA\|_{l_2(S^p)} \le \|R\|_{B(H^\infty)} \|A\|_{l_p(S^p)} .$$

(ii) *Let $p \in [2, \infty)$ and $A \in l_2(S^p)$. Then $RA \in l_p(S^p)$ and*

$$\|RA\|_{l_p(S^p)} \le \|R\|_{B(H^\infty)} \|A\|_{l_2(S^p)} .$$

Proof. (i) Let $p \in [1, 2]$ and $A \in l_p(S^p)$. It follows from Theorem 2.4(i) that $A \in S^p(H, H^\infty)$. Therefore, by (2.3), RA belongs to $S^p(H, H^\infty)$ and $\|RA\|_p \le \|R\|_{B(H^\infty)} \|A\|_p$. We have from Theorem 2.4(i) that $RA \in l_2(S^p)$, $\|RA\|_{l_2(S^p)} \le \|RA\|_p$ and $\|A\|_p \le \|A\|_{l_p(S^p)}$. Hence

$$\|RA\|_{l_2(S^p)} \le \|RA\|_p \le \|R\|_{B(H^\infty)} \|A\|_p \le \|R\|_{B(H^\infty)} \|A\|_{l_p(S^p)} .$$

Using part (ii) of Theorem 2.4 instead of (i), we obtain similarly the proof of (ii). □

We can use inequality (1.9) to obtain some analogues of McCarthy inequality (1.1) for $l_q(S^p)$ spaces. Let $\{n_k\}_{k=1}^{\infty}$ be positive integers. For $A = (A_n)_{n=1}^{\infty} \in l_q(S^p)$, set

$$B_1 = (A_1, \ldots, A_{n_1}), B_2 = (A_{n_1+1}, \ldots, A_{n_1+n_2}), \ldots,$$
$$B_k = (A_{n_1+\ldots+n_{k-1}+1}, \ldots, A_{n_1+\ldots+n_k}), \ldots$$

Then $B_k \in l_q^{n_k}(S^p)$,

$$A = (B_k)_{k=1}^{\infty} \text{ and } \|A\|_{l_q(S^p)} = \left(\sum_{k=1}^{\infty} \|B_k\|^q_{l_q^{n_k}(S^p)} \right)^{1/q} . \tag{3.3}$$

For each k, let H^{n_k} be the orthogonal sum of n_k copies of H. Then $H^\infty = \oplus_{k=1}^\infty H^{n_k}$. Consider the block-diagonal operator $R = \{R_k\}_{k=1}^\infty$ with the operators $R_k \in B(H^{n_k})$ on the diagonal and $\mathbf{0}$ off the diagonal and suppose that $\alpha := \sup n_k^{\left|\frac{1}{p}-\frac{1}{2}\right|} \|R_k\| < \infty$. Then $R \in B(H^\infty)$.

Theorem 3.3. *Let $p \in [1,\infty)$ and $q \in [\min(p,2), \max(p,2)]$. Then*

$$\|RA\|_{l_q(S^p)} \le \alpha \|A\|_{l_q(S^p)} \text{ for all } A \in l_q(S^p).$$

If $n_k = N$, for some N and all k, then

$$\|RA\|_{l_q(S^p)} \le N^{\left|\frac{1}{p}-\frac{1}{2}\right|} (\sup \|R_k\|) \|A\|_{l_q(S^p)}.$$

Proof. It follows from the block-diagonal structure of the operator R and from (3.3) that

$$RA = (R_k B_k)_{k=1}^\infty \text{ and } \|RA\|_{l_q(S^p)}^q = \sum_{k=1}^\infty \|R_k B_k\|_{l_q^{n_k}(S^p)}^q.$$

By (1.9),

$$\|R_k B_k\|_{l_q^{n_k}(S^p)} \le n_k^{\left|\frac{1}{p}-\frac{1}{2}\right|} \|R_k\| \|B_k\|_{l_q^{n_k}(S^p)}.$$

Substituting this in the above formula, we have

$$\|RA\|_{l_q(S^p)}^q \le \sum_{k=1}^\infty n_k^{q\left|\frac{1}{p}-\frac{1}{2}\right|} \|R_k\|^q \|B_k\|_{l_q^{n_k}(S^p)}^q \overset{(3.3)}{\le} \alpha^q \|A\|_{l_q(S^p)}^q$$

which completes the proof. ∎

Let $n_k = 2$ and $R_k = 2^{-1/2}\begin{pmatrix} \mathbf{1}_H & \mathbf{1}_H \\ \mathbf{1}_H & -\mathbf{1}_H \end{pmatrix}$ be a unitary operator on $H \oplus H$, for all k. Then, for $A = (A_n)_{n=1}^\infty$,

$$RA = 2^{-1/2}(A_1 + A_2, A_1 - A_2, \ldots, A_{2n-1} + A_{2n}, A_{2n-1} - A_{2n}, \ldots).$$

Set $X = (X_n)_{n=1}^\infty$ and $Y = (Y_n)_{n=1}^\infty$, where $X_n = A_{2n-1}$ and $Y_n = A_{2n}$. Then

$$\|A\|_{l_q(S^p)}^q = \sum_{n=1}^\infty \|A_{2n-1}\|_p^q + \sum_{n=1}^\infty \|A_{2n}\|_p^q = \|X\|_{l_q(S^p)}^q + \|Y\|_{l_q(S^p)}^q,$$

$$\begin{aligned}\|RA\|_{l_q(S^p)} &= 2^{-1/2}\left(\sum_{n=1}^\infty \|A_{2n-1} + A_{2n}\|_p^q + \sum_{n=1}^\infty \|A_{2n-1} - A_{2n}\|_p^q\right)^{1/q} \\ &= 2^{-1/2}(\|X+Y\|_{l_q(S^p)}^q + \|X-Y\|_{l_q(S^p)}^q)^{1/q}.\end{aligned}$$

Taking into account that $\|R\| = 1$ and substituting the above formulas in Theorems 3.2 and 3.3, we have the following analogue of McCarthy inequality (1.1) for spaces $l_q(S^p)$.

Corollary 3.4. (i) *Let $p \in [1,2]$ and $X, Y \in l_p(S^p)$. Then*

$$\left(\|X+Y\|_{l_2(S^p)}^2 + \|X-Y\|_{l_2(S^p)}^2\right)^{1/2} \le 2^{\frac{1}{2}} \left(\|X\|_{l_p(S^p)}^p + \|Y\|_{l_p(S^p)}^p\right)^{1/p}.$$

Let $p \in [2,\infty)$ *and* $X, Y \in l_2(S^p)$. *Then*

$$\left(\|X+Y\|^p_{l_p(S^p)} + \|X-Y\|^p_{l_p(S^p)}\right)^{1/p} \le 2^{\frac{1}{2}}\left(\|X\|^2_{l_2(S^p)} + \|Y\|^2_{l_2(S^p)}\right)^{1/2}.$$

(ii) *Let* $p \in [1,\infty), q \in [\min(p,2), \max(p,2)]$ *and* $X, Y \in l_q(S^p)$. *Then*

$$\left(\|X+Y\|^q_{l_q(S^p)} + \|X-Y\|^q_{l_q(S^p)}\right)^{\frac{1}{q}} \le 2^{\left|\frac{1}{p}-\frac{1}{2}\right|+\frac{1}{2}}\left(\|X\|^q_{l_q(S^p)} + \|Y\|^q_{l_q(S^p)}\right)^{\frac{1}{q}}.$$

For a Banach space $(B, \|\cdot\|)$, the modulus of convexity δ_B (see [2,15]) is defined by

$$\delta_B(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|X+Y\| \colon X, Y \in B, \|X\| = \|Y\| = 1, \|X-Y\| \ge \varepsilon\right\},$$

for $0 < \varepsilon \le 2$; and the modulus of smoothness ρ_B by

$$\rho_B(\tau) = \sup\left\{\frac{\|X+\tau Y\| + \|X-\tau Y\|}{2} - 1 \colon X, Y \in B, \|X\| = \|Y\| = 1\right\},$$

for $\tau > 0$. The space B is called *r-uniformly convex*, for $2 \le r < \infty$, if $\delta_B(\varepsilon) \ge C\varepsilon^r$ for some $C > 0$ and all $0 < \varepsilon \le 2$. It is called *r-uniformly smooth*, for $1 < r \le 2$, if $\rho_B(\tau) \le C\tau^r$ for some $C > 0$ and all $\tau > 0$.

Corollary 3.5. (i) *The space* $l_p(S^p)$, *for* $p \in [2,\infty)$, *is p-uniformly convex.*

(ii) *The space* $l_p(S^p)$, *for* $p \in (1,2]$, *is p-uniformly smooth.*

Proof. Set $B = l_p(S^p)$ and $\|\cdot\| = \|\cdot\|_{l_p(S^p)}$.

(i) Let $2 \le p < \infty$ and $X, Y \in B$. Setting $q = p$ in Corollary 3.4(ii), we obtain the p-uniform convexity inequality

$$(\|X+Y\|^p + \|X-Y\|^p)^{\frac{1}{p}} \le 2^{1-\frac{1}{p}}(\|X\|^p + \|Y\|^p)^{\frac{1}{p}}$$

which yields p-uniform convexity of B (see [2,11]). Indeed, (we prove it for the convenience of the reader) let $\|X\| = \|Y\| = 1$ and $\|X-Y\| = \varepsilon > 0$. Then $(\|X+Y\|^p + \varepsilon^p)^{\frac{1}{p}} \le 2$. As $\frac{1}{p} < 1$, we get

$$\left\|\frac{X+Y}{2}\right\| \le \left(1 - \frac{\varepsilon^p}{2^p}\right)^{1/p} \le 1 - \frac{\varepsilon^p}{p2^p}, \text{ so that } \delta_B(\varepsilon) \ge \frac{\varepsilon^p}{p2^p}.$$

(ii) Let $1 < p \le 2$ and $X, Y \in B$. Setting $q = p$ in Corollary 3.4(ii), we obtain the p-uniform smoothness inequality

$$\|X+Y\|^p + \|X-Y\|^p \le 2(\|X\|^p + \|Y\|^p)$$

which yields p-uniform smoothness of B (see [2,11]). Indeed, let $\|X\| = 1$ and $Y = \tau Z$ with $\|Z\| = 1$ and $\tau > 0$. Then

$$\frac{\|X+\tau Z\|^p + \|X-\tau Z\|^p}{2} \le 1 + \tau^p.$$

As $\frac{a+b}{2} \le \left(\frac{a^p+b^p}{2}\right)^{1/p}$ for non-negative a, b, it follows that

$$\frac{\|X+\tau Z\| + \|X-\tau Z\|}{2} \le (1+\tau^p)^{1/p} \le 1 + \frac{\tau^p}{p}.$$

Hence $\rho_B(\tau) \le \frac{\tau^p}{p}$. □

4. Inequality for Partitions of Operators from S^p

A family $\{\mathcal{P}_n\}_{n=1}^N$ of mutually orthogonal projections in $B(H)$ is a *partition of the identity operator* $\mathbf{1}_H$ if

$$\sum_{n=1}^{N} \mathcal{P}_n = \mathbf{1}_H. \tag{4.1}$$

Let $\{\mathcal{P}_n\}_{n=1}^N$ and $\{\mathcal{Q}_m\}_{m=1}^M$ be partitions of $\mathbf{1}_H$. It was proved in [15] that, for $M, N < \infty$, the partition $\mathcal{A} = \{\mathcal{P}_n A \mathcal{Q}_m\}$ of an operator A in $S^p(H)$ satisfies inequalities (1.11), that is, $\mathcal{A}$ belongs to $l_q^{NM}(S^p)$ and

$$(NM)^{\frac{1}{p}-\frac{1}{q}} \|\mathcal{A}\|_{l_q^{MN}(S^p)} \le \|A\|_p \le (NM)^{\frac{1}{2}-\frac{1}{q}} \|\mathcal{A}\|_{l_q^{MN}(S^p)},$$

for $2 \le q \le p < \infty$. For $1 < p \le q \le 2$, the inequalities are reversed.

In this section we study infinite partitions $\mathcal{A} = \{\mathcal{P}_n A \mathcal{Q}_m\}_{n,m=1}^\infty$.

Proposition 4.1. *Let $\{\mathcal{P}_n\}_{n=1}^\infty$ be a partition of $\mathbf{1}_H$. For $A \in S^p(H)$,*

$$\left(\sum_{n=1}^{\infty} \|\mathcal{P}_n A\|_p^2\right)^{1/2} \le \|A\|_p \le \left(\sum_{n=1}^{\infty} \|\mathcal{P}_n A\|_p^p\right)^{1/p}, \ \text{if } 1 \le p \le 2,$$

where the last series may diverge. For $2 \le p < \infty$, the inequalities are reversed.

Proof. Set $A_n = \mathcal{P}_n A$. It follows from (4.1) that

$$\sum_{n=1}^{m} \mathcal{P}_n \overset{\text{s.o.t.}}{\to} \mathbf{1}_H, \text{ as } m \to \infty, \text{ and } \|x\|^2 = \sum_{n=1}^{\infty} \|\mathcal{P}_n x\|^2 \text{ for } x \in H. \tag{4.2}$$

As $A \in S^p(H)$, all A_n belong to $S^p(H)$ and have mutually orthogonal ranges: $A_k^* A_n = A^* \mathcal{P}_k \mathcal{P}_n A = 0$ if $k \ne n$. Consider the operator $\mathcal{A} = (A_n)_{n=1}^\infty$ from H to H^∞. Then

$$\|\mathcal{A}x\|^2 = \sum_{n=1}^{\infty} \|\mathcal{P}_n A x\|^2 \overset{(4.2)}{=} \|Ax\|^2 \text{ for } x \in H.$$

Hence $\mathcal{A} \in B(H, H^\infty)$ and $\|\mathcal{A}\| = \|A\|$. For all $x, y \in H$ and $m \in \mathbb{N}$, we have

$$|(\mathcal{A}^*\mathcal{A}x - A^*Ax, y)| \le \left|\left(\mathcal{A}^*\mathcal{A}x - \sum_{n=1}^{m} A_n^* A_n x, y\right)\right| + |(\mathcal{R}_m Ax - Ax, Ay)|,$$

where $\mathcal{R}_m = \sum_{n=1}^m \mathcal{P}_n$. Since

$$\left(\mathcal{A}^*\mathcal{A}x - \sum_{n=1}^{m} A_n^* A_n x, y\right) \overset{(2.6)}{\longrightarrow} 0 \text{ and } (\mathcal{R}_m Ax - Ax, Ay) \overset{(4.2)}{\longrightarrow} 0,$$

as $m \to \infty$, we have $\mathcal{A}^*\mathcal{A} = A^*A \in S^{p/2}(H)$, so that $\mathcal{A} \in S^p(H, H^\infty)$ and $\|\mathcal{A}\|_p = \|A\|_p$.

Let $1 \le p < 2$. Then it follows from Theorem 2.4(i) that $\mathcal{A} \in l_2(S^p)$ and

$$\left(\sum_{n=1}^{\infty} \|\mathcal{P}_n A\|_p^2\right)^{1/2} \overset{(2.14)}{=} \|\mathcal{A}\|_{l_2(S^p)} \overset{(2.16)}{\le} \|\mathcal{A}\|_p$$

$$= \|A\|_p \overset{(2.16)}{\le} \|\mathcal{A}\|_{l_p(S^p)} \overset{(2.14)}{=} \left(\sum_{n=1}^{\infty} \|\mathcal{P}_n A\|_p^p\right)^{1/p},$$

where the last series above may diverge if $\mathcal{A} \notin l_p(S^p)$.

For $2 \le p < \infty$, using (2.14), (2.17), we obtain the reversed inequalities. □

We consider now partitions of operators. If $A \notin l_q(S^p)$, we assume that $\|A\|_{l_q(S^p)} = \infty$.

Theorem 4.2. *Let $\{\mathcal{P}_n\}_{n=1}^{\infty}$ and $\{\mathcal{Q}_k\}_{k=1}^{\infty}$ be partitions of $\mathbf{1}_H$. For $A \in S^p(H)$, let $\mathcal{A} = \{\mathcal{P}_n A \mathcal{Q}_k\}$ be the corresponding partition of A.*

(i) *If $1 \le p < 2$ then $\mathcal{A} \in l_2(S^p)$ and*

$$\|\mathcal{A}\|_{l_2(S^p)} = \left(\sum_{n,k=1}^{\infty} \|\mathcal{P}_n A \mathcal{Q}_k\|_p^2\right)^{1/2} \le \|A\|_p$$

$$\le \left(\sum_{n,k=1}^{\infty} \|\mathcal{P}_n A \mathcal{Q}_k\|_p^p\right)^{1/p} = \|\mathcal{A}\|_{l_p(S^p)}.$$

(ii) *If $2 \le p$ then $\mathcal{A} \in l_p(S^p)$ and*

$$\|\mathcal{A}\|_{l_p(S^p)} = \left(\sum_{n,k=1}^{\infty} \|\mathcal{P}_n A \mathcal{Q}_k\|_p^p\right)^{1/p} \le \|A\|_p$$

$$\le \left(\sum_{n,k=1}^{\infty} \|\mathcal{P}_n A \mathcal{Q}_k\|_p^2\right)^{1/2} = \|\mathcal{A}\|_{l_2(S^p)}.$$

Proof. (i) Let $1 \le p \le 2$. It follows from Proposition 4.1 that

$$\left(\sum_{n=1}^{\infty} \|\mathcal{P}_n A\|_p^2\right)^{1/2} \le \|A\|_p \le \left(\sum_{n=1}^{\infty} \|\mathcal{P}_n A\|_p^p\right)^{1/p}, \tag{4.3}$$

where the last series above may diverge. Fix n and set $B_n = A^*\mathcal{P}_n$. Then $B_n \in S^p(H)$. Replacing in (4.3), A by B_n and $\{\mathcal{P}_n\}_{n=1}^{\infty}$ by $\{\mathcal{Q}_k\}_{k=1}^{\infty}$, we obtain

$$\left(\sum_{k=1}^{\infty} \|\mathcal{Q}_k B_n\|_p^2\right)^{1/2} \le \|B_n\|_p \le \left(\sum_{k=1}^{\infty} \|\mathcal{Q}_k B_n\|_p^p\right)^{1/p}, \tag{4.4}$$

where the last series above may diverge. Since, by (2.3), $\|B_n\|_p = \|B_n^*\|_p = \|\mathcal{P}_n A\|_p$ and

$$\|\mathcal{Q}_k B_n\|_p = \|\mathcal{Q}_k A^*\mathcal{P}_n\|_p = \|(\mathcal{Q}_k A^*\mathcal{P}_n)^*\|_p = \|\mathcal{P}_n A \mathcal{Q}_k\|_p,$$

we can rewrite (4.4) as follows

$$\left(\sum_{k=1}^{\infty}\|\mathcal{P}_nA\mathcal{Q}_k\|_p^2\right)^{1/2} \le \|\mathcal{P}_nA\|_p \le \left(\sum_{k=1}^{\infty}\|\mathcal{P}_nA\mathcal{Q}_k\|_p^p\right)^{1/p}, \text{ for each } n.$$

Substituting this into (4.3) and using (2.14), we complete the proof of (i).

(ii) Let $2 \le p$. From Proposition 4.1 we have

$$\left(\sum_{n=1}^{\infty}\|\mathcal{P}_nA\|_p^p\right)^{1/p} \le \|A\|_p \le \left(\sum_{n=1}^{\infty}\|\mathcal{P}_nA\|_p^2\right)^{1/2}.$$

Proceeding now, as in part (i), we complete the proof. □

5. Cartesian Decomposition and Schatten Norms

Define the following natural involution $\sharp$ on $l_\infty(B(H))$:

$$A^\sharp = (A_n^*)_{n=1}^\infty \text{ for each } A = (A_n)_{n=1}^\infty \in l_\infty(B(H)).$$

It follows from (2.3) that $\sharp$ preserves all spaces $l_q(S^p)$, as $\sharp^2 = \mathbf{1}$. Moreover, all $l_q(S^p)$ are symmetrically normed ideals of the C*-algebra $l_\infty(B(H))$ and

$$\|A\|_{l_q(S^p)} = \left\|A^\sharp\right\|_{l_q(S^p)}, \text{ for all } A \in l_q(S^p). \tag{5.1}$$

For each n, consider the selfadjoint operators $X_n = \frac{1}{2}(A_n + A_n^*)$ and $Y_n = \frac{1}{2i}(A_n - A_n^*)$. Set $X = (X_n)_{n=1}^\infty$ and $Y = (Y_n)_{n=1}^\infty$, so that

$$X = (A + A^\sharp)/2 \text{ and } Y = (A - A^\sharp)/2i. \tag{5.2}$$

Then $A = X + iY$ is the "Cartesian decomposition" of A. If $A \in l_q(S^p)$ then $X, Y \in l_q(S^p)$.

Corollary 5.1. *Let $A \in l_q(S^p)$, where $p \in [1,\infty)$ and $q \in [\min(p,2), \max(p,2)]$. Then*

$$2^{\frac{1}{q}-\frac{1}{2}-\left|\frac{1}{p}-\frac{1}{2}\right|}\|A\|_{l_q(S^p)} \le \left(\|X\|_{l_q(S^p)}^q + \|Y\|_{l_q(S^p)}^q\right)^{1/q} \le 2^{\frac{1}{q}-\frac{1}{2}+\left|\frac{1}{p}-\frac{1}{2}\right|}\|A\|_{l_q(S^p)}. \tag{5.3}$$

Proof. Replace Y by iY in Corollary 3.4(ii) and replace consequently $X+iY$ by A and $X - iY$ by $A^\sharp$. Using (5.1), we obtain the left-hand side inequality in (5.3). Replace now X by A and Y by $A^\sharp$ in Corollary 3.4(ii). Using (5.2) and (5.1), we obtain the right-hand side inequality in (5.3). □

Remark 5.2. *Doing the same replacements in Corollary 3.4(i) as in Corollary 5.1, we obtain*

$$\|A\|_{l_2(S^p)}^p \le \|X\|_{l_p(S^p)}^p + \|Y\|_{l_p(S^p)}^p, \tag{5.4}$$

$$\|X\|_{l_2(S^p)}^2 + \|Y\|_{l_2(S^p)}^2 \le 2^{\frac{2}{p}-1}\|A\|_{l_p(S^p)}^2, \tag{5.5}$$

for $p \in [1,2]$ *and* $A \in l_p(S^p)$. *If* $p \in [2,\infty)$ *and* $A \in l_2(S^p)$, *then*

$$\|A\|^2_{l_p(S^p)} \le 2^{1-\frac{2}{p}} \left(\|X\|^2_{l_2(S^p)} + \|Y\|^2_{l_2(S^p)}\right), \tag{5.6}$$

$$\|X\|^p_{l_p(S^p)} + \|Y\|^p_{l_p(S^p)} \le \|A\|^p_{l_2(S^p)}. \tag{5.7}$$

However, they can be deduced from (5.3). *For* $p \in [1,2]$, *set* $q = p$ *in the first inequality in* (5.3) *and* $q = 2$ *in the second inequality in* (5.3). *Using that* $\|A\|_{l_2(S^p)} \le \|A\|_{l_p(S^p)}$, *by* (1.6), *we get* (5.4) *and* (5.5).

If $p \in [2,\infty)$ *then* $\|A\|_{l_p(S^p)} \le \|A\|_{l_2(S^p)}$, *by* (1.6). *Setting* $q = 2$ *in the first inequality in* (5.3) *and* $q = p$ *in the second inequality, we obtain* (5.6) *and* (5.7). □

Although the involution $\sharp$ preserves all spaces $l_q(S^p)$, it does not preserve $S^p(H,H^\infty)$, if $p \ne 2$. Set $S^{\mathrm{b}}(H,H^\infty) = B(H,H^\infty)$ and $S^\infty(H,H^\infty) = C(H,H^\infty)$. Set also

$$D_p(\sharp) = \{A \in S^p(H,H^\infty) \colon A^\sharp \in S^p(H,H^\infty)\}, \text{ for each } p \in [1,\infty] \cup \mathrm{b}.$$

Then $\sharp$ preserves $D_p(\sharp)$. Indeed, if $A \in D_p(\sharp)$ then $A^\sharp \in S^p(H,H^\infty)$ and $A^{\sharp\sharp} = A \in S^p(H,H^\infty)$. Thus $A^\sharp \in D_p(\sharp)$.

Proposition 5.3. (i) *If* $1 \le p < 2$ *then* $l_p(S^p) \subset D_p(\sharp) \subset S^p(H,H^\infty)$.

(ii) $S^2(H,H^\infty) = D_2(\sharp)$. *If* $2 < p$ *then* $D_p(\sharp) \subset S^p(H,H^\infty) \not\subseteq D_{\mathrm{b}}(\sharp)$.

Proof. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis in H, let P_{e_n} be the projections on $\mathbb{C}e_n$ and let $\{V_n\}_{n=1}^\infty$ be the partial isometries from $\mathbb{C}e_n$ on $\mathbb{C}e_1$: $V_n e_n = e_1$ and $V_n e_j = 0$, for $j \ne n$. Then

$$P_{e_n} = V_n^* V_n \text{ and } P_{e_1} = V_n V_n^*. \tag{5.8}$$

(i) Let $1 \le p < 2$. By Theorem 2.4(i), $l_p(S^p) \subset S^p(H,H^\infty)$ and $l_2(C(H)) \subset C(H,H^\infty)$. As $\sharp$ preserves all $l_q(S^p)$, we get all the inclusions and only need to prove $l_p(S^p) \ne D_p(\sharp) \ne S^p(H,H^\infty)$.

To prove that $l_p(S^p) \ne D_p(\sharp)$, set $A_n = n^{-\frac{1}{p}} P_{e_1}$ and $A = (A_n)_{n=1}^\infty$. Then $A \notin l_p(S^p)$, since $\|A\|^p_{l_p(S^p)} = \sum_{n=1}^\infty n^{-1} = \infty$. On the other hand, for $x = \sum_{n=1}^\infty \alpha_n e_n \in H$,

$$Ax = \sum_{n=1}^\infty \oplus n^{-\frac{1}{p}} P_{e_1} x = \alpha_1 \sum_{n=1}^\infty \oplus n^{-\frac{1}{p}} e_1 = \alpha_1 u, \text{ where } u = \sum_{n=1}^\infty \oplus n^{-\frac{1}{p}} e_1$$

and each $n^{-\frac{1}{p}} e_1$ lies in the n-th component of H^∞. Then u belongs to H^∞, as $\|u\|^2 = \sum_{n=1}^\infty n^{-\frac{2}{p}} < \infty$. Hence $A = e_1 \otimes u$ is a rank one operator. Thus $A \in S^p(H,H^\infty)$ (see (2.15)) and $A \in D_p(\sharp)$, as $A^\sharp = A$.

To prove that $D_p(\sharp) \ne S^p(H,H^\infty)$, set $A_n = n^{-\frac{1}{p}} V_n^*$ and $A = (A_n)_{n=1}^\infty$. By (5.8), for all $m \in \mathbb{N}$,

$$\|P_m A\|_p^2 \overset{(2.7)}{=} \left\|\sum_{n=1}^m A_n^* A_n\right\|_{p/2} = \left\|\sum_{n=1}^m n^{-\frac{2}{p}} V_n V_n^*\right\|_{p/2}$$

$$= \left\|\left(\sum_{n=1}^m n^{-\frac{2}{p}}\right) P_{e_1}\right\|_{p/2} = \sum_{n=1}^m n^{-\frac{2}{p}} < \infty.$$

Hence it follows from Lemma 2.2(ii) that $A \in S^p(H, H^\infty)$. On the other hand,

$$\left\|P_m A^\sharp\right\|_p^p \overset{(2.7)}{=} \left\|\sum_{n=1}^m n^{-\frac{2}{p}} V_n^* V_n\right\|_{p/2}^{p/2} \overset{(5.8)}{=} \left\|\sum_{n=1}^m n^{-\frac{2}{p}} P_{e_n}\right\|_{p/2}^{p/2} = \sum_{n=1}^m n^{-1} \to \infty,$$

as $m \to \infty$. Hence, by Lemma 2.2(ii), $A^\sharp \notin S^p(H, H^\infty)$.

(ii) As $S^2(H, H^\infty) = l_2(S^2)$, we have $S^2(H, H^\infty) = D_2(\sharp)$.

Let $p > 2$. Set $A_n = n^{-\frac{1}{2}} V_n$ and consider $A = (A_n)_{n=1}^\infty$. By (5.8),

$$\|P_m A\|_p^2 \overset{(2.7)}{=} \left\|\sum_{n=1}^m n^{-1} V_n^* V_n\right\|_{p/2} = \left\|\sum_{n=1}^m n^{-1} P_{e_n}\right\|_{p/2} = \left(\sum_{n=1}^m n^{-\frac{p}{2}}\right)^{2/p} < \infty,$$

for all m. Therefore, by Lemma 2.2(ii), $A \in S^p(H, H^\infty)$. On the other hand, if $m \to \infty$ then

$$\begin{aligned}\left\|P_m A^\sharp\right\|_p^2 \overset{(2.7)}{=} \left\|\sum_{n=1}^m A_n A_n^*\right\|_{p/2} &= \left\|\sum_{n=1}^m n^{-1} V_n V_n^*\right\|_{p/2} \\ &= \left\|\left(\sum_{n=1}^m n^{-1}\right) P_{e_1}\right\|_{p/2} = \sum_{n=1}^m n^{-1} \to \infty.\end{aligned}$$

Therefore, by Lemma 2.2(ii), $A^\sharp \notin S^p(H, H^\infty)$. Thus $A \notin D_p(\sharp)$, so that $D_p(\sharp) \subsetneqq S^p(H, H^\infty)$. Making use of Lemma 2.2(i), one can show that, in fact, $A^\sharp \notin B(H, H^\infty)$. Hence $S^p(H, H^\infty) \nsubseteq D_{\mathrm{b}}(\sharp)$. □

Let $A = (A_n)_{n=1}^\infty \in D_p(\sharp)$. Then $A^\sharp \in D_p(\sharp)$ and (see (5.2)) $X, Y \in D_p(\sharp)$. As $A_n = X_n + iY_n$,

$$\begin{aligned}|A|^2 + \left|A^\sharp\right|^2 = A^*A + (A^\sharp)^* A^\sharp &\overset{(2.13)}{=} \lim_{m\to\infty} \sum_{n=1}^m (A_n^* A_n + A_n A_n^*) \\ &= 2 \lim_{m\to\infty} \sum_{n=1}^m (X_n^2 + Y_n^2) \overset{(2.13)}{=} 2(X^*X + Y^*Y).\end{aligned} \tag{5.9}$$

Theorem 5.4. *Let $A = (A_n)_{n=1}^\infty \in D_p(\sharp)$ and let all $A_n = X_n + iY_n$. If $1 \le p \le 2$ then*

$$\begin{aligned}\|A\|_{l_2(S^p)} &\le \lim_{m\to\infty} \left\|\left(\sum_{n=1}^m (X_n^2 + Y_n^2)\right)^{1/2}\right\|_p \\ &= 2^{-\frac{1}{2}} \left\|\left(|A|^2 + \left|A^\sharp\right|^2\right)^{1/2}\right\|_p \le 2^{\frac{1}{p}-\frac{1}{2}} \|A\|_{l_p(S^p)},\end{aligned}$$

where $\|A\|_{l_p(S^p)} = \infty$ if $A \notin l_p(S^p)$. For $2 \le p < \infty$, the inequalities are reversed.

Proof. Consider $B = (B_n)_{n=1}^\infty$ with $B_{2j} = A_j$ and $B_{2j-1} = A_j^*$. Let H_k be the k-th component H in H^∞. Set $\mathcal{H}_1 = \oplus_{j=1}^\infty H_{2j-1}$ and $\mathcal{H}_2 = \oplus_{j=1}^\infty H_{2j}$. Then $H^\infty = \mathcal{H}_1 \oplus \mathcal{H}_2$ and with respect to this decomposition, $B = A^\sharp \oplus A$

and $B^\sharp = A \oplus A^\sharp$. As $A \in D_p(\sharp))$, we have $A, A^\sharp \in S^p(H, H^\infty)$. Hence $A^\sharp \in S^p(H, \mathcal{H}_1), A \in S^p(H, \mathcal{H}_2)$. Then, by (2.1), $(A^\sharp)^* A^\sharp, A^* A \in S^{p/2}(H)$. Therefore $B^* B = (A^\sharp)^* A^\sharp + A^* A \in S^{p/2}(H)$. Then, by (2.1), $B \in S^p(H, H^\infty)$.

Similarly, we have $B^\sharp \in S^p(H, H^\infty)$. Hence $B \in D_p(\sharp)$. We also have

$$B^*_{2n-1} B_{2n-1} + B^*_{2n} B_{2n} = A_n A_n^* + A_n^* A_n = 2(X_n^2 + Y_n^2) \in S^{p/2}(H).$$

Set $T_m^2 = \sum_{n=1}^m (X_n^2 + Y_n^2)$. Then $T_m \in S^p(H)$ and, by (2.1), $\left\|T_m^2\right\|_{p/2} = \|T_m\|_p^2$. This yields

$$\|B\|_p^2 \overset{(2.1)}{=} \|B^* B\|_{p/2} \overset{(2.13)}{=} \lim_{m\to\infty} \left\| \sum_{n=1}^m (B^*_{2n-1} B_{2n-1} + B^*_{2n} B_{2n}) \right\|_{p/2}$$
$$= 2 \lim_{m\to\infty} \left\| \sum_{n=1}^m (X_n^2 + Y_n^2) \right\|_{p/2} = 2 \lim_{m\to\infty} \left\|T_m^2\right\|_{p/2} = 2 \lim_{m\to\infty} \|T_m\|_p^2 . \tag{5.10}$$

As $\|B_{2n}\|_p = \|A_n\|_p = \|A_n^*\|_p = \|B_{2n-1}\|_p$, we have, for each q,

$$\|B\|^q_{l_q(S^p)} = \sum_{n=1}^\infty \left(\|B_{2n}\|_p^q + \|B_{2n-1}\|_p^q \right)$$
$$= \sum_{j=1}^\infty \|A_n\|_p^q + \sum_{j=1}^\infty \|A_n^*\|_p^q = 2\, \|A\|^q_{l_q(S^p)} . \tag{5.11}$$

Let $1 \le p \le 2$. Substituting $q = 2$ and $q = p$ in (5.11), we obtain

$$2^{\frac{1}{2}} \|A\|_{l_2(S^p)} \overset{(5.11)}{=} \|B\|_{l_2(S^p)} \overset{(2.16)}{\le} \|B\|_p \overset{(5.10)}{=} 2^{1/2} \lim_{m\to\infty} \|T_m\|_p$$
$$= 2^{1/2} \lim_{m\to\infty} \left\| \left(\sum_{j=1}^m (X_j^2 + Y_j^2) \right)^{1/2} \right\|_p \overset{(5.10)}{=} \|B\|_p \overset{(2.16)}{\le} \|B\|_{l_p(S^p)}$$
$$\overset{(5.11)}{=} 2^{\frac{1}{p}} \|A\|_{l_p(S^p)} .$$

Making use of (5.9), we complete the proof in the case when $1 \le p \le 2$.

To prove the reversed inequality in the case $2 \le p < \infty$, use (2.17) instead of (2.16). □

Acknowledgements

The authors are extremely grateful to the reviewer for his valuable and perceptive comments. They are also grateful for bringing to their attention the fact (see Corollary 3.5(ii)) that Corollary 3.4(ii) implies that the spaces $l_p(S^p), p \in (1, 2]$, are p-uniformly smooth.

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Received: January 16, 2013.
Revised: March 19, 2014.